INSTRUCTOR'S SOLUTIONS MANUAL
MULTIVARIABLE

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THOMAS' CALCULUS
TWELFTH EDITION

BASED ON THE ORIGINAL WORK BY

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PREFACE TO THE INSTRUCTOR

This Instructor's Solutions Manual contains the solutions to every exercise in the 12th Edition of THOMAS' CALCULUS by Maurice Weir and Joel Hass, including the Computer Algebra System (CAS) exercises. The corresponding Student's Solutions Manual omits the solutions to the even-numbered exercises as well as the solutions to the CAS exercises (because the CAS command templates would give them all away).

In addition to including the solutions to all of the new exercises in this edition of Thomas, we have carefully revised or rewritten every solution which appeared in previous solutions manuals to ensure that each solution

- conforms exactly to the methods, procedures and steps presented in the text
- is mathematically correct
- includes all of the steps necessary so a typical calculus student can follow the logical argument and algebra
- includes a graph or figure whenever called for by the exercise, or if needed to help with the explanation
- is formatted in an appropriate style to aid in its understanding

Every CAS exercise is solved in both the MAPLE and *MATHEMATICA* computer algebra systems. A template showing an example of the CAS commands needed to execute the solution is provided for each exercise type. Similar exercises within the text grouping require a change only in the input function or other numerical input parameters associated with the problem (such as the interval endpoints or the number of iterations).

For more information about other resources available with Thomas' Calculus, visit http://pearsonhighered.com.

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CHAPTER 10 INFINITE SEQUENCES AND SERIES

10.1 SEQUENCES

1. $a_1 = \frac{1-1}{1^2} = 0$, $a_2 = \frac{1-2}{2^2} = -\frac{1}{4}$, $a_3 = \frac{1-3}{3^2} = -\frac{2}{9}$, $a_4 = \frac{1-4}{4^2} = -\frac{3}{16}$

2. $a_1 = \frac{1}{1!} = 1$, $a_2 = \frac{1}{2!} = \frac{1}{2}$, $a_3 = \frac{1}{3!} = \frac{1}{6}$, $a_4 = \frac{1}{4!} = \frac{1}{24}$

3. $a_1 = \frac{(-1)^2}{2-1} = 1$, $a_2 = \frac{(-1)^3}{4-1} = -\frac{1}{3}$, $a_3 = \frac{(-1)^4}{6-1} = \frac{1}{5}$, $a_4 = \frac{(-1)^5}{8-1} = -\frac{1}{7}$

4. $a_1 = 2 + (-1)^1 = 1$, $a_2 = 2 + (-1)^2 = 3$, $a_3 = 2 + (-1)^3 = 1$, $a_4 = 2 + (-1)^4 = 3$

5. $a_1 = \frac{2}{2^2} = \frac{1}{2}$, $a_2 = \frac{2^2}{2^3} = \frac{1}{2}$, $a_3 = \frac{2^3}{2^4} = \frac{1}{2}$, $a_4 = \frac{2^4}{2^5} = \frac{1}{2}$

6. $a_1 = \frac{2-1}{2} = \frac{1}{2}$, $a_2 = \frac{2^2-1}{2^2} = \frac{3}{4}$, $a_3 = \frac{2^3-1}{2^3} = \frac{7}{8}$, $a_4 = \frac{2^4-1}{2^4} = \frac{15}{16}$

7. $a_1 = 1$, $a_2 = 1 + \frac{1}{2} = \frac{3}{2}$, $a_3 = \frac{3}{2} + \frac{1}{2^2} = \frac{7}{4}$, $a_4 = \frac{7}{4} + \frac{1}{2^3} = \frac{15}{8}$, $a_5 = \frac{15}{8} + \frac{1}{2^4} = \frac{31}{16}$, $a_6 = \frac{63}{32}$,
   $a_7 = \frac{127}{64}$, $a_8 = \frac{255}{128}$, $a_9 = \frac{511}{256}$, $a_{10} = \frac{1023}{512}$

8. $a_1 = 1$, $a_2 = \frac{1}{2}$, $a_3 = \frac{\left(\frac{1}{2}\right)}{3} = \frac{1}{6}$, $a_4 = \frac{\left(\frac{1}{6}\right)}{4} = \frac{1}{24}$, $a_5 = \frac{\left(\frac{1}{24}\right)}{5} = \frac{1}{120}$, $a_6 = \frac{1}{720}$, $a_7 = \frac{1}{5040}$, $a_8 = \frac{1}{40{,}320}$,
   $a_9 = \frac{1}{362{,}880}$, $a_{10} = \frac{1}{3{,}628{,}800}$

9. $a_1 = 2$, $a_2 = \frac{(-1)^2(2)}{2} = 1$, $a_3 = \frac{(-1)^3(1)}{2} = -\frac{1}{2}$, $a_4 = \frac{(-1)^4\left(-\frac{1}{2}\right)}{2} = -\frac{1}{4}$, $a_5 = \frac{(-1)^5\left(-\frac{1}{4}\right)}{2} = \frac{1}{8}$,
   $a_6 = \frac{1}{16}$, $a_7 = -\frac{1}{32}$, $a_8 = -\frac{1}{64}$, $a_9 = \frac{1}{128}$, $a_{10} = \frac{1}{256}$

10. $a_1 = -2$, $a_2 = \frac{1\cdot(-2)}{2} = -1$, $a_3 = \frac{2\cdot(-1)}{3} = -\frac{2}{3}$, $a_4 = \frac{3\cdot\left(-\frac{2}{3}\right)}{4} = -\frac{1}{2}$, $a_5 = \frac{4\cdot\left(-\frac{1}{2}\right)}{5} = -\frac{2}{5}$, $a_6 = -\frac{1}{3}$,
    $a_7 = -\frac{2}{7}$, $a_8 = -\frac{1}{4}$, $a_9 = -\frac{2}{9}$, $a_{10} = -\frac{1}{5}$

11. $a_1 = 1$, $a_2 = 1$, $a_3 = 1 + 1 = 2$, $a_4 = 2 + 1 = 3$, $a_5 = 3 + 2 = 5$, $a_6 = 8$, $a_7 = 13$, $a_8 = 21$, $a_9 = 34$, $a_{10} = 55$

12. $a_1 = 2$, $a_2 = -1$, $a_3 = -\frac{1}{2}$, $a_4 = \frac{\left(-\frac{1}{2}\right)}{-1} = \frac{1}{2}$, $a_5 = \frac{\left(\frac{1}{2}\right)}{\left(-\frac{1}{2}\right)} = -1$, $a_6 = -2$, $a_7 = 2$, $a_8 = -1$, $a_9 = -\frac{1}{2}$, $a_{10} = \frac{1}{2}$

13. $a_n = (-1)^{n+1}$, $n = 1, 2, \ldots$

14. $a_n = (-1)^n$, $n = 1, 2, \ldots$

15. $a_n = (-1)^{n+1} n^2$, $n = 1, 2, \ldots$

16. $a_n = \frac{(-1)^{n+1}}{n^2}$, $n = 1, 2, \ldots$

17. $a_n = \frac{2^{n-1}}{3(n+2)}$, $n = 1, 2, \ldots$

18. $a_n = \frac{2n-5}{n(n+1)}$, $n = 1, 2, \ldots$

19. $a_n = n^2 - 1$, $n = 1, 2, \ldots$

20. $a_n = n - 4$, $n = 1, 2, \ldots$

21. $a_n = 4n - 3$, $n = 1, 2, \ldots$

22. $a_n = 4n - 2$, $n = 1, 2, \ldots$

23. $a_n = \frac{3n+2}{n!}$, $n = 1, 2, \ldots$

24. $a_n = \frac{n^3}{5^{n+1}}$, $n = 1, 2, \ldots$

25. $a_n = \frac{1+(-1)^{n+1}}{2}$, n = 1, 2, ...

26. $a_n = \frac{n - \frac{1}{2} + (-1)^n \left(\frac{1}{2}\right)}{2} = \lfloor \frac{n}{2} \rfloor$, n = 1, 2, ...

27. $\lim_{n \to \infty} 2 + (0.1)^n = 2 \Rightarrow$ converges (Theorem 5, #4)

28. $\lim_{n \to \infty} \frac{n+(-1)^n}{n} = \lim_{n \to \infty} 1 + \frac{(-1)^n}{n} = 1 \Rightarrow$ converges

29. $\lim_{n \to \infty} \frac{1-2n}{1+2n} = \lim_{n \to \infty} \frac{\left(\frac{1}{n}\right)-2}{\left(\frac{1}{n}\right)+2} = \lim_{n \to \infty} \frac{-2}{2} = -1 \Rightarrow$ converges

30. $\lim_{n \to \infty} \frac{2n+1}{1-3\sqrt{n}} = \lim_{n \to \infty} \frac{2\sqrt{n} + \left(\frac{1}{\sqrt{n}}\right)}{\left(\frac{1}{\sqrt{n}} - 3\right)} = -\infty \Rightarrow$ diverges

31. $\lim_{n \to \infty} \frac{1-5n^4}{n^4+8n^3} = \lim_{n \to \infty} \frac{\left(\frac{1}{n^4}\right)-5}{1+\left(\frac{8}{n}\right)} = -5 \Rightarrow$ converges

32. $\lim_{n \to \infty} \frac{n+3}{n^2+5n+6} = \lim_{n \to \infty} \frac{n+3}{(n+3)(n+2)} = \lim_{n \to \infty} \frac{1}{n+2} = 0 \Rightarrow$ converges

33. $\lim_{n \to \infty} \frac{n^2-2n+1}{n-1} = \lim_{n \to \infty} \frac{(n-1)(n-1)}{n-1} = \lim_{n \to \infty} (n-1) = \infty \Rightarrow$ diverges

34 $\lim_{n \to \infty} \frac{1-n^3}{70-4n^2} = \lim_{n \to \infty} \frac{\left(\frac{1}{n^2}\right)-n}{\left(\frac{70}{n^2}\right)-4} = \infty \Rightarrow$ diverges

35. $\lim_{n \to \infty} (1+(-1)^n)$ does not exist $\Rightarrow$ diverges

36. $\lim_{n \to \infty} (-1)^n \left(1-\frac{1}{n}\right)$ does not exist $\Rightarrow$ diverges

37. $\lim_{n \to \infty} \left(\frac{n+1}{2n}\right)\left(1-\frac{1}{n}\right) = \lim_{n \to \infty} \left(\frac{1}{2}+\frac{1}{2n}\right)\left(1-\frac{1}{n}\right) = \frac{1}{2} \Rightarrow$ converges

38. $\lim_{n \to \infty} \left(2-\frac{1}{2^n}\right)\left(3+\frac{1}{2^n}\right) = 6 \Rightarrow$ converges

39. $\lim_{n \to \infty} \frac{(-1)^{n+1}}{2n-1} = 0 \Rightarrow$ converges

40. $\lim_{n \to \infty} \left(-\frac{1}{2}\right)^n = \lim_{n \to \infty} \frac{(-1)^n}{2^n} = 0 \Rightarrow$ converges

41. $\lim_{n \to \infty} \sqrt{\frac{2n}{n+1}} = \sqrt{\lim_{n \to \infty} \frac{2n}{n+1}} = \sqrt{\lim_{n \to \infty} \left(\frac{2}{1+\frac{1}{n}}\right)} = \sqrt{2} \Rightarrow$ converges

42. $\lim_{n \to \infty} \frac{1}{(0.9)^n} = \lim_{n \to \infty} \left(\frac{10}{9}\right)^n = \infty \Rightarrow$ diverges

43. $\lim_{n \to \infty} \sin\left(\frac{\pi}{2}+\frac{1}{n}\right) = \sin\left(\lim_{n \to \infty} \left(\frac{\pi}{2}+\frac{1}{n}\right)\right) = \sin\frac{\pi}{2} = 1 \Rightarrow$ converges

44. $\lim_{n \to \infty} n\pi \cos(n\pi) = \lim_{n \to \infty} (n\pi)(-1)^n$ does not exist $\Rightarrow$ diverges

45. $\lim_{n \to \infty} \frac{\sin n}{n} = 0$ because $-\frac{1}{n} \le \frac{\sin n}{n} \le \frac{1}{n} \Rightarrow$ converges by the Sandwich Theorem for sequences

46. $\lim_{n \to \infty} \frac{\sin^2 n}{2^n} = 0$ because $0 \le \frac{\sin^2 n}{2^n} \le \frac{1}{2^n} \Rightarrow$ converges by the Sandwich Theorem for sequences

47. $\lim_{n \to \infty} \frac{n}{2^n} = \lim_{n \to \infty} \frac{1}{2^n \ln 2} = 0 \Rightarrow$ converges (using l'Hôpital's rule)

48. $\lim_{n\to\infty} \frac{3^n}{n^3} = \lim_{n\to\infty} \frac{3^n \ln 3}{3n^2} = \lim_{n\to\infty} \frac{3^n(\ln 3)^2}{6n} = \lim_{n\to\infty} \frac{3^n(\ln 3)^3}{6} = \infty \Rightarrow$ diverges (using l'Hôpital's rule)

49. $\lim_{n\to\infty} \frac{\ln(n+1)}{\sqrt{n}} = \lim_{n\to\infty} \frac{\left(\frac{1}{n+1}\right)}{\left(\frac{1}{2\sqrt{n}}\right)} = \lim_{n\to\infty} \frac{2\sqrt{n}}{n+1} = \lim_{n\to\infty} \frac{\left(\frac{2}{\sqrt{n}}\right)}{1+\left(\frac{1}{n}\right)} = 0 \Rightarrow$ converges

50. $\lim_{n\to\infty} \frac{\ln n}{\ln 2n} = \lim_{n\to\infty} \frac{\left(\frac{1}{n}\right)}{\left(\frac{2}{2n}\right)} = 1 \Rightarrow$ converges

51. $\lim_{n\to\infty} 8^{1/n} = 1 \Rightarrow$ converges (Theorem 5, #3)

52. $\lim_{n\to\infty} (0.03)^{1/n} = 1 \Rightarrow$ converges (Theorem 5, #3)

53. $\lim_{n\to\infty} \left(1+\frac{7}{n}\right)^n = e^7 \Rightarrow$ converges (Theorem 5, #5)

54. $\lim_{n\to\infty} \left(1-\frac{1}{n}\right)^n = \lim_{n\to\infty} \left[1+\frac{(-1)}{n}\right]^n = e^{-1} \Rightarrow$ converges (Theorem 5, #5)

55. $\lim_{n\to\infty} \sqrt[n]{10n} = \lim_{n\to\infty} 10^{1/n} \cdot n^{1/n} = 1 \cdot 1 = 1 \Rightarrow$ converges (Theorem 5, #3 and #2)

56. $\lim_{n\to\infty} \sqrt[n]{n^2} = \lim_{n\to\infty} \left(\sqrt[n]{n}\right)^2 = 1^2 = 1 \Rightarrow$ converges (Theorem 5, #2)

57. $\lim_{n\to\infty} \left(\frac{3}{n}\right)^{1/n} = \frac{\lim_{n\to\infty} 3^{1/n}}{\lim_{n\to\infty} n^{1/n}} = \frac{1}{1} = 1 \Rightarrow$ converges (Theorem 5, #3 and #2)

58. $\lim_{n\to\infty} (n+4)^{1/(n+4)} = \lim_{x\to\infty} x^{1/x} = 1 \Rightarrow$ converges; (let $x = n+4$, then use Theorem 5, #2)

59. $\lim_{n\to\infty} \frac{\ln n}{n^{1/n}} = \frac{\lim_{n\to\infty} \ln n}{\lim_{n\to\infty} n^{1/n}} = \frac{\infty}{1} = \infty \Rightarrow$ diverges (Theorem 5, #2)

60. $\lim_{n\to\infty} [\ln n - \ln(n+1)] = \lim_{n\to\infty} \ln\left(\frac{n}{n+1}\right) = \ln\left(\lim_{n\to\infty} \frac{n}{n+1}\right) = \ln 1 = 0 \Rightarrow$ converges

61. $\lim_{n\to\infty} \sqrt[n]{4^n n} = \lim_{n\to\infty} 4\sqrt[n]{n} = 4 \cdot 1 = 4 \Rightarrow$ converges (Theorem 5, #2)

62. $\lim_{n\to\infty} \sqrt[n]{3^{2n+1}} = \lim_{n\to\infty} 3^{2+(1/n)} = \lim_{n\to\infty} 3^2 \cdot 3^{1/n} = 9 \cdot 1 = 9 \Rightarrow$ converges (Theorem 5, #3)

63. $\lim_{n\to\infty} \frac{n!}{n^n} = \lim_{n\to\infty} \frac{1\cdot 2\cdot 3\cdots(n-1)(n)}{n\cdot n\cdot n\cdots n\cdot n} \le \lim_{n\to\infty} \left(\frac{1}{n}\right) = 0$ and $\frac{n!}{n^n} \ge 0 \Rightarrow \lim_{n\to\infty} \frac{n!}{n^n} = 0 \Rightarrow$ converges

64. $\lim_{n\to\infty} \frac{(-4)^n}{n!} = 0 \Rightarrow$ converges (Theorem 5, #6)

65. $\lim_{n\to\infty} \frac{n!}{10^{6n}} = \lim_{n\to\infty} \frac{1}{\left(\frac{(10^6)^n}{n!}\right)} = \infty \Rightarrow$ diverges (Theorem 5, #6)

66. $\lim_{n\to\infty} \frac{n!}{2^n 3^n} = \lim_{n\to\infty} \frac{1}{\left(\frac{6^n}{n!}\right)} = \infty \Rightarrow$ diverges (Theorem 5, #6)

67. $\lim_{n\to\infty} \left(\frac{1}{n}\right)^{1/(\ln n)} = \lim_{n\to\infty} \exp\left(\frac{1}{\ln n} \ln\left(\frac{1}{n}\right)\right) = \lim_{n\to\infty} \exp\left(\frac{\ln 1 - \ln n}{\ln n}\right) = e^{-1} \Rightarrow$ converges

68. $\lim_{n \to \infty} \ln\left(1 + \frac{1}{n}\right)^n = \ln\left(\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n\right) = \ln e = 1 \Rightarrow$ converges (Theorem 5, #5)

69. $\lim_{n \to \infty} \left(\frac{3n+1}{3n-1}\right)^n = \lim_{n \to \infty} \exp\left(n \ln\left(\frac{3n+1}{3n-1}\right)\right) = \lim_{n \to \infty} \exp\left(\frac{\ln(3n+1) - \ln(3n-1)}{\frac{1}{n}}\right)$
$= \lim_{n \to \infty} \exp\left(\frac{\frac{3}{3n+1} - \frac{3}{3n-1}}{\left(-\frac{1}{n^2}\right)}\right) = \lim_{n \to \infty} \exp\left(\frac{6n^2}{(3n+1)(3n-1)}\right) = \exp\left(\frac{6}{9}\right) = e^{2/3} \Rightarrow$ converges

70. $\lim_{n \to \infty} \left(\frac{n}{n+1}\right)^n = \lim_{n \to \infty} \exp\left(n \ln\left(\frac{n}{n+1}\right)\right) = \lim_{n \to \infty} \exp\left(\frac{\ln n - \ln(n+1)}{\left(\frac{1}{n}\right)}\right) = \lim_{n \to \infty} \exp\left(\frac{\frac{1}{n} - \frac{1}{n+1}}{\left(-\frac{1}{n^2}\right)}\right)$
$= \lim_{n \to \infty} \exp\left(-\frac{n^2}{n(n+1)}\right) = e^{-1} \Rightarrow$ converges

71. $\lim_{n \to \infty} \left(\frac{x^n}{2n+1}\right)^{1/n} = \lim_{n \to \infty} x\left(\frac{1}{2n+1}\right)^{1/n} = x \lim_{n \to \infty} \exp\left(\frac{1}{n} \ln\left(\frac{1}{2n+1}\right)\right) = x \lim_{n \to \infty} \exp\left(\frac{-\ln(2n+1)}{n}\right)$
$= x \lim_{n \to \infty} \exp\left(\frac{-2}{2n+1}\right) = xe^0 = x,\ x > 0 \Rightarrow$ converges

72. $\lim_{n \to \infty} \left(1 - \frac{1}{n^2}\right)^n = \lim_{n \to \infty} \exp\left(n \ln\left(1 - \frac{1}{n^2}\right)\right) = \lim_{n \to \infty} \exp\left(\frac{\ln\left(1 - \frac{1}{n^2}\right)}{\left(\frac{1}{n}\right)}\right) = \lim_{n \to \infty} \exp\left[\frac{\left(\frac{2}{n^3}\right)/\left(1 - \frac{1}{n^2}\right)}{\left(-\frac{1}{n^2}\right)}\right]$
$= \lim_{n \to \infty} \exp\left(\frac{-2n}{n^2-1}\right) = e^0 = 1 \Rightarrow$ converges

73. $\lim_{n \to \infty} \frac{3^n \cdot 6^n}{2^{-n} \cdot n!} = \lim_{n \to \infty} \frac{36^n}{n!} = 0 \Rightarrow$ converges (Theorem 5, #6)

74. $\lim_{n \to \infty} \frac{\left(\frac{10}{11}\right)^n}{\left(\frac{9}{10}\right)^n + \left(\frac{11}{12}\right)^n} = \lim_{n \to \infty} \frac{\left(\frac{12}{11}\right)^n \left(\frac{10}{11}\right)^n}{\left(\frac{12}{11}\right)^n \left(\frac{9}{10}\right)^n + \left(\frac{12}{11}\right)^n \left(\frac{11}{12}\right)^n} = \lim_{n \to \infty} \frac{\left(\frac{120}{121}\right)^n}{\left(\frac{108}{110}\right)^n + 1} = 0 \Rightarrow$ converges
(Theorem 5, #4)

75. $\lim_{n \to \infty} \tanh n = \lim_{n \to \infty} \frac{e^n - e^{-n}}{e^n + e^{-n}} = \lim_{n \to \infty} \frac{e^{2n} - 1}{e^{2n} + 1} = \lim_{n \to \infty} \frac{2e^{2n}}{2e^{2n}} = \lim_{n \to \infty} 1 = 1 \Rightarrow$ converges

76. $\lim_{n \to \infty} \sinh(\ln n) = \lim_{n \to \infty} \frac{e^{\ln n} - e^{-\ln n}}{2} = \lim_{n \to \infty} \frac{n - \left(\frac{1}{n}\right)}{2} = \infty \Rightarrow$ diverges

77. $\lim_{n \to \infty} \frac{n^2 \sin\left(\frac{1}{n}\right)}{2n-1} = \lim_{n \to \infty} \frac{\sin\left(\frac{1}{n}\right)}{\left(\frac{2}{n} - \frac{1}{n^2}\right)} = \lim_{n \to \infty} \frac{-\left(\cos\left(\frac{1}{n}\right)\right)\left(\frac{1}{n^2}\right)}{\left(-\frac{2}{n^2} + \frac{2}{n^3}\right)} = \lim_{n \to \infty} \frac{-\cos\left(\frac{1}{n}\right)}{-2 + \left(\frac{2}{n}\right)} = \frac{1}{2} \Rightarrow$ converges

78. $\lim_{n \to \infty} n\left(1 - \cos\frac{1}{n}\right) = \lim_{n \to \infty} \frac{\left(1 - \cos\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} = \lim_{n \to \infty} \frac{\left[\sin\left(\frac{1}{n}\right)\right]\left(\frac{1}{n^2}\right)}{\left(\frac{1}{n^2}\right)} = \lim_{n \to \infty} \sin\left(\frac{1}{n}\right) = 0 \Rightarrow$ converges

79. $\lim_{n \to \infty} \sqrt{n} \sin\left(\frac{1}{\sqrt{n}}\right) = \lim_{n \to \infty} \frac{\sin\left(\frac{1}{\sqrt{n}}\right)}{\frac{1}{\sqrt{n}}} = \lim_{n \to \infty} \frac{\cos\left(\frac{1}{\sqrt{n}}\right)\left(-\frac{1}{2n^{3/2}}\right)}{-\frac{1}{2n^{3/2}}} = \lim_{n \to \infty} \cos\left(\frac{1}{\sqrt{n}}\right) = \cos 0 = 1 \Rightarrow$ converges

80. $\lim_{n \to \infty} (3^n + 5^n)^{1/n} = \lim_{n \to \infty} \exp\left[\ln(3^n + 5^n)^{1/n}\right] = \lim_{n \to \infty} \exp\left[\frac{\ln(3^n + 5^n)}{n}\right] = \lim_{n \to \infty} \exp\left[\frac{\frac{3^n \ln 3 + 5^n \ln 5}{3^n + 5^n}}{1}\right]$
$= \lim_{n \to \infty} \exp\left[\frac{\left(\frac{3^n}{5^n}\right)\ln 3 + \ln 5}{\left(\frac{3^n}{5^n}\right) + 1}\right] = \lim_{n \to \infty} \exp\left[\frac{\left(\frac{3}{5}\right)^n \ln 3 + \ln 5}{\left(\frac{3}{5}\right)^n + 1}\right] = \exp(\ln 5) = 5$

81. $\lim_{n \to \infty} \tan^{-1} n = \frac{\pi}{2} \Rightarrow$ converges

82. $\lim_{n \to \infty} \frac{1}{\sqrt{n}} \tan^{-1} n = 0 \cdot \frac{\pi}{2} = 0 \Rightarrow$ converges

83. $\lim_{n\to\infty} \left(\frac{1}{3}\right)^n + \frac{1}{\sqrt{2^n}} = \lim_{n\to\infty} \left(\left(\frac{1}{3}\right)^n + \left(\frac{1}{\sqrt{2}}\right)^n\right) = 0 \Rightarrow$ converges (Theorem 5, #4)

84. $\lim_{n\to\infty} \sqrt[n]{n^2+n} = \lim_{n\to\infty} \exp\left[\frac{\ln(n^2+n)}{n}\right] = \lim_{n\to\infty} \exp\left(\frac{2n+1}{n^2+n}\right) = e^0 = 1 \Rightarrow$ converges

85. $\lim_{n\to\infty} \frac{(\ln n)^{200}}{n} = \lim_{n\to\infty} \frac{200(\ln n)^{199}}{n} = \lim_{n\to\infty} \frac{200\cdot199(\ln n)^{198}}{n} = \ldots = \lim_{n\to\infty} \frac{200!}{n} = 0 \Rightarrow$ converges

86. $\lim_{n\to\infty} \frac{(\ln n)^5}{\sqrt{n}} = \lim_{n\to\infty} \left[\frac{\left(\frac{5(\ln n)^4}{n}\right)}{\left(\frac{1}{2\sqrt{n}}\right)}\right] = \lim_{n\to\infty} \frac{10(\ln n)^4}{\sqrt{n}} = \lim_{n\to\infty} \frac{80(\ln n)^3}{\sqrt{n}} = \ldots = \lim_{n\to\infty} \frac{3840}{\sqrt{n}} = 0 \Rightarrow$ converges

87. $\lim_{n\to\infty} \left(n - \sqrt{n^2-n}\right) = \lim_{n\to\infty} \left(n - \sqrt{n^2-n}\right)\left(\frac{n+\sqrt{n^2-n}}{n+\sqrt{n^2-n}}\right) = \lim_{n\to\infty} \frac{n}{n+\sqrt{n^2-n}} = \lim_{n\to\infty} \frac{1}{1+\sqrt{1-\frac{1}{n}}}$
$= \frac{1}{2} \Rightarrow$ converges

88. $\lim_{n\to\infty} \frac{1}{\sqrt{n^2-1}-\sqrt{n^2+n}} = \lim_{n\to\infty} \left(\frac{1}{\sqrt{n^2-1}-\sqrt{n^2+n}}\right)\left(\frac{\sqrt{n^2-1}+\sqrt{n^2+n}}{\sqrt{n^2-1}+\sqrt{n^2+n}}\right) = \lim_{n\to\infty} \frac{\sqrt{n^2-1}+\sqrt{n^2+n}}{-1-n}$
$= \lim_{n\to\infty} \frac{\sqrt{1-\frac{1}{n^2}}+\sqrt{1+\frac{1}{n}}}{\left(-\frac{1}{n}-1\right)} = -2 \Rightarrow$ converges

89. $\lim_{n\to\infty} \frac{1}{n}\int_1^n \frac{1}{x}\,dx = \lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{1}{n} = 0 \Rightarrow$ converges (Theorem 5, #1)

90. $\lim_{n\to\infty} \int_1^n \frac{1}{x^p}\,dx = \lim_{n\to\infty} \left[\frac{1}{1-p}\,\frac{1}{x^{p-1}}\right]_1^n = \lim_{n\to\infty} \frac{1}{1-p}\left(\frac{1}{n^{p-1}}-1\right) = \frac{1}{p-1}$ if $p > 1 \Rightarrow$ converges

91. Since a_n converges $\Rightarrow \lim_{n\to\infty} a_n = L \Rightarrow \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{72}{1+a_n} \Rightarrow L = \frac{72}{1+L} \Rightarrow L(1+L) = 72 \Rightarrow L^2 + L - 72 = 0$
$\Rightarrow L = -9$ or $L = 8$; since $a_n > 0$ for $n \ge 1 \Rightarrow L = 8$

92. Since a_n converges $\Rightarrow \lim_{n\to\infty} a_n = L \Rightarrow \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{a_n+6}{a_n+2} \Rightarrow L = \frac{L+6}{L+2} \Rightarrow L(L+2) = L+6 \Rightarrow L^2 + L - 6 = 0$
$\Rightarrow L = -3$ or $L = 2$; since $a_n > 0$ for $n \ge 2 \Rightarrow L = 2$

93. Since a_n converges $\Rightarrow \lim_{n\to\infty} a_n = L \Rightarrow \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{8+2a_n} \Rightarrow L = \sqrt{8+2L} \Rightarrow L^2 - 2L - 8 = 0 \Rightarrow L = -2$ or $L = 4$; since $a_n > 0$ for $n \ge 3 \Rightarrow L = 4$

94. Since a_n converges $\Rightarrow \lim_{n\to\infty} a_n = L \Rightarrow \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{8+2a_n} \Rightarrow L = \sqrt{8+2L} \Rightarrow L^2 - 2L - 8 = 0 \Rightarrow L = -2$ or $L = 4$; since $a_n > 0$ for $n \ge 2 \Rightarrow L = 4$

95. Since a_n converges $\Rightarrow \lim_{n\to\infty} a_n = L \Rightarrow \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{5a_n} \Rightarrow L = \sqrt{5L} \Rightarrow L^2 - 5L = 0 \Rightarrow L = 0$ or $L = 5$; since $a_n > 0$ for $n \ge 1 \Rightarrow L = 5$

96. Since a_n converges $\Rightarrow \lim_{n\to\infty} a_n = L \Rightarrow \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \left(12 - \sqrt{a_n}\right) \Rightarrow L = \left(12 - \sqrt{L}\right) \Rightarrow L^2 - 25L + 144 = 0$
$\Rightarrow L = 9$ or $L = 16$; since $12 - \sqrt{a_n} < 12$ for $n \ge 1 \Rightarrow L = 9$

97. $a_{n+1} = 2 + \frac{1}{a_n}$, $n \ge 1$, $a_1 = 2$. Since a_n converges $\Rightarrow \lim_{n\to\infty} a_n = L \Rightarrow \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \left(2 + \frac{1}{a_n}\right) \Rightarrow L = 2 + \frac{1}{L}$
$\Rightarrow L^2 - 2L - 1 = 0 \Rightarrow L = 1 \pm \sqrt{2}$; since $a_n > 0$ for $n \ge 1 \Rightarrow L = 1 + \sqrt{2}$

98. $a_{n+1} = \sqrt{1 + a_n}$, $n \ge 1$, $a_1 = \sqrt{1}$. Since a_n converges $\Rightarrow \lim_{n \to \infty} a_n = L \Rightarrow \lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} \sqrt{1 + a_n} \Rightarrow L = \sqrt{1 + L}$ $\Rightarrow L^2 - L - 1 = 0 \Rightarrow L = \frac{1 \pm \sqrt{5}}{2}$; since $a_n > 0$ for $n \ge 1 \Rightarrow L = \frac{1+\sqrt{5}}{2}$

99. $1, 1, 2, 4, 8, 16, 32, \ldots = 1, 2^0, 2^1, 2^2, 2^3, 2^4, 2^5, \ldots \Rightarrow x_1 = 1$ and $x_n = 2^{n-2}$ for $n \ge 2$

100. (a) $1^2 - 2(1)^2 = -1$, $3^2 - 2(2)^2 = 1$; let $f(a, b) = (a + 2b)^2 - 2(a + b)^2 = a^2 + 4ab + 4b^2 - 2a^2 - 4ab - 2b^2$ $= 2b^2 - a^2$; $a^2 - 2b^2 = -1 \Rightarrow f(a, b) = 2b^2 - a^2 = 1$; $a^2 - 2b^2 = 1 \Rightarrow f(a, b) = 2b^2 - a^2 = -1$

(b) $r_n^2 - 2 = \left(\frac{a+2b}{a+b}\right)^2 - 2 = \frac{a^2 + 4ab + 4b^2 - 2a^2 - 4ab - 2b^2}{(a+b)^2} = \frac{-(a^2 - 2b^2)}{(a+b)^2} = \frac{\pm 1}{y_n^2} \Rightarrow r_n = \sqrt{2 \pm \left(\frac{1}{y_n}\right)^2}$

In the first and second fractions, $y_n \ge n$. Let $\frac{a}{b}$ represent the $(n-1)$th fraction where $\frac{a}{b} \ge 1$ and $b \ge n - 1$ for n a positive integer ≥ 3. Now the nth fraction is $\frac{a+2b}{a+b}$ and $a + b \ge 2b \ge 2n - 2 \ge n \Rightarrow y_n \ge n$. Thus, $\lim_{n \to \infty} r_n = \sqrt{2}$.

101. (a) $f(x) = x^2 - 2$; the sequence converges to $1.414213562 \approx \sqrt{2}$

(b) $f(x) = \tan(x) - 1$; the sequence converges to $0.7853981635 \approx \frac{\pi}{4}$

(c) $f(x) = e^x$; the sequence $1, 0, -1, -2, -3, -4, -5, \ldots$ diverges

102. (a) $\lim_{n \to \infty} nf\left(\frac{1}{n}\right) = \lim_{\Delta x \to 0^+} \frac{f(\Delta x)}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{f(0+\Delta x) - f(0)}{\Delta x} = f'(0)$, where $\Delta x = \frac{1}{n}$

(b) $\lim_{n \to \infty} n \tan^{-1}\left(\frac{1}{n}\right) = f'(0) = \frac{1}{1+0^2} = 1$, $f(x) = \tan^{-1} x$

(c) $\lim_{n \to \infty} n\left(e^{1/n} - 1\right) = f'(0) = e^0 = 1$, $f(x) = e^x - 1$

(d) $\lim_{n \to \infty} n \ln\left(1 + \frac{2}{n}\right) = f'(0) = \frac{2}{1+2(0)} = 2$, $f(x) = \ln(1 + 2x)$

103. (a) If $a = 2n + 1$, then $b = \left\lfloor \frac{a^2}{2} \right\rfloor = \left\lfloor \frac{4n^2+4n+1}{2} \right\rfloor = \left\lfloor 2n^2 + 2n + \frac{1}{2} \right\rfloor = 2n^2 + 2n$, $c = \left\lceil \frac{a^2}{2} \right\rceil = \left\lceil 2n^2 + 2n + \frac{1}{2} \right\rceil$ $= 2n^2 + 2n + 1$ and $a^2 + b^2 = (2n+1)^2 + \left(2n^2 + 2n\right)^2 = 4n^2 + 4n + 1 + 4n^4 + 8n^3 + 4n^2$ $= 4n^4 + 8n^3 + 8n^2 + 4n + 1 = \left(2n^2 + 2n + 1\right)^2 = c^2$.

(b) $\lim_{a \to \infty} \frac{\left\lfloor \frac{a^2}{2} \right\rfloor}{\left\lceil \frac{a^2}{2} \right\rceil} = \lim_{a \to \infty} \frac{2n^2+2n}{2n^2+2n+1} = 1$ or $\lim_{a \to \infty} \frac{\left\lfloor \frac{a^2}{2} \right\rfloor}{\left\lceil \frac{a^2}{2} \right\rceil} = \lim_{a \to \infty} \sin\theta = \lim_{\theta \to \pi/2} \sin\theta = 1$

104. (a) $\lim_{n \to \infty} (2n\pi)^{1/(2n)} = \lim_{n \to \infty} \exp\left(\frac{\ln 2n\pi}{2n}\right) = \lim_{n \to \infty} \exp\left(\frac{\left(\frac{2\pi}{2n\pi}\right)}{2}\right) = \lim_{n \to \infty} \exp\left(\frac{1}{2n}\right) = e^0 = 1$;

$n! \approx \left(\frac{n}{e}\right) \sqrt[n]{2n\pi}$, Stirlings approximation $\Rightarrow \sqrt[n]{n!} \approx \left(\frac{n}{e}\right)(2n\pi)^{1/(2n)} \approx \frac{n}{e}$ for large values of n

(b)

n	$\sqrt[n]{n!}$	$\frac{n}{e}$
40	15.76852702	14.71517765
50	19.48325423	18.39397206
60	23.19189561	22.07276647

105. (a) $\lim_{n \to \infty} \frac{\ln n}{n^c} = \lim_{n \to \infty} \frac{\left(\frac{1}{n}\right)}{cn^{c-1}} = \lim_{n \to \infty} \frac{1}{cn^c} = 0$

(b) For all $\epsilon > 0$, there exists an $N = e^{-(\ln\epsilon)/c}$ such that $n > e^{-(\ln\epsilon)/c} \Rightarrow \ln n > -\frac{\ln\epsilon}{c} \Rightarrow \ln n^c > \ln\left(\frac{1}{\epsilon}\right)$ $\Rightarrow n^c > \frac{1}{\epsilon} \Rightarrow \frac{1}{n^c} < \epsilon \Rightarrow \left|\frac{1}{n^c} - 0\right| < \epsilon \Rightarrow \lim_{n \to \infty} \frac{1}{n^c} = 0$

106. Let $\{a_n\}$ and $\{b_n\}$ be sequences both converging to L. Define $\{c_n\}$ by $c_{2n} = b_n$ and $c_{2n-1} = a_n$, where $n = 1, 2, 3, \ldots$. For all $\epsilon > 0$ there exists N_1 such that when $n > N_1$ then $|a_n - L| < \epsilon$ and there exists N_2 such that when $n > N_2$ then $|b_n - L| < \epsilon$. If $n > 1 + 2\max\{N_1, N_2\}$, then $|c_n - L| < \epsilon$, so $\{c_n\}$ converges to L.

107. $\lim_{n \to \infty} n^{1/n} = \lim_{n \to \infty} \exp\left(\frac{1}{n} \ln n\right) = \lim_{n \to \infty} \exp\left(\frac{1}{n}\right) = e^0 = 1$

108. $\lim_{n \to \infty} x^{1/n} = \lim_{n \to \infty} \exp\left(\frac{1}{n} \ln x\right) = e^0 = 1$, because x remains fixed while n gets large

109. Assume the hypotheses of the theorem and let ϵ be a positive number. For all ϵ there exists a N_1 such that when $n > N_1$ then $|a_n - L| < \epsilon \Rightarrow -\epsilon < a_n - L < \epsilon \Rightarrow L - \epsilon < a_n$, and there exists a N_2 such that when $n > N_2$ then $|c_n - L| < \epsilon \Rightarrow -\epsilon < c_n - L < \epsilon \Rightarrow c_n < L + \epsilon$. If $n > \max\{N_1, N_2\}$, then $L - \epsilon < a_n \le b_n \le c_n < L + \epsilon \Rightarrow |b_n - L| < \epsilon \Rightarrow \lim_{n \to \infty} b_n = L$.

110. Let $\epsilon > 0$. We have f continuous at $L \Rightarrow$ there exists δ so that $|x - L| < \delta \Rightarrow |f(x) - f(L)| < \epsilon$. Also, $a_n \to L \Rightarrow$ there exists N so that for $n > N$ $|a_n - L| < \delta$. Thus for $n > N$, $|f(a_n) - f(L)| < \epsilon \Rightarrow f(a_n) \to f(L)$.

111. $a_{n+1} \ge a_n \Rightarrow \frac{3(n+1)+1}{(n+1)+1} > \frac{3n+1}{n+1} \Rightarrow \frac{3n+4}{n+2} > \frac{3n+1}{n+1} \Rightarrow 3n^2 + 3n + 4n + 4 > 3n^2 + 6n + n + 2$ $\Rightarrow 4 > 2$; the steps are reversible so the sequence is nondecreasing; $\frac{3n+1}{n+1} < 3 \Rightarrow 3n + 1 < 3n + 3$ $\Rightarrow 1 < 3$; the steps are reversible so the sequence is bounded above by 3

112. $a_{n+1} \ge a_n \Rightarrow \frac{(2(n+1)+3)!}{((n+1)+1)!} > \frac{(2n+3)!}{(n+1)!} \Rightarrow \frac{(2n+5)!}{(n+2)!} > \frac{(2n+3)!}{(n+1)!} \Rightarrow \frac{(2n+5)!}{(2n+3)!} > \frac{(n+2)!}{(n+1)!}$ $\Rightarrow (2n+5)(2n+4) > n + 2$; the steps are reversible so the sequence is nondecreasing; the sequence is not bounded since $\frac{(2n+3)!}{(n+1)!} = (2n+3)(2n+2)\cdots(n+2)$ can become as large as we please

113. $a_{n+1} \le a_n \Rightarrow \frac{2^{n+1}3^{n+1}}{(n+1)!} \le \frac{2^n 3^n}{n!} \Rightarrow \frac{2^{n+1}3^{n+1}}{2^n 3^n} \le \frac{(n+1)!}{n!} \Rightarrow 2 \cdot 3 \le n + 1$ which is true for $n \ge 5$; the steps are reversible so the sequence is decreasing after a_5, but it is not nondecreasing for all its terms; $a_1 = 6$, $a_2 = 18$, $a_3 = 36$, $a_4 = 54$, $a_5 = \frac{324}{5} = 64.8 \Rightarrow$ the sequence is bounded from above by 64.8

114. $a_{n+1} \ge a_n \Rightarrow 2 - \frac{2}{n+1} - \frac{1}{2^{n+1}} \ge 2 - \frac{2}{n} - \frac{1}{2^n} \Rightarrow \frac{2}{n} - \frac{2}{n+1} \ge \frac{1}{2^{n+1}} - \frac{1}{2^n} \Rightarrow \frac{2}{n(n+1)} \ge -\frac{1}{2^{n+1}}$; the steps are reversible so the sequence is nondecreasing; $2 - \frac{2}{n} - \frac{1}{2^n} \le 2 \Rightarrow$ the sequence is bounded from above

115. $a_n = 1 - \frac{1}{n}$ converges because $\frac{1}{n} \to 0$ by Example 1; also it is a nondecreasing sequence bounded above by 1

116. $a_n = n - \frac{1}{n}$ diverges because $n \to \infty$ and $\frac{1}{n} \to 0$ by Example 1, so the sequence is unbounded

117. $a_n = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n}$ and $0 < \frac{1}{2^n} < \frac{1}{n}$; since $\frac{1}{n} \to 0$ (by Example 1) $\Rightarrow \frac{1}{2^n} \to 0$, the sequence converges; also it is a nondecreasing sequence bounded above by 1

118. $a_n = \frac{2^n - 1}{3^n} = \left(\frac{2}{3}\right)^n - \frac{1}{3^n}$; the sequence converges to 0 by Theorem 5, #4

119. $a_n = ((-1)^n + 1)\left(\frac{n+1}{n}\right)$ diverges because $a_n = 0$ for n odd, while for n even $a_n = 2\left(1 + \frac{1}{n}\right)$ converges to 2; it diverges by definition of divergence

120. $x_n = \max\{\cos 1, \cos 2, \cos 3, \ldots, \cos n\}$ and $x_{n+1} = \max\{\cos 1, \cos 2, \cos 3, \ldots, \cos(n+1)\} \ge x_n$ with $x_n \le 1$ so the sequence is nondecreasing and bounded above by 1 $\Rightarrow$ the sequence converges.

121. $a_n \ge a_{n+1} \Leftrightarrow \frac{1+\sqrt{2n}}{\sqrt{n}} \ge \frac{1+\sqrt{2(n+1)}}{\sqrt{n+1}} \Leftrightarrow \sqrt{n+1} + \sqrt{2n^2 + 2n} \ge \sqrt{n} + \sqrt{2n^2 + 2n} \Leftrightarrow \sqrt{n+1} \ge \sqrt{n}$ and $\frac{1+\sqrt{2n}}{\sqrt{n}} \ge \sqrt{2}$; thus the sequence is nonincreasing and bounded below by $\sqrt{2} \Rightarrow$ it converges

122. $a_n \ge a_{n+1} \Leftrightarrow \frac{n+1}{n} \ge \frac{(n+1)+1}{n+1} \Leftrightarrow n^2+2n+1 \ge n^2+2n \Leftrightarrow 1 \ge 0$ and $\frac{n+1}{n} \ge 1$; thus the sequence is nonincreasing and bounded below by 1 $\Rightarrow$ it converges

123. $\frac{4^{n+1}+3^n}{4^n} = 4 + \left(\frac{3}{4}\right)^n$ so $a_n \ge a_{n+1} \Leftrightarrow 4 + \left(\frac{3}{4}\right)^n \ge 4 + \left(\frac{3}{4}\right)^{n+1} \Leftrightarrow \left(\frac{3}{4}\right)^n \ge \left(\frac{3}{4}\right)^{n+1} \Leftrightarrow 1 \ge \frac{3}{4}$ and $4 + \left(\frac{3}{4}\right)^n \ge 4$; thus the sequence is nonincreasing and bounded below by 4 $\Rightarrow$ it converges

124. $a_1 = 1$, $a_2 = 2-3$, $a_3 = 2(2-3)-3 = 2^2-(2^2-1)\cdot 3$, $a_4 = 2\left(2^2-(2^2-1)\cdot 3\right)-3 = 2^3-(2^3-1)\,3$, $a_5 = 2\left[2^3-(2^3-1)\,3\right]-3 = 2^4-(2^4-1)\,3, \dots, a_n = 2^{n-1}-(2^{n-1}-1)\,3 = 2^{n-1}-3\cdot 2^{n-1}+3$ $= 2^{n-1}(1-3)+3 = -2^n+3$; $a_n \ge a_{n+1} \Leftrightarrow -2^n+3 \ge -2^{n+1}+3 \Leftrightarrow -2^n \ge -2^{n+1} \Leftrightarrow 1 \le 2$ so the sequence is nonincreasing but not bounded below and therefore diverges

125. Let $0 < M < 1$ and let N be an integer greater than $\frac{M}{1-M}$. Then $n > N \Rightarrow n > \frac{M}{1-M} \Rightarrow n - nM > M$ $\Rightarrow n > M + nM \Rightarrow n > M(n+1) \Rightarrow \frac{n}{n+1} > M$.

126. Since M_1 is a least upper bound and M_2 is an upper bound, $M_1 \le M_2$. Since M_2 is a least upper bound and M_1 is an upper bound, $M_2 \le M_1$. We conclude that $M_1 = M_2$ so the least upper bound is unique.

127. The sequence $a_n = 1 + \frac{(-1)^n}{2}$ is the sequence $\frac{1}{2}, \frac{3}{2}, \frac{1}{2}, \frac{3}{2}, \dots$. This sequence is bounded above by $\frac{3}{2}$, but it clearly does not converge, by definition of convergence.

128. Let L be the limit of the convergent sequence $\{a_n\}$. Then by definition of convergence, for $\frac{\epsilon}{2}$ there corresponds an N such that for all m and n, $m > N \Rightarrow |a_m - L| < \frac{\epsilon}{2}$ and $n > N \Rightarrow |a_n - L| < \frac{\epsilon}{2}$. Now $|a_m - a_n| = |a_m - L + L - a_n| \le |a_m - L| + |L - a_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ whenever $m > N$ and $n > N$.

129. Given an $\epsilon > 0$, by definition of convergence there corresponds an N such that for all $n > N$, $|L_1 - a_n| < \epsilon$ and $|L_2 - a_n| < \epsilon$. Now $|L_2 - L_1| = |L_2 - a_n + a_n - L_1| \le |L_2 - a_n| + |a_n - L_1| < \epsilon + \epsilon = 2\epsilon$. $|L_2 - L_1| < 2\epsilon$ says that the difference between two fixed values is smaller than any positive number 2ϵ. The only nonnegative number smaller than every positive number is 0, so $|L_1 - L_2| = 0$ or $L_1 = L_2$.

130. Let k(n) and i(n) be two order-preserving functions whose domains are the set of positive integers and whose ranges are a subset of the positive integers. Consider the two subsequences $a_{k(n)}$ and $a_{i(n)}$, where $a_{k(n)} \to L_1$, $a_{i(n)} \to L_2$ and $L_1 \ne L_2$. Thus $\left|a_{k(n)} - a_{i(n)}\right| \to |L_1 - L_2| > 0$. So there does not exist N such that for all $m, n > N$ $\Rightarrow |a_m - a_n| < \epsilon$. So by Exercise 128, the sequence $\{a_n\}$ is not convergent and hence diverges.

131. $a_{2k} \to L \Leftrightarrow$ given an $\epsilon > 0$ there corresponds an N_1 such that $[2k > N_1 \Rightarrow |a_{2k} - L| < \epsilon]$. Similarly, $a_{2k+1} \to L \Leftrightarrow [2k+1 > N_2 \Rightarrow |a_{2k+1} - L| < \epsilon]$. Let $N = \max\{N_1, N_2\}$. Then $n > N \Rightarrow |a_n - L| < \epsilon$ whether n is even or odd, and hence $a_n \to L$.

132. Assume $a_n \to 0$. This implies that given an $\epsilon > 0$ there corresponds an N such that $n > N \Rightarrow |a_n - 0| < \epsilon$ $\Rightarrow |a_n| < \epsilon \Rightarrow ||a_n|| < \epsilon \Rightarrow ||a_n| - 0| < \epsilon \Rightarrow |a_n| \to 0$. On the other hand, assume $|a_n| \to 0$. This implies that given an $\epsilon > 0$ there corresponds an N such that for $n > N$, $||a_n| - 0| < \epsilon \Rightarrow ||a_n|| < \epsilon \Rightarrow |a_n| < \epsilon$ $\Rightarrow |a_n - 0| < \epsilon \Rightarrow a_n \to 0$.

133. (a) $f(x) = x^2 - a \Rightarrow f'(x) = 2x \Rightarrow x_{n+1} = x_n - \frac{x_n^2 - a}{2x_n} \Rightarrow x_{n+1} = \frac{2x_n^2 - (x_n^2 - a)}{2x_n} = \frac{x_n^2 + a}{2x_n} = \frac{\left(x_n + \frac{a}{x_n}\right)}{2}$

(b) $x_1 = 2$, $x_2 = 1.75$, $x_3 = 1.732142857$, $x_4 = 1.73205081$, $x_5 = 1.732050808$; we are finding the positive number where $x^2 - 3 = 0$; that is, where $x^2 = 3$, $x > 0$, or where $x = \sqrt{3}$.

134. $x_1 = 1$, $x_2 = 1 + \cos(1) = 1.540302306$, $x_3 = 1.540302306 + \cos(1 + \cos(1)) = 1.570791601$, $x_4 = 1.570791601 + \cos(1.570791601) = 1.570796327 = \frac{\pi}{2}$ to 9 decimal places. After a few steps, the arc (x_{n-1}) and line segment $\cos(x_{n-1})$ are nearly the same as the quarter circle.

135-146. Example CAS Commands:

Mathematica: (sequence functions may vary):

```
Clear[a, n]
a[n_]; = n^(1 / n)
first25= Table[N[a[n]],{n, 1, 25}]
Limit[a[n], n → 8]
```

Mathematica: (sequence functions may vary):

```
Clear[a, n]
a[n_]; = n^(1 / n)
first25= Table[N[a[n]],{n, 1, 25}]
Limit[a[n], n → 8]
```


The last command (Limit) will not always work in Mathematica. You could also explore the limit by enlarging your table to more than the first 25 values.

If you know the limit (1 in the above example), to determine how far to go to have all further terms within 0.01 of the limit, do the following.

```
Clear[minN, lim]
lim= 1
Do[{diff=Abs[a[n] − lim], If[diff < .01, {minN= n, Abort[]}]}, {n, 2, 1000}]
minN
```

For sequences that are given recursively, the following code is suggested. The portion of the command a[n_]:=a[n] stores the elements of the sequence and helps to streamline computation.

```
Clear[a, n]
a[1]= 1;
a[n_]; = a[n]= a[n − 1] + (1/5)^(n−1)
first25= Table[N[a[n]], {n, 1, 25}]
```

The limit command does not work in this case, but the limit can be observed as 1.25.

```
Clear[minN, lim]
lim= 1.25
Do[{diff=Abs[a[n] − lim], If[diff < .01, {minN= n, Abort[]}]}, {n, 2, 1000}]
minN
```

10.2 INFINITE SERIES

1. $s_n = \frac{a(1-r^n)}{(1-r)} = \frac{2\left(1-\left(\frac{1}{3}\right)^n\right)}{1-\left(\frac{1}{3}\right)} \Rightarrow \lim_{n\to\infty} s_n = \frac{2}{1-\left(\frac{1}{3}\right)} = 3$

2. $s_n = \frac{a(1-r^n)}{(1-r)} = \frac{\left(\frac{9}{100}\right)\left(1-\left(\frac{1}{100}\right)^n\right)}{1-\left(\frac{1}{100}\right)} \Rightarrow \lim_{n\to\infty} s_n = \frac{\left(\frac{9}{100}\right)}{1-\left(\frac{1}{100}\right)} = \frac{1}{11}$

3. $s_n = \frac{a(1-r^n)}{(1-r)} = \frac{1-\left(-\frac{1}{2}\right)^n}{1-\left(-\frac{1}{2}\right)} \Rightarrow \lim_{n\to\infty} s_n = \frac{1}{\left(\frac{3}{2}\right)} = \frac{2}{3}$

4. $s_n = \frac{1-(-2)^n}{1-(-2)}$, a geometric series where $|r| > 1 \Rightarrow$ divergence

5. $\frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2} \Rightarrow s_n = \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{n+1} - \frac{1}{n+2}\right) = \frac{1}{2} - \frac{1}{n+2} \Rightarrow \lim_{n\to\infty} s_n = \frac{1}{2}$

6. $\frac{5}{n(n+1)} = \frac{5}{n} - \frac{5}{n+1} \Rightarrow s_n = \left(5 - \frac{5}{2}\right) + \left(\frac{5}{2} - \frac{5}{3}\right) + \left(\frac{5}{3} - \frac{5}{4}\right) + \ldots + \left(\frac{5}{n-1} - \frac{5}{n}\right) + \left(\frac{5}{n} - \frac{5}{n+1}\right) = 5 - \frac{5}{n+1}$
$\Rightarrow \lim_{n \to \infty} s_n = 5$

7. $1 - \frac{1}{4} + \frac{1}{16} - \frac{1}{64} + \ldots$, the sum of this geometric series is $\frac{1}{1-\left(-\frac{1}{4}\right)} = \frac{1}{1+\left(\frac{1}{4}\right)} = \frac{4}{5}$

8. $\frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \ldots$, the sum of this geometric series is $\frac{\left(\frac{1}{16}\right)}{1-\left(\frac{1}{4}\right)} = \frac{1}{12}$

9. $\frac{7}{4} + \frac{7}{16} + \frac{7}{64} + \ldots$, the sum of this geometric series is $\frac{\left(\frac{7}{4}\right)}{1-\left(\frac{1}{4}\right)} = \frac{7}{3}$

10. $5 - \frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \ldots$, the sum of this geometric series is $\frac{5}{1-\left(-\frac{1}{4}\right)} = 4$

11. $(5+1) + \left(\frac{5}{2} + \frac{1}{3}\right) + \left(\frac{5}{4} + \frac{1}{9}\right) + \left(\frac{5}{8} + \frac{1}{27}\right) + \ldots$, is the sum of two geometric series; the sum is
$\frac{5}{1-\left(\frac{1}{2}\right)} + \frac{1}{1-\left(\frac{1}{3}\right)} = 10 + \frac{3}{2} = \frac{23}{2}$

12. $(5-1) + \left(\frac{5}{2} - \frac{1}{3}\right) + \left(\frac{5}{4} - \frac{1}{9}\right) + \left(\frac{5}{8} - \frac{1}{27}\right) + \ldots$, is the difference of two geometric series; the sum is
$\frac{5}{1-\left(\frac{1}{2}\right)} - \frac{1}{1-\left(\frac{1}{3}\right)} = 10 - \frac{3}{2} = \frac{17}{2}$

13. $(1+1) + \left(\frac{1}{2} - \frac{1}{5}\right) + \left(\frac{1}{4} + \frac{1}{25}\right) + \left(\frac{1}{8} - \frac{1}{125}\right) + \ldots$, is the sum of two geometric series; the sum is
$\frac{1}{1-\left(\frac{1}{2}\right)} + \frac{1}{1+\left(\frac{1}{5}\right)} = 2 + \frac{5}{6} = \frac{17}{6}$

14. $2 + \frac{4}{5} + \frac{8}{25} + \frac{16}{125} + \ldots = 2\left(1 + \frac{2}{5} + \frac{4}{25} + \frac{8}{125} + \ldots\right)$; the sum of this geometric series is $2\left(\frac{1}{1-\left(\frac{2}{5}\right)}\right) = \frac{10}{3}$

15. Series is geometric with $r = \frac{2}{5} \Rightarrow \left|\frac{2}{5}\right| < 1 \Rightarrow$ Converges to $\frac{1}{1-\frac{2}{5}} = \frac{5}{3}$

16. Series is geometric with $r = -3 \Rightarrow \left|-3\right| > 1 \Rightarrow$ Diverges

17. Series is geometric with $r = \frac{1}{8} \Rightarrow \left|\frac{1}{8}\right| < 1 \Rightarrow$ Converges to $\frac{\frac{1}{8}}{1-\frac{1}{8}} = \frac{1}{7}$

18. Series is geometric with $r = -\frac{2}{3} \Rightarrow \left|-\frac{2}{3}\right| < 1 \Rightarrow$ Converges to $\frac{-\frac{2}{3}}{1-\left(-\frac{2}{3}\right)} = -\frac{2}{5}$

19. $0.\overline{23} = \sum_{n=0}^{\infty} \frac{23}{100}\left(\frac{1}{10^2}\right)^n = \frac{\left(\frac{23}{100}\right)}{1-\left(\frac{1}{100}\right)} = \frac{23}{99}$

20. $0.\overline{234} = \sum_{n=0}^{\infty} \frac{234}{1000}\left(\frac{1}{10^3}\right)^n = \frac{\left(\frac{234}{1000}\right)}{1-\left(\frac{1}{1000}\right)} = \frac{234}{999}$

21. $0.\overline{7} = \sum_{n=0}^{\infty} \frac{7}{10}\left(\frac{1}{10}\right)^n = \frac{\left(\frac{7}{10}\right)}{1-\left(\frac{1}{10}\right)} = \frac{7}{9}$

22. $0.\overline{d} = \sum_{n=0}^{\infty} \frac{d}{10}\left(\frac{1}{10}\right)^n = \frac{\left(\frac{d}{10}\right)}{1-\left(\frac{1}{10}\right)} = \frac{d}{9}$

23. $0.0\overline{6} = \sum_{n=0}^{\infty} \left(\frac{1}{10}\right)\left(\frac{6}{10}\right)\left(\frac{1}{10}\right)^n = \frac{\left(\frac{6}{100}\right)}{1-\left(\frac{1}{10}\right)} = \frac{6}{90} = \frac{1}{15}$

24. $1.\overline{414} = 1 + \sum_{n=0}^{\infty} \frac{414}{1000}\left(\frac{1}{10^3}\right)^n = 1 + \frac{\left(\frac{414}{1000}\right)}{1-\left(\frac{1}{1000}\right)} = 1 + \frac{414}{999} = \frac{1413}{999}$

25. $1.24\overline{123} = \frac{124}{100} + \sum_{n=0}^{\infty} \frac{123}{10^5}\left(\frac{1}{10^3}\right)^n = \frac{124}{100} + \frac{\left(\frac{123}{10^5}\right)}{1-\left(\frac{1}{10^3}\right)} = \frac{124}{100} + \frac{123}{10^5 - 10^2} = \frac{124}{100} + \frac{123}{99{,}900} = \frac{123{,}999}{99{,}900} = \frac{41{,}333}{33{,}300}$

26. $3.\overline{142857} = 3 + \sum_{n=0}^{\infty} \frac{142{,}857}{10^6}\left(\frac{1}{10^6}\right)^n = 3 + \frac{\left(\frac{142{,}857}{10^6}\right)}{1-\left(\frac{1}{10^6}\right)} = 3 + \frac{142{,}857}{10^6 - 1} = \frac{3{,}142{,}854}{999{,}999} = \frac{116{,}402}{37{,}037}$

27. $\lim_{n\to\infty} \frac{n}{n+10} = \lim_{n\to\infty} \frac{1}{1} = 1 \neq 0 \Rightarrow$ diverges

28. $\lim_{n\to\infty} \frac{n(n+1)}{(n+2)(n+3)} = \lim_{n\to\infty} \frac{n^2+n}{n^2+5n+6} = \lim_{n\to\infty} \frac{2n+1}{2n+5} = \lim_{n\to\infty} \frac{2}{2} = 1 \neq 0 \Rightarrow$ diverges

29. $\lim_{n\to\infty} \frac{1}{n+4} = 0 \Rightarrow$ test inconclusive

30. $\lim_{n\to\infty} \frac{n}{n^2+3} = \lim_{n\to\infty} \frac{1}{2n} = 0 \Rightarrow$ test inconclusive

31. $\lim_{n\to\infty} \cos\frac{1}{n} = \cos 0 = 1 \neq 0 \Rightarrow$ diverges

32. $\lim_{n\to\infty} \frac{e^n}{e^n+n} = \lim_{n\to\infty} \frac{e^n}{e^n+1} = \lim_{n\to\infty} \frac{e^n}{e^n} = \lim_{n\to\infty} \frac{1}{1} = 1 \neq 0 \Rightarrow$ diverges

33. $\lim_{n\to\infty} \ln\frac{1}{n} = -\infty \neq 0 \Rightarrow$ diverges

34. $\lim_{n\to\infty} \cos n\pi =$ does not exist $\Rightarrow$ diverges

35. $s_k = \left(1-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \ldots + \left(\frac{1}{k-1}-\frac{1}{k}\right) + \left(\frac{1}{k}-\frac{1}{k+1}\right) = 1 - \frac{1}{k+1} \Rightarrow \lim_{k\to\infty} s_k$
$= \lim_{k\to\infty} \left(1-\frac{1}{k+1}\right) = 1$, series converges to 1

36. $s_k = \left(\frac{3}{1}-\frac{3}{4}\right) + \left(\frac{3}{4}-\frac{3}{9}\right) + \left(\frac{3}{9}-\frac{3}{16}\right) + \ldots + \left(\frac{3}{(k-1)^2}-\frac{3}{k^2}\right) + \left(\frac{3}{k^2}-\frac{3}{(k+1)^2}\right) = 3 - \frac{3}{(k+1)^2} \Rightarrow \lim_{k\to\infty} s_k$
$= \lim_{k\to\infty} \left(3-\frac{3}{(k+1)^2}\right) = 3$, series converges to 3

37. $s_k = \left(\ln\sqrt{2}-\ln\sqrt{1}\right) + \left(\ln\sqrt{3}-\ln\sqrt{2}\right) + \left(\ln\sqrt{4}-\ln\sqrt{3}\right) + \ldots + \left(\ln\sqrt{k}-\ln\sqrt{k-1}\right) + \left(\ln\sqrt{k+1}-\ln\sqrt{k}\right)$
$= \ln\sqrt{k+1} - \ln\sqrt{1} = \ln\sqrt{k+1} \Rightarrow \lim_{k\to\infty} s_k = \lim_{k\to\infty} \ln\sqrt{k+1} = \infty$; series diverges

38. $s_k = (\tan 1 - \tan 0) + (\tan 2 - \tan 1) + (\tan 3 - \tan 2) + \ldots + (\tan k - \tan(k-1)) + (\tan(k+1) - \tan k)$
$= \tan(k+1) - \tan 0 = \tan(k+1) \Rightarrow \lim_{k\to\infty} sk = \lim_{k\to\infty} \tan(k+1) =$ does not exist; series diverges

39. $s_k = \left(\cos^{-1}\left(\frac{1}{2}\right) - \cos^{-1}\left(\frac{1}{3}\right)\right) + \left(\cos^{-1}\left(\frac{1}{3}\right) - \cos^{-1}\left(\frac{1}{4}\right)\right) + \left(\cos^{-1}\left(\frac{1}{4}\right) - \cos^{-1}\left(\frac{1}{5}\right)\right) + \ldots$
$+ \left(\cos^{-1}\left(\frac{1}{k}\right) - \cos^{-1}\left(\frac{1}{k+1}\right)\right) + \left(\cos^{-1}\left(\frac{1}{k+1}\right) - \cos^{-1}\left(\frac{1}{k+2}\right)\right) = \frac{\pi}{3} - \cos^{-1}\left(\frac{1}{k+2}\right)$
$\Rightarrow \lim_{k\to\infty} s_k = \lim_{k\to\infty} \left[\frac{\pi}{3} - \cos^{-1}\left(\frac{1}{k+2}\right)\right] = \frac{\pi}{3} - \frac{\pi}{2} = \frac{\pi}{6}$, series converges to $\frac{\pi}{6}$

40. $s_k = \left(\sqrt{5}-\sqrt{4}\right) + \left(\sqrt{6}-\sqrt{5}\right) + \left(\sqrt{7}-\sqrt{6}\right) + \ldots + \left(\sqrt{k+3}-\sqrt{k+2}\right) + \left(\sqrt{k+4}-\sqrt{k+3}\right)$
$= \sqrt{k+4} - 2 \Rightarrow \lim_{k\to\infty} s_k = \lim_{k\to\infty} \left[\sqrt{k+4} - 2\right] = \infty$; series diverges

41. $\frac{4}{(4n-3)(4n+1)} = \frac{1}{4n-3} - \frac{1}{4n+1} \Rightarrow s_k = \left(1 - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{9}\right) + \left(\frac{1}{9} - \frac{1}{13}\right) + \ldots + \left(\frac{1}{4k-7} - \frac{1}{4k-3}\right)$
$+ \left(\frac{1}{4k-3} - \frac{1}{4k+1}\right) = 1 - \frac{1}{4k+1} \Rightarrow \lim_{k\to\infty} s_k = \lim_{k\to\infty} \left(1 - \frac{1}{4k+1}\right) = 1$

42. $\frac{6}{(2n-1)(2n+1)} = \frac{A}{2n-1} + \frac{B}{2n+1} = \frac{A(2n+1)+B(2n-1)}{(2n-1)(2n+1)} \Rightarrow A(2n+1) + B(2n-1) = 6 \Rightarrow (2A+2B)n + (A-B) = 6$
$\Rightarrow \begin{cases} 2A+2B=0 \\ A-\ \ B=6 \end{cases} \Rightarrow \begin{cases} A+B=0 \\ A-B=6 \end{cases} \Rightarrow 2A = 6 \Rightarrow A = 3$ and $B = -3$. Hence, $\sum_{n=1}^{k} \frac{6}{(2n-1)(2n+1)} = 3\sum_{n=1}^{k}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$
$= 3\left(\frac{1}{1} - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} + \frac{1}{5} - \frac{1}{7} + \ldots - \frac{1}{2(k-1)+1} + \frac{1}{2k-1} - \frac{1}{2k+1}\right) = 3\left(1 - \frac{1}{2k+1}\right) \Rightarrow$ the sum is
$\lim_{k\to\infty} 3\left(1 - \frac{1}{2k+1}\right) = 3$

43. $\frac{40n}{(2n-1)^2(2n+1)^2} = \frac{A}{(2n-1)} + \frac{B}{(2n-1)^2} + \frac{C}{(2n+1)} + \frac{D}{(2n+1)^2} = \frac{A(2n-1)(2n+1)^2 + B(2n+1)^2 + C(2n+1)(2n-1)^2 + D(2n-1)^2}{(2n-1)^2(2n+1)^2}$
$\Rightarrow A(2n-1)(2n+1)^2 + B(2n+1)^2 + C(2n+1)(2n-1)^2 + D(2n-1)^2 = 40n$
$\Rightarrow A\left(8n^3 + 4n^2 - 2n - 1\right) + B\left(4n^2 + 4n + 1\right) + C\left(8n^3 - 4n^2 - 2n + 1\right) = D\left(4n^2 - 4n + 1\right) = 40n$
$\Rightarrow (8A + 8C)n^3 + (4A + 4B - 4C + 4D)n^2 + (-2A + 4B - 2C - 4D)n + (-A + B + C + D) = 40n$
$\Rightarrow \begin{cases} 8A+8C=0 \\ 4A+4B-4C+4D=0 \\ -2A+4B-2C-4D=40 \\ -A+B+C+D=0 \end{cases} \Rightarrow \begin{cases} 8A+8C=0 \\ A+B-C+D=0 \\ -A+2B-C-2D=20 \\ -A+B+C+D=0 \end{cases} \Rightarrow \begin{cases} B+D=0 \\ 2B-2D=20 \end{cases} \Rightarrow 4B = 20 \Rightarrow B = 5$
and $D = -5 \Rightarrow \begin{cases} A+C=0 \\ -A+5+C-5=0 \end{cases} \Rightarrow C = 0$ and $A = 0$. Hence, $\sum_{n=1}^{k}\left[\frac{40n}{(2n-1)^2(2n+1)^2}\right]$
$= 5\sum_{n=1}^{k}\left[\frac{1}{(2n-1)^2} - \frac{1}{(2n+1)^2}\right] = 5\left(\frac{1}{1} - \frac{1}{9} + \frac{1}{9} - \frac{1}{25} + \frac{1}{25} - \ldots - \frac{1}{(2(k-1)+1)^2} + \frac{1}{(2k-1)^2} - \frac{1}{(2k+1)^2}\right)$
$= 5\left(1 - \frac{1}{(2k+1)^2}\right) \Rightarrow$ the sum is $\lim_{n\to\infty} 5\left(1 - \frac{1}{(2k+1)^2}\right) = 5$

44. $\frac{2n+1}{n^2(n+1)^2} = \frac{1}{n^2} - \frac{1}{(n+1)^2} \Rightarrow s_k = \left(1 - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{9}\right) + \left(\frac{1}{9} - \frac{1}{16}\right) + \ldots + \left[\frac{1}{(k-1)^2} - \frac{1}{k^2}\right] + \left[\frac{1}{k^2} - \frac{1}{(k+1)^2}\right]$
$\Rightarrow \lim_{k\to\infty} s_k = \lim_{k\to\infty}\left[1 - \frac{1}{(k+1)^2}\right] = 1$

45. $s_k = \left(1 - \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}}\right) + \ldots + \left(\frac{1}{\sqrt{k-1}} + \frac{1}{\sqrt{k}}\right) + \left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}\right) = 1 - \frac{1}{\sqrt{k+1}}$
$\Rightarrow \lim_{k\to\infty} s_k = \lim_{k\to\infty}\left(1 - \frac{1}{\sqrt{k+1}}\right) = 1$

46. $s_k = \left(\frac{1}{2} - \frac{1}{2^{1/2}}\right) + \left(\frac{1}{2^{1/2}} - \frac{1}{2^{1/3}}\right) + \left(\frac{1}{2^{1/3}} - \frac{1}{2^{1/4}}\right) + \ldots + \left(\frac{1}{2^{1/(k-1)}} - \frac{1}{2^{1/k}}\right) + \left(\frac{1}{2^{1/k}} - \frac{1}{2^{1/(k+1)}}\right) = \frac{1}{2} - \frac{1}{2^{1/(k+1)}}$
$\Rightarrow \lim_{k\to\infty} s_k = \frac{1}{2} - \frac{1}{1} = -\frac{1}{2}$

47. $s_k = \left(\frac{1}{\ln 3} - \frac{1}{\ln 2}\right) + \left(\frac{1}{\ln 4} - \frac{1}{\ln 3}\right) + \left(\frac{1}{\ln 5} - \frac{1}{\ln 4}\right) + \ldots + \left(\frac{1}{\ln(k+1)} - \frac{1}{\ln k}\right) + \left(\frac{1}{\ln(k+2)} - \frac{1}{\ln(k+1)}\right)$
$= -\frac{1}{\ln 2} + \frac{1}{\ln(k+2)} \Rightarrow \lim_{k\to\infty} s_k = -\frac{1}{\ln 2}$

48. $s_k = \left[\tan^{-1}(1) - \tan^{-1}(2)\right] + \left[\tan^{-1}(2) - \tan^{-1}(3)\right] + \ldots + \left[\tan^{-1}(k-1) - \tan^{-1}(k)\right]$
$+ \left[\tan^{-1}(k) - \tan^{-1}(k+1)\right] = \tan^{-1}(1) - \tan^{-1}(k+1) \Rightarrow \lim_{k\to\infty} s_k = \tan^{-1}(1) - \frac{\pi}{2} = \frac{\pi}{4} - \frac{\pi}{2} = -\frac{\pi}{4}$

49. convergent geometric series with sum $\frac{1}{1 - \left(\frac{1}{\sqrt{2}}\right)} = \frac{\sqrt{2}}{\sqrt{2}-1} = 2 + \sqrt{2}$

50. divergent geometric series with $|r| = \sqrt{2} > 1$

51. convergent geometric series with sum $\frac{\left(\frac{3}{2}\right)}{1 - \left(-\frac{1}{2}\right)} = 1$

52. $\lim_{n \to \infty} (-1)^{n+1} n \neq 0 \Rightarrow$ diverges

53. $\lim_{n \to \infty} \cos(n\pi) = \lim_{n \to \infty} (-1)^n \neq 0 \Rightarrow$ diverges

54. $\cos(n\pi) = (-1)^n \Rightarrow$ convergent geometric series with sum $\frac{1}{1-\left(-\frac{1}{5}\right)} = \frac{5}{6}$

55. convergent geometric series with sum $\frac{1}{1-\left(\frac{1}{e^2}\right)} = \frac{e^2}{e^2-1}$

56. $\lim_{n \to \infty} \ln \frac{1}{3^n} = -\infty \neq 0 \Rightarrow$ diverges

57. convergent geometric series with sum $\frac{2}{1-\left(\frac{1}{10}\right)} - 2 = \frac{20}{9} - \frac{18}{9} = \frac{2}{9}$

58. convergent geometric series with sum $\frac{1}{1-\left(\frac{1}{x}\right)} = \frac{x}{x-1}$

59. difference of two geometric series with sum $\frac{1}{1-\left(\frac{2}{3}\right)} - \frac{1}{1-\left(\frac{1}{3}\right)} = 3 - \frac{3}{2} = \frac{3}{2}$

60. $\lim_{n \to \infty} \left(1 - \frac{1}{n}\right)^n = \lim_{n \to \infty} \left(1 + \frac{-1}{n}\right)^n = e^{-1} \neq 0 \Rightarrow$ diverges

61. $\lim_{n \to \infty} \frac{n!}{1000^n} = \infty \neq 0 \Rightarrow$ diverges

62. $\lim_{n \to \infty} \frac{n^n}{n!} = \lim_{n \to \infty} \frac{n \cdot n \cdots n}{1 \cdot 2 \cdots n} > \lim_{n \to \infty} n = \infty \Rightarrow$ diverges

63. $\sum_{n=1}^{\infty} \frac{2^n + 3^n}{4^n} = \sum_{n=1}^{\infty} \frac{2^n}{4^n} + \sum_{n=1}^{\infty} \frac{3^n}{4^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n + \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$; both $= \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ and $\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$ are geometric series, and both converge since $r = \frac{1}{2} \Rightarrow \left|\frac{1}{2}\right| < 1$ and $r = \frac{3}{4} \Rightarrow \left|\frac{3}{4}\right| < 1$, respectivley $\Rightarrow \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$ and $\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n = \frac{\frac{3}{4}}{1-\frac{3}{4}} = 3 \Rightarrow$ $\sum_{n=1}^{\infty} \frac{2^n + 3^n}{4^n} = 1 + 3 = 4$ by Theorem 8, part (1)

64. $\lim_{n \to \infty} \frac{2^n + 4^n}{3^n + 4^n} = \lim_{n \to \infty} \frac{\frac{2^n}{4^n} + 1}{\frac{3^n}{4^n} + 1} = \lim_{n \to \infty} \frac{\left(\frac{1}{2}\right)^n + 1}{\left(\frac{3}{4}\right)^n + 1} = \frac{1}{1} = 1 \neq 0 \Rightarrow$ diverges by n^{th} term test for divergence

65. $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right) = \sum_{n=1}^{\infty} [\ln(n) - \ln(n+1)] \Rightarrow s_k = [\ln(1) - \ln(2)] + [\ln(2) - \ln(3)] + [\ln(3) - \ln(4)] + \ldots$ $+ [\ln(k-1) - \ln(k)] + [\ln(k) - \ln(k+1)] = -\ln(k+1) \Rightarrow \lim_{k \to \infty} s_k = -\infty, \Rightarrow$ diverges

66. $\lim_{n \to \infty} a_n = \lim_{n \to \infty} \ln\left(\frac{n}{2n+1}\right) = \ln\left(\frac{1}{2}\right) \neq 0 \Rightarrow$ diverges

67. convergent geometric series with sum $\frac{1}{1-\left(\frac{e}{\pi}\right)} = \frac{\pi}{\pi - e}$

68. divergent geometric series with $|r| = \frac{e^\pi}{\pi^e} \approx \frac{23.141}{22.459} > 1$

69. $\sum_{n=0}^{\infty} (-1)^n x^n = \sum_{n=0}^{\infty} (-x)^n$; $a = 1$, $r = -x$; converges to $\frac{1}{1-(-x)} = \frac{1}{1+x}$ for $|x| < 1$

70. $\sum_{n=0}^{\infty} (-1)^n x^{2n} = \sum_{n=0}^{\infty} \left(-x^2\right)^n$; $a = 1$, $r = -x^2$; converges to $\frac{1}{1+x^2}$ for $|x| < 1$

71. $a = 3, r = \frac{x-1}{2}$; converges to $\frac{3}{1-\left(\frac{x-1}{2}\right)} = \frac{6}{3-x}$ for $-1 < \frac{x-1}{2} < 1$ or $-1 < x < 3$

72. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2}\left(\frac{1}{3+\sin x}\right)^n = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{-1}{3+\sin x}\right)^n$; $a = \frac{1}{2}$, $r = \frac{-1}{3+\sin x}$; converges to $\frac{\left(\frac{1}{2}\right)}{1-\left(\frac{-1}{3+\sin x}\right)}$
$= \frac{3+\sin x}{2(4+\sin x)} = \frac{3+\sin x}{8+2\sin x}$ for all x $\left(\text{since } \frac{1}{4} \le \frac{1}{3+\sin x} \le \frac{1}{2} \text{ for all x}\right)$

73. $a = 1, r = 2x$; converges to $\frac{1}{1-2x}$ for $|2x| < 1$ or $|x| < \frac{1}{2}$

74. $a = 1, r = -\frac{1}{x^2}$; converges to $\frac{1}{1-\left(\frac{-1}{x^2}\right)} = \frac{x^2}{x^2+1}$ for $\left|\frac{1}{x^2}\right| < 1$ or $|x| > 1$.

75. $a = 1, r = -(x+1)^n$; converges to $\frac{1}{1+(x+1)} = \frac{1}{2+x}$ for $|x+1| < 1$ or $-2 < x < 0$

76. $a = 1, r = \frac{3-x}{2}$; converges to $\frac{1}{1-\left(\frac{3-x}{2}\right)} = \frac{2}{x-1}$ for $\left|\frac{3-x}{2}\right| < 1$ or $1 < x < 5$

77. $a = 1, r = \sin x$; converges to $\frac{1}{1-\sin x}$ for $x \ne (2k+1)\frac{\pi}{2}$, k an integer

78. $a = 1, r = \ln x$; converges to $\frac{1}{1-\ln x}$ for $|\ln x| < 1$ or $e^{-1} < x < e$

79. (a) $\sum_{n=-2}^{\infty} \frac{1}{(n+4)(n+5)}$ (b) $\sum_{n=0}^{\infty} \frac{1}{(n+2)(n+3)}$ (c) $\sum_{n=5}^{\infty} \frac{1}{(n-3)(n-2)}$

80. (a) $\sum_{n=-1}^{\infty} \frac{5}{(n+2)(n+3)}$ (b) $\sum_{n=3}^{\infty} \frac{5}{(n-2)(n-1)}$ (c) $\sum_{n=20}^{\infty} \frac{5}{(n-19)(n-18)}$

81. (a) one example is $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots = \frac{\left(\frac{1}{2}\right)}{1-\left(\frac{1}{2}\right)} = 1$

(b) one example is $-\frac{3}{2} - \frac{3}{4} - \frac{3}{8} - \frac{3}{16} - \ldots = \frac{\left(-\frac{3}{2}\right)}{1-\left(\frac{1}{2}\right)} = -3$

(c) one example is $1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{8} - \frac{1}{16} - \ldots = 1 - \frac{\left(\frac{1}{2}\right)}{1-\left(\frac{1}{2}\right)} = 0$.

82. The series $\sum_{n=0}^{\infty} k\left(\frac{1}{2}\right)^{n+1}$ is a geometric series whose sum is $\frac{\left(\frac{k}{2}\right)}{1-\left(\frac{1}{2}\right)} = k$ where k can be any positive or negative number.

83. Let $a_n = b_n = \left(\frac{1}{2}\right)^n$. Then $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 1$, while $\sum_{n=1}^{\infty} \left(\frac{a_n}{b_n}\right) = \sum_{n=1}^{\infty} (1)$ diverges.

84. Let $a_n = b_n = \left(\frac{1}{2}\right)^n$. Then $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 1$, while $\sum_{n=1}^{\infty} (a_n b_n) = \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n = \frac{1}{3} \ne AB$.

85. Let $a_n = \left(\frac{1}{4}\right)^n$ and $b_n = \left(\frac{1}{2}\right)^n$. Then $A = \sum_{n=1}^{\infty} a_n = \frac{1}{3}$, $B = \sum_{n=1}^{\infty} b_n = 1$ and $\sum_{n=1}^{\infty} \left(\frac{a_n}{b_n}\right) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 1 \ne \frac{A}{B}$.

86. Yes: $\sum \left(\frac{1}{a_n}\right)$ diverges. The reasoning: $\sum a_n$ converges $\Rightarrow a_n \to 0 \Rightarrow \frac{1}{a_n} \to \infty \Rightarrow \sum \left(\frac{1}{a_n}\right)$ diverges by the nth-Term Test.

87. Since the sum of a finite number of terms is finite, adding or subtracting a finite number of terms from a series that diverges does not change the divergence of the series.

88. Let $A_n = a_1 + a_2 + \ldots + a_n$ and $\lim_{n\to\infty} A_n = A$. Assume $\sum (a_n + b_n)$ converges to S. Let $S_n = (a_1+b_1) + (a_2+b_2) + \ldots + (a_n+b_n) \Rightarrow S_n = (a_1+a_2+\ldots+a_n) + (b_1+b_2+\ldots+b_n)$ $\Rightarrow b_1+b_2+\ldots+b_n = S_n - A_n \Rightarrow \lim_{n\to\infty} (b_1+b_2+\ldots+b_n) = S - A \Rightarrow \sum b_n$ converges. This contradicts the assumption that $\sum b_n$ diverges; therefore, $\sum (a_n+b_n)$ diverges.

89. (a) $\frac{2}{1-r} = 5 \Rightarrow \frac{2}{5} = 1 - r \Rightarrow r = \frac{3}{5}$; $2 + 2\left(\frac{3}{5}\right) + 2\left(\frac{3}{5}\right)^2 + \ldots$

(b) $\frac{\left(\frac{13}{2}\right)}{1-r} = 5 \Rightarrow \frac{13}{10} = 1 - r \Rightarrow r = -\frac{3}{10}$; $\frac{13}{2} - \frac{13}{2}\left(\frac{3}{10}\right) + \frac{13}{2}\left(\frac{3}{10}\right)^2 - \frac{13}{2}\left(\frac{3}{10}\right)^3 + \ldots$

90. $1 + e^b + e^{2b} + \ldots = \frac{1}{1-e^b} = 9 \Rightarrow \frac{1}{9} = 1 - e^b \Rightarrow e^b = \frac{8}{9} \Rightarrow b = \ln\left(\frac{8}{9}\right)$

91. $s_n = 1 + 2r + r^2 + 2r^3 + r^4 + 2r^5 + \ldots + r^{2n} + 2r^{2n+1}$, $n = 0, 1, \ldots$
$\Rightarrow s_n = (1 + r^2 + r^4 + \ldots + r^{2n}) + (2r + 2r^3 + 2r^5 + \ldots + 2r^{2n+1}) \Rightarrow \lim_{n\to\infty} s_n = \frac{1}{1-r^2} + \frac{2r}{1-r^2}$
$= \frac{1+2r}{1-r^2}$, if $|r^2| < 1$ or $|r| < 1$

92. $L - s_n = \frac{a}{1-r} - \frac{a(1-r^n)}{1-r} = \frac{ar^n}{1-r}$

93. $\text{area} = 2^2 + \left(\sqrt{2}\right)^2 + (1)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \ldots = 4 + 2 + 1 + \frac{1}{2} + \ldots = \frac{4}{1-\frac{1}{2}} = 8 \text{ m}^2$

94. (a) $L_1 = 3, L_2 = 3\left(\frac{4}{3}\right), L_3 = 3\left(\frac{4}{3}\right)^2, \ldots, L_n = 3\left(\frac{4}{3}\right)^{n-1} \Rightarrow \lim_{n\to\infty} L_n = \lim_{n\to\infty} 3\left(\frac{4}{3}\right)^{n-1} = \infty$

(b) Using the fact that the area of an equilateral triangle of side length s is $\frac{\sqrt{3}}{4}s^2$, we see that $A_1 = \frac{\sqrt{3}}{4}$,
$A_2 = A_1 + 3\left(\frac{\sqrt{3}}{4}\right)\left(\frac{1}{3}\right)^2 = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12}$, $A_3 = A_2 + 3(4)\left(\frac{\sqrt{3}}{4}\right)\left(\frac{1}{3^2}\right)^2 = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12} + \frac{\sqrt{3}}{27}$,
$A_4 = A_3 + 3(4)^2\left(\frac{\sqrt{3}}{4}\right)\left(\frac{1}{3^3}\right)^2$, $A_5 = A_4 + 3(4)^3\left(\frac{\sqrt{3}}{4}\right)\left(\frac{1}{3^4}\right)^2, \ldots,$
$A_n = \frac{\sqrt{3}}{4} + \sum_{k=2}^{n} 3(4)^{k-2}\left(\frac{\sqrt{3}}{4}\right)\left(\frac{1}{3^2}\right)^{k-1} = \frac{\sqrt{3}}{4} + \sum_{k=2}^{n} 3\sqrt{3}(4)^{k-3}\left(\frac{1}{9}\right)^{k-1} = \frac{\sqrt{3}}{4} + 3\sqrt{3}\left(\sum_{k=2}^{n} \frac{4^{k-3}}{9^{k-1}}\right)$.
$\lim_{n\to\infty} A_n = \lim_{n\to\infty}\left(\frac{\sqrt{3}}{4} + 3\sqrt{3}\left(\sum_{k=2}^{n}\frac{4^{k-3}}{9^{k-1}}\right)\right) = \frac{\sqrt{3}}{4} + 3\sqrt{3}\left(\frac{\frac{1}{36}}{1-\frac{4}{9}}\right) = \frac{\sqrt{3}}{4} + 3\sqrt{3}\left(\frac{1}{20}\right) = \frac{\sqrt{3}}{4}\left(1 + \frac{3}{5}\right)$
$= \frac{\sqrt{3}}{4}\left(\frac{8}{5}\right) = \frac{8}{5}A_1$

10.3 THE INTEGRAL TEST

1. $f(x) = \frac{1}{x^2}$ is positive, continuous, and decreasing for $x \ge 1$; $\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty}\left[-\frac{1}{x}\right]_1^b$
$= \lim_{b\to\infty}\left(-\frac{1}{b} + 1\right) = 1 \Rightarrow \int_1^\infty \frac{1}{x^2}\,dx$ converges $\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2}$ converges

2. $f(x) = \frac{1}{x^{0.2}}$ is positive, continuous, and decreasing for $x \ge 1$; $\int_1^\infty \frac{1}{x^{0.2}}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^{0.2}}\,dx = \lim_{b\to\infty}\left[\frac{5}{4}x^{0.8}\right]_1^b$
$= \lim_{b\to\infty}\left(\frac{5}{4}b^{0.8} - \frac{5}{4}\right) = \infty \Rightarrow \int_1^\infty \frac{1}{x^{0.2}}\,dx$ diverges $\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^{0.2}}$ diverges

3. $f(x) = \frac{1}{x^2+4}$ is positive, continuous, and decreasing for $x \ge 1$; $\int_1^\infty \frac{1}{x^2+4}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2+4}\,dx = \lim_{b\to\infty} \left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_1^b$
$= \lim_{b\to\infty} \left(\frac{1}{2}\tan^{-1}\frac{b}{2} - \frac{1}{2}\tan^{-1}\frac{1}{2}\right) = \frac{\pi}{4} - \frac{1}{2}\tan^{-1}\frac{1}{2} \Rightarrow \int_1^\infty \frac{1}{x^2+4}\,dx$ converges $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2+4}$ converges

4. $f(x) = \frac{1}{x+4}$ is positive, continuous, and decreasing for $x \ge 1$; $\int_1^\infty \frac{1}{x+4}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x+4}\,dx = \lim_{b\to\infty} \left[\ln|x+4|\right]_1^b$
$= \lim_{b\to\infty} (\ln|b+4| - \ln 5) = \infty \Rightarrow \int_1^\infty \frac{1}{x+4}\,dx$ diverges $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n+4}$ diverges

5. $f(x) = e^{-2x}$ is positive, continuous, and decreasing for $x \ge 1$; $\int_1^\infty e^{-2x}\,dx = \lim_{b\to\infty} \int_1^b e^{-2x}\,dx = \lim_{b\to\infty} \left[-\frac{1}{2}e^{-2x}\right]_1^b$
$= \lim_{b\to\infty} \left(-\frac{1}{2e^{2b}} + \frac{1}{2e^2}\right) = \frac{1}{2e^2} \Rightarrow \int_1^\infty e^{-2x}\,dx$ converges $\Rightarrow \sum_{n=1}^{\infty} e^{-2n}$ converges

6. $f(x) = \frac{1}{x(\ln x)^2}$ is positive, continuous, and decreasing for $x \ge 2$; $\int_2^\infty \frac{1}{x(\ln x)^2}\,dx = \lim_{b\to\infty} \int_2^b \frac{1}{x(\ln x)^2}\,dx = \lim_{b\to\infty} \left[-\frac{1}{\ln x}\right]_2^b$
$= \lim_{b\to\infty} \left(-\frac{1}{\ln b} + \frac{1}{\ln 2}\right) = \frac{1}{\ln 2} \Rightarrow \int_2^\infty \frac{1}{x(\ln x)^2}\,dx$ converges $\Rightarrow \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges

7. $f(x) = \frac{x}{x^2+4}$ is positive and continuous for $x \ge 1$, $f'(x) = \frac{4-x^2}{(x^2+4)^2} < 0$ for $x > 2$, thus f is decreasing for $x \ge 3$;
$\int_3^\infty \frac{x}{x^2+4}\,dx = \lim_{b\to\infty} \int_3^b \frac{x}{x^2+4}\,dx = \lim_{b\to\infty} \left[\frac{1}{2}\ln(x^2+4)\right]_3^b = \lim_{b\to\infty} \left(\frac{1}{2}\ln(b^2+4) - \frac{1}{2}\ln(13)\right) = \infty \Rightarrow \int_3^\infty \frac{x}{x^2+4}\,dx$
diverges $\Rightarrow \sum_{n=3}^{\infty} \frac{n}{n^2+4}$ diverges $\Rightarrow \sum_{n=1}^{\infty} \frac{n}{n^2+4} = \frac{1}{5} + \frac{2}{8} + \sum_{n=3}^{\infty} \frac{n}{n^2+4}$ diverges

8. $f(x) = \frac{\ln x^2}{x}$ is positive and continuous for $x \ge 2$, $f'(x) = \frac{2-\ln x^2}{x^2} < 0$ for $x > e$, thus f is decreasing for $x \ge 3$;
$\int_3^\infty \frac{\ln x^2}{x}\,dx = \lim_{b\to\infty} \int_3^b \frac{\ln x^2}{x}\,dx = \lim_{b\to\infty} \left[2(\ln x)\right]_3^b = \lim_{b\to\infty} (2(\ln b) - 2(\ln 3)) = \infty \Rightarrow \int_3^\infty \frac{\ln x^2}{x}\,dx$
diverges $\Rightarrow \sum_{n=3}^{\infty} \frac{\ln n^2}{n}$ diverges $\Rightarrow \sum_{n=2}^{\infty} \frac{\ln n^2}{n} = \frac{\ln 4}{2} + \sum_{n=3}^{\infty} \frac{\ln n^2}{n}$ diverges

9. $f(x) = \frac{x^2}{e^{x/3}}$ is positive and continuous for $x \ge 1$, $f'(x) = \frac{-x(x-6)}{3e^{x/3}} < 0$ for $x > 6$, thus f is decreasing for $x \ge 7$;
$\int_7^\infty \frac{x^2}{e^{x/3}}\,dx = \lim_{b\to\infty} \int_7^b \frac{x^2}{e^{x/3}}\,dx = \lim_{b\to\infty} \left[-\frac{3x^2}{e^{x/3}} - \frac{18x}{e^{x/3}} - \frac{54}{e^{x/3}}\right]_7^b = \lim_{b\to\infty} \left(\frac{-3b^2-18b-54}{e^{b/3}} + \frac{327}{e^{7/3}}\right) =$
$= \lim_{b\to\infty} \left(\frac{3(-6b-18)}{e^{b/3}}\right) + \frac{327}{e^{7/3}} = \lim_{b\to\infty} \left(\frac{-54}{e^{b/3}}\right) + \frac{327}{e^{7/3}} = \frac{327}{e^{7/3}} \Rightarrow \int_7^\infty \frac{x^2}{e^{x/3}}\,dx$ converges $\Rightarrow \sum_{n=7}^{\infty} \frac{n^2}{e^{n/3}}$ converges
$\Rightarrow \sum_{n=1}^{\infty} \frac{n^2}{e^{n/3}} = \frac{1}{e^{1/3}} + \frac{4}{e^{2/3}} + \frac{9}{e^1} + \frac{16}{e^{4/3}} + \frac{25}{e^{5/3}} + \frac{36}{e^2} + \sum_{n=7}^{\infty} \frac{n^2}{e^{n/3}}$ converges

10. $f(x) = \frac{x-4}{x^2-2x+1} = \frac{x-4}{(x-1)^2}$ is continuous for $x \ge 2$, f is positive for $x > 4$, and $f'(x) = \frac{7-x}{(x-1)^3} < 0$ for $x > 7$, thus f is decreasing for $x \ge 8$; $\int_8^\infty \frac{x-4}{(x-1)^2}\,dx = \lim_{b\to\infty} \left[\int_8^b \frac{x-1}{(x-1)^2}\,dx - \int_8^b \frac{3}{(x-1)^2}\,dx\right] = \lim_{b\to\infty} \left[\int_8^b \frac{1}{x-1}\,dx - \int_8^b \frac{3}{(x-1)^2}\,dx\right]$
$= \lim_{b\to\infty} \left[\ln|x-1| + \frac{3}{x-1}\right]_8^b = \lim_{b\to\infty} \left(\ln|b-1| + \frac{3}{b-1} - \ln 7 - \frac{3}{7}\right) = \infty \Rightarrow \int_8^\infty \frac{x-4}{(x-1)^2}\,dx$ diverges
$\Rightarrow \sum_{n=8}^{\infty} \frac{n-4}{n^2-2n+1}$ diverges $\Rightarrow \sum_{n=2}^{\infty} \frac{n-4}{n^2-2n+1} = -2 - \frac{1}{4} + 0 + \frac{1}{16} + \frac{2}{25} + \frac{3}{36} + \sum_{n=8}^{\infty} \frac{n-4}{n^2-2n+1}$ diverges

11. converges; a geometric series with $r = \frac{1}{10} < 1$

12. converges; a geometric series with $r = \frac{1}{e} < 1$

13. diverges; by the nth-Term Test for Divergence, $\lim_{n\to\infty} \frac{n}{n+1} = 1 \ne 0$

14. diverges by the Integral Test; $\int_1^n \frac{5}{x+1}\,dx = 5\ln(n+1) - 5\ln 2 \Rightarrow \int_1^\infty \frac{5}{x+1}\,dx \to \infty$

15. diverges; $\sum_{n=1}^{\infty} \frac{3}{\sqrt{n}} = 3\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which is a divergent p-series $\left(p = \frac{1}{2}\right)$

16. converges; $\sum_{n=1}^{\infty} \frac{-2}{n\sqrt{n}} = -2\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$, which is a convergent p-series $\left(p = \frac{3}{2}\right)$

17. converges; a geometric series with $r = \frac{1}{8} < 1$

18. diverges; $\sum_{n=1}^{\infty} \frac{-8}{n} = -8\sum_{n=1}^{\infty} \frac{1}{n}$ and since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, $-8\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

19. diverges by the Integral Test: $\int_2^n \frac{\ln x}{x}\,dx = \frac{1}{2}\left(\ln^2 n - \ln 2\right) \Rightarrow \int_2^\infty \frac{\ln x}{x}\,dx \to \infty$

20. diverges by the Integral Test: $\int_2^\infty \frac{\ln x}{\sqrt{x}}\,dx$; $\begin{bmatrix} t = \ln x \\ dt = \frac{dx}{x} \\ dx = e^t\,dt \end{bmatrix} \to \int_{\ln 2}^\infty te^{t/2}\,dt = \lim_{b\to\infty} \left[2te^{t/2} - 4e^{t/2}\right]_{\ln 2}^{b}$
$= \lim_{b\to\infty} \left[2e^{b/2}(b-2) - 2e^{(\ln 2)/2}(\ln 2 - 2)\right] = \infty$

21. converges; a geometric series with $r = \frac{2}{3} < 1$

22. diverges; $\lim_{n\to\infty} \frac{5^n}{4^n+3} = \lim_{n\to\infty} \frac{5^n \ln 5}{4^n \ln 4} = \lim_{n\to\infty} \left(\frac{\ln 5}{\ln 4}\right)\left(\frac{5}{4}\right)^n = \infty \neq 0$

23. diverges; $\sum_{n=0}^{\infty} \frac{-2}{n+1} = -2\sum_{n=0}^{\infty} \frac{1}{n+1}$, which diverges by the Integral Test

24. diverges by the Integral Test: $\int_1^n \frac{dx}{2x-1} = \frac{1}{2}\ln(2n-1) \to \infty$ as $n \to \infty$

25. diverges; $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2^n}{n+1} = \lim_{n\to\infty} \frac{2^n \ln 2}{1} = \infty \neq 0$

26. diverges by the Integral Test: $\int_1^n \frac{dx}{\sqrt{x}\left(\sqrt{x}+1\right)}$; $\begin{bmatrix} u = \sqrt{x}+1 \\ du = \frac{dx}{\sqrt{x}} \end{bmatrix} \to \int_2^{\sqrt{n}+1} \frac{du}{u} = \ln\left(\sqrt{n}+1\right) - \ln 2 \to \infty$ as $n \to \infty$

27. diverges; $\lim_{n\to\infty} \frac{\sqrt{n}}{\ln n} = \lim_{n\to\infty} \frac{\left(\frac{1}{2\sqrt{n}}\right)}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{\sqrt{n}}{2} = \infty \neq 0$

28. diverges; $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e \neq 0$

29. diverges; a geometric series with $r = \frac{1}{\ln 2} \approx 1.44 > 1$

30. converges; a geometric series with $r = \frac{1}{\ln 3} \approx 0.91 < 1$

31. converges by the Integral Test: $\int_3^\infty \frac{\left(\frac{1}{x}\right)}{(\ln x)\sqrt{(\ln x)^2 - 1}}\,dx$; $\begin{bmatrix} u = \ln x \\ du = \frac{1}{x}\,dx \end{bmatrix} \to \int_{\ln 3}^\infty \frac{1}{u\sqrt{u^2-1}}\,du$

$= \lim_{b \to \infty} \left[\sec^{-1} |u|\right]_{\ln 3}^{b} = \lim_{b \to \infty} \left[\sec^{-1} b - \sec^{-1} (\ln 3)\right] = \lim_{b \to \infty} \left[\cos^{-1}\left(\frac{1}{b}\right) - \sec^{-1} (\ln 3)\right]$
$= \cos^{-1}(0) - \sec^{-1}(\ln 3) = \frac{\pi}{2} - \sec^{-1}(\ln 3) \approx 1.1439$

32. converges by the Integral Test: $\int_1^\infty \frac{1}{x(1+\ln^2 x)}\, dx = \int_1^\infty \frac{\left(\frac{1}{x}\right)}{1+(\ln x)^2}\, dx$; $\begin{bmatrix} u = \ln x \\ du = \frac{1}{x}\, dx \end{bmatrix} \to \int_0^\infty \frac{1}{1+u^2}\, du$
$= \lim_{b \to \infty} \left[\tan^{-1} u\right]_0^b = \lim_{b \to \infty} \left(\tan^{-1} b - \tan^{-1} 0\right) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

33. diverges by the nth-Term Test for divergence; $\lim_{n \to \infty} n \sin\left(\frac{1}{n}\right) = \lim_{n \to \infty} \frac{\sin\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} = \lim_{x \to 0} \frac{\sin x}{x} = 1 \neq 0$

34. diverges by the nth-Term Test for divergence; $\lim_{n \to \infty} n \tan\left(\frac{1}{n}\right) = \lim_{n \to \infty} \frac{\tan\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} = \lim_{n \to \infty} \frac{\left(-\frac{1}{n^2}\right)\sec^2\left(\frac{1}{n}\right)}{\left(-\frac{1}{n^2}\right)}$
$= \lim_{n \to \infty} \sec^2\left(\frac{1}{n}\right) = \sec^2 0 = 1 \neq 0$

35. converges by the Integral Test: $\int_1^\infty \frac{e^x}{1+e^{2x}}\, dx$; $\begin{bmatrix} u = e^x \\ du = e^x\, dx \end{bmatrix} \to \int_e^\infty \frac{1}{1+u^2}\, du = \lim_{n \to \infty} \left[\tan^{-1} u\right]_e^b$
$= \lim_{b \to \infty} \left(\tan^{-1} b - \tan^{-1} e\right) = \frac{\pi}{2} - \tan^{-1} e \approx 0.35$

36. converges by the Integral Test: $\int_1^\infty \frac{2}{1+e^x}\, dx$; $\begin{bmatrix} u = e^x \\ du = e^x\, dx \\ dx = \frac{1}{u}\, du \end{bmatrix} \to \int_e^\infty \frac{2}{u(1+u)}\, du = \int_e^\infty \left(\frac{2}{u} - \frac{2}{u+1}\right) du$
$= \lim_{b \to \infty} \left[2 \ln \frac{u}{u+1}\right]_e^b = \lim_{b \to \infty} 2 \ln\left(\frac{b}{b+1}\right) - 2 \ln\left(\frac{e}{e+1}\right) = 2 \ln 1 - 2 \ln\left(\frac{e}{e+1}\right) = -2 \ln\left(\frac{e}{e+1}\right) \approx 0.63$

37. converges by the Integral Test: $\int_1^\infty \frac{8 \tan^{-1} x}{1+x^2}\, dx$; $\begin{bmatrix} u = \tan^{-1} x \\ du = \frac{dx}{1+x^2} \end{bmatrix} \to \int_{\pi/4}^{\pi/2} 8u\, du = \left[4u^2\right]_{\pi/4}^{\pi/2} = 4\left(\frac{\pi^2}{4} - \frac{\pi^2}{16}\right) = \frac{3\pi^2}{4}$

38. diverges by the Integral Test: $\int_1^\infty \frac{x}{x^2+1}\, dx$; $\begin{bmatrix} u = x^2+1 \\ du = 2x\, dx \end{bmatrix} \to \frac{1}{2}\int_2^\infty \frac{du}{4} = \lim_{b \to \infty} \left[\frac{1}{2} \ln u\right]_2^b = \lim_{b \to \infty} \frac{1}{2}(\ln b - \ln 2) = \infty$

39. converges by the Integral Test: $\int_1^\infty \operatorname{sech} x\, dx = 2 \lim_{b \to \infty} \int_1^b \frac{e^x}{1+(e^x)^2}\, dx = 2 \lim_{b \to \infty} \left[\tan^{-1} e^x\right]_1^b$
$= 2 \lim_{b \to \infty} \left(\tan^{-1} e^b - \tan^{-1} e\right) = \pi - 2 \tan^{-1} e \approx 0.71$

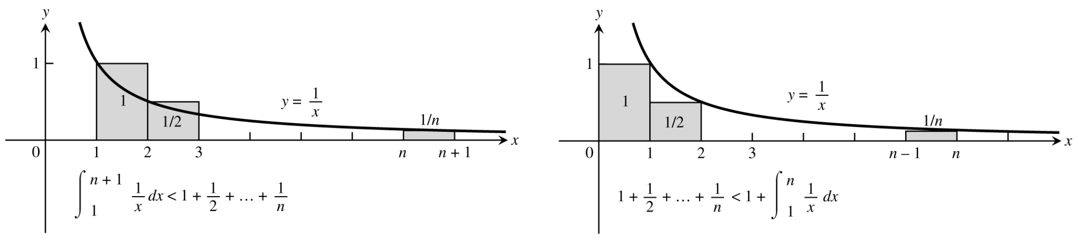
40. converges by the Integral Test: $\int_1^\infty \operatorname{sech}^2 x\, dx = \lim_{b \to \infty} \int_1^b \operatorname{sech}^2 x\, dx = \lim_{b \to \infty} \left[\tanh x\right]_1^b = \lim_{b \to \infty} (\tanh b - \tanh 1)$
$= 1 - \tanh 1 \approx 0.76$

41. $\int_1^\infty \left(\frac{a}{x+2} - \frac{1}{x+4}\right) dx = \lim_{b \to \infty} \left[a \ln|x+2| - \ln|x+4|\right]_1^b = \lim_{b \to \infty} \ln \frac{(b+2)^a}{b+4} - \ln\left(\frac{3^a}{5}\right)$;
$\lim_{b \to \infty} \frac{(b+2)^a}{b+4} = a \lim_{b \to \infty} (b+2)^{a-1} = \begin{cases} \infty, & a > 1 \\ 1, & a = 1 \end{cases} \Rightarrow$ the series converges to $\ln\left(\frac{5}{3}\right)$ if $a = 1$ and diverges to ∞ if $a > 1$. If $a < 1$, the terms of the series eventually become negative and the Integral Test does not apply. From that point on, however, the series behaves like a negative multiple of the harmonic series, and so it diverges.

42. $\int_3^\infty \left(\frac{1}{x-1} - \frac{2a}{x+1}\right) dx = \lim_{b \to \infty} \left[\ln\left|\frac{x-1}{(x+1)^{2a}}\right|\right]_3^b = \lim_{b \to \infty} \ln \frac{b-1}{(b+1)^{2a}} - \ln\left(\frac{2}{4^{2a}}\right)$; $\lim_{b \to \infty} \frac{b-1}{(b+1)^{2a}}$
$= \lim_{b \to \infty} \frac{1}{2a(b+1)^{2a-1}} = \begin{cases} 1, & a = \frac{1}{2} \\ \infty, & a < \frac{1}{2} \end{cases} \Rightarrow$ the series converges to $\ln\left(\frac{4}{2}\right) = \ln 2$ if $a = \frac{1}{2}$ and diverges to ∞ if

if $a < \frac{1}{2}$. If $a > \frac{1}{2}$, the terms of the series eventually become negative and the Integral Test does not apply. From that point on, however, the series behaves like a negative multiple of the harmonic series, and so it diverges.

43. (a)



(b) There are $(13)(365)(24)(60)(60)\,(10^9)$ seconds in 13 billion years; by part (a) $s_n \le 1 + \ln n$ where $n = (13)(365)(24)(60)(60)\,(10^9) \Rightarrow s_n \le 1 + \ln\left((13)(365)(24)(60)(60)\,(10^9)\right)$
$= 1 + \ln(13) + \ln(365) + \ln(24) + 2\ln(60) + 9\ln(10) \approx 41.55$

44. No, because $\sum_{n=1}^{\infty} \frac{1}{nx} = \frac{1}{x}\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

45. Yes. If $\sum_{n=1}^{\infty} a_n$ is a divergent series of positive numbers, then $\left(\frac{1}{2}\right)\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\left(\frac{a_n}{2}\right)$ also diverges and $\frac{a_n}{2} < a_n$.
There is no "smallest" divergent series of positive numbers: for any divergent series $\sum_{n=1}^{\infty} a_n$ of positive numbers $\sum_{n=1}^{\infty}\left(\frac{a_n}{2}\right)$ has smaller terms and still diverges.

46. No, if $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive numbers, then $2\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} 2a_n$ also converges, and $2a_n \ge a_n$.
There is no "largest" convergent series of positive numbers.

47. (a) Both integrals can represent the area under the curve $f(x) = \frac{1}{\sqrt{x+1}}$, and the sum s_{50} can be considered an approximation of either integral using rectangles with $\Delta x = 1$. The sum $s_{50} = \sum_{n=1}^{50} \frac{1}{\sqrt{n+1}}$ is an overestimate of the integral $\int_1^{51} \frac{1}{\sqrt{x+1}}dx$. The sum s_{50} represents a left-hand sum (that is, the we are choosing the left-hand endpoint of each subinterval for c_i) and because f is a decreasing function, the value of f is a maximum at the left-hand endpoint of each sub interval. The area of each rectangle overestimates the true area, thus $\int_1^{51} \frac{1}{\sqrt{x+1}}dx < \sum_{n=1}^{50} \frac{1}{\sqrt{n+1}}$. In a similar manner, s_{50} underestimates the integral $\int_0^{50} \frac{1}{\sqrt{x+1}}dx$. In this case, the sum s_{50} represents a right-hand sum and because f is a decreasing function, the value of f is aminimum at the right-hand endpoint of each subinterval. The area of each rectangle underestimates the true area, thus $\sum_{n=1}^{50} \frac{1}{\sqrt{n+1}} < \int_0^{50} \frac{1}{\sqrt{x+1}}dx$. Evaluating the integrals we find $\int_1^{51} \frac{1}{\sqrt{x+1}}dx$ $= \left[2\sqrt{x+1}\right]_1^{51} = 2\sqrt{52} - 2\sqrt{2} \approx 11.6$ and $\int_0^{50} \frac{1}{\sqrt{x+1}}dx = \left[2\sqrt{x+1}\right]_0^{50} = 2\sqrt{51} - 2\sqrt{1} \approx 12.3$. Thus, $11.6 < \sum_{n=1}^{50} \frac{1}{\sqrt{n+1}} < 12.3$.

(b) $s_n > 1000 \Rightarrow \int_1^{n+1} \frac{1}{\sqrt{x+1}}dx = \left[2\sqrt{x+1}\right]_1^{n+1} = 2\sqrt{n+1} - 2\sqrt{2} > 1000 \Rightarrow n > \left(500 + 2\sqrt{2}\right)^2 - \approx 251414.2$
$\Rightarrow n \ge 251415.$

48. (a) Since we are using $s_{30} = \sum_{n=1}^{30} \frac{1}{n^4}$ to estimate $\sum_{n=1}^{\infty} \frac{1}{n^4}$, the error is given by $\sum_{n=31}^{\infty} \frac{1}{n^4}$. We can consider this sum as an estimate of the area under the curve $f(x) = \frac{1}{x^4}$ when $x \ge 30$ using rectangles with $\Delta x = 1$ and c_i is the right-hand endpoint of each subinterval. Since f is a decreasing function, the value of f is a minimum at the right-hand endpoint of each subinterval, thus $\sum_{n=31}^{\infty} \frac{1}{n^4} < \int_{30}^{\infty} \frac{1}{x^4} dx = \lim_{b\to\infty} \int_{30}^{b} \frac{1}{x^4} dx = \lim_{b\to\infty} \left[-\frac{1}{3x^3}\right]_{30}^{b} = \lim_{b\to\infty} \left(-\frac{1}{3b^3} + \frac{1}{3(30)^3}\right) \approx 1.23 \times 10^{-5}$. Thus the error $< 1.23 \times 10^{-5}$.

(b) We want $S - s_n < 0.000001 \Rightarrow \int_{n}^{\infty} \frac{1}{x^4} dx < 0.000001 \Rightarrow \int_{n}^{\infty} \frac{1}{x^4} dx = \lim_{b\to\infty} \int_{n}^{b} \frac{1}{x^4} dx = \lim_{b\to\infty} \left[-\frac{1}{3x^3}\right]_{n}^{b}$
$= \lim_{b\to\infty} \left(-\frac{1}{3b^3} + \frac{1}{3n^3}\right) = \frac{1}{3n^3} < 0.000001 \Rightarrow n > \sqrt[3]{\frac{1000000}{3}} \approx 69.336 \Rightarrow n \ge 70.$

49. We want $S - s_n < 0.01 \Rightarrow \int_{n}^{\infty} \frac{1}{x^3} dx < 0.01 \Rightarrow \int_{n}^{\infty} \frac{1}{x^3} dx = \lim_{b\to\infty} \int_{n}^{b} \frac{1}{x^3} dx = \lim_{b\to\infty} \left[-\frac{1}{2x^2}\right]_{n}^{b} = \lim_{b\to\infty} \left(-\frac{1}{2b^2} + \frac{1}{2n^2}\right)$
$= \frac{1}{2n^2} < 0.01 \Rightarrow n > \sqrt{50} \approx 7.071 \Rightarrow n \ge 8 \Rightarrow S \approx s_8 = \sum_{n=1}^{8} \frac{1}{n^3} \approx 1.195$

50. We want $S - s_n < 0.1 \Rightarrow \int_{n}^{\infty} \frac{1}{x^2+4} dx < 0.1 \Rightarrow \lim_{b\to\infty} \int_{n}^{b} \frac{1}{x^2+4} dx = \lim_{b\to\infty} \left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_{n}^{b}$
$= \lim_{b\to\infty} \left(\frac{1}{2}\tan^{-1}\left(\frac{b}{2}\right) - \frac{1}{2}\tan^{-1}\left(\frac{n}{2}\right)\right) = \frac{\pi}{4} - \frac{1}{2}\tan^{-1}\left(\frac{n}{2}\right) < 0.1 \Rightarrow n > 2\tan\left(\frac{\pi}{2} - 0.2\right) \approx 9.867 \Rightarrow n \ge 10 \Rightarrow S \approx s_{10}$
$= \sum_{n=1}^{10} \frac{1}{n^2+4} \approx 0.57$

51. $S - s_n < 0.00001 \Rightarrow \int_{n}^{\infty} \frac{1}{x^{1.1}} dx < 0.00001 \Rightarrow \int_{n}^{\infty} \frac{1}{x^{1.1}} dx = \lim_{b\to\infty} \int_{n}^{b} \frac{1}{x^{1.1}} dx = \lim_{b\to\infty} \left[-\frac{10}{x^{0.1}}\right]_{n}^{b} = \lim_{b\to\infty} \left(-\frac{10}{b^{0.1}} + \frac{10}{n^{0.1}}\right)$
$= \frac{10}{n^{0.1}} < 0.00001 \Rightarrow n > 1000000^{10} \Rightarrow n > 10^{60}$

52. $S - s_n < 0.01 \Rightarrow \int_{n}^{\infty} \frac{1}{x(\ln x)^3} dx < 0.01 \Rightarrow \int_{n}^{\infty} \frac{1}{x(\ln x)^3} dx = \lim_{b\to\infty} \int_{n}^{b} \frac{1}{x(\ln x)^3} dx = \lim_{b\to\infty} \left[-\frac{1}{2(\ln x)^2}\right]_{n}^{b}$
$= \lim_{b\to\infty} \left(-\frac{1}{2(\ln b)^2} + \frac{1}{2(\ln n)^2}\right) = \frac{1}{2(\ln n)^2} < 0.01 \Rightarrow n > e^{\sqrt{50}} \approx 1177.405 \Rightarrow n \ge 1178$

53. Let $A_n = \sum_{k=1}^{n} a_k$ and $B_n = \sum_{k=1}^{n} 2^k a_{(2^k)}$, where $\{a_k\}$ is a nonincreasing sequence of positive terms converging to 0. Note that $\{A_n\}$ and $\{B_n\}$ are nondecreasing sequences of positive terms. Now,
$B_n = 2a_2 + 4a_4 + 8a_8 + \ldots + 2^n a_{(2^n)} = 2a_2 + (2a_4 + 2a_4) + (2a_8 + 2a_8 + 2a_8 + 2a_8) + \ldots$
$+ \underbrace{\left(2a_{(2^n)} + 2a_{(2^n)} + \ldots + 2a_{(2^n)}\right)}_{2^{n-1} \text{ terms}} \le 2a_1 + 2a_2 + (2a_3 + 2a_4) + (2a_5 + 2a_6 + 2a_7 + 2a_8) + \ldots$
$+ \left(2a_{(2^{n-1})} + 2a_{(2^{n-1}+1)} + \ldots + 2a_{(2^n)}\right) = 2A_{(2^n)} \le 2\sum_{k=1}^{\infty} a_k$. Therefore if $\sum a_k$ converges,
then $\{B_n\}$ is bounded above $\Rightarrow \sum 2^k a_{(2^k)}$ converges. Conversely,
$A_n = a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \ldots + a_n < a_1 + 2a_2 + 4a_4 + \ldots + 2^n a_{(2^n)} = a_1 + B_n < a_1 + \sum_{k=1}^{\infty} 2^k a_{(2^k)}$.
Therefore, if $\sum_{k=1}^{\infty} 2^k a_{(2^k)}$ converges, then $\{A_n\}$ is bounded above and hence converges.

54. (a) $a_{(2^n)} = \frac{1}{2^n \ln(2^n)} = \frac{1}{2^n \cdot n(\ln 2)} \Rightarrow \sum_{n=2}^{\infty} 2^n a_{(2^n)} = \sum_{n=2}^{\infty} 2^n \frac{1}{2^n \cdot n(\ln 2)} = \frac{1}{\ln 2} \sum_{n=2}^{\infty} \frac{1}{n}$, which diverges
$\Rightarrow \sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges.

(b) $a_{(2^n)} = \frac{1}{2^{np}} \Rightarrow \sum_{n=1}^{\infty} 2^n a_{(2^n)} = \sum_{n=1}^{\infty} 2^n \cdot \frac{1}{2^{np}} = \sum_{n=1}^{\infty} \frac{1}{(2^n)^{p-1}} = \sum_{n=1}^{\infty} \left(\frac{1}{2^{p-1}}\right)^n$, a geometric series that converges if $\frac{1}{2^{p-1}} < 1$ or $p > 1$, but diverges if $p \le 1$.

55. (a) $\int_2^{\infty} \frac{dx}{x(\ln x)^p}$; $\begin{bmatrix} u = \ln x \\ du = \frac{dx}{x} \end{bmatrix} \to \int_{\ln 2}^{\infty} u^{-p}\, du = \lim_{b\to\infty} \left[\frac{u^{-p+1}}{-p+1}\right]_{\ln 2}^{b} = \lim_{b\to\infty} \left(\frac{1}{1-p}\right)\left[b^{-p+1} - (\ln 2)^{-p+1}\right]$

$= \begin{cases} \frac{1}{p-1}(\ln 2)^{-p+1}, & p > 1 \\ \infty, & p < 1 \end{cases} \Rightarrow$ the improper integral converges if $p > 1$ and diverges if $p < 1$.

For $p = 1$: $\int_2^{\infty} \frac{dx}{x \ln x} = \lim_{b\to\infty} \left[\ln(\ln x)\right]_2^b = \lim_{b\to\infty} \left[\ln(\ln b) - \ln(\ln 2)\right] = \infty$, so the improper integral diverges if $p = 1$.

(b) Since the series and the integral converge or diverge together, $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges if and only if $p > 1$.

56. (a) $p = 1 \Rightarrow$ the series diverges

(b) $p = 1.01 \Rightarrow$ the series converges

(c) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n^3)} = \frac{1}{3} \sum_{n=2}^{\infty} \frac{1}{n(\ln n)}$; $p = 1 \Rightarrow$ the series diverges

(d) $p = 3 \Rightarrow$ the series converges

57. (a) From Fig. 10.11(a) in the text with $f(x) = \frac{1}{x}$ and $a_k = \frac{1}{k}$, we have $\int_1^{n+1} \frac{1}{x}\, dx \le 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$

$\le 1 + \int_1^n f(x)\, dx \Rightarrow \ln(n+1) \le 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \le 1 + \ln n \Rightarrow 0 \le \ln(n+1) - \ln n$

$\le \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right) - \ln n \le 1$. Therefore the sequence $\left\{\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right) - \ln n\right\}$ is bounded above by 1 and below by 0.

(b) From the graph in Fig. 10.11(b) with $f(x) = \frac{1}{x}$, $\frac{1}{n+1} < \int_n^{n+1} \frac{1}{x}\, dx = \ln(n+1) - \ln n$

$\Rightarrow 0 > \frac{1}{n+1} - \left[\ln(n+1) - \ln n\right] = \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n+1} - \ln(n+1)\right) - \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln n\right)$.

If we define $a_n = 1 + \frac{1}{2} = \frac{1}{3} + \frac{1}{n} - \ln n$, then $0 > a_{n+1} - a_n \Rightarrow a_{n+1} < a_n \Rightarrow \{a_n\}$ is a decreasing sequence of nonnegative terms.

58. $e^{-x^2} \le e^{-x}$ for $x \ge 1$, and $\int_1^{\infty} e^{-x}\, dx = \lim_{b\to\infty} \left[-e^{-x}\right]_1^b = \lim_{b\to\infty} \left(-e^{-b} + e^{-1}\right) = e^{-1} \Rightarrow \int_1^{\infty} e^{-x^2}\, dx$ converges by the Comparison Test for improper integrals $\Rightarrow \sum_{n=0}^{\infty} e^{-n^2} = 1 + \sum_{n=1}^{\infty} e^{-n^2}$ converges by the Integral Test.

59. (a) $s_{10} = \sum_{n=1}^{10} \frac{1}{n^3} = 1.97531986$; $\int_{11}^{\infty} \frac{1}{x^3}\, dx = \lim_{b\to\infty} \int_{11}^{b} x^{-3}\, dx = \lim_{b\to\infty} \left[-\frac{x^{-2}}{2}\right]_{11}^{b} = \lim_{b\to\infty} \left(-\frac{1}{2b^2} + \frac{1}{242}\right) = \frac{1}{242}$ and

$\int_{10}^{\infty} \frac{1}{x^3}\, dx = \lim_{b\to\infty} \int_{10}^{b} x^{-3}\, dx = \lim_{b\to\infty} \left[-\frac{x^{-2}}{2}\right]_{10}^{b} = \lim_{b\to\infty} \left(-\frac{1}{2b^2} + \frac{1}{200}\right) = \frac{1}{200}$

$\Rightarrow 1.97531986 + \frac{1}{242} < s < 1.97531986 + \frac{1}{200} \Rightarrow 1.20166 < s < 1.20253$

(b) $s = \sum_{n=1}^{\infty} \frac{1}{n^3} \approx \frac{1.20166 + 1.20253}{2} = 1.202095$; error $\le \frac{1.20253 - 1.20166}{2} = 0.000435$

60. (a) $s_{10} = \sum_{n=1}^{10} \frac{1}{n^4} = 1.082036583$; $\int_{11}^{\infty} \frac{1}{x^4}\, dx = \lim_{b\to\infty} \int_{11}^{b} x^{-4}\, dx = \lim_{b\to\infty} \left[-\frac{x^{-3}}{3}\right]_{11}^{b} = \lim_{b\to\infty} \left(-\frac{1}{3b^3} + \frac{1}{3993}\right) = \frac{1}{3993}$ and

$\int_{10}^{\infty} \frac{1}{x^4}\, dx = \lim_{b\to\infty} \int_{10}^{b} x^{-4}\, dx = \lim_{b\to\infty} \left[-\frac{x^{-3}}{3}\right]_{10}^{b} = \lim_{b\to\infty} \left(-\frac{1}{3b^3} + \frac{1}{3000}\right) = \frac{1}{3000}$

$\Rightarrow 1.082036583 + \frac{1}{3993} < s < 1.082036583 + \frac{1}{3000} \Rightarrow 1.08229 < s < 1.08237$

(b) $s = \sum_{n=1}^{\infty} \frac{1}{n^4} \approx \frac{1.08229 + 1.08237}{2} = 1.08233$; error $\le \frac{1.08237 - 1.08229}{2} = 0.00004$

10.4 COMPARISON TESTS

1. Compare with $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which is a convergent p-series, since $p = 2 > 1$. Both series have nonnegative terms for $n \geq 1$. For $n \geq 1$, we have $n^2 \leq n^2 + 30 \Rightarrow \frac{1}{n^2} \geq \frac{1}{n^2+30}$. Then by Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{n^2+30}$ converges.

2. Compare with $\sum_{n=1}^{\infty} \frac{1}{n^3}$, which is a convergent p-series, since $p = 3 > 1$. Both series have nonnegative terms for $n \geq 1$. For $n \geq 1$, we have $n^4 \leq n^4 + 2 \Rightarrow \frac{1}{n^4} \geq \frac{1}{n^4+2} \Rightarrow \frac{n}{n^4} \geq \frac{n}{n^4+2} \Rightarrow \frac{1}{n^3} \geq \frac{n}{n^4+2} \geq \frac{n-1}{n^4+2}$. Then by Comparison Test, $\sum_{n=1}^{\infty} \frac{n-1}{n^4+2}$ converges.

3. Compare with $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$, which is a divergent p-series, since $p = \frac{1}{2} \leq 1$. Both series have nonnegative terms for $n \geq 2$. For $n \geq 2$, we have $\sqrt{n} - 1 \leq \sqrt{n} \Rightarrow \frac{1}{\sqrt{n}-1} \geq \frac{1}{\sqrt{n}}$. Then by Comparison Test, $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$ diverges.

4. Compare with $\sum_{n=2}^{\infty} \frac{1}{n}$, which is a divergent p-series, since $p = 1 \leq 1$. Both series have nonnegative terms for $n \geq 2$. For $n \geq 2$, we have $n^2 - n \leq n^2 \Rightarrow \frac{1}{n^2-n} \geq \frac{1}{n^2} \Rightarrow \frac{n}{n^2-n} \geq \frac{n}{n^2} = \frac{1}{n} \Rightarrow \frac{n+2}{n^2-n} \geq \frac{n}{n^2-n} \geq \frac{1}{n}$. Thus $\sum_{n=2}^{\infty} \frac{n+2}{n^2-n}$ diverges.

5. Compare with $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$, which is a convergent p-series, since $p = \frac{3}{2} > 1$. Both series have nonnegative terms for $n \geq 1$. For $n \geq 1$, we have $0 \leq \cos^2 n \leq 1 \Rightarrow \frac{\cos^2 n}{n^{3/2}} \leq \frac{1}{n^{3/2}}$. Then by Comparison Test, $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^{3/2}}$ converges.

6. Compare with $\sum_{n=1}^{\infty} \frac{1}{3^n}$, which is a convergent geometric series, since $|r| = \left|\frac{1}{3}\right| < 1$. Both series have nonnegative terms for $n \geq 1$. For $n \geq 1$, we have $n \cdot 3^n \geq 3^n \Rightarrow \frac{1}{n \cdot 3^n} \leq \frac{1}{3^n}$. Then by Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{n \cdot 3^n}$ converges.

7. Compare with $\sum_{n=1}^{\infty} \frac{\sqrt{5}}{n^{3/2}}$. The series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is a convergent p-series, since $p = \frac{3}{2} > 1$, and the series $\sum_{n=1}^{\infty} \frac{\sqrt{5}}{n^{3/2}} = \sqrt{5} \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges by Theorem 8 part 3. Both series have nonnegative terms for $n \geq 1$. For $n \geq 1$, we have $n^3 \leq n^4 \Rightarrow 4n^3 \leq 4n^4 \Rightarrow n^4 + 4n^3 \leq n^4 + 4n^4 = 5n^4 \Rightarrow n^4 + 4n^3 \leq 5n^4 + 20 = 5(n^4 + 4) \Rightarrow \frac{n^4+4n^3}{n^4+4} \leq 5$.
$\Rightarrow \frac{n^3(n+4)}{n^4+4} \leq 5 \Rightarrow \frac{n+4}{n^4+4} \leq \frac{5}{n^3} \Rightarrow \sqrt{\frac{n+4}{n^4+4}} \leq \sqrt{\frac{5}{n^3}} = \frac{\sqrt{5}}{n^{3/2}}$ Then by Comparison Test, $\sum_{n=1}^{\infty} \sqrt{\frac{n+4}{n^4+4}}$ converges.

8. Compare with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which is a divergent p-series, since $p = \frac{1}{2} \leq 1$. Both series have nonnegative terms for $n \geq 1$. For $n \geq 1$, we have $\sqrt{n} \geq 1 \Rightarrow 2\sqrt{n} \geq 2 \Rightarrow 2\sqrt{n} + 1 \geq 3 \Rightarrow n\left(2\sqrt{n} + 1\right) \geq 3n \geq 3 \Rightarrow 2n\sqrt{n} + n \geq 3$
$\Rightarrow n^2 + 2n\sqrt{n} + n \geq n^2 + 3 \Rightarrow \frac{n\left(n+2\sqrt{n}+1\right)}{n^2+3} \geq 1 \Rightarrow \frac{n+2\sqrt{n}+1}{n^2+3} \geq \frac{1}{n} \Rightarrow \frac{\left(\sqrt{n}+1\right)^2}{n^2+3} \geq \frac{1}{n} \Rightarrow \sqrt{\frac{\left(\sqrt{n}+1\right)^2}{n^2+3}} \geq \sqrt{\frac{1}{n}}$
$\Rightarrow \frac{\sqrt{n}+1}{\sqrt{n^2+3}} \geq \frac{1}{\sqrt{n}}$. Then by Comparison Test, $\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{\sqrt{n^2+3}}$ diverges.

9. Compare with $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which is a convergent p-series, since $p = 2 > 1$. Both series have positive terms for $n \geq 1$. $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n-2}{n^3-n^2+3}}{1/n^2} = \lim_{n\to\infty} \frac{n^3-2n^2}{n^3-n^2+3} = \lim_{n\to\infty} \frac{3n^2-4n}{3n^2-2n} = \lim_{n\to\infty} \frac{6n-4}{6n-2} = \lim_{n\to\infty} \frac{6}{6} = 1 > 0$. Then by Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{n-2}{n^3-n^2+3}$ converges.

10. Compare with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which is a divergent p-series, since $p = \frac{1}{2} \leq 1$. Both series have positive terms for $n \geq 1$. $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{\frac{n+1}{n^2+2}}}{1/\sqrt{n}} = \lim_{n\to\infty} \sqrt{\frac{n^2+n}{n^2+2}} = \sqrt{\lim_{n\to\infty} \frac{n^2+n}{n^2+2}} = \sqrt{\lim_{n\to\infty} \frac{2n+1}{2n}} = \sqrt{\lim_{n\to\infty} \frac{2}{2}} = \sqrt{1} = 1 > 0$. Then by Limit Comparison Test, $\sum_{n=1}^{\infty} \sqrt{\frac{n+1}{n^2+2}}$ diverges.

11. Compare with $\sum_{n=2}^{\infty} \frac{1}{n}$, which is a divergent p-series, since $p = 1 \leq 1$. Both series have positive terms for $n \geq 2$. $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n(n+1)}{(n^2+1)(n-1)}}{1/n} = \lim_{n\to\infty} \frac{n^3+n^2}{n^3-n^2+n-1} = \lim_{n\to\infty} \frac{3n^2+2n}{3n^2-2n+1} = \lim_{n\to\infty} \frac{6n+2}{6n-2} = \lim_{n\to\infty} \frac{6}{6} = 1 > 0$. Then by Limit Comparison Test, $\sum_{n=2}^{\infty} \frac{n(n+1)}{(n^2+1)(n-1)}$ diverges.

12. Compare with $\sum_{n=1}^{\infty} \frac{1}{2^n}$, which is a convergent geometric series, since $|r| = \left|\frac{1}{2}\right| < 1$. Both series have positive terms for $n \geq 1$. $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{2^n}{3+4^n}}{1/2^n} = \lim_{n\to\infty} \frac{4^n}{3+4^n} = \lim_{n\to\infty} \frac{4^n \ln 4}{4^n \ln 4} = 1 > 0$. Then by Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{2^n}{3+4^n}$ converges.

13. Compare with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which is a divergent p-series, since $p = \frac{1}{2} \leq 1$. Both series have positive terms for $n \geq 1$. $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{5^n}{\sqrt{n}\cdot 4^n}}{1/\sqrt{n}} = \lim_{n\to\infty} \frac{5^n}{4^n} = \lim_{n\to\infty} \left(\frac{5}{4}\right)^n = \infty$. Then by Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{5^n}{\sqrt{n}\cdot 4^n}$ diverges.

14. Compare with $\sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^n$, which is a convergent geometric series, since $|r| = \left|\frac{2}{5}\right| < 1$. Both series have positive terms for $n \geq 1$. $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{2n+3}{5n+4}\right)^n}{(2/5)^n} = \lim_{n\to\infty} \left(\frac{10n+15}{10n+8}\right)^n = \exp \lim_{n\to\infty} \ln\left(\frac{10n+15}{10n+8}\right)^n = \exp \lim_{n\to\infty} n \ln\left(\frac{10n+15}{10n+8}\right)$
$= \exp \lim_{n\to\infty} \frac{\ln\left(\frac{10n+15}{10n+8}\right)}{1/n} = \exp \lim_{n\to\infty} \frac{\frac{10}{10n+15} - \frac{10}{10n+8}}{-1/n^2} = \exp \lim_{n\to\infty} \frac{70n^2}{(10n+15)(10n+8)} = \exp \lim_{n\to\infty} \frac{70n^2}{100n^2+230n+120}$
$= \exp \lim_{n\to\infty} \frac{140n}{200n+230} = \exp \lim_{n\to\infty} \frac{140}{200} = e^{7/10} > 0$. Then by Limit Comparison Test, $\sum_{n=1}^{\infty} \left(\frac{2n+3}{5n+4}\right)^n$ converges.

15. Compare with $\sum_{n=2}^{\infty} \frac{1}{n}$, which is a divergent p-series, since $p = 1 \leq 1$. Both series have positive terms for $n \geq 2$. $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{\ln n}}{1/n} = \lim_{n\to\infty} \frac{n}{\ln n} = \lim_{n\to\infty} \frac{1}{1/n} = \lim_{n\to\infty} n = \infty$. Then by Limit Comparison Test, $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ diverges.

16. Compare with $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which is a convergent p-series, since $p = 2 > 1$. Both series have positive terms for $n \geq 1$. $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n^2}\right)}{1/n^2} = \lim_{n\to\infty} \frac{\frac{1}{1+\frac{1}{n^2}}\left(-\frac{2}{n^3}\right)}{\left(-\frac{2}{n^3}\right)} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^2}} = 1 > 0$. Then by Limit Comparison Test, $\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n^2}\right)$ converges.

17. diverges by the Limit Comparison Test (part 1) when compared with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, a divergent p-series:

$$\lim_{n\to\infty} \frac{\left(\frac{1}{2\sqrt{n}+\sqrt[3]{n}}\right)}{\left(\frac{1}{\sqrt{n}}\right)} = \lim_{n\to\infty} \frac{\sqrt{n}}{2\sqrt{n}+\sqrt[3]{n}} = \lim_{n\to\infty} \left(\frac{1}{2+n^{-1/6}}\right) = \frac{1}{2}$$

18. diverges by the Direct Comparison Test since $n+n+n > n+\sqrt{n}+0 \Rightarrow \frac{3}{n+\sqrt{n}} > \frac{1}{n}$, which is the nth term of the divergent series $\sum_{n=1}^{\infty} \frac{1}{n}$ or use Limit Comparison Test with $b_n = \frac{1}{n}$

19. converges by the Direct Comparison Test; $\frac{\sin^2 n}{2^n} \le \frac{1}{2^n}$, which is the nth term of a convergent geometric series

20. converges by the Direct Comparison Test; $\frac{1+\cos n}{n^2} \le \frac{2}{n^2}$ and the p-series $\sum \frac{1}{n^2}$ converges

21. diverges since $\lim_{n\to\infty} \frac{2n}{3n-1} = \frac{2}{3} \ne 0$

22. converges by the Limit Comparison Test (part 1) with $\frac{1}{n^{3/2}}$, the nth term of a convergent p-series:

$$\lim_{n\to\infty} \frac{\left(\frac{n+1}{n^2\sqrt{n}}\right)}{\left(\frac{1}{n^{3/2}}\right)} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right) = 1$$

23. converges by the Limit Comparison Test (part 1) with $\frac{1}{n^2}$, the nth term of a convergent p-series:

$$\lim_{n\to\infty} \frac{\left(\frac{10n+1}{n(n+1)(n+2)}\right)}{\left(\frac{1}{n^2}\right)} = \lim_{n\to\infty} \frac{10n^2+n}{n^2+3n+2} = \lim_{n\to\infty} \frac{20n+1}{2n+3} = \lim_{n\to\infty} \frac{20}{2} = 10$$

24. converges by the Limit Comparison Test (part 1) with $\frac{1}{n^2}$, the nth term of a convergent p-series:

$$\lim_{n\to\infty} \frac{\left(\frac{5n^3-3n}{n^2(n-2)\left(n^2+5\right)}\right)}{\left(\frac{1}{n^2}\right)} = \lim_{n\to\infty} \frac{5n^3-3n}{n^3-2n^2+5n-10} = \lim_{n\to\infty} \frac{15n^2-3}{3n^2-4n+5} = \lim_{n\to\infty} \frac{30n}{6n-4} = 5$$

25. converges by the Direct Comparison Test; $\left(\frac{n}{3n+1}\right)^n < \left(\frac{n}{3n}\right)^n = \left(\frac{1}{3}\right)^n$, the nth term of a convergent geometric series

26. converges by the Limit Comparison Test (part 1) with $\frac{1}{n^{3/2}}$, the nth term of a convergent p-series:

$$\lim_{n\to\infty} \frac{\left(\frac{1}{n^{3/2}}\right)}{\left(\frac{1}{\sqrt{n^3+2}}\right)} = \lim_{n\to\infty} \sqrt{\frac{n^3+2}{n^3}} = \lim_{n\to\infty} \sqrt{1+\frac{2}{n^3}} = 1$$

27. diverges by the Direct Comparison Test; $n > \ln n \Rightarrow \ln n > \ln \ln n \Rightarrow \frac{1}{n} < \frac{1}{\ln n} < \frac{1}{\ln(\ln n)}$ and $\sum_{n=3}^{\infty} \frac{1}{n}$ diverges

28. converges by the Limit Comparison Test (part 2) when compared with $\sum_{n=1}^{\infty} \frac{1}{n^2}$, a convergent p-series:

$$\lim_{n\to\infty} \frac{\left[\frac{(\ln n)^2}{n^3}\right]}{\left(\frac{1}{n^2}\right)} = \lim_{n\to\infty} \frac{(\ln n)^2}{n} = \lim_{n\to\infty} \frac{2(\ln n)\left(\frac{1}{n}\right)}{1} = 2\lim_{n\to\infty} \frac{\ln n}{n} = 0$$

29. diverges by the Limit Comparison Test (part 3) with $\frac{1}{n}$, the nth term of the divergent harmonic series:

$$\lim_{n\to\infty} \frac{\left[\frac{1}{\sqrt{n}\ln n}\right]}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{\sqrt{n}}{\ln n} = \lim_{n\to\infty} \frac{\left(\frac{1}{2\sqrt{n}}\right)}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{\sqrt{n}}{2} = \infty$$

30. converges by the Limit Comparison Test (part 2) with $\frac{1}{n^{5/4}}$, the nth term of a convergent p-series:

$$\lim_{n \to \infty} \frac{\left[\frac{(\ln n)^2}{n^{3/2}}\right]}{\left(\frac{1}{n^{5/4}}\right)} = \lim_{n \to \infty} \frac{(\ln n)^2}{n^{1/4}} = \lim_{n \to \infty} \frac{\left(\frac{2 \ln n}{n}\right)}{\left(\frac{1}{4n^{3/4}}\right)} = 8 \lim_{n \to \infty} \frac{\ln n}{n^{1/4}} = 8 \lim_{n \to \infty} \frac{\left(\frac{1}{n}\right)}{\left(\frac{1}{4n^{3/4}}\right)} = 32 \lim_{n \to \infty} \frac{1}{n^{1/4}} = 32 \cdot 0 = 0$$

31. diverges by the Limit Comparison Test (part 3) with $\frac{1}{n}$, the nth term of the divergent harmonic series:

$$\lim_{n \to \infty} \frac{\left(\frac{1}{1+\ln n}\right)}{\left(\frac{1}{n}\right)} = \lim_{n \to \infty} \frac{n}{1+\ln n} = \lim_{n \to \infty} \frac{1}{\left(\frac{1}{n}\right)} = \lim_{n \to \infty} n = \infty$$

32. diverges by the Integral Test: $\int_2^\infty \frac{\ln(x+1)}{x+1}\,dx = \int_{\ln 3}^\infty u\,du = \lim_{b \to \infty} \left[\frac{1}{2}u^2\right]_{\ln 3}^{b} = \lim_{b \to \infty} \frac{1}{2}\left(b^2 - \ln^2 3\right) = \infty$

33. converges by the Direct Comparison Test with $\frac{1}{n^{3/2}}$, the nth term of a convergent p-series: $n^2 - 1 > n$ for $n \ge 2 \Rightarrow n^2\left(n^2 - 1\right) > n^3 \Rightarrow n\sqrt{n^2 - 1} > n^{3/2} \Rightarrow \frac{1}{n^{3/2}} > \frac{1}{n\sqrt{n^2-1}}$ or use Limit Comparison Test with $\frac{1}{n^2}$.

34. converges by the Direct Comparison Test with $\frac{1}{n^{3/2}}$, the nth term of a convergent p-series: $n^2 + 1 > n^2$ $\Rightarrow n^2 + 1 > \sqrt{n}n^{3/2} \Rightarrow \frac{n^2+1}{\sqrt{n}} > n^{3/2} \Rightarrow \frac{\sqrt{n}}{n^2+1} < \frac{1}{n^{3/2}}$ or use Limit Comparison Test with $\frac{1}{n^{3/2}}$.

35. converges because $\sum_{n=1}^{\infty} \frac{1-n}{n2^n} = \sum_{n=1}^{\infty} \frac{1}{n2^n} + \sum_{n=1}^{\infty} \frac{-1}{2^n}$ which is the sum of two convergent series: $\sum_{n=1}^{\infty} \frac{1}{n2^n}$ converges by the Direct Comparison Test since $\frac{1}{n2^n} < \frac{1}{2^n}$, and $\sum_{n=1}^{\infty} \frac{-1}{2^n}$ is a convergent geometric series

36. converges by the Direct Comparison Test: $\sum_{n=1}^{\infty} \frac{n+2^n}{n^2 2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{n2^n} + \frac{1}{n^2}\right)$ and $\frac{1}{n2^n} + \frac{1}{n^2} \le \frac{1}{2^n} + \frac{1}{n^2}$, the sum of the nth terms of a convergent geometric series and a convergent p-series

37. converges by the Direct Comparison Test: $\frac{1}{3^{n-1}+1} < \frac{1}{3^{n-1}}$, which is the nth term of a convergent geometric series

38. diverges; $\lim_{n \to \infty} \left(\frac{3^{n-1}+1}{3^n}\right) = \lim_{n \to \infty} \left(\frac{1}{3} + \frac{1}{3^n}\right) = \frac{1}{3} \ne 0$

39. converges by Limit Comparison Test: compare with $\sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^n$, which is a convergent geometric series with $|r| = \frac{1}{5} < 1$,

$$\lim_{n \to \infty} \frac{\left(\frac{n+1}{n^2+3n} \cdot \frac{1}{5^n}\right)}{(1/5)^n} = \lim_{n \to \infty} \frac{n+1}{n^2+3n} = \lim_{n \to \infty} \frac{1}{2n+3} = 0.$$

40. converges by Limit Comparison Test: compare with $\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$, which is a convergent geometric series with $|r| = \frac{1}{5} < 1$,

$$\lim_{n \to \infty} \frac{\left(\frac{2^n+3^n}{3^n+4^n}\right)}{(3/4)^n} = \lim_{n \to \infty} \frac{8^n+12^n}{9^n+12^n} = \lim_{n \to \infty} \frac{\left(\frac{8}{12}\right)^n+1}{\left(\frac{9}{12}\right)^n+1} = \frac{1}{1} = 1 > 0.$$

41. diverges by Limit Comparison Test: compare with $\sum_{n=1}^{\infty} \frac{1}{n}$, which is a divergent p-series, $\lim_{n \to \infty} \frac{\left(\frac{2^n-n}{n \cdot 2^n}\right)}{1/n} = \lim_{n \to \infty} \frac{2^n-n}{2^n}$ $= \lim_{n \to \infty} \frac{2^n \ln 2 - 1}{2^n \ln 2} = \lim_{n \to \infty} \frac{2^n(\ln 2)^2}{2^n(\ln 2)^2} = 1 > 0.$

42. diverges by the definition of an infinite series: $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right) = \sum_{n=1}^{\infty} \left[\ln n - \ln(n+1)\right]$, $s_k = (\ln 1 - \ln 2) + (\ln 2 - \ln 3)$ $+ \ldots + (\ln(k-1) - \ln k) + (\ln k - \ln(k+1)) = -\ln(k+1) \Rightarrow \lim_{k \to \infty} s_k = -\infty$

43. converges by Comparison Test with $\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$ which converges since $\sum_{n=2}^{\infty} \frac{1}{n(n-1)} = \sum_{n=2}^{\infty} \left[\frac{1}{n-1} - \frac{1}{n}\right]$, and $s_k = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots + \left(\frac{1}{k-2} - \frac{1}{k-1}\right) + \left(\frac{1}{k-1} - \frac{1}{k}\right) = 1 - \frac{1}{k} \Rightarrow \lim_{k \to \infty} s_k = 1$; for $n \geq 2$, $(n-2)! \geq 1$ $\Rightarrow n(n-1)(n-2)! \geq n(n-1) \Rightarrow n! \geq n(n-1) \Rightarrow \frac{1}{n!} \leq \frac{1}{n(n-1)}$

44. converges by Limit Comparison Test: compare with $\sum_{n=1}^{\infty} \frac{1}{n^3}$, which is a convergent p-series, $\lim_{n \to \infty} \frac{\frac{(n-1)!}{(n+2)!}}{1/n^3}$ $= \lim_{n \to \infty} \frac{n^3(n-1)!}{(n+2)(n+1)n(n-1)!} = \lim_{n \to \infty} \frac{n^2}{n^2+3n+2} = \lim_{n \to \infty} \frac{2n}{2n+3} = \lim_{n \to \infty} \frac{2}{2} = 1 > 0$

45. diverges by the Limit Comparison Test (part 1) with $\frac{1}{n}$, the nth term of the divergent harmonic series: $\lim_{n \to \infty} \frac{\left(\sin \frac{1}{n}\right)}{\left(\frac{1}{n}\right)} = \lim_{x \to 0} \frac{\sin x}{x} = 1$

46. diverges by the Limit Comparison Test (part 1) with $\frac{1}{n}$, the nth term of the divergent harmonic series: $\lim_{n \to \infty} \frac{\left(\tan \frac{1}{n}\right)}{\left(\frac{1}{n}\right)} = \lim_{n \to \infty} \left(\frac{1}{\cos \frac{1}{n}}\right) \frac{\left(\sin \frac{1}{n}\right)}{\left(\frac{1}{n}\right)} = \lim_{x \to 0} \left(\frac{1}{\cos x}\right)\left(\frac{\sin x}{x}\right) = 1 \cdot 1 = 1$

47. converges by the Direct Comparison Test: $\frac{\tan^{-1} n}{n^{1.1}} < \frac{\frac{\pi}{2}}{n^{1.1}}$ and $\sum_{n=1}^{\infty} \frac{\frac{\pi}{2}}{n^{1.1}} = \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^{1.1}}$ is the product of a convergent p-series and a nonzero constant

48. converges by the Direct Comparison Test: $\sec^{-1} n < \frac{\pi}{2} \Rightarrow \frac{\sec^{-1} n}{n^{1.3}} < \frac{\left(\frac{\pi}{2}\right)}{n^{1.3}}$ and $\sum_{n=1}^{\infty} \frac{\left(\frac{\pi}{2}\right)}{n^{1.3}} = \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^{1.3}}$ is the product of a convergent p-series and a nonzero constant

49. converges by the Limit Comparison Test (part 1) with $\frac{1}{n^2}$: $\lim_{n \to \infty} \frac{\left(\frac{\coth n}{n^2}\right)}{\left(\frac{1}{n^2}\right)} = \lim_{n \to \infty} \coth n = \lim_{n \to \infty} \frac{e^n + e^{-n}}{e^n - e^{-n}}$ $= \lim_{n \to \infty} \frac{1 + e^{-2n}}{1 - e^{-2n}} = 1$

50. converges by the Limit Comparison Test (part 1) with $\frac{1}{n^2}$: $\lim_{n \to \infty} \frac{\left(\frac{\tanh n}{n^2}\right)}{\left(\frac{1}{n^2}\right)} = \lim_{n \to \infty} \tanh n = \lim_{n \to \infty} \frac{e^n - e^{-n}}{e^n + e^{-n}}$ $= \lim_{n \to \infty} \frac{1 - e^{-2n}}{1 + e^{-2n}} = 1$

51. diverges by the Limit Comparison Test (part 1) with $\frac{1}{n}$: $\lim_{n \to \infty} \frac{\left(\frac{1}{n\sqrt[n]{n}}\right)}{\left(\frac{1}{n}\right)} = \lim_{n \to \infty} \frac{1}{\sqrt[n]{n}} = 1.$

52. converges by the Limit Comparison Test (part 1) with $\frac{1}{n^2}$: $\lim_{n \to \infty} \frac{\left(\frac{\sqrt[n]{n}}{n^2}\right)}{\left(\frac{1}{n^2}\right)} = \lim_{n \to \infty} \sqrt[n]{n} = 1$

53. $\frac{1}{1+2+3+\ldots+n} = \frac{1}{\left(\frac{n(n+1)}{2}\right)} = \frac{2}{n(n+1)}$. The series converges by the Limit Comparison Test (part 1) with $\frac{1}{n^2}$: $\lim_{n \to \infty} \frac{\left(\frac{2}{n(n+1)}\right)}{\left(\frac{1}{n^2}\right)} = \lim_{n \to \infty} \frac{2n^2}{n^2+n} = \lim_{n \to \infty} \frac{4n}{2n+1} = \lim_{n \to \infty} \frac{4}{2} = 2.$

54. $\frac{1}{1+2^2+3^2+\ldots+n^2} = \frac{1}{\frac{n(n+1)(2n+1)}{6}} = \frac{6}{n(n+1)(2n+1)} \leq \frac{6}{n^3} \Rightarrow$ the series converges by the Direct Comparison Test

55. (a) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, then there exists an integer N such that for all $n > N$, $\left|\frac{a_n}{b_n} - 0\right| < 1 \Rightarrow -1 < \frac{a_n}{b_n} < 1$ $\Rightarrow a_n < b_n$. Thus, if $\sum b_n$ converges, then $\sum a_n$ converges by the Direct Comparison Test.

(b) If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, then there exists an integer N such that for all $n > N$, $\frac{a_n}{b_n} > 1 \Rightarrow a_n > b_n$. Thus, if $\sum b_n$ diverges, then $\sum a_n$ diverges by the Direct Comparison Test.

56. Yes, $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges by the Direct Comparison Test because $\frac{a_n}{n} < a_n$

57. $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty \Rightarrow$ there exists an integer N such that for all $n > N$, $\frac{a_n}{b_n} > 1 \Rightarrow a_n > b_n$. If $\sum a_n$ converges, then $\sum b_n$ converges by the Direct Comparison Test

58. $\sum a_n$ converges $\Rightarrow \lim_{n\to\infty} a_n = 0 \Rightarrow$ there exists an integer N such that for all $n > N$, $0 \le a_n < 1 \Rightarrow a_n^2 < a_n$ $\Rightarrow \sum a_n^2$ converges by the Direct Comparison Test

59. Since $a_n > 0$ and $\lim_{n\to\infty} a_n = \infty \ne 0$, by n^{th} term test for divergence, $\sum a_n$ diverges.

60. Since $a_n > 0$ and $\lim_{n\to\infty} (n^2 \cdot a_n) = 0$, compare $\sum a_n$ with $\sum \frac{1}{n^2}$, which is a convergent p-series; $\lim_{n\to\infty} \frac{a_n}{1/n^2}$ $= \lim_{n\to\infty} (n^2 \cdot a_n) = 0 \Rightarrow \sum a_n$ converges by Limit Comparison Test

61. Let $-\infty < q < \infty$ and $p > 1$. If $q = 0$, then $\sum_{n=2}^{\infty} \frac{(\ln n)^q}{n^p} = \sum_{n=2}^{\infty} \frac{1}{n^p}$, which is a convergent p-series. If $q \ne 0$, compare with $\sum_{n=2}^{\infty} \frac{1}{n^r}$ where $1 < r < p$, then $\lim_{n\to\infty} \frac{\frac{(\ln n)^q}{n^p}}{1/n^r} = \lim_{n\to\infty} \frac{(\ln n)^q}{n^{p-r}}$, and $p - r > 0$. If $q < 0 \Rightarrow -q > 0$ and $\lim_{n\to\infty} \frac{(\ln n)^q}{n^{p-r}}$ $= \lim_{n\to\infty} \frac{1}{(\ln n)^{-q} n^{p-r}} = 0$. If $q > 0$, $\lim_{n\to\infty} \frac{(\ln n)^q}{n^{p-r}} = \lim_{n\to\infty} \frac{q(\ln n)^{q-1}\left(\frac{1}{n}\right)}{(p-r)n^{p-r-1}} = \lim_{n\to\infty} \frac{q(\ln n)^{q-1}}{(p-r)n^{p-r}}$. If $q - 1 \le 0 \Rightarrow 1 - q \ge 0$ and $\lim_{n\to\infty} \frac{q(\ln n)^{q-1}}{(p-r)n^{p-r}} = \lim_{n\to\infty} \frac{q}{(p-r)n^{p-r}(\ln n)^{1-q}} = 0$, otherwise, we apply L'Hopital's Rule again. $\lim_{n\to\infty} \frac{q(q-1)(\ln n)^{q-2}\left(\frac{1}{n}\right)}{(p-r)^2 n^{p-r-1}}$ $= \lim_{n\to\infty} \frac{q(q-1)(\ln n)^{q-2}}{(p-r)^2 n^{p-r}}$. If $q - 2 \le 0 \Rightarrow 2 - q \ge 0$ and $\lim_{n\to\infty} \frac{q(q-1)(\ln n)^{q-2}}{(p-r)^2 n^{p-r}} = \lim_{n\to\infty} \frac{q(q-1)}{(p-r)^2 n^{p-r}(\ln n)^{2-q}} = 0$; otherwise, we apply L'Hopital's Rule again. Since q is finite, there is a positive integer k such that $q - k \le 0 \Rightarrow k - q \ge 0$. Thus, after k applications of L'Hopital's Rule we obtain $\lim_{n\to\infty} \frac{q(q-1)\cdots(q-k+1)(\ln n)^{q-k}}{(p-r)^k n^{p-r}} = \lim_{n\to\infty} \frac{q(q-1)\cdots(q-k+1)}{(p-r)^k n^{p-r}(\ln n)^{k-q}} = 0$. Since the limit is 0 in every case, by Limit Comparison Test, the series $\sum_{n=1}^{\infty} \frac{(\ln n)^q}{n^p}$ converges.

62. Let $-\infty < q < \infty$ and $p \le 1$. If $q = 0$, then $\sum_{n=2}^{\infty} \frac{(\ln n)^q}{n^p} = \sum_{n=2}^{\infty} \frac{1}{n^p}$, which is a divergent p-series. If $q > 0$, compare with $\sum_{n=2}^{\infty} \frac{1}{n^p}$, which is a divergent p-series. Then $\lim_{n\to\infty} \frac{\frac{(\ln n)^q}{n^p}}{1/n^p} = \lim_{n\to\infty} (\ln n)^q = \infty$. If $q < 0 \Rightarrow -q > 0$, compare with $\sum_{n=2}^{\infty} \frac{1}{n^r}$, where $0 < p < r \le 1$. $\lim_{n\to\infty} \frac{\frac{(\ln n)^q}{n^p}}{1/n^r} = \lim_{n\to\infty} \frac{(\ln n)^q}{n^{p-r}} = \lim_{n\to\infty} \frac{n^{r-p}}{(\ln n)^{-q}}$ since $r - p > 0$. Apply L'Hopital's to obtain $\lim_{n\to\infty} \frac{(r-p)n^{r-p-1}}{(-q)(\ln n)^{-q-1}\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{(r-p)n^{r-p}}{(-q)(\ln n)^{-q-1}}$. If $-q - 1 \le 0 \Rightarrow q + 1 \ge 0$ and $\lim_{n\to\infty} \frac{(r-p)n^{r-p}(\ln n)^{q+1}}{(-q)} = \infty$, otherwise, we apply L'Hopital's Rule again to obtain $\lim_{n\to\infty} \frac{(r-p)^2 n^{r-p-1}}{(-q)(-q-1)(\ln n)^{-q-2}\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{(r-p)^2 n^{r-p}}{(-q)(-q-1)(\ln n)^{-q-2}}$. If $-q - 2 \le 0 \Rightarrow q + 2 \ge 0$ and $\lim_{n\to\infty} \frac{(r-p)^2 n^{r-p}}{(-q)(-q-1)(\ln n)^{-q-2}} = \lim_{n\to\infty} \frac{(r-p)^2 n^{r-p}(\ln n)^{q+2}}{(-q)(-q-1)} = \infty$, otherwise, we apply L'Hopital's Rule again. Since q is finite, there is a positive integer k such that $-q - k \le 0 \Rightarrow q + k \ge 0$. Thus, after k applications of L'Hopital's Rule we obtain $\lim_{n\to\infty} \frac{(r-p)^k n^{r-p}}{(-q)(-q-1)\cdots(-q-k+1)(\ln n)^{-q-k}} = \lim_{n\to\infty} \frac{(r-p)^k n^{r-p}(\ln n)^{q+k}}{(-q)(-q-1)\cdots(-q-k+1)} = \infty$.

Since the limit is ∞ if $q > 0$ or if $q < 0$ and $p < 1$, by Limit comparison test, the series $\sum_{n=1}^{\infty} \frac{(\ln n)^q}{n^{p-r}}$ diverges. Finally if $q < 0$ and $p = 1$ then $\sum_{n=2}^{\infty} \frac{(\ln n)^q}{n^p} = \sum_{n=2}^{\infty} \frac{(\ln n)^q}{n}$. Compare with $\sum_{n=2}^{\infty} \frac{1}{n}$, which is a divergent p-series. For $n \geq 3$, $\ln n \geq 1$ $\Rightarrow (\ln n)^q \geq 1 \Rightarrow \frac{(\ln n)^q}{n} \geq \frac{1}{n}$. Thus $\sum_{n=2}^{\infty} \frac{(\ln n)^q}{n}$ diverges by Comparison Test. Thus, if $-\infty < q < \infty$ and $p \leq 1$, the series $\sum_{n=1}^{\infty} \frac{(\ln n)^q}{n^{p-r}}$ diverges.

63. Converges by Exercise 61 with $q = 3$ and $p = 4$.

64. Diverges by Exercise 62 with $q = \frac{1}{2}$ and $p = \frac{1}{2}$.

65. Converges by Exercise 61 with $q = 1000$ and $p = 1.001$.

66. Diverges by Exercise 62 with $q = \frac{1}{5}$ and $p = 0.99$.

67. Converges by Exercise 61 with $q = -3$ and $p = 1.1$.

68. Diverges by Exercise 62 with $q = -\frac{1}{2}$ and $p = \frac{1}{2}$.

69. Example CAS commands:

Maple:

```
a := n -> 1./n^3/sin(n)^2;
s := k -> sum( a(n), n=1..k );                                  # (a)]
limit( s(k), k=infinity );
pts := [seq( [k,s(k)], k=1..100 )]:                              # (b)
plot( pts, style=point, title="#69(b) (Section 10.4)" );
pts := [seq( [k,s(k)], k=1..200 )]:                              # (c)
plot( pts, style=point, title="#69(c) (Section 10.4)" );
pts := [seq( [k,s(k)], k=1..400 )]:                              # (d)
plot( pts, style=point, title="#69(d) (Section 10.4)" );
evalf( 355/113 );
```

Mathematica:

```
Clear[a, n, s, k, p]
a[n_]:= 1 / ( n^3 Sin[n]^2 )
s[k_]= Sum[ a[n], {n, 1, k}]
points[p_]:= Table[{k, N[s[k]]}, {k, 1, p}]
points[100]
ListPlot[points[100]]
points[200]
ListPlot[points[200]
points[400]
ListPlot[points[400], PlotRange → All]
```

To investigate what is happening around k = 355, you could do the following.

```
N[355/113]
N[π − 355/113]
Sin[355]//N
a[355]//N
N[s[354]]
```

N[s[355]]
N[s[356]]

70. (a) Let $S = \sum_{n=1}^{\infty} \frac{1}{n^2}$, which is a convergent p-series. By Example 5 in Section 10.2, $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges to 1. By Theorem 8, $S = \sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{1}{n(n+1)}\right)$ also converges.

(b) Since $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges to 1 (from Example 5 in Section 10.2), $S = 1 + \sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{1}{n(n+1)}\right) = 1 + \sum_{n=1}^{\infty} \frac{1}{n^2(n+1)}$

(c) The new series is comparible to $\sum_{n=1}^{\infty} \frac{1}{n^3}$, so it will converge faster because its terms $\to 0$ faster than the terms of $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

(d) The series $1 + \sum_{n=1}^{1000} \frac{1}{n^2(n+1)}$ gives a better approximation. Using Mathematica, $1 + \sum_{n=1}^{1000} \frac{1}{n^2(n+1)} = 1.644933568$, while $\sum_{n=1}^{1000000} \frac{1}{n^2} = 1.644933067$. Note that $\frac{\pi^2}{6} = 1.644934067$. The error is 4.99×10^{-7} compared with 1×10^{-6}.

10.5 THE RATIO AND ROOT TESTS

1. $\frac{2^n}{n!} > 0$ for all $n \geq 1$; $\lim_{n\to\infty} \left(\frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}}\right) = \lim_{n\to\infty} \left(\frac{2^n \cdot 2}{(n+1)\cdot n!} \cdot \frac{n!}{2^n}\right) = \lim_{n\to\infty} \left(\frac{2}{n+1}\right) = 0 < 1 \Rightarrow \sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges

2. $\frac{n+2}{3^n} > 0$ for all $n \geq 1$; $\lim_{n\to\infty} \left(\frac{\frac{(n+1)+2}{3^{n+1}}}{\frac{n+2}{3^n}}\right) = \lim_{n\to\infty} \left(\frac{n+3}{3^n \cdot 3} \cdot \frac{3^n}{n+2}\right) = \lim_{n\to\infty} \left(\frac{n+3}{3n+6}\right) = \lim_{n\to\infty} \left(\frac{1}{3}\right) = \frac{1}{3} < 1 \Rightarrow \sum_{n=1}^{\infty} \frac{n+2}{3^n}$ converges

3. $\frac{(n-1)!}{(n+1)^2} > 0$ for all $n \geq 1$; $\lim_{n\to\infty} \left(\frac{\frac{((n+1)-1)!}{((n+1)+1)^2}}{\frac{(n-1)!}{(n+1)^2}}\right) = \lim_{n\to\infty} \left(\frac{n\cdot(n-1)!}{(n+2)^2} \cdot \frac{(n+1)^2}{(n-1)!}\right) = \lim_{n\to\infty} \left(\frac{n^3+2n^2+n}{n^2+4n+4}\right) = \lim_{n\to\infty} \left(\frac{3n^2+4n+1}{2n+4}\right)$
$= \lim_{n\to\infty} \left(\frac{6n+4}{2}\right) = \infty > 1 \Rightarrow \sum_{n=1}^{\infty} \frac{(n-1)!}{(n+1)^2}$ diverges

4. $\frac{2^{n+1}}{n\cdot 3^{n-1}} > 0$ for all $n \geq 1$; $\lim_{n\to\infty} \left(\frac{\frac{2^{(n+1)+1}}{(n+1)\cdot 3^{(n+1)-1}}}{\frac{2^{n+1}}{n\cdot 3^{n-1}}}\right) = \lim_{n\to\infty} \left(\frac{2^{n+1}\cdot 2}{(n+1)\cdot 3^{n-1}\cdot 3} \cdot \frac{n\cdot 3^{n-1}}{2^{n+1}}\right) = \lim_{n\to\infty} \left(\frac{2n}{3n+3}\right) = \lim_{n\to\infty} \left(\frac{2}{3}\right) = \frac{2}{3} < 1$
$\Rightarrow \sum_{n=1}^{\infty} \frac{2^{n+1}}{n\cdot 3^{n-1}}$ converges

5. $\frac{n^4}{4^n} > 0$ for all $n \geq 1$; $\lim_{n\to\infty} \left(\frac{\frac{(n+1)^4}{4^{n+1}}}{\frac{n^4}{4^n}}\right) = \lim_{n\to\infty} \left(\frac{(n+1)^4}{4^n \cdot 4} \cdot \frac{4^n}{n^4}\right) = \lim_{n\to\infty} \left(\frac{n^4+4n^3+6n^2+4n+1}{4n^4}\right)$
$= \lim_{n\to\infty} \left(\frac{1}{4} + \frac{1}{n} + \frac{3}{2n^2} + \frac{1}{n^3} + \frac{1}{4n^4}\right) = \frac{1}{4} < 1 \Rightarrow \sum_{n=1}^{\infty} \frac{n^4}{4^n}$ converges

6. $\frac{3^{n+2}}{\ln n} > 0$ for all $n \geq 2$; $\lim_{n\to\infty} \left(\frac{\frac{3^{(n+1)+2}}{\ln(n+1)}}{\frac{3^{n+2}}{\ln n}}\right) = \lim_{n\to\infty} \left(\frac{3^{n+2}\cdot 3}{\ln(n+1)} \cdot \frac{\ln n}{3^{n+2}}\right) = \lim_{n\to\infty} \left(\frac{3\ln n}{\ln(n+1)}\right) = \lim_{n\to\infty} \left(\frac{\frac{3}{n}}{\frac{1}{n+1}}\right) = \lim_{n\to\infty} \left(\frac{3n+3}{n}\right)$
$= \lim_{n\to\infty} \left(\frac{3}{1}\right) = 3 > 1 \Rightarrow \sum_{n=2}^{\infty} \frac{3^{n+2}}{\ln n}$ diverges

7. $\frac{n^2(n+2)!}{n!3^{2n}} > 0$ for all $n \geq 1$; $\lim_{n\to\infty} \left(\frac{\frac{(n+1)^2((n+1)+2)!}{(n+1)!3^{2(n+1)}}}{\frac{n^2(n+2)!}{n!3^{2n}}}\right) = \lim_{n\to\infty} \left(\frac{(n+1)^2(n+3)(n+2)!}{(n+1)\cdot n!3^{2n}\cdot 3^2} \cdot \frac{n!3^{2n}}{n^2(n+2)!}\right) = \lim_{n\to\infty} \left(\frac{n^3+5n^2+7n+3}{9n^3+9n^2}\right)$
$= \lim_{n\to\infty} \left(\frac{3n^2+15n+7}{27n^2+18n}\right) = \lim_{n\to\infty} \left(\frac{6n+15}{54n+18}\right) = \lim_{n\to\infty} \left(\frac{6}{54}\right) = \frac{1}{9} < 1 \Rightarrow \sum_{n=1}^{\infty} \frac{n^2(n+2)!}{n!3^{2n}}$ converges

8. $\frac{n\cdot 5^n}{(2n+3)\ln(n+1)} > 0$ for all $n \ge 1$; $\lim\limits_{n\to\infty}\left(\frac{\frac{(n+1)\cdot 5^{n+1}}{(2(n+1)+3)\ln((n+1)+1)}}{\frac{n\cdot 5^n}{(2n+3)\ln(n+1)}}\right) = \lim\limits_{n\to\infty}\left(\frac{(n+1)\cdot 5^n\cdot 5}{(2n+5)\ln(n+2)}\cdot\frac{(2n+3)\ln(n+1)}{n\cdot 5^n}\right)$
$= \lim\limits_{n\to\infty}\left(\frac{5(n+1)\cdot(2n+3)}{n(2n+5)}\cdot\frac{\ln(n+1)}{\ln(n+2)}\right) = \lim\limits_{n\to\infty}\left(\frac{10n^2+25n+15}{2n^2+5n}\right)\cdot \lim\limits_{n\to\infty}\left(\frac{\ln(n+1)}{\ln(n+2)}\right) = \lim\limits_{n\to\infty}\left(\frac{20n+25}{4n+5}\right)\cdot \lim\limits_{n\to\infty}\left(\frac{\frac{1}{n+1}}{\frac{1}{n+2}}\right)$
$= \lim\limits_{n\to\infty}\left(\frac{20}{4}\right)\cdot \lim\limits_{n\to\infty}\left(\frac{n+2}{n+1}\right) = 5\cdot \lim\limits_{n\to\infty}\left(\frac{1}{1}\right) = 5\cdot 1 = 5 > 1 \Rightarrow \sum\limits_{n=2}^{\infty}\frac{n\cdot 5^n}{(2n+3)\ln(n+1)}$ diverges

9. $\frac{7}{(2n+5)^n} \ge 0$ for all $n \ge 1$; $\lim\limits_{n\to\infty}\sqrt[n]{\frac{7}{(2n+5)^n}} = \lim\limits_{n\to\infty}\left(\frac{\sqrt[n]{7}}{2n+5}\right) = 0 < 1 \Rightarrow \sum\limits_{n=1}^{\infty}\frac{7}{(2n+5)^n}$ converges

10. $\frac{4^n}{(3n)^n} \ge 0$ for all $n \ge 1$; $\lim\limits_{n\to\infty}\sqrt[n]{\frac{4^n}{(3n)^n}} = \lim\limits_{n\to\infty}\left(\frac{4}{3n}\right) = 0 < 1 \Rightarrow \sum\limits_{n=1}^{\infty}\frac{4^n}{(3n)^n}$ converges

11. $\left(\frac{4n+3}{3n-5}\right)^n \ge 0$ for all $n \ge 2$; $\lim\limits_{n\to\infty}\sqrt[n]{\left(\frac{4n+3}{3n-5}\right)^n} = \lim\limits_{n\to\infty}\left(\frac{4n+3}{3n-5}\right) = \lim\limits_{n\to\infty}\left(\frac{4}{3}\right) = \frac{4}{3} > 1 \Rightarrow \sum\limits_{n=1}^{\infty}\left(\frac{4n+3}{3n-5}\right)^n$ diverges

12. $\left[\ln\left(e^2+\frac{1}{n}\right)\right]^{n+1} \ge 0$ for all $n \ge 1$; $\lim\limits_{n\to\infty}\sqrt[n]{\left[\ln\left(e^2+\frac{1}{n}\right)\right]^{n+1}} = \lim\limits_{n\to\infty}\left[\ln\left(e^2+\frac{1}{n}\right)\right]^{1+1/n} = \ln(e^2) = 2 > 1$
$\Rightarrow \sum\limits_{n=1}^{\infty}\left[\ln\left(e^2+\frac{1}{n}\right)\right]^{n+1}$ diverges

13. $\frac{8}{\left(3+\frac{1}{n}\right)^{2n}} \ge 0$ for all $n \ge 1$; $\lim\limits_{n\to\infty}\sqrt[n]{\frac{8}{\left(3+\frac{1}{n}\right)^{2n}}} = \lim\limits_{n\to\infty}\left(\frac{\sqrt[n]{8}}{\left(3+\frac{1}{n}\right)^2}\right) = \frac{1}{9} < 1 \Rightarrow \sum\limits_{n=1}^{\infty}\frac{8}{\left(3+\frac{1}{n}\right)^{2n}}$ converges

14. $\left[\sin\left(\frac{1}{\sqrt{n}}\right)\right]^n \ge 0$ for all $n \ge 1$; $\lim\limits_{n\to\infty}\sqrt[n]{\left[\sin\left(\frac{1}{\sqrt{n}}\right)\right]^n} = \lim\limits_{n\to\infty}\sin\left(\frac{1}{\sqrt{n}}\right) = \sin(0) = 0 < 1 \Rightarrow \sum\limits_{n=1}^{\infty}\left[\sin\left(\frac{1}{\sqrt{n}}\right)\right]^n$ converges

15. $\left(1-\frac{1}{n}\right)^{n^2} \ge 0$ for all $n \ge 1$; $\lim\limits_{n\to\infty}\sqrt[n]{\left(1-\frac{1}{n}\right)^{n^2}} = \lim\limits_{n\to\infty}\left(1-\frac{1}{n}\right)^n = e^{-1} < 1 \Rightarrow \sum\limits_{n=1}^{\infty}\left(1-\frac{1}{n}\right)^{n^2}$ converges

16. $\frac{1}{n^{1+n}} \ge 0$ for all $n \ge 2$; $\lim\limits_{n\to\infty}\sqrt[n]{\frac{1}{n^{1+n}}} = \lim\limits_{n\to\infty}\left(\frac{\sqrt[n]{1}}{n^{1/n+1}}\right) = \lim\limits_{n\to\infty}\left(\frac{\sqrt[n]{1}}{n\sqrt[n]{n}}\right) = 0 < 1 \Rightarrow \sum\limits_{n=2}^{\infty}\frac{1}{n^{1+n}}$ converges

17. converges by the Ratio Test: $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty}\frac{\left[\frac{(n+1)^{\sqrt{2}}}{2^{n+1}}\right]}{\left[\frac{n^{\sqrt{2}}}{2^n}\right]} = \lim\limits_{n\to\infty}\frac{(n+1)^{\sqrt{2}}}{2^{n+1}}\cdot\frac{2^n}{n^{\sqrt{2}}} = \lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^{\sqrt{2}}\left(\frac{1}{2}\right) = \frac{1}{2} < 1$

18. converges by the Ratio Test: $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty}\frac{\left(\frac{(n+1)^2}{e^{n+1}}\right)}{\left(\frac{n^2}{e^n}\right)} = \lim\limits_{n\to\infty}\frac{(n+1)^2}{e^{n+1}}\cdot\frac{e^n}{n^2} = \lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^2\left(\frac{1}{e}\right) = \frac{1}{e} < 1$

19. diverges by the Ratio Test: $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty}\frac{\left(\frac{(n+1)!}{e^{n+1}}\right)}{\left(\frac{n!}{e^n}\right)} = \lim\limits_{n\to\infty}\frac{(n+1)!}{e^{n+1}}\cdot\frac{e^n}{n!} = \lim\limits_{n\to\infty}\frac{n+1}{e} = \infty$

20. diverges by the Ratio Test: $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty}\frac{\left(\frac{(n+1)!}{10^{n+1}}\right)}{\left(\frac{n!}{10^n}\right)} = \lim\limits_{n\to\infty}\frac{(n+1)!}{10^{n+1}}\cdot\frac{10^n}{n!} = \lim\limits_{n\to\infty}\frac{n}{10} = \infty$

21. converges by the Ratio Test: $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty}\frac{\left(\frac{(n+1)^{10}}{10^{n+1}}\right)}{\left(\frac{n^{10}}{10^n}\right)} = \lim\limits_{n\to\infty}\frac{(n+1)^{10}}{10^{n+1}}\cdot\frac{10^n}{n^{10}} = \lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^{10}\left(\frac{1}{10}\right) = \frac{1}{10} < 1$

22. diverges; $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{n-2}{n}\right)^n = \lim_{n\to\infty} \left(1 + \frac{-2}{n}\right)^n = e^{-2} \neq 0$

23. converges by the Direct Comparison Test: $\frac{2+(-1)^n}{(1.25)^n} = \left(\frac{4}{5}\right)^n [2 + (-1)^n] \le \left(\frac{4}{5}\right)^n (3)$ which is the n^{th} term of a convergent geometric series

24. converges; a geometric series with $|r| = \left|-\frac{2}{3}\right| < 1$

25. diverges; $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1 - \frac{3}{n}\right)^n = \lim_{n\to\infty} \left(1 + \frac{-3}{n}\right)^n = e^{-3} \approx 0.05 \neq 0$

26. diverges; $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1 - \frac{1}{3n}\right)^n = \lim_{n\to\infty} \left(1 + \frac{\left(-\frac{1}{3}\right)}{n}\right)^n = e^{-1/3} \approx 0.72 \neq 0$

27. converges by the Direct Comparison Test: $\frac{\ln n}{n^3} < \frac{n}{n^3} = \frac{1}{n^2}$ for $n \ge 2$, the n^{th} term of a convergent p-series.

28. converges by the nth-Root Test: $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{(\ln n)^n}{n^n}} = \lim_{n\to\infty} \frac{((\ln n)^n)^{1/n}}{(n^n)^{1/n}} = \lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{\left(\frac{1}{n}\right)}{1} = 0 < 1$

29. diverges by the Direct Comparison Test: $\frac{1}{n} - \frac{1}{n^2} = \frac{n-1}{n^2} > \frac{1}{2}\left(\frac{1}{n}\right)$ for $n > 2$ or by the Limit Comparison Test (part 1) with $\frac{1}{n}$.

30. converges by the nth-Root Test: $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{1}{n} - \frac{1}{n^2}\right)^n} = \lim_{n\to\infty} \left(\left(\frac{1}{n} - \frac{1}{n^2}\right)^n\right)^{1/n} = \lim_{n\to\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right) = 0 < 1$

31. diverges by the Direct Comparison Test: $\frac{\ln n}{n} > \frac{1}{n}$ for $n \ge 3$

32. converges by the Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)\ln(n+1)}{2^{n+1}} \cdot \frac{2^n}{n \ln(n)} = \frac{1}{2} < 1$

33. converges by the Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+2)(n+3)}{(n+1)!} \cdot \frac{n!}{(n+1)(n+2)} = 0 < 1$

34. converges by the Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^3}{e^{n+1}} \cdot \frac{e^n}{n^3} = \frac{1}{e} < 1$

35. converges by the Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+4)!}{3!\,(n+1)!\,3^{n+1}} \cdot \frac{3!\,n!\,3^n}{(n+3)!} = \lim_{n\to\infty} \frac{n+4}{3(n+1)} = \frac{1}{3} < 1$

36. converges by the Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)2^{n+1}(n+2)!}{3^{n+1}(n+1)!} \cdot \frac{3^n n!}{n2^n(n+1)!} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)\left(\frac{2}{3}\right)\left(\frac{n+2}{n+1}\right) = \frac{2}{3} < 1$

37. converges by the Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!}{(2n+3)!} \cdot \frac{(2n+1)!}{n!} = \lim_{n\to\infty} \frac{n+1}{(2n+3)(2n+2)} = 0 < 1$

38. converges by the Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \frac{1}{\left(\frac{n+1}{n}\right)^n}$
$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1$

39. converges by the Root Test: $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{n}{(\ln n)^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{n}}{\ln n} = \lim_{n\to\infty} \frac{1}{\ln n} = 0 < 1$

40. converges by the Root Test: $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{n}{(\ln n)^{n/2}}} = \lim_{n\to\infty} \frac{\sqrt[n]{n}}{\sqrt{\ln n}} = \frac{\lim_{n\to\infty} \sqrt[n]{n}}{\lim_{n\to\infty} \sqrt{\ln n}} = 0 < 1$
$\left(\lim_{n\to\infty} \sqrt[n]{n} = 1\right)$

41. converges by the Direct Comparison Test: $\frac{n!\ln n}{n(n+2)!} = \frac{\ln n}{n(n+1)(n+2)} < \frac{n}{n(n+1)(n+2)} = \frac{1}{(n+1)(n+2)} < \frac{1}{n^2}$ which is the nth-term of a convergent p-series

42. diverges by the Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{3^{n+1}}{(n+1)^3\, 2^{n+1}} \cdot \frac{n^3 2^n}{3^n} = \lim_{n\to\infty} \frac{n^3}{(n+1)^3}\left(\frac{3}{2}\right) = \frac{3}{2} > 1$

43. converges by the Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{[(n+1)!]^2}{[2(n+1)]!} \cdot \frac{(2n)!}{[n!]^2} = \lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)} = \lim_{n\to\infty} \frac{n^2+2n+1}{4n^2+6n+2} = \frac{1}{4} < 1$

44. converges by the Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(2n+5)(2^{n+1}+3)}{3^{n+1}+2} \cdot \frac{3^n+2}{(2n+3)(2^n+3)} = \lim_{n\to\infty} \left[\frac{2n+5}{2n+3} \cdot \frac{2\cdot 6^n+4\cdot 2^n+3\cdot 3^n+6}{3\cdot 6^n+9\cdot 3^n+2\cdot 2^n+6}\right]$
$= \lim_{n\to\infty} \left[\frac{2n+5}{2n+3}\right] \cdot \lim_{n\to\infty} \left[\frac{2\cdot 6^n+4\cdot 2^n+3\cdot 3^n+6}{3\cdot 6^n+9\cdot 3^n+2\cdot 2^n+6}\right] = 1 \cdot \frac{2}{3} = \frac{2}{3} < 1$

45. converges by the Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left(\frac{1+\sin n}{n}\right) a_n}{a_n} = 0 < 1$

46. converges by the Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left(\frac{1+\tan^{-1} n}{n}\right) a_n}{a_n} = \lim_{n\to\infty} \frac{1+\tan^{-1} n}{n} = 0$ since the numerator approaches $1 + \frac{\pi}{2}$ while the denominator tends to ∞

47. diverges by the Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left(\frac{3n-1}{2n+5}\right) a_n}{a_n} = \lim_{n\to\infty} \frac{3n-1}{2n+5} = \frac{3}{2} > 1$

48. diverges; $a_{n+1} = \frac{n}{n+1} a_n \Rightarrow a_{n+1} = \left(\frac{n}{n+1}\right)\left(\frac{n-1}{n} a_{n-1}\right) \Rightarrow a_{n+1} = \left(\frac{n}{n+1}\right)\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n-1} a_{n-2}\right)$
$\Rightarrow a_{n+1} = \left(\frac{n}{n+1}\right)\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n-1}\right)\cdots\left(\frac{1}{2}\right) a_1 \Rightarrow a_{n+1} = \frac{a_1}{n+1} \Rightarrow a_{n+1} = \frac{3}{n+1}$, which is a constant times the general term of the diverging harmonic series

49. converges by the Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left(\frac{2}{n}\right) a_n}{a_n} = \lim_{n\to\infty} \frac{2}{n} = 0 < 1$

50. converges by the Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left(\frac{\sqrt[n]{n}}{2}\right) a_n}{a_n} = \lim_{n\to\infty} \frac{\sqrt[n]{n}}{n} = \frac{1}{2} < 1$

51. converges by the Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left(\frac{1+\ln n}{n}\right) a_n}{a_n} = \lim_{n\to\infty} \frac{1+\ln n}{n} = \lim_{n\to\infty} \frac{1}{n} = 0 < 1$

52. $\frac{n+\ln n}{n+10} > 0$ and $a_1 = \frac{1}{2} \Rightarrow a_n > 0$; $\ln n > 10$ for $n > e^{10} \Rightarrow n + \ln n > n + 10 \Rightarrow \frac{n+\ln n}{n+10} > 1$
$\Rightarrow a_{n+1} = \frac{n+\ln n}{n+10} a_n > a_n$; thus $a_{n+1} > a_n \ge \frac{1}{2} \Rightarrow \lim_{n\to\infty} a_n \ne 0$, so the series diverges by the nth-Term Test

53. diverges by the nth-Term Test: $a_1 = \frac{1}{3}$, $a_2 = \sqrt[2]{\frac{1}{3}}$, $a_3 = \sqrt[3]{\sqrt[2]{\frac{1}{3}}} = \sqrt[6]{\frac{1}{3}}$, $a_4 = \sqrt[4]{\sqrt[3]{\sqrt[2]{\frac{1}{3}}}} = \sqrt[4!]{\frac{1}{3}}, \ldots,$
$a_n = \sqrt[n!]{\frac{1}{3}} \Rightarrow \lim_{n\to\infty} a_n = 1$ because $\left\{\sqrt[n!]{\frac{1}{3}}\right\}$ is a subsequence of $\left\{\sqrt[n]{\frac{1}{3}}\right\}$ whose limit is 1 by Table 8.1

54. converges by the Direct Comparison Test: $a_1 = \frac{1}{2}$, $a_2 = \left(\frac{1}{2}\right)^2$, $a_3 = \left(\left(\frac{1}{2}\right)^2\right)^3 = \left(\frac{1}{2}\right)^6$, $a_4 = \left(\left(\frac{1}{2}\right)^6\right)^4 = \left(\frac{1}{2}\right)^{24}, \ldots$
$\Rightarrow a_n = \left(\frac{1}{2}\right)^{n!} < \left(\frac{1}{2}\right)^n$ which is the nth-term of a convergent geometric series

55. converges by the Ratio Test: $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{2^{n+1}(n+1)!\,(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{2^n n!\,n!} = \lim_{n \to \infty} \frac{2(n+1)(n+1)}{(2n+2)(2n+1)}$
$= \lim_{n \to \infty} \frac{n+1}{2n+1} = \frac{1}{2} < 1$

56. diverges by the Ratio Test: $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{(3n+3)!}{(n+1)!\,(n+2)!\,(n+3)!} \cdot \frac{n!\,(n+1)!\,(n+2)!}{(3n)!}$
$= \lim_{n \to \infty} \frac{(3n+3)(3+2)(3n+1)}{(n+1)(n+2)(n+3)} = \lim_{n \to \infty} 3\left(\frac{3n+2}{n+2}\right)\left(\frac{3n+1}{n+3}\right) = 3 \cdot 3 \cdot 3 = 27 > 1$

57. diverges by the Root Test: $\lim_{n \to \infty} \sqrt[n]{a_n} \equiv \lim_{n \to \infty} \sqrt[n]{\frac{(n!)^n}{(n^n)^2}} = \lim_{n \to \infty} \frac{n!}{n^2} = \infty > 1$

58. converges by the Root Test: $\lim_{n \to \infty} \sqrt[n]{\frac{(n!)^n}{n^{n^2}}} = \lim_{n \to \infty} \sqrt[n]{\frac{(n!)^n}{(n^n)^n}} = \lim_{n \to \infty} \frac{n!}{n^n} = \lim_{n \to \infty} \left(\frac{1}{n}\right)\left(\frac{2}{n}\right)\left(\frac{3}{n}\right)\cdots\left(\frac{n-1}{n}\right)\left(\frac{n}{n}\right)$
$\le \lim_{n \to \infty} \frac{1}{n} = 0 < 1$

59. converges by the Root Test: $\lim_{n \to \infty} \sqrt[n]{a_n} = \lim_{n \to \infty} \sqrt[n]{\frac{n^n}{2^{n^2}}} = \lim_{n \to \infty} \frac{n}{2^n} = \lim_{n \to \infty} \frac{1}{2^n \ln 2} = 0 < 1$

60. diverges by the Root Test: $\lim_{n \to \infty} \sqrt[n]{a_n} = \lim_{n \to \infty} \sqrt[n]{\frac{n^n}{(2^n)^2}} = \lim_{n \to \infty} \frac{n}{4} = \infty > 1$

61. converges by the Ratio Test: $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{1 \cdot 3 \cdot \cdots \cdot (2n-1)(2n+1)}{4^{n+1} 2^{n+1} (n+1)!} \cdot \frac{4^n 2^n n!}{1 \cdot 3 \cdot \cdots \cdot (2n-1)} = \lim_{n \to \infty} \frac{2n+1}{(4 \cdot 2)(n+1)} = \frac{1}{4} < 1$

62. converges by the Ratio Test: $a_n = \frac{1 \cdot 3 \cdots (2n-1)}{(2 \cdot 4 \cdots 2n)\,(3^n+1)} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdots (2n-1)(2n)}{(2 \cdot 4 \cdots 2n)^2\,(3^n+1)} = \frac{(2n)!}{(2^n n!)^2\,(3^n+1)}$
$\Rightarrow \lim_{n \to \infty} \frac{(2n+2)!}{[2^{n+1}(n+1)!]^2\,(3^{n+1}+1)} \cdot \frac{(2^n n!)^2\,(3^n+1)}{(2n)!} = \lim_{n \to \infty} \frac{(2n+1)(2n+2)\,(3^n+1)}{2^2(n+1)^2\,(3^{n+1}+1)}$
$= \lim_{n \to \infty} \left(\frac{4n^2+6n+2}{4n^2+8n+4}\right) \frac{(1+3^{-n})}{(3+3^{-n})} = 1 \cdot \frac{1}{3} = \frac{1}{3} < 1$

63. Ratio: $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{1}{(n+1)^p} \cdot \frac{n^p}{1} = \lim_{n \to \infty} \left(\frac{n}{n+1}\right)^p = 1^p = 1 \Rightarrow$ no conclusion

Root: $\lim_{n \to \infty} \sqrt[n]{a_n} = \lim_{n \to \infty} \sqrt[n]{\frac{1}{n^p}} = \lim_{n \to \infty} \frac{1}{(\sqrt[n]{n})^p} = \frac{1}{(1)^p} = 1 \Rightarrow$ no conclusion

64. Ratio: $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{1}{(\ln(n+1))^p} \cdot \frac{(\ln n)^p}{1} = \left[\lim_{n \to \infty} \frac{\ln n}{\ln(n+1)}\right]^p = \left[\lim_{n \to \infty} \frac{\left(\frac{1}{n}\right)}{\left(\frac{1}{n+1}\right)}\right]^p = \left(\lim_{n \to \infty} \frac{n+1}{n}\right)^p$
$= (1)^p = 1 \Rightarrow$ no conclusion

Root: $\lim_{n \to \infty} \sqrt[n]{a_n} = \lim_{n \to \infty} \sqrt[n]{\frac{1}{(\ln n)^p}} = \frac{1}{\left(\lim_{n \to \infty} (\ln n)^{1/n}\right)^p}$; let $f(n) = (\ln n)^{1/n}$, then $\ln f(n) = \frac{\ln(\ln n)}{n}$
$\Rightarrow \lim_{n \to \infty} \ln f(n) = \lim_{n \to \infty} \frac{\ln(\ln n)}{n} = \lim_{n \to \infty} \frac{\left(\frac{1}{n \ln n}\right)}{1} = \lim_{n \to \infty} \frac{1}{n \ln n} = 0 \Rightarrow \lim_{n \to \infty} (\ln n)^{1/n}$
$= \lim_{n \to \infty} e^{\ln f(n)} = e^0 = 1$; therefore $\lim_{n \to \infty} \sqrt[n]{a_n} = \frac{1}{\left(\lim_{n \to \infty} (\ln n)^{1/n}\right)^p} = \frac{1}{(1)^p} = 1 \Rightarrow$ no conclusion

65. $a_n \le \frac{n}{2^n}$ for every n and the series $\sum_{n=1}^{\infty} \frac{n}{2^n}$ converges by the Ratio Test since $\lim_{n \to \infty} \frac{(n+1)}{2^{n+1}} \cdot \frac{2^n}{n} = \frac{1}{2} < 1$
$\Rightarrow \sum_{n=1}^{\infty} a_n$ converges by the Direct Comparison Test

66. $\frac{2^{n^2}}{n!} > 0$ for all $n \geq 1$; $\lim_{n\to\infty} \left(\frac{\frac{2^{(n+1)^2}}{(n+1)!}}{\frac{2^{n^2}}{n!}} \right) = \lim_{n\to\infty} \left(\frac{2^{n^2+2n+1}}{(n+1)\cdot n!} \cdot \frac{n!}{2^{n^2}} \right) = \lim_{n\to\infty} \left(\frac{2^{2n+1}}{n+1} \right) = \lim_{n\to\infty} \left(\frac{2\cdot 4^n}{n+1} \right) = \lim_{n\to\infty} \left(\frac{2\cdot 4^n \ln 4}{1} \right)$
$= \infty > 1 \Rightarrow \sum_{n=1}^{\infty} \frac{2^{n^2}}{n!}$ diverges

10.6 ALTERNATING SERIES, ABSOLUTE AND CONDITIONAL CONVERGENCE

1. converges by the Alternating Convergence Test since: $u_n = \frac{1}{\sqrt{n}} > 0$ for all $n \geq 1$; $n \geq 1 \Rightarrow n+1 \geq n \Rightarrow \sqrt{n+1} \geq \sqrt{n}$
$\Rightarrow \frac{1}{\sqrt{n+1}} \leq \frac{1}{\sqrt{n}} \Rightarrow u_{n+1} \leq u_n$; $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.

2. converges absolutely $\Rightarrow$ converges by the Alternating Convergence Test since $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ which is a convergent p-series

3. converges $\Rightarrow$ converges by Alternating Series Test since: $u_n = \frac{1}{n3^n} > 0$ for all $n \geq 1$; $n \geq 1 \Rightarrow n+1 \geq n \Rightarrow 3^{n+1} \geq 3^n$
$\Rightarrow (n+1)3^{n+1} \geq n\,3^n \Rightarrow \frac{1}{(n+1)3^{n+1}} \leq \frac{1}{n\,3^n} \Rightarrow u_{n+1} \leq u_n$; $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n\,3^n} = 0$.

4. converges $\Rightarrow$ converges by Alternating Series Test since: $u_n = \frac{4}{(\ln n)^2} > 0$ for all $n \geq 2$; $n \geq 2 \Rightarrow n+1 \geq n$
$\Rightarrow \ln(n+1) \geq \ln n \Rightarrow (\ln(n+1))^2 \geq (\ln n)^2 \Rightarrow \frac{1}{(\ln(n+1))^2} \leq \frac{1}{(\ln n)^2} \Rightarrow \frac{4}{(\ln(n+1))^2} \leq \frac{4}{(\ln n)^2} \Rightarrow u_{n+1} \leq u_n$;
$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{4}{(\ln n)^2} = 0$.

5. converges $\Rightarrow$ converges by Alternating Series Test since: $u_n = \frac{n}{n^2+1} > 0$ for all $n \geq 1$; $n \geq 1 \Rightarrow 2n^2 + 2n \geq n^2 + n + 1$
$\Rightarrow n^3 + 2n^2 + 2n \geq n^3 + n^2 + n + 1 \Rightarrow n(n^2 + 2n + 2) \geq n^3 + n^2 + n + 1 \Rightarrow n\left((n+1)^2 + 1\right) \geq (n^2+1)(n+1)$
$\Rightarrow \frac{n}{n^2+1} \geq \frac{n+1}{(n+1)^2+1} \Rightarrow u_{n+1} \leq u_n$; $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{n^2+1} = 0$.

6. diverges $\Rightarrow$ diverges by n^{th} Term Test for Divergence since: $\lim_{n\to\infty} \frac{n^2+5}{n^2+4} = 1 \Rightarrow \lim_{n\to\infty} (-1)^{n+1} \frac{n^2+5}{n^2+4} =$ does not exist

7. diverges $\Rightarrow$ diverges by n^{th} Term Test for Divergence since: $\lim_{n\to\infty} \frac{2^n}{n^2} = \infty \Rightarrow \lim_{n\to\infty} (-1)^{n+1} \frac{2^n}{n^2} =$ does not exist

8. converges absolutely $\Rightarrow$ converges by the Absolute Convergence Test since $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{10^n}{(n+1)!}$, which converges by the Ratio Test, since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{10}{n+2} = 0 < 1$

9. diverges by the nth-Term Test since for $n > 10 \Rightarrow \frac{n}{10} > 1 \Rightarrow \lim_{n\to\infty} \left(\frac{n}{10}\right)^n \neq 0 \Rightarrow \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{n}{10}\right)^n$ diverges

10. converges by the Alternating Series Test because $f(x) = \ln x$ is an increasing function of $x \Rightarrow \frac{1}{\ln x}$ is decreasing
$\Rightarrow u_n \geq u_{n+1}$ for $n \geq 1$; also $u_n \geq 0$ for $n \geq 1$ and $\lim_{n\to\infty} \frac{1}{\ln n} = 0$

11. converges by the Alternating Series Test since $f(x) = \frac{\ln x}{x} \Rightarrow f'(x) = \frac{1-\ln x}{x^2} < 0$ when $x > e \Rightarrow f(x)$ is decreasing
$\Rightarrow u_n \geq u_{n+1}$; also $u_n \geq 0$ for $n \geq 1$ and $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{\left(\frac{1}{n}\right)}{1} = 0$

12. converges by the Alternating Series Test since $f(x) = \ln\left(1 + x^{-1}\right) \Rightarrow f'(x) = \frac{-1}{x(x+1)} < 0$ for $x > 0 \Rightarrow f(x)$ is decreasing $\Rightarrow u_n \ge u_{n+1}$; also $u_n \ge 0$ for $n \ge 1$ and $\lim_{n \to \infty} u_n = \lim_{n \to \infty} \ln\left(1 + \frac{1}{n}\right) = \ln\left(\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)\right) = \ln 1 = 0$

13. converges by the Alternating Series Test since $f(x) = \frac{\sqrt{x}+1}{x+1} \Rightarrow f'(x) = \frac{1 - x - 2\sqrt{x}}{2\sqrt{x}\,(x+1)^2} < 0 \Rightarrow f(x)$ is decreasing $\Rightarrow u_n \ge u_{n+1}$; also $u_n \ge 0$ for $n \ge 1$ and $\lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{\sqrt{n}+1}{n+1} = 0$

14. diverges by the nth-Term Test since $\lim_{n \to \infty} \frac{3\sqrt{n+1}}{\sqrt{n}+1} = \lim_{n \to \infty} \frac{3\sqrt{1+\frac{1}{n}}}{1+\left(\frac{1}{\sqrt{n}}\right)} = 3 \ne 0$

15. converges absolutely since $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^n$ a convergent geometric series

16. converges absolutely by the Direct Comparison Test since $\left|\frac{(-1)^{n+1}(0.1)^n}{n}\right| = \frac{1}{(10)^n n} < \left(\frac{1}{10}\right)^n$ which is the nth term of a convergent geometric series

17. converges conditionally since $\frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}} > 0$ and $\lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0 \Rightarrow$ convergence; but $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ is a divergent p-series

18. converges conditionally since $\frac{1}{1+\sqrt{n}} > \frac{1}{1+\sqrt{n+1}} > 0$ and $\lim_{n \to \infty} \frac{1}{1+\sqrt{n}} = 0 \Rightarrow$ convergence; but $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$ is a divergent series since $\frac{1}{1+\sqrt{n}} \ge \frac{1}{2\sqrt{n}}$ and $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ is a divergent p-series

19. converges absolutely since $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{n}{n^3+1}$ and $\frac{n}{n^3+1} < \frac{1}{n^2}$ which is the nth-term of a converging p-series

20. diverges by the nth-Term Test since $\lim_{n \to \infty} \frac{n!}{2^n} = \infty$

21. converges conditionally since $\frac{1}{n+3} > \frac{1}{(n+1)+3} > 0$ and $\lim_{n \to \infty} \frac{1}{n+3} = 0 \Rightarrow$ convergence; but $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n+3}$ diverges because $\frac{1}{n+3} \ge \frac{1}{4n}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent series

22. converges absolutely because the series $\sum_{n=1}^{\infty} \left|\frac{\sin n}{n^2}\right|$ converges by the Direct Comparison Test since $\left|\frac{\sin n}{n^2}\right| \le \frac{1}{n^2}$

23. diverges by the nth-Term Test since $\lim_{n \to \infty} \frac{3+n}{5+n} = 1 \ne 0$

24. converges absolutely by the Direct Comparison Test since $\left|\frac{(-2)^{n+1}}{n+5^n}\right| = \frac{2^{n+1}}{n+5^n} < 2\left(\frac{2}{5}\right)^n$ which is the nth term of a convergent geometric series

25. converges conditionally since $f(x) = \frac{1}{x^2} + \frac{1}{x} \Rightarrow f'(x) = -\left(\frac{2}{x^3} + \frac{1}{x^2}\right) < 0 \Rightarrow f(x)$ is decreasing and hence $u_n > u_{n+1} > 0$ for $n \ge 1$ and $\lim_{n \to \infty} \left(\frac{1}{n^2} + \frac{1}{n}\right) = 0 \Rightarrow$ convergence; but $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1+n}{n^2}$ $= \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{1}{n}$ is the sum of a convergent and divergent series, and hence diverges

26. diverges by the nth-Term Test since $\lim_{n \to \infty} a_n = \lim_{n \to \infty} 10^{1/n} = 1 \neq 0$

27. converges absolutely by the Ratio Test: $\lim_{n \to \infty} \left(\frac{u_{n+1}}{u_n}\right) = \lim_{n \to \infty} \left[\frac{(n+1)^2 \left(\frac{2}{3}\right)^{n+1}}{n^2 \left(\frac{2}{3}\right)^n}\right] = \frac{2}{3} < 1$

28. converges conditionally since $f(x) = \frac{1}{x \ln x} \Rightarrow f'(x) = -\frac{[\ln(x) + 1]}{(x \ln x)^2} < 0 \Rightarrow f(x)$ is decreasing $\Rightarrow u_n > u_{n+1} > 0$ for $n \geq 2$ and $\lim_{n \to \infty} \frac{1}{n \ln n} = 0 \Rightarrow$ convergence; but by the Integral Test, $\int_2^{\infty} \frac{dx}{x \ln x} = \lim_{b \to \infty} \int_2^b \left(\frac{\left(\frac{1}{x}\right)}{\ln x}\right) dx = \lim_{b \to \infty} [\ln(\ln x)]_2^b = \lim_{b \to \infty} [\ln(\ln b) - \ln(\ln 2)] = \infty$ $\Rightarrow \sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n \ln n}$ diverges

29. converges absolutely by the Integral Test since $\int_1^{\infty} (\tan^{-1} x)\left(\frac{1}{1+x^2}\right) dx = \lim_{b \to \infty} \left[\frac{(\tan^{-1} x)^2}{2}\right]_1^b$ $= \lim_{b \to \infty} \left[(\tan^{-1} b)^2 - (\tan^{-1} 1)^2\right] = \frac{1}{2}\left[\left(\frac{\pi}{2}\right)^2 - \left(\frac{\pi}{4}\right)^2\right] = \frac{3\pi^2}{32}$

30. converges conditionally since $f(x) = \frac{\ln x}{x - \ln x} \Rightarrow f'(x) = \frac{\left(\frac{1}{x}\right)(x - \ln x) - (\ln x)\left(1 - \frac{1}{x}\right)}{(x - \ln x)^2}$ $= \frac{1 - \left(\frac{\ln x}{x}\right) - \ln x + \left(\frac{\ln x}{x}\right)}{(x - \ln x)^2} = \frac{1 - \ln x}{(x - \ln x)^2} < 0 \Rightarrow u_n \geq u_{n+1} > 0$ when $n > e$ and $\lim_{n \to \infty} \frac{\ln n}{n - \ln n}$ $= \lim_{n \to \infty} \frac{\left(\frac{1}{n}\right)}{1 - \left(\frac{1}{n}\right)} = 0 \Rightarrow$ convergence; but $n - \ln n < n \Rightarrow \frac{1}{n - \ln n} > \frac{1}{n} \Rightarrow \frac{\ln n}{n - \ln n} > \frac{1}{n}$ so that $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{\ln n}{n - \ln n}$ diverges by the Direct Comparison Test

31. diverges by the nth-Term Test since $\lim_{n \to \infty} \frac{n}{n+1} = 1 \neq 0$

32. converges absolutely since $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^n$ is a convergent geometric series

33. converges absolutely by the Ratio Test: $\lim_{n \to \infty} \left(\frac{u_{n+1}}{u_n}\right) = \lim_{n \to \infty} \frac{(100)^{n+1}}{(n+1)!} \cdot \frac{n!}{(100)^n} = \lim_{n \to \infty} \frac{100}{n+1} = 0 < 1$

34. converges absolutely by the Direct Comparison Test since $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^2 + 2n + 1}$ and $\frac{1}{n^2 + 2n + 1} < \frac{1}{n^2}$ which is the nth-term of a convergent p-series

35. converges absolutely since $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n\sqrt{n}}\right| = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is a convergent p-series

36. converges conditionally since $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is the convergent alternating harmonic series, but $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges

37. converges absolutely by the Root Test: $\lim_{n \to \infty} \sqrt[n]{|a_n|} = \lim_{n \to \infty} \left(\frac{(n+1)^n}{(2n)^n}\right)^{1/n} = \lim_{n \to \infty} \frac{n+1}{2n} = \frac{1}{2} < 1$

38. converges absolutely by the Ratio Test: $\lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n \to \infty} \frac{((n+1)!)^2}{((2n+2)!)} \cdot \frac{(2n)!}{(n!)^2} = \lim_{n \to \infty} \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{1}{4} < 1$

39. diverges by the nth-Term Test since $\lim_{n \to \infty} |a_n| = \lim_{n \to \infty} \frac{(2n)!}{2^n n!\, n} = \lim_{n \to \infty} \frac{(n+1)(n+2)\cdots(2n)}{2^n n}$
$= \lim_{n \to \infty} \frac{(n+1)(n+2)\cdots(n+(n-1))}{2^{n-1}} > \lim_{n \to \infty} \left(\frac{n+1}{2}\right)^{n-1} = \infty \neq 0$

40. converges absolutely by the Ratio Test: $\lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n \to \infty} \frac{(n+1)!\,(n+1)!\,3^{n+1}}{(2n+3)!} \cdot \frac{(2n+1)!}{n!\,n!\,3^n}$
$= \lim_{n \to \infty} \frac{(n+1)^2\, 3}{(2n+2)(2n+3)} = \frac{3}{4} < 1$

41. converges conditionally since $\frac{\sqrt{n+1}-\sqrt{n}}{1} \cdot \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}}$ and $\left\{\frac{1}{\sqrt{n+1}+\sqrt{n}}\right\}$ is a decreasing sequence of positive terms which converges to 0 $\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}+\sqrt{n}}$ converges; but $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$ diverges by the Limit Comparison Test (part 1) with $\frac{1}{\sqrt{n}}$; a divergent p-series:
$\lim_{n \to \infty} \left(\frac{\frac{1}{\sqrt{n+1}+\sqrt{n}}}{\frac{1}{\sqrt{n}}}\right) = \lim_{n \to \infty} \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \lim_{n \to \infty} \frac{1}{\sqrt{1+\frac{1}{n}}+1} = \frac{1}{2}$

42. diverges by the nth-Term Test since $\lim_{n \to \infty} \left(\sqrt{n^2+n}-n\right) = \lim_{n \to \infty} \left(\sqrt{n^2+n}-n\right) \cdot \left(\frac{\sqrt{n^2+n}+n}{\sqrt{n^2+n}+n}\right)$
$= \lim_{n \to \infty} \frac{n}{\sqrt{n^2+n}+n} = \lim_{n \to \infty} \frac{1}{\sqrt{1+\frac{1}{n}}+1} = \frac{1}{2} \neq 0$

43. diverges by the nth-Term Test since $\lim_{n \to \infty} \left(\sqrt{n+\sqrt{n}}-\sqrt{n}\right) = \lim_{n \to \infty} \left[\left(\sqrt{n+\sqrt{n}}-\sqrt{n}\right)\left(\frac{\sqrt{n+\sqrt{n}}+\sqrt{n}}{\sqrt{n+\sqrt{n}}+\sqrt{n}}\right)\right]$
$= \lim_{n \to \infty} \frac{\sqrt{n}}{\sqrt{n+\sqrt{n}}+\sqrt{n}} = \lim_{n \to \infty} \frac{1}{\sqrt{1+\frac{1}{\sqrt{n}}}+1} = \frac{1}{2} \neq 0$

44. converges conditionally since $\left\{\frac{1}{\sqrt{n}+\sqrt{n+1}}\right\}$ is a decreasing sequence of positive terms converging to 0
$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}+\sqrt{n+1}}$ converges; but $\lim_{n \to \infty} \frac{\left(\frac{1}{\sqrt{n}+\sqrt{n+1}}\right)}{\left(\frac{1}{\sqrt{n}}\right)} = \lim_{n \to \infty} \frac{\sqrt{n}}{\sqrt{n}+\sqrt{n+1}} = \lim_{n \to \infty} \frac{1}{1+\sqrt{1+\frac{1}{n}}} = \frac{1}{2}$
so that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{n+1}}$ diverges by the Limit Comparison Test with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which is a divergent p-series

45. converges absolutely by the Direct Comparison Test since $\operatorname{sech}(n) = \frac{2}{e^n+e^{-n}} = \frac{2e^n}{e^{2n}+1} < \frac{2e^n}{e^{2n}} = \frac{2}{e^n}$ which is the nth term of a convergent geometric series

46. converges absolutely by the Limit Comparison Test (part 1): $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{2}{e^n-e^{-n}}$
Apply the Limit Comparison Test with $\frac{1}{e^n}$, the n-th term of a convergent geometric series:
$\lim_{n \to \infty} \left(\frac{\frac{2}{e^n-e^{-n}}}{\frac{1}{e^n}}\right) = \lim_{n \to \infty} \frac{2e^n}{e^n-e^{-n}} = \lim_{n \to \infty} \frac{2}{1-e^{-2n}} = 2$

47. $\frac{1}{4} - \frac{1}{6} + \frac{1}{8} - \frac{1}{10} + \frac{1}{12} - \frac{1}{14} + \ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2(n+1)}$; converges by Alternating Series Test since: $u_n = \frac{1}{2(n+1)} > 0$ for all $n \geq 1$; $n+2 \geq n+1 \Rightarrow 2(n+2) \geq 2(n+1) \Rightarrow \frac{1}{2((n+1)+1)} \leq \frac{1}{2(n+1)} \Rightarrow u_{n+1} \leq u_n$; $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{2(n+1)} = 0.$

48. $1 + \frac{1}{4} - \frac{1}{9} - \frac{1}{16} + \frac{1}{25} + \frac{1}{36} - \frac{1}{49} - \frac{1}{64} + \ldots = \sum_{n=1}^{\infty} a_n$; converges by the Absolute Convergence Test since $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ which is a convergent p-series

49. $|\text{error}| < \left|(-1)^6 \left(\frac{1}{5}\right)\right| = 0.2$

50. $|\text{error}| < \left|(-1)^6 \left(\frac{1}{10^5}\right)\right| = 0.00001$

51. $|\text{error}| < \left|(-1)^6 \frac{(0.01)^5}{5}\right| = 2 \times 10^{-11}$

52. $|\text{error}| < |(-1)^4 t^4| = t^4 < 1$

53. $|\text{error}| < 0.001 \Rightarrow u_{n+1} < 0.001 \Rightarrow \frac{1}{(n+1)^2+3} < 0.001 \Rightarrow (n+1)^2 + 3 > 1000 \Rightarrow n > -1 + \sqrt{997} \approx 30.5753 \Rightarrow n \geq 31$

54. $|\text{error}| < 0.001 \Rightarrow u_{n+1} < 0.001 \Rightarrow \frac{n+1}{(n+1)^2+1} < 0.001 \Rightarrow (n+1)^2 + 1 > 1000(n+1) \Rightarrow n > \frac{998+\sqrt{998^2+4(998)}}{2}$
$\approx 998.9999 \Rightarrow n \geq 999$

55. $|\text{error}| < 0.001 \Rightarrow u_{n+1} < 0.001 \Rightarrow \frac{1}{\left((n+1)+3\sqrt{n+1}\right)^3} < 0.001 \Rightarrow \left((n+1) + 3\sqrt{n+1}\right)^3 > 1000$
$\Rightarrow \left(\sqrt{n+1}\right)^2 + 3\sqrt{n+1} - 10 > 0 \Rightarrow \sqrt{n+1} = -\frac{3+\sqrt{9+40}}{2} = 2 \Rightarrow n = 3 \Rightarrow n \geq 4$

56. $|\text{error}| < 0.001 \Rightarrow u_{n+1} < 0.001 \Rightarrow \frac{1}{\ln(\ln(n+3))} < 0.001 \Rightarrow \ln(\ln(n+3)) > 1000 \Rightarrow n > -3 + e^{e^{1000}} \approx 5.297 \times 10^{323228467}$ which is the maximum arbitrary-precision number represented by Mathematica on the particular computer solving this problem..

57. $\frac{1}{(2n)!} < \frac{5}{10^6} \Rightarrow (2n)! > \frac{10^6}{5} = 200{,}000 \Rightarrow n \geq 5 \Rightarrow 1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \frac{1}{8!} \approx 0.54030$

58. $\frac{1}{n!} < \frac{5}{10^6} \Rightarrow \frac{10^6}{5} < n! \Rightarrow n \geq 9 \Rightarrow 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \frac{1}{8!} \approx 0.367881944$

59. (a) $a_n \geq a_{n+1}$ fails since $\frac{1}{3} < \frac{1}{2}$

(b) Since $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left[\left(\frac{1}{3}\right)^n + \left(\frac{1}{2}\right)^n\right] = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n + \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ is the sum of two absolutely convergent series, we can rearrange the terms of the original series to find its sum:
$\left(\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots\right) - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\right) = \frac{\left(\frac{1}{3}\right)}{1-\left(\frac{1}{3}\right)} - \frac{\left(\frac{1}{2}\right)}{1-\left(\frac{1}{2}\right)} = \frac{1}{2} - 1 = -\frac{1}{2}$

60. $s_{20} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{19} - \frac{1}{20} \approx 0.6687714032 \Rightarrow s_{20} + \frac{1}{2} \cdot \frac{1}{21} \approx 0.692580927$

61. The unused terms are $\sum_{j=n+1}^{\infty} (-1)^{j+1} a_j = (-1)^{n+1}(a_{n+1} - a_{n+2}) + (-1)^{n+3}(a_{n+3} - a_{n+4}) + \ldots$
$= (-1)^{n+1}\left[(a_{n+1} - a_{n+2}) + (a_{n+3} - a_{n+4}) + \ldots\right]$. Each grouped term is positive, so the remainder has the same sign as $(-1)^{n+1}$, which is the sign of the first unused term.

62. $s_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots + \frac{1}{n(n+1)} = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right)$
$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \ldots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$ which are the first 2n terms of the first series, hence the two series are the same. Yes, for
$s_n = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \ldots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}$

$\Rightarrow \lim_{n \to \infty} s_n = \lim_{n \to \infty} \left(1 - \frac{1}{n+1}\right) = 1 \Rightarrow$ both series converge to 1. The sum of the first $2n + 1$ terms of the first series is $\left(1 - \frac{1}{n+1}\right) + \frac{1}{n+1} = 1$. Their sum is $\lim_{n \to \infty} s_n = \lim_{n \to \infty} \left(1 - \frac{1}{n+1}\right) = 1$.

63. Theorem 16 states that $\sum_{n=1}^{\infty} |a_n|$ converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges. But this is equivalent to $\sum_{n=1}^{\infty} a_n$ diverges $\Rightarrow \sum_{n=1}^{\infty} |a_n|$ diverges

64. $|a_1 + a_2 + \ldots + a_n| \le |a_1| + |a_2| + \ldots + |a_n|$ for all n; then $\sum_{n=1}^{\infty} |a_n|$ converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges and these imply that $\left|\sum_{n=1}^{\infty} a_n\right| \le \sum_{n=1}^{\infty} |a_n|$

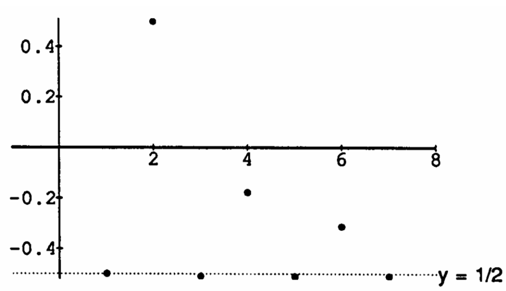
65. (a) $\sum_{n=1}^{\infty} |a_n + b_n|$ converges by the Direct Comparison Test since $|a_n + b_n| \le |a_n| + |b_n|$ and hence $\sum_{n=1}^{\infty} (a_n + b_n)$ converges absolutely

(b) $\sum_{n=1}^{\infty} |b_n|$ converges $\Rightarrow \sum_{n=1}^{\infty} -b_n$ converges absolutely; since $\sum_{n=1}^{\infty} a_n$ converges absolutely and $\sum_{n=1}^{\infty} -b_n$ converges absolutely, we have $\sum_{n=1}^{\infty} [a_n + (-b_n)] = \sum_{n=1}^{\infty} (a_n - b_n)$ converges absolutely by part (a)

(c) $\sum_{n=1}^{\infty} |a_n|$ converges $\Rightarrow |k| \sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} |ka_n|$ converges $\Rightarrow \sum_{n=1}^{\infty} ka_n$ converges absolutely

66. If $a_n = b_n = (-1)^n \frac{1}{\sqrt{n}}$, then $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$ converges, but $\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges

67. $s_1 = -\frac{1}{2}$, $s_2 = -\frac{1}{2} + 1 = \frac{1}{2}$,
$s_3 = -\frac{1}{2} + 1 - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} - \frac{1}{10} - \frac{1}{12} - \frac{1}{14} - \frac{1}{16} - \frac{1}{18} - \frac{1}{20} - \frac{1}{22} \approx -0.5099$,
$s_4 = s_3 + \frac{1}{3} \approx -0.1766$,
$s_5 = s_4 - \frac{1}{24} - \frac{1}{26} - \frac{1}{28} - \frac{1}{30} - \frac{1}{32} - \frac{1}{34} - \frac{1}{36} - \frac{1}{38} - \frac{1}{40} - \frac{1}{42} - \frac{1}{44} \approx -0.512$,
$s_6 = s_5 + \frac{1}{5} \approx -0.312$,
$s_7 = s_6 - \frac{1}{46} - \frac{1}{48} - \frac{1}{50} - \frac{1}{52} - \frac{1}{54} - \frac{1}{56} - \frac{1}{58} - \frac{1}{60} - \frac{1}{62} - \frac{1}{64} - \frac{1}{66} \approx -0.51106$



68. (a) Since $\sum |a_n|$ converges, say to M, for $\epsilon > 0$ there is an integer N_1 such that $\left|\sum_{n=1}^{N_1-1} |a_n| - M\right| < \frac{\epsilon}{2}$
$\Leftrightarrow \left|\sum_{n=1}^{N_1-1} |a_n| - \left(\sum_{n=1}^{N_1-1} |a_n| + \sum_{n=N_1}^{\infty} |a_n|\right)\right| < \frac{\epsilon}{2} \Leftrightarrow \left|-\sum_{n=N_1}^{\infty} |a_n|\right| < \frac{\epsilon}{2} \Leftrightarrow \sum_{n=N_1}^{\infty} |a_n| < \frac{\epsilon}{2}$. Also, $\sum a_n$ converges to L $\Leftrightarrow$ for $\epsilon > 0$ there is an integer N_2 (which we can choose greater than or equal to N_1) such that $|s_{N_2} - L| < \frac{\epsilon}{2}$. Therefore, $\sum_{n=N_1}^{\infty} |a_n| < \frac{\epsilon}{2}$ and $|s_{N_2} - L| < \frac{\epsilon}{2}$.

(b) The series $\sum_{n=1}^{\infty} |a_n|$ converges absolutely, say to M. Thus, there exists N_1 such that $\left|\sum_{n=1}^{k} |a_n| - M\right| < \epsilon$ whenever $k > N_1$. Now all of the terms in the sequence $\{|b_n|\}$ appear in $\{|a_n|\}$. Sum together all of the terms in $\{|b_n|\}$, in order, until you include all of the terms $\{|a_n|\}_{n=1}^{N_1}$, and let N_2 be the largest index in the sum $\sum_{n=1}^{N_2} |b_n|$ so obtained. Then $\left|\sum_{n=1}^{N_2} |b_n| - M\right| < \epsilon$ as well $\Rightarrow \sum_{n=1}^{\infty} |b_n|$ converges to M.

10.7 POWER SERIES

1. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{x^{n+1}}{x^n}\right| < 1 \Rightarrow |x| < 1 \Rightarrow -1 < x < 1$; when $x = -1$ we have $\sum_{n=1}^{\infty} (-1)^n$, a divergent series; when $x = 1$ we have $\sum_{n=1}^{\infty} 1$, a divergent series
   (a) the radius is 1; the interval of convergence is $-1 < x < 1$
   (b) the interval of absolute convergence is $-1 < x < 1$
   (c) there are no values for which the series converges conditionally

2. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(x+5)^{n+1}}{(x+5)^n}\right| < 1 \Rightarrow |x+5| < 1 \Rightarrow -6 < x < -4$; when $x = -6$ we have $\sum_{n=1}^{\infty} (-1)^n$, a divergent series; when $x = -4$ we have $\sum_{n=1}^{\infty} 1$, a divergent series
   (a) the radius is 1; the interval of convergence is $-6 < x < -4$
   (b) the interval of absolute convergence is $-6 < x < -4$
   (c) there are no values for which the series converges conditionally

3. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(4x+1)^{n+1}}{(4x+1)^n}\right| < 1 \Rightarrow |4x+1| < 1 \Rightarrow -1 < 4x+1 < 1 \Rightarrow -\frac{1}{2} < x < 0$; when $x = -\frac{1}{2}$ we have $\sum_{n=1}^{\infty} (-1)^n(-1)^n = \sum_{n=1}^{\infty} (-1)^{2n} = \sum_{n=1}^{\infty} 1^n$, a divergent series; when $x = 0$ we have $\sum_{n=1}^{\infty} (-1)^n(1)^n = \sum_{n=1}^{\infty} (-1)^n$, a divergent series
   (a) the radius is $\frac{1}{4}$; the interval of convergence is $-\frac{1}{2} < x < 0$
   (b) the interval of absolute convergence is $-\frac{1}{2} < x < 0$
   (c) there are no values for which the series converges conditionally

4. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(3x-2)^{n+1}}{n+1} \cdot \frac{n}{(3x-2)^n}\right| < 1 \Rightarrow |3x-2| \lim_{n\to\infty} \left(\frac{n}{n+1}\right) < 1 \Rightarrow |3x-2| < 1$ $\Rightarrow -1 < 3x-2 < 1 \Rightarrow \frac{1}{3} < x < 1$; when $x = \frac{1}{3}$ we have $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which is the alternating harmonic series and is conditionally convergent; when $x = 1$ we have $\sum_{n=1}^{\infty} \frac{1}{n}$, the divergent harmonic series
   (a) the radius is $\frac{1}{3}$; the interval of convergence is $\frac{1}{3} \le x < 1$
   (b) the interval of absolute convergence is $\frac{1}{3} < x < 1$
   (c) the series converges conditionally at $x = \frac{1}{3}$

5. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(x-2)^{n+1}}{10^{n+1}} \cdot \frac{10^n}{(x-2)^n}\right| < 1 \Rightarrow \frac{|x-2|}{10} < 1 \Rightarrow |x-2| < 10 \Rightarrow -10 < x-2 < 10$ $\Rightarrow -8 < x < 12$; when $x = -8$ we have $\sum_{n=1}^{\infty} (-1)^n$, a divergent series; when $x = 12$ we have $\sum_{n=1}^{\infty} 1$, a divergent series
   (a) the radius is 10; the interval of convergence is $-8 < x < 12$
   (b) the interval of absolute convergence is $-8 < x < 12$
   (c) there are no values for which the series converges conditionally

6. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(2x)^{n+1}}{(2x)^n}\right| < 1 \Rightarrow \lim_{n\to\infty} |2x| < 1 \Rightarrow |2x| < 1 \Rightarrow -\frac{1}{2} < x < \frac{1}{2}$; when $x = -\frac{1}{2}$ we have $\sum_{n=1}^{\infty} (-1)^n$, a divergent series; when $x = \frac{1}{2}$ we have $\sum_{n=1}^{\infty} 1$, a divergent series
   (a) the radius is $\frac{1}{2}$; the interval of convergence is $-\frac{1}{2} < x < \frac{1}{2}$
   (b) the interval of absolute convergence is $-\frac{1}{2} < x < \frac{1}{2}$
   (c) there are no values for which the series converges conditionally

7. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(n+1)x^{n+1}}{(n+3)} \cdot \frac{(n+2)}{nx^n}\right| < 1 \Rightarrow |x| \lim_{n\to\infty} \frac{(n+1)(n+2)}{(n+3)(n)} < 1 \Rightarrow |x| < 1$
   $\Rightarrow -1 < x < 1$; when $x = -1$ we have $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+2}$, a divergent series by the nth-term Test; when $x = 1$ we have $\sum_{n=1}^{\infty} \frac{n}{n+2}$, a divergent series
   (a) the radius is 1; the interval of convergence is $-1 < x < 1$
   (b) the interval of absolute convergence is $-1 < x < 1$
   (c) there are no values for which the series converges conditionally

8. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(x+2)^{n+1}}{n+1} \cdot \frac{n}{(x+2)^n}\right| < 1 \Rightarrow |x+2| \lim_{n\to\infty} \left(\frac{n}{n+1}\right) < 1 \Rightarrow |x+2| < 1$
   $\Rightarrow -1 < x+2 < 1 \Rightarrow -3 < x < -1$; when $x = -3$ we have $\sum_{n=1}^{\infty} \frac{1}{n}$, a divergent series; when $x = -1$ we have $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, a convergent series
   (a) the radius is 1; the interval of convergence is $-3 < x \le -1$
   (b) the interval of absolute convergence is $-3 < x < -1$
   (c) the series converges conditionally at $x = -1$

9. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)\sqrt{n+1}\,3^{n+1}} \cdot \frac{n\sqrt{n}\,3^n}{x^n}\right| < 1 \Rightarrow \frac{|x|}{3}\left(\lim_{n\to\infty} \frac{n}{n+1}\right)\left(\sqrt{\lim_{n\to\infty} \frac{n}{n+1}}\right) < 1$
   $\Rightarrow \frac{|x|}{3}(1)(1) < 1 \Rightarrow |x| < 3 \Rightarrow -3 < x < 3$; when $x = -3$ we have $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/2}}$, an absolutely convergent series; when $x = 3$ we have $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$, a convergent p-series
   (a) the radius is 3; the interval of convergence is $-3 \le x \le 3$
   (b) the interval of absolute convergence is $-3 \le x \le 3$
   (c) there are no values for which the series converges conditionally

10. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(x-1)^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{(x-1)^n}\right| < 1 \Rightarrow |x-1| \sqrt{\lim_{n\to\infty} \frac{n}{n+1}} < 1 \Rightarrow |x-1| < 1$
   $\Rightarrow -1 < x-1 < 1 \Rightarrow 0 < x < 2$; when $x = 0$ we have $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/2}}$, a conditionally convergent series; when $x = 2$ we have $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$, a divergent series
   (a) the radius is 1; the interval of convergence is $0 \le x < 2$
   (b) the interval of absolute convergence is $0 < x < 2$
   (c) the series converges conditionally at $x = 0$

11. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| < 1 \Rightarrow |x| \lim_{n\to\infty} \left(\frac{1}{n+1}\right) < 1$ for all x
   (a) the radius is ∞; the series converges for all x

(b) the series converges absolutely for all x
(c) there are no values for which the series converges conditionally

12. $\lim_{n \to \infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n \to \infty} \left|\frac{3^{n+1} x^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n x^n}\right| < 1 \Rightarrow 3|x| \lim_{n \to \infty} \left(\frac{1}{n+1}\right) < 1$ for all x
(a) the radius is ∞; the series converges for all x
(b) the series converges absolutely for all x
(c) there are no values for which the series converges conditionally

13. $\lim_{n \to \infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n \to \infty} \left|\frac{4^{n+1} x^{2n+2}}{n+1} \cdot \frac{n}{4^n x^{2n}}\right| < 1 \Rightarrow x^2 \lim_{n \to \infty} \left(\frac{4n}{n+1}\right) = 4x^2 < 1 \Rightarrow x^2 < \frac{1}{4}$
$\Rightarrow -\frac{1}{2} < x < \frac{1}{2}$; when $x = -\frac{1}{2}$ we have $\sum_{n=1}^{\infty} \frac{4^n}{n}\left(-\frac{1}{2}\right)^{2n} = \sum_{n=1}^{\infty} \frac{1}{n}$, a divergent p-series; when $x = \frac{1}{2}$ we have $\sum_{n=1}^{\infty} \frac{4^n}{n}\left(\frac{1}{2}\right)^{2n} = \sum_{n=1}^{\infty} \frac{1}{n}$, a divergent p-series
(a) the radius is $\frac{1}{2}$; the interval of convergence is $-\frac{1}{2} < x < \frac{1}{2}$
(b) the interval of absolute convergence is $-\frac{1}{2} < x < \frac{1}{2}$
(c) there are no values for which the series converges conditionally

14. $\lim_{n \to \infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n \to \infty} \left|\frac{(x-1)^{n+1}}{(n+1)^2 3^{n+1}} \cdot \frac{n^2 3^n}{(x-1)^n}\right| < 1 \Rightarrow |x-1| \lim_{n \to \infty} \left(\frac{n^2}{3(n+1)^2}\right) = \frac{1}{3}|x-1| < 1$
$\Rightarrow -2 < x < 4$; when $x = -2$ we have $\sum_{n=1}^{\infty} \frac{(-3)^n}{n^2 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$, an absolutely convergent series; when $x = 4$ we have $\sum_{n=1}^{\infty} \frac{(3)^n}{n^2 3^n} = \sum_{n=1}^{\infty} \frac{1}{n^2}$, an absolutely convergent series.
(a) the radius is 3; the interval of convergence is $-2 \le x \le 4$
(b) the interval of absolute convergence is $-2 \le x \le 4$
(c) there are no values for which the series converges conditionally

15. $\lim_{n \to \infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n \to \infty} \left|\frac{x^{n+1}}{\sqrt{(n+1)^2+3}} \cdot \frac{\sqrt{n^2+3}}{x^n}\right| < 1 \Rightarrow |x| \sqrt{\lim_{n \to \infty} \frac{n^2+3}{n^2+2n+4}} < 1 \Rightarrow |x| < 1$
$\Rightarrow -1 < x < 1$; when $x = -1$ we have $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+3}}$, a conditionally convergent series; when $x = 1$ we have $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+3}}$, a divergent series
(a) the radius is 1; the interval of convergence is $-1 \le x < 1$
(b) the interval of absolute convergence is $-1 < x < 1$
(c) the series converges conditionally at $x = -1$

16. $\lim_{n \to \infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n \to \infty} \left|\frac{x^{n+1}}{\sqrt{(n+1)^2+3}} \cdot \frac{\sqrt{n^2+3}}{x^n}\right| < 1 \Rightarrow |x| \sqrt{\lim_{n \to \infty} \frac{n^2+3}{n^2+2n+4}} < 1 \Rightarrow |x| < 1$
$\Rightarrow -1 < x < 1$; when $x = -1$ we have $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+3}}$, a divergent series; when $x = 1$ we have $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+3}}$, a conditionally convergent series
(a) the radius is 1; the interval of convergence is $-1 < x \le 1$
(b) the interval of absolute convergence is $-1 < x < 1$
(c) the series converges conditionally at $x = 1$

17. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(n+1)(x+3)^{n+1}}{5^{n+1}} \cdot \frac{5^n}{n(x+3)^n}\right| < 1 \Rightarrow \frac{|x+3|}{5} \lim_{n\to\infty} \left(\frac{n+1}{n}\right) < 1 \Rightarrow \frac{|x+3|}{5} < 1$
$\Rightarrow |x+3| < 5 \Rightarrow -5 < x+3 < 5 \Rightarrow -8 < x < 2$; when $x = -8$ we have $\sum_{n=1}^{\infty} \frac{n(-5)^n}{5^n} = \sum_{n=1}^{\infty} (-1)^n n$, a divergent series; when $x = 2$ we have $\sum_{n=1}^{\infty} \frac{n5^n}{5^n} = \sum_{n=1}^{\infty} n$, a divergent series

(a) the radius is 5; the interval of convergence is $-8 < x < 2$
(b) the interval of absolute convergence is $-8 < x < 2$
(c) there are no values for which the series converges conditionally

18. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(n+1)x^{n+1}}{4^{n+1}(n^2+2n+2)} \cdot \frac{4^n(n^2+1)}{nx^n}\right| < 1 \Rightarrow \frac{|x|}{4} \lim_{n\to\infty} \left|\frac{(n+1)(n^2+1)}{n(n^2+2n+2)}\right| < 1 \Rightarrow |x| < 4$
$\Rightarrow -4 < x < 4$; when $x = -4$ we have $\sum_{n=1}^{\infty} \frac{n(-1)^n}{n^2+1}$, a conditionally convergent series; when $x = 4$ we have $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$, a divergent series

(a) the radius is 4; the interval of convergence is $-4 \le x < 4$
(b) the interval of absolute convergence is $-4 < x < 4$
(c) the series converges conditionally at $x = -4$

19. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{\sqrt{n+1}\,x^{n+1}}{3^{n+1}} \cdot \frac{3^n}{\sqrt{n}\,x^n}\right| < 1 \Rightarrow \frac{|x|}{3} \sqrt{\lim_{n\to\infty} \left(\frac{n+1}{n}\right)} < 1 \Rightarrow \frac{|x|}{3} < 1 \Rightarrow |x| < 3$
$\Rightarrow -3 < x < 3$; when $x = -3$ we have $\sum_{n=1}^{\infty} (-1)^n \sqrt{n}$, a divergent series; when $x = 3$ we have $\sum_{n=1}^{\infty} \sqrt{n}$, a divergent series

(a) the radius is 3; the interval of convergence is $-3 < x < 3$
(b) the interval of absolute convergence is $-3 < x < 3$
(c) there are no values for which the series converges conditionally

20. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{\sqrt[n+1]{n+1}\,(2x+5)^{n+1}}{\sqrt[n]{n}\,(2x+5)^n}\right| < 1 \Rightarrow |2x+5| \lim_{n\to\infty} \left(\frac{\sqrt[n+1]{n+1}}{\sqrt[n]{n}}\right) < 1$
$\Rightarrow |2x+5| \left(\frac{\lim_{t\to\infty} \sqrt[t]{t}}{\lim_{n\to\infty} \sqrt[n]{n}}\right) < 1 \Rightarrow |2x+5| < 1 \Rightarrow -1 < 2x+5 < 1 \Rightarrow -3 < x < -2$; when $x = -3$ we have $\sum_{n=1}^{\infty} (-1)\sqrt[n]{n}$, a divergent series since $\lim_{n\to\infty} \sqrt[n]{n} = 1$; when $x = -2$ we have $\sum_{n=1}^{\infty} \sqrt[n]{n}$, a divergent series

(a) the radius is $\frac{1}{2}$; the interval of convergence is $-3 < x < -2$
(b) the interval of absolute convergence is $-3 < x < -2$
(c) there are no values for which the series converges conditionally

21. First, rewrite the series as $\sum_{n=1}^{\infty} (2+(-1)^n)(x+1)^{n-1} = \sum_{n=1}^{\infty} 2(x+1)^{n-1} + \sum_{n=1}^{\infty} (-1)^n(x+1)^{n-1}$. For the series $\sum_{n=1}^{\infty} 2(x+1)^{n-1}$: $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{2(x+1)^n}{2(x+1)^{n-1}}\right| < 1 \Rightarrow |x+1| \lim_{n\to\infty} 1 = |x+1| < 1 \Rightarrow -2 < x < 0$; For the series $\sum_{n=1}^{\infty} (-1)^n(x+1)^{n-1}$: $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(-1)^{n+1}(x+1)^n}{(-1)^n(x+1)^{n-1}}\right| < 1 \Rightarrow |x+1| \lim_{n\to\infty} 1 = |x+1| < 1$
$\Rightarrow -2 < x < 0$; when $x = -2$ we have $\sum_{n=1}^{\infty} (2+(-1)^n)(-1)^{n-1}$, a divergent series; when $x = 0$ we have $\sum_{n=1}^{\infty} (2+(-1)^n)$, a divergent series

(a) the radius is 1; the interval of convergence is $-2 < x < 0$
(b) the interval of absolute convergence is $-2 < x < 0$
(c) there are no values for which the series converges conditionally

22. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{(-1)^{n+1}3^{2n+2}(x-2)^{n+1}}{3(n+1)}\cdot\frac{3n}{(-1)^n 3^{2n}(x-2)^n}\right| < 1 \Rightarrow |x-2|\lim_{n\to\infty}\frac{9n}{n+1} = 9|x-2| < 1$
$\Rightarrow \frac{17}{9} < x < \frac{19}{9}$; when $x = \frac{17}{9}$ we have $\sum_{n=1}^{\infty}\frac{(-1)^n 3^{2n}}{3n}\left(-\frac{1}{9}\right)^n = \sum_{n=1}^{\infty}\frac{1}{3n}$, a divergent series; when $x = \frac{19}{9}$ we have $\sum_{n=1}^{\infty}\frac{(-1)^n 3^{2n}}{3n}\left(\frac{1}{9}\right)^n = \sum_{n=1}^{\infty}\frac{(-1)^n}{3n}$, a conditionally convergent series.
(a) the radius is $\frac{1}{9}$; the interval of convergence is $\frac{17}{9} < x \le \frac{19}{9}$
(b) the interval of absolute convergence is $\frac{17}{9} < x < \frac{19}{9}$
(c) the series converges conditionally at $x = \frac{19}{9}$

23. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{\left(1+\frac{1}{n+1}\right)^{n+1}x^{n+1}}{\left(1+\frac{1}{n}\right)^n x^n}\right| < 1 \Rightarrow |x|\left(\frac{\lim_{t\to\infty}\left(1+\frac{1}{t}\right)^t}{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n}\right) < 1 \Rightarrow |x|\left(\frac{e}{e}\right) < 1 \Rightarrow |x| < 1$
$\Rightarrow -1 < x < 1$; when $x = -1$ we have $\sum_{n=1}^{\infty}(-1)^n\left(1+\frac{1}{n}\right)^n$, a divergent series by the nth-Term Test since $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \ne 0$; when $x = 1$ we have $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^n$, a divergent series
(a) the radius is 1; the interval of convergence is $-1 < x < 1$
(b) the interval of absolute convergence is $-1 < x < 1$
(c) there are no values for which the series converges conditionally

24. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{\ln(n+1)x^{n+1}}{x^n \ln n}\right| < 1 \Rightarrow |x|\lim_{n\to\infty}\left|\frac{\left(\frac{1}{n+1}\right)}{\left(\frac{1}{n}\right)}\right| < 1 \Rightarrow |x|\lim_{n\to\infty}\left(\frac{n}{n+1}\right) < 1 \Rightarrow |x| < 1$
$\Rightarrow -1 < x < 1$; when $x = -1$ we have $\sum_{n=1}^{\infty}(-1)^n \ln n$, a divergent series by the nth-Term Test since $\lim_{n\to\infty}\ln n \ne 0$; when $x = 1$ we have $\sum_{n=1}^{\infty}\ln n$, a divergent series
(a) the radius is 1; the interval of convergence is $-1 < x < 1$
(b) the interval of absolute convergence is $-1 < x < 1$
(c) there are no values for which the series converges conditionally

25. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{(n+1)^{n+1}x^{n+1}}{n^n x^n}\right| < 1 \Rightarrow |x|\left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right)\left(\lim_{n\to\infty}(n+1)\right) < 1$
$\Rightarrow e|x|\lim_{n\to\infty}(n+1) < 1 \Rightarrow$ only $x = 0$ satisfies this inequality
(a) the radius is 0; the series converges only for $x = 0$
(b) the series converges absolutely only for $x = 0$
(c) there are no values for which the series converges conditionally

26. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{(n+1)!\,(x-4)^{n+1}}{n!\,(x-4)^n}\right| < 1 \Rightarrow |x-4|\lim_{n\to\infty}(n+1) < 1 \Rightarrow$ only $x = 4$ satisfies this inequality
(a) the radius is 0; the series converges only for $x = 4$
(b) the series converges absolutely only for $x = 4$
(c) there are no values for which the series converges conditionally

27. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{(x+2)^{n+1}}{(n+1)2^{n+1}}\cdot\frac{n2^n}{(x+2)^n}\right| < 1 \Rightarrow \frac{|x+2|}{2}\lim_{n\to\infty}\left(\frac{n}{n+1}\right) < 1 \Rightarrow \frac{|x+2|}{2} < 1 \Rightarrow |x+2| < 2$
$\Rightarrow -2 < x+2 < 2 \Rightarrow -4 < x < 0$; when $x = -4$ we have $\sum_{n=1}^{\infty}\frac{-1}{n}$, a divergent series; when $x = 0$ we have $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$, the alternating harmonic series which converges conditionally
(a) the radius is 2; the interval of convergence is $-4 < x \le 0$
(b) the interval of absolute convergence is $-4 < x < 0$
(c) the series converges conditionally at $x = 0$

28. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{(-2)^{n+1}(n+2)(x-1)^{n+1}}{(-2)^n(n+1)(x-1)^n}\right| < 1 \Rightarrow 2|x-1|\lim_{n\to\infty}\left(\frac{n+2}{n+1}\right) < 1 \Rightarrow 2|x-1| < 1$

$\Rightarrow |x-1| < \frac{1}{2} \Rightarrow -\frac{1}{2} < x-1 < \frac{1}{2} \Rightarrow \frac{1}{2} < x < \frac{3}{2}$; when $x = \frac{1}{2}$ we have $\sum_{n=1}^{\infty}(n+1)$, a divergent series; when $x = \frac{3}{2}$ we have $\sum_{n=1}^{\infty}(-1)^n(n+1)$, a divergent series

(a) the radius is $\frac{1}{2}$; the interval of convergence is $\frac{1}{2} < x < \frac{3}{2}$

(b) the interval of absolute convergence is $\frac{1}{2} < x < \frac{3}{2}$

(c) there are no values for which the series converges conditionally

29. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)(\ln(n+1))^2}\cdot\frac{n(\ln n)^2}{x^n}\right| < 1 \Rightarrow |x|\left(\lim_{n\to\infty}\frac{n}{n+1}\right)\left(\lim_{n\to\infty}\frac{\ln n}{\ln(n+1)}\right)^2 < 1$

$\Rightarrow |x|(1)\left(\lim_{n\to\infty}\frac{\left(\frac{1}{n}\right)}{\left(\frac{1}{n+1}\right)}\right)^2 < 1 \Rightarrow |x|\left(\lim_{n\to\infty}\frac{n+1}{n}\right)^2 < 1 \Rightarrow |x| < 1 \Rightarrow -1 < x < 1$; when $x = -1$ we have $\sum_{n=1}^{\infty}\frac{(-1)^n}{n(\ln n)^2}$ which converges absolutely; when $x = 1$ we have $\sum_{n=1}^{\infty}\frac{1}{n(\ln n)^2}$ which converges

(a) the radius is 1; the interval of convergence is $-1 \le x \le 1$

(b) the interval of absolute convergence is $-1 \le x \le 1$

(c) there are no values for which the series converges conditionally

30. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)\ln(n+1)}\cdot\frac{n\ln(n)}{x^n}\right| < 1 \Rightarrow |x|\left(\lim_{n\to\infty}\frac{n}{n+1}\right)\left(\lim_{n\to\infty}\frac{\ln(n)}{\ln(n+1)}\right) < 1$

$\Rightarrow |x|(1)(1) < 1 \Rightarrow |x| < 1 \Rightarrow -1 < x < 1$; when $x = -1$ we have $\sum_{n=2}^{\infty}\frac{(-1)^n}{n\ln n}$, a convergent alternating series; when $x = 1$ we have $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$ which diverges by Exercise 38, Section 9.3

(a) the radius is 1; the interval of convergence is $-1 \le x < 1$

(b) the interval of absolute convergence is $-1 < x < 1$

(c) the series converges conditionally at $x = -1$

31. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{(4x-5)^{2n+3}}{(n+1)^{3/2}}\cdot\frac{n^{3/2}}{(4x-5)^{2n+1}}\right| < 1 \Rightarrow (4x-5)^2\left(\lim_{n\to\infty}\frac{n}{n+1}\right)^{3/2} < 1 \Rightarrow (4x-5)^2 < 1$

$\Rightarrow |4x-5| < 1 \Rightarrow -1 < 4x-5 < 1 \Rightarrow 1 < x < \frac{3}{2}$; when $x = 1$ we have $\sum_{n=1}^{\infty}\frac{(-1)^{2n+1}}{n^{3/2}} = \sum_{n=1}^{\infty}\frac{-1}{n^{3/2}}$ which is absolutely convergent; when $x = \frac{3}{2}$ we have $\sum_{n=1}^{\infty}\frac{(1)^{2n+1}}{n^{3/2}}$, a convergent p-series

(a) the radius is $\frac{1}{4}$; the interval of convergence is $1 \le x \le \frac{3}{2}$

(b) the interval of absolute convergence is $1 \le x \le \frac{3}{2}$

(c) there are no values for which the series converges conditionally

32. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{(3x+1)^{n+2}}{2n+4}\cdot\frac{2n+2}{(3x+1)^{n+1}}\right| < 1 \Rightarrow |3x+1|\lim_{n\to\infty}\left(\frac{2n+2}{2n+4}\right) < 1 \Rightarrow |3x+1| < 1$

$\Rightarrow -1 < 3x+1 < 1 \Rightarrow -\frac{2}{3} < x < 0$; when $x = -\frac{2}{3}$ we have $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n+1}$, a conditionally convergent series; when $x = 0$ we have $\sum_{n=1}^{\infty}\frac{(1)^{n+1}}{2n+1} = \sum_{n=1}^{\infty}\frac{1}{2n+1}$, a divergent series

(a) the radius is $\frac{1}{3}$; the interval of convergence is $-\frac{2}{3} \le x < 0$

(b) the interval of absolute convergence is $-\frac{2}{3} < x < 0$

(c) the series converges conditionally at $x = -\frac{2}{3}$

33. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{x^{n+1}}{2\cdot4\cdot6\cdots(2n)(2(n+1))}\cdot\frac{2\cdot4\cdot6\cdots(2n)}{x^n}\right| < 1 \Rightarrow |x|\lim_{n\to\infty}\left(\frac{1}{2n+2}\right) < 1$ for all x
(a) the radius is ∞; the series converges for all x
(b) the series converges absolutely for all x
(c) there are no values for which the series converges conditionally

34. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{3\cdot5\cdot7\cdots(2n+1)(2(n+1)+1)x^{n+2}}{(n+1)^2 2^{n+1}}\cdot\frac{n^2 2^n}{3\cdot5\cdot7\cdots(2n+1)x^{n+1}}\right| < 1 \Rightarrow |x|\lim_{n\to\infty}\left(\frac{(2n+3)n^2}{2(n+1)^2}\right) < 1 \Rightarrow$ only $x = 0$ satisfies this inequality
(a) the radius is 0; the series converges only for $x = 0$
(b) the series converges absolutely only for $x = 0$
(c) there are no values for which the series converges conditionally

35. For the series $\sum_{n=1}^{\infty}\frac{1+2+\cdots+n}{1^2+2^2+\cdots+n^2}x^n$, recall $1+2+\cdots+n = \frac{n(n+1)}{2}$ and $1^2+2^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$ so that we can rewrite the series as $\sum_{n=1}^{\infty}\left(\frac{\frac{n(n+1)}{2}}{\frac{n(n+1)(2n+1)}{6}}\right)x^n = \sum_{n=1}^{\infty}\left(\frac{3}{2n+1}\right)x^n$; then $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{3x^{n+1}}{(2(n+1)+1)}\cdot\frac{(2n+1)}{3x^n}\right| < 1$ $\Rightarrow |x|\lim_{n\to\infty}\left|\frac{(2n+1)}{(2n+3)}\right| < 1 \Rightarrow |x| < 1 \Rightarrow -1 < x < 1$; when $x = -1$ we have $\sum_{n=1}^{\infty}\left(\frac{3}{2n+1}\right)(-1)^n$, a conditionally convergent series; when $x = 1$ we have $\sum_{n=1}^{\infty}\left(\frac{3}{2n+1}\right)$, a divergent series.
(a) the radius is 1; the interval of convergence is $-1 \le x < 1$
(b) the interval of absolute convergence is $-1 < x < 1$
(c) the series converges conditionally at $x = -1$

36. For the series $\sum_{n=1}^{\infty}\left(\sqrt{n+1}-\sqrt{n}\right)(x-3)^n$, note that $\sqrt{n+1}-\sqrt{n} = \frac{\sqrt{n+1}-\sqrt{n}}{1}\cdot\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}}$ so that we can rewrite the series as $\sum_{n=1}^{\infty}\frac{(x-3)^n}{\sqrt{n+1}+\sqrt{n}}$; then $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{(x-3)^{n+1}}{\sqrt{n+2}+\sqrt{n+1}}\cdot\frac{\sqrt{n+1}+\sqrt{n}}{(x-3)^n}\right| < 1$ $\Rightarrow |x-3|\lim_{n\to\infty}\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+2}+\sqrt{n+1}} < 1 \Rightarrow |x-3| < 1 \Rightarrow 2 < x < 4$; when $x = 2$ we have $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n+1}+\sqrt{n}}$, a conditionally convergent series; when $x = 4$ we have $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}}$, a divergent series;
(a) the radius is 1; the interval of convergence is $2 \le x < 4$
(b) the interval of absolute convergence is $2 < x < 4$
(c) the series converges conditionally at $x = 2$

37. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{3\cdot6\cdot9\cdots(3n)(3(n+1))}\cdot\frac{3\cdot6\cdot9\cdots(3n)}{n!\,x^n}\right| < 1 \Rightarrow |x|\lim_{n\to\infty}\left|\frac{(n+1)}{3(n+1)}\right| < 1 \Rightarrow \frac{|x|}{3} < 1 \Rightarrow |x| < 3 \Rightarrow R = 3$

38. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{(2\cdot4\cdot6\cdots(2n)(2(n+1)))^2x^{n+1}}{(2\cdot5\cdot8\cdots(3n-1)(3(n+1)-1))^2}\cdot\frac{(2\cdot5\cdot8\cdots(3n-1))^2}{(2\cdot4\cdot6\cdots(2n))^2x^n}\right| < 1 \Rightarrow |x|\lim_{n\to\infty}\left|\frac{(2n+2)^2}{(3n+2)^2}\right| < 1 \Rightarrow \frac{4|x|}{9} < 1$ $\Rightarrow |x| < \frac{9}{4} \Rightarrow R = \frac{9}{4}$

39. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{((n+1)!)^2x^{n+1}}{2^{n+1}(2(n+1))!}\cdot\frac{2^n(2n)!}{(n!)^2x^n}\right| < 1 \Rightarrow |x|\lim_{n\to\infty}\left|\frac{(n+1)^2}{2(2n+2)(2n+1)}\right| < 1 \Rightarrow \frac{|x|}{8} < 1 \Rightarrow |x| < 8 \Rightarrow R = 8$

40. $\lim_{n\to\infty}\sqrt[n]{u_n} < 1 \Rightarrow \lim_{n\to\infty}\sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2}x^n} < 1 \Rightarrow |x|\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n < 1 \Rightarrow |x|e^{-1} < 1 \Rightarrow |x| < e \Rightarrow R = e$

41. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{3^{n+1}x^{n+1}}{3^n x^n}\right| < 1 \Rightarrow |x|\lim_{n\to\infty} 3 < 1 \Rightarrow |x| < \frac{1}{3} \Rightarrow -\frac{1}{3} < x < \frac{1}{3}$; at $x = -\frac{1}{3}$ we have $\sum_{n=0}^{\infty} 3^n\left(-\frac{1}{3}\right)^n = \sum_{n=0}^{\infty}(-1)^n$, which diverges; at $x = \frac{1}{3}$ we have $\sum_{n=0}^{\infty} 3^n\left(\frac{1}{3}\right)^n = \sum_{n=0}^{\infty} 1$, which diverges. The series $\sum_{n=0}^{\infty} 3^n x^n = \sum_{n=0}^{\infty}(3x)^n$ is a convergent geometric series when $-\frac{1}{3} < x < \frac{1}{3}$ and the sum is $\frac{1}{1-3x}$.

42. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{(e^x-4)^{n+1}}{(e^x-4)^n}\right| < 1 \Rightarrow |e^x - 4|\lim_{n\to\infty} 1 < 1 \Rightarrow |e^x - 4| < 1 \Rightarrow 3 < e^x < 5 \Rightarrow \ln 3 < x < \ln 5$; at $x = \ln 3$ we have $\sum_{n=0}^{\infty}\left(e^{\ln 3} - 4\right)^n = \sum_{n=0}^{\infty}(-1)^n$, which diverges; at $x = \ln 5$ we have $\sum_{n=0}^{\infty}\left(e^{\ln 5} - 4\right)^n = \sum_{n=0}^{\infty} 1$, which diverges. The series $\sum_{n=0}^{\infty}(e^x - 4)^n$ is a convergent geometric series when $\ln 3 < x < \ln 5$ and the sum is $\frac{1}{1-(e^x-4)} = \frac{1}{5-e^x}$.

43. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{(x-1)^{2n+2}}{4^{n+1}}\cdot\frac{4^n}{(x-1)^{2n}}\right| < 1 \Rightarrow \frac{(x-1)^2}{4}\lim_{n\to\infty}|1| < 1 \Rightarrow (x-1)^2 < 4 \Rightarrow |x-1| < 2$ $\Rightarrow -2 < x - 1 < 2 \Rightarrow -1 < x < 3$; at $x = -1$ we have $\sum_{n=0}^{\infty}\frac{(-2)^{2n}}{4^n} = \sum_{n=0}^{\infty}\frac{4^n}{4^n} = \sum_{n=0}^{\infty} 1$, which diverges; at $x = 3$ we have $\sum_{n=0}^{\infty}\frac{2^{2n}}{4^n} = \sum_{n=0}^{\infty}\frac{4^n}{4^n} = \sum_{n=0}^{\infty} 1$, a divergent series; the interval of convergence is $-1 < x < 3$; the series $\sum_{n=0}^{\infty}\frac{(x-1)^{2n}}{4^n} = \sum_{n=0}^{\infty}\left(\left(\frac{x-1}{2}\right)^2\right)^n$ is a convergent geometric series when $-1 < x < 3$ and the sum is $\frac{1}{1-\left(\frac{x-1}{2}\right)^2} = \frac{1}{\left[\frac{4-(x-1)^2}{4}\right]} = \frac{4}{4-x^2+2x-1} = \frac{4}{3+2x-x^2}$

44. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{(x+1)^{2n+2}}{9^{n+1}}\cdot\frac{9^n}{(x+1)^{2n}}\right| < 1 \Rightarrow \frac{(x+1)^2}{9}\lim_{n\to\infty}|1| < 1 \Rightarrow (x+1)^2 < 9 \Rightarrow |x+1| < 3$ $\Rightarrow -3 < x + 1 < 3 \Rightarrow -4 < x < 2$; when $x = -4$ we have $\sum_{n=0}^{\infty}\frac{(-3)^{2n}}{9^n} = \sum_{n=0}^{\infty} 1$ which diverges; at $x = 2$ we have $\sum_{n=0}^{\infty}\frac{3^{2n}}{9^n} = \sum_{n=0}^{\infty} 1$ which also diverges; the interval of convergence is $-4 < x < 2$; the series $\sum_{n=0}^{\infty}\frac{(x+1)^{2n}}{9^n} = \sum_{n=0}^{\infty}\left(\left(\frac{x+1}{3}\right)^2\right)^n$ is a convergent geometric series when $-4 < x < 2$ and the sum is $\frac{1}{1-\left(\frac{x+1}{3}\right)^2} = \frac{1}{\left[\frac{9-(x+1)^2}{9}\right]} = \frac{9}{9-x^2-2x-1} = \frac{9}{8-2x-x^2}$

45. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{\left(\sqrt{x}-2\right)^{n+1}}{2^{n+1}}\cdot\frac{2^n}{\left(\sqrt{x}-2\right)^n}\right| < 1 \Rightarrow \left|\sqrt{x}-2\right| < 2 \Rightarrow -2 < \sqrt{x} - 2 < 2 \Rightarrow 0 < \sqrt{x} < 4$ $\Rightarrow 0 < x < 16$; when $x = 0$ we have $\sum_{n=0}^{\infty}(-1)^n$, a divergent series; when $x = 16$ we have $\sum_{n=0}^{\infty}(1)^n$, a divergent series; the interval of convergence is $0 < x < 16$; the series $\sum_{n=0}^{\infty}\left(\frac{\sqrt{x}-2}{2}\right)^n$ is a convergent geometric series when $0 < x < 16$ and its sum is $\frac{1}{1-\left(\frac{\sqrt{x}-2}{2}\right)} = \frac{1}{\left(\frac{2-\sqrt{x}+2}{2}\right)} = \frac{2}{4-\sqrt{x}}$

46. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{(\ln x)^{n+1}}{(\ln x)^n}\right| < 1 \Rightarrow |\ln x| < 1 \Rightarrow -1 < \ln x < 1 \Rightarrow e^{-1} < x < e$; when $x = e^{-1}$ or e we obtain the series $\sum_{n=0}^{\infty} 1^n$ and $\sum_{n=0}^{\infty}(-1)^n$ which both diverge; the interval of convergence is $e^{-1} < x < e$; $\sum_{n=0}^{\infty}(\ln x)^n = \frac{1}{1-\ln x}$ when $e^{-1} < x < e$

47. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\left(\frac{x^2+1}{3}\right)^{n+1}\cdot\left(\frac{3}{x^2+1}\right)^n\right| < 1 \Rightarrow \frac{(x^2+1)}{3}\lim_{n\to\infty}|1| < 1 \Rightarrow \frac{x^2+1}{3} < 1 \Rightarrow x^2 < 2$ $\Rightarrow |x| < \sqrt{2} \Rightarrow -\sqrt{2} < x < \sqrt{2}$; at $x = \pm\sqrt{2}$ we have $\sum_{n=0}^{\infty}(1)^n$ which diverges; the interval of convergence is $-\sqrt{2} < x < \sqrt{2}$; the series $\sum_{n=0}^{\infty}\left(\frac{x^2+1}{3}\right)^n$ is a convergent geometric series when $-\sqrt{2} < x < \sqrt{2}$ and its sum is $\frac{1}{1-\left(\frac{x^2+1}{3}\right)} = \frac{1}{\left(\frac{3-x^2-1}{3}\right)} = \frac{3}{2-x^2}$

48. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty}\left|\frac{(x^2-1)^{n+1}}{2^{n+1}}\cdot\frac{2^n}{(x^2+1)^n}\right| < 1 \Rightarrow |x^2-1| < 2 \Rightarrow -\sqrt{3} < x < \sqrt{3}$; when $x = \pm\sqrt{3}$ we have $\sum_{n=0}^{\infty}1^n$, a divergent series; the interval of convergence is $-\sqrt{3} < x < \sqrt{3}$; the series $\sum_{n=0}^{\infty}\left(\frac{x^2-1}{2}\right)^n$ is a convergent geometric series when $-\sqrt{3} < x < \sqrt{3}$ and its sum is $\frac{1}{1-\left(\frac{x^2-1}{2}\right)} = \frac{1}{\left(\frac{2-\left(x^2-1\right)}{2}\right)} = \frac{2}{3-x^2}$

49. $\lim_{n\to\infty}\left|\frac{(x-3)^{n+1}}{2^{n+1}}\cdot\frac{2^n}{(x-3)^n}\right| < 1 \Rightarrow |x-3| < 2 \Rightarrow 1 < x < 5$; when $x = 1$ we have $\sum_{n=1}^{\infty}(1)^n$ which diverges; when $x = 5$ we have $\sum_{n=1}^{\infty}(-1)^n$ which also diverges; the interval of convergence is $1 < x < 5$; the sum of this convergent geometric series is $\frac{1}{1+\left(\frac{x-3}{2}\right)} = \frac{2}{x-1}$. If $f(x) = 1 - \frac{1}{2}(x-3) + \frac{1}{4}(x-3)^2 + \ldots + \left(-\frac{1}{2}\right)^n(x-3)^n + \ldots$ $= \frac{2}{x-1}$ then $f'(x) = -\frac{1}{2} + \frac{1}{2}(x-3) + \ldots + \left(-\frac{1}{2}\right)^n n(x-3)^{n-1} + \ldots$ is convergent when $1 < x < 5$, and diverges when $x = 1$ or 5. The sum for $f'(x)$ is $\frac{-2}{(x-1)^2}$, the derivative of $\frac{2}{x-1}$.

50. If $f(x) = 1 - \frac{1}{2}(x-3) + \frac{1}{4}(x-3)^2 + \ldots + \left(-\frac{1}{2}\right)^n(x-3)^n + \ldots = \frac{2}{x-1}$ then $\int f(x)\,dx$ $= x - \frac{(x-3)^2}{4} + \frac{(x-3)^3}{12} + \ldots + \left(-\frac{1}{2}\right)^n\frac{(x-3)^{n+1}}{n+1} + \ldots$. At $x = 1$ the series $\sum_{n=1}^{\infty}\frac{-2}{n+1}$ diverges; at $x = 5$ the series $\sum_{n=1}^{\infty}\frac{(-1)^n 2}{n+1}$ converges. Therefore the interval of convergence is $1 < x \le 5$ and the sum is $2\ln|x-1| + (3 - \ln 4)$, since $\int\frac{2}{x-1}\,dx = 2\ln|x-1| + C$, where $C = 3 - \ln 4$ when $x = 3$.

51. (a) Differentiate the series for sin x to get $\cos x = 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \frac{9x^8}{9!} - \frac{11x^{10}}{11!} + \ldots$ $= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \ldots$. The series converges for all values of x since $\lim_{n\to\infty}\left|\frac{x^{2n+2}}{(2n+2)!}\cdot\frac{(2n)!}{x^{2n}}\right| = x^2\lim_{n\to\infty}\left(\frac{1}{(2n+1)(2n+2)}\right) = 0 < 1$ for all x.

(b) $\sin 2x = 2x - \frac{2^3x^3}{3!} + \frac{2^5x^5}{5!} - \frac{2^7x^7}{7!} + \frac{2^9x^9}{9!} - \frac{2^{11}x^{11}}{11!} + \ldots = 2x - \frac{8x^3}{3!} + \frac{32x^5}{5!} - \frac{128x^7}{7!} + \frac{512x^9}{9!} - \frac{2048x^{11}}{11!} + \ldots$

(c) $2\sin x\cos x = 2\left[(0\cdot 1) + (0\cdot 0 + 1\cdot 1)x + \left(0\cdot\frac{-1}{2} + 1\cdot 0 + 0\cdot 1\right)x^2 + \left(0\cdot 0 - 1\cdot\frac{1}{2} + 0\cdot 0 - 1\cdot\frac{1}{3!}\right)x^3\right.$ $+ \left(0\cdot\frac{1}{4!} + 1\cdot 0 - 0\cdot\frac{1}{2} - 0\cdot\frac{1}{3!} + 0\cdot 1\right)x^4 + \left(0\cdot 0 + 1\cdot\frac{1}{4!} + 0\cdot 0 + \frac{1}{2}\cdot\frac{1}{3!} + 0\cdot 0 + 1\cdot\frac{1}{5!}\right)x^5$ $\left. + \left(0\cdot\frac{1}{6!} + 1\cdot 0 + 0\cdot\frac{1}{4!} + 0\cdot\frac{1}{3!} + 0\cdot\frac{1}{2} + 0\cdot\frac{1}{5!} + 0\cdot 1\right)x^6 + \ldots\right] = 2\left[x - \frac{4x^3}{3!} + \frac{16x^5}{5!} - \ldots\right]$ $= 2x - \frac{2^3x^3}{3!} + \frac{2^5x^5}{5!} - \frac{2^7x^7}{7!} + \frac{2^9x^9}{9!} - \frac{2^{11}x^{11}}{11!} + \ldots$

52. (a) $\frac{d}{x}(e^x) = 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \frac{5x^4}{5!} + \ldots = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots = e^x$; thus the derivative of e^x is e^x itself

(b) $\int e^x\,dx = e^x + C = x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots + C$, which is the general antiderivative of e^x

(c) $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \ldots$; $e^{-x}\cdot e^x = 1\cdot 1 + (1\cdot 1 - 1\cdot 1)x + \left(1\cdot\frac{1}{2!} - 1\cdot 1 + \frac{1}{2!}\cdot 1\right)x^2$ $+ \left(1\cdot\frac{1}{3!} - 1\cdot\frac{1}{2!} + \frac{1}{2!}\cdot 1 - \frac{1}{3!}\cdot 1\right)x^3 + \left(1\cdot\frac{1}{4!} - 1\cdot\frac{1}{3!} + \frac{1}{2!}\cdot\frac{1}{2!} - \frac{1}{3!}\cdot 1 + \frac{1}{4!}\cdot 1\right)x^4$ $+ \left(1\cdot\frac{1}{5!} - 1\cdot\frac{1}{4!} + \frac{1}{2!}\cdot\frac{1}{3!} - \frac{1}{3!}\cdot\frac{1}{2!} + \frac{1}{4!}\cdot 1 - \frac{1}{5!}\cdot 1\right)x^5 + \ldots = 1 + 0 + 0 + 0 + 0 + 0 + \ldots$

53. (a) $\ln|\sec x| + C = \int \tan x\, dx = \int \left(x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \ldots\right) dx$
$= \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \frac{17x^8}{2520} + \frac{31x^{10}}{14{,}175} + \ldots + C;\ x = 0 \Rightarrow C = 0 \Rightarrow \ln|\sec x| = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \frac{17x^8}{2520} + \frac{31x^{10}}{14{,}175} + \ldots,$
converges when $-\frac{\pi}{2} < x < \frac{\pi}{2}$

(b) $\sec^2 x = \frac{d(\tan x)}{dx} = \frac{d}{dx}\left(x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \ldots\right) = 1 + x^2 + \frac{2x^4}{3} + \frac{17x^6}{45} + \frac{62x^8}{315} + \ldots$, converges when $-\frac{\pi}{2} < x < \frac{\pi}{2}$

(c) $\sec^2 x = (\sec x)(\sec x) = \left(1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \ldots\right)\left(1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \ldots\right)$
$= 1 + \left(\frac{1}{2} + \frac{1}{2}\right)x^2 + \left(\frac{5}{24} + \frac{1}{4} + \frac{5}{24}\right)x^4 + \left(\frac{61}{720} + \frac{5}{48} + \frac{5}{48} + \frac{61}{720}\right)x^6 + \ldots$
$= 1 + x^2 + \frac{2x^4}{3} + \frac{17x^6}{45} + \frac{62x^8}{315} + \ldots,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$

54. (a) $\ln|\sec x + \tan x| + C = \int \sec x\, dx = \int \left(1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \ldots\right) dx$
$= x + \frac{x^3}{6} + \frac{x^5}{24} + \frac{61x^7}{5040} + \frac{277x^9}{72{,}576} + \ldots + C;\ x = 0 \Rightarrow C = 0 \Rightarrow \ln|\sec x + \tan x|$
$= x + \frac{x^3}{6} + \frac{x^5}{24} + \frac{61x^7}{5040} + \frac{277x^9}{72{,}576} + \ldots$, converges when $-\frac{\pi}{2} < x < \frac{\pi}{2}$

(b) $\sec x \tan x = \frac{d(\sec x)}{dx} = \frac{d}{dx}\left(1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \ldots\right) = x + \frac{5x^3}{6} + \frac{61x^5}{120} + \frac{277x^7}{1008} + \ldots$, converges when $-\frac{\pi}{2} < x < \frac{\pi}{2}$

(c) $(\sec x)(\tan x) = \left(1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \ldots\right)\left(x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \ldots\right)$
$= x + \left(\frac{1}{3} + \frac{1}{2}\right)x^3 + \left(\frac{2}{15} + \frac{1}{6} + \frac{5}{24}\right)x^5 + \left(\frac{17}{315} + \frac{1}{15} + \frac{5}{72} + \frac{61}{720}\right)x^7 + \ldots = x + \frac{5x^3}{6} + \frac{61x^5}{120} + \frac{277x^7}{1008} + \ldots,$
$-\frac{\pi}{2} < x < \frac{\pi}{2}$

55. (a) If $f(x) = \sum_{n=0}^{\infty} a_n x^n$, then $f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)(n-2)\cdots(n-(k-1))\, a_n x^{n-k}$ and $f^{(k)}(0) = k!a_k$
$\Rightarrow a_k = \frac{f^{(k)}(0)}{k!}$; likewise if $f(x) = \sum_{n=0}^{\infty} b_n x^n$, then $b_k = \frac{f^{(k)}(0)}{k!} \Rightarrow a_k = b_k$ for every nonnegative integer k

(b) If $f(x) = \sum_{n=0}^{\infty} a_n x^n = 0$ for all x, then $f^{(k)}(x) = 0$ for all x $\Rightarrow$ from part (a) that $a_k = 0$ for every nonnegative integer k

10.8 TAYLOR AND MACLAURIN SERIES

1. $f(x) = e^{2x}$, $f'(x) = 2e^{2x}$, $f''(x) = 4e^{2x}$, $f'''(x) = 8e^{2x}$; $f(0) = e^{2(0)} = 1$, $f'(0) = 2$, $f''(0) = 4$, $f'''(0) = 8 \Rightarrow P_0(x) = 1$, $P_1(x) = 1 + 2x$, $P_2(x) = 1 + x + 2x^2$, $P_3(x) = 1 + x + 2x^2 + \frac{4}{3}x^3$

2. $f(x) = \sin x$, $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$; $f(0) = \sin 0 = 0$, $f'(0) = 1$, $f''(0) = 0$, $f'''(0) = -1$ $\Rightarrow P_0(x) = 0$, $P_1(x) = x$, $P_2(x) = x$, $P_3(x) = x - \frac{1}{6}x^3$

3. $f(x) = \ln x$, $f'(x) = \frac{1}{x}$, $f''(x) = -\frac{1}{x^2}$, $f'''(x) = \frac{2}{x^3}$; $f(1) = \ln 1 = 0$, $f'(1) = 1$, $f''(1) = -1$, $f'''(1) = 2 \Rightarrow P_0(x) = 0$, $P_1(x) = (x-1)$, $P_2(x) = (x-1) - \frac{1}{2}(x-1)^2$, $P_3(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$

4. $f(x) = \ln(1+x)$, $f'(x) = \frac{1}{1+x} = (1+x)^{-1}$, $f''(x) = -(1+x)^{-2}$, $f'''(x) = 2(1+x)^{-3}$; $f(0) = \ln 1 = 0$, $f'(0) = \frac{1}{1} = 1$, $f''(0) = -(1)^{-2} = -1$, $f'''(0) = 2(1)^{-3} = 2 \Rightarrow P_0(x) = 0$, $P_1(x) = x$, $P_2(x) = x - \frac{x^2}{2}$, $P_3(x) = x - \frac{x^2}{2} + \frac{x^3}{3}$

5. $f(x) = \frac{1}{x} = x^{-1}$, $f'(x) = -x^{-2}$, $f''(x) = 2x^{-3}$, $f'''(x) = -6x^{-4}$; $f(2) = \frac{1}{2}$, $f'(2) = -\frac{1}{4}$, $f''(2) = \frac{1}{4}$, $f'''(x) = -\frac{3}{8}$
$\Rightarrow P_0(x) = \frac{1}{2}$, $P_1(x) = \frac{1}{2} - \frac{1}{4}(x-2)$, $P_2(x) = \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2$,
$P_3(x) = \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2 - \frac{1}{16}(x-2)^3$

6. $f(x) = (x+2)^{-1}$, $f'(x) = -(x+2)^{-2}$, $f''(x) = 2(x+2)^{-3}$, $f'''(x) = -6(x+2)^{-4}$; $f(0) = (2)^{-1} = \frac{1}{2}$, $f'(0) = -(2)^{-2} = -\frac{1}{4}$, $f''(0) = 2(2)^{-3} = \frac{1}{4}$, $f'''(0) = -6(2)^{-4} = -\frac{3}{8} \Rightarrow P_0(x) = \frac{1}{2}$, $P_1(x) = \frac{1}{2} - \frac{x}{4}$, $P_2(x) = \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8}$, $P_3(x) = \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16}$

7. $f(x) = \sin x$, $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$; $f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, $f'\left(\frac{\pi}{4}\right) = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, $f''\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$, $f'''\left(\frac{\pi}{4}\right) = -\cos\frac{\pi}{4} = -\frac{\sqrt{2}}{2} \Rightarrow P_0 = \frac{\sqrt{2}}{2}$, $P_1(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right)$, $P_2(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x - \frac{\pi}{4}\right)^2$, $P_3(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x - \frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{12}\left(x - \frac{\pi}{4}\right)^3$

8. $f(x) = \tan x$, $f'(x) = \sec^2 x$, $f''(x) = 2\sec^2 x \tan x$, $f'''(x) = 2\sec^4 x + 4\sec^2 x \tan^2 x$; $f\left(\frac{\pi}{4}\right) = \tan\frac{\pi}{4} = 1$, $f'\left(\frac{\pi}{4}\right) = \sec^2\left(\frac{\pi}{4}\right) = 2$, $f''\left(\frac{\pi}{4}\right) = 2\sec^2\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{4}\right) = 4$, $f'''\left(\frac{\pi}{4}\right) = 2\sec^4\left(\frac{\pi}{4}\right) + 4\sec^2\left(\frac{\pi}{4}\right)\tan^2\left(\frac{\pi}{4}\right) = 16 \Rightarrow P_0(x) = 1$, $P_1(x) = 1 + 2\left(x - \frac{\pi}{4}\right)$, $P_2(x) = 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2$, $P_3(x) = 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3$

9. $f(x) = \sqrt{x} = x^{1/2}$, $f'(x) = \left(\frac{1}{2}\right)x^{-1/2}$, $f''(x) = \left(-\frac{1}{4}\right)x^{-3/2}$, $f'''(x) = \left(\frac{3}{8}\right)x^{-5/2}$; $f(4) = \sqrt{4} = 2$, $f'(4) = \left(\frac{1}{2}\right)4^{-1/2} = \frac{1}{4}$, $f''(4) = \left(-\frac{1}{4}\right)4^{-3/2} = -\frac{1}{32}$, $f'''(4) = \left(\frac{3}{8}\right)4^{-5/2} = \frac{3}{256} \Rightarrow P_0(x) = 2$, $P_1(x) = 2 + \frac{1}{4}(x-4)$, $P_2(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2$, $P_3(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3$

10. $f(x) = (1-x)^{1/2}$, $f'(x) = -\frac{1}{2}(1-x)^{-1/2}$, $f''(x) = -\frac{1}{4}(1-x)^{-3/2}$, $f'''(x) = -\frac{3}{8}(1-x)^{-5/2}$; $f(0) = (1)^{1/2} = 1$, $f'(0) = -\frac{1}{2}(1)^{-1/2} = -\frac{1}{2}$, $f''(0) = -\frac{1}{4}(1)^{-3/2} = -\frac{1}{4}$, $f'''(0) = -\frac{3}{8}(1)^{-5/2} = -\frac{3}{8} \Rightarrow P_0(x) = 1$, $P_1(x) = 1 - \frac{1}{2}x$, $P_2(x) = 1 - \frac{1}{2}x - \frac{1}{8}x^2$, $P_3(x) = 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3$

11. $f(x) = e^{-x}$, $f'(x) = -e^{-x}$, $f''(x) = e^{-x}$, $f'''(x) = -e^{-x} \Rightarrow \ldots f^{(k)}(x) = (-1)^k e^{-x}$; $f(0) = e^{-(0)} = 1$, $f'(0) = -1$, $f''(0) = 1$, $f'''(0) = -1, \ldots, f^{(k)}(0) = (-1)^k \Rightarrow e^{-x} = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}x^n$

12. $f(x) = x e^x$, $f'(x) = x e^x + e^x$, $f''(x) = x e^x + 2e^x$, $f'''(x) = x e^x + 3e^x \Rightarrow \ldots f^{(k)}(x) = x e^x + k e^x$; $f(0) = (0)e^{(0)} = 0$, $f'(0) = 1$, $f''(0) = 2$, $f'''(0) = 3, \ldots, f^{(k)}(0) = k \Rightarrow x + x^2 + \frac{1}{2}x^3 + \ldots = \sum_{n=0}^{\infty} \frac{1}{(n-1)!}x^n$

13. $f(x) = (1+x)^{-1} \Rightarrow f'(x) = -(1+x)^{-2}$, $f''(x) = 2(1+x)^{-3}$, $f'''(x) = -3!(1+x)^{-4} \Rightarrow \ldots f^{(k)}(x) = (-1)^k k!(1+x)^{-k-1}$; $f(0) = 1$, $f'(0) = -1$, $f''(0) = 2$, $f'''(0) = -3!, \ldots, f^{(k)}(0) = (-1)^k k!$ $\Rightarrow 1 - x + x^2 - x^3 + \ldots = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n$

14. $f(x) = \frac{2+x}{1-x} \Rightarrow f'(x) = \frac{3}{(1-x)^2}$, $f''(x) = 6(1-x)^{-3}$, $f'''(x) = 18(1-x)^{-4} \Rightarrow \ldots f^{(k)}(x) = 3(k!)(1-x)^{-k-1}$; $f(0) = 2$, $f'(0) = 3$, $f''(0) = 6$, $f'''(0) = 18, \ldots, f^{(k)}(0) = 3(k!) \Rightarrow 2 + 3x + 3x^2 + 3x^3 + \ldots = 2 + \sum_{n=1}^{\infty} 3x^n$

15. $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \Rightarrow \sin 3x = \sum_{n=0}^{\infty} \frac{(-1)^n (3x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1} x^{2n+1}}{(2n+1)!} = 3x - \frac{3^3x^3}{3!} + \frac{3^5x^5}{5!} - \ldots$

16. $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \Rightarrow \sin\frac{x}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^{2n+1}(2n+1)!} = \frac{x}{2} - \frac{x^3}{2^3 \cdot 3!} + \frac{x^5}{2^5 \cdot 5!} + \ldots$

17. $7\cos(-x) = 7\cos x = 7\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 7 - \frac{7x^2}{2!} + \frac{7x^4}{4!} - \frac{7x^6}{6!} + \ldots$, since the cosine is an even function

18. $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \Rightarrow 5\cos \pi x = 5 \sum_{n=0}^{\infty} \frac{(-1)^n (\pi x)^{2n}}{(2n)!} = 5 - \frac{5\pi^2 x^2}{2!} + \frac{5\pi^4 x^4}{4!} - \frac{5\pi^6 x^6}{6!} + \ldots$

19. $\cosh x = \frac{e^x + e^{-x}}{2} = \frac{1}{2}\left[\left(1 + x^2 + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\right) + \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots\right)\right] = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots$
$= \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

20. $\sinh x = \frac{e^x - e^{-x}}{2} = \frac{1}{2}\left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\right) - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots\right)\right] = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \ldots$
$= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

21. $f(x) = x^4 - 2x^3 - 5x + 4 \Rightarrow f'(x) = 4x^3 - 6x^2 - 5,\ f''(x) = 12x^2 - 12x,\ f'''(x) = 24x - 12,\ f^{(4)}(x) = 24$
$\Rightarrow f^{(n)}(x) = 0$ if $n \ge 5$; $f(0) = 4,\ f'(0) = -5,\ f''(0) = 0,\ f'''(0) = -12,\ f^{(4)}(0) = 24,\ f^{(n)}(0) = 0$ if $n \ge 5$
$\Rightarrow x^4 - 2x^3 - 5x + 4 = 4 - 5x - \frac{12}{3!}x^3 + \frac{24}{4!}x^4 = x^4 - 2x^3 - 5x + 4$

22. $f(x) = \frac{x^2}{x+1} \Rightarrow f'(x) = \frac{2x + x^2}{(x+1)^2};\ f''(x) = \frac{2}{(x+1)^3};\ f'''(x) = \frac{-6}{(x+1)^4} \Rightarrow f^{(n)}(x) = \frac{(-1)^n n!}{(x+1)^{n+1}};\ f(0) = 0,\ f'(0) = 0,\ f''(0) = 2,$
$f'''(0) = -6,\ f^{(n)}(0) = (-1)^n n!$ if $n \ge 2 \Rightarrow x^2 - x^3 + x^4 - x^5 + \ldots = \sum_{n=2}^{\infty} (-1)^n x^n$

23. $f(x) = x^3 - 2x + 4 \Rightarrow f'(x) = 3x^2 - 2,\ f''(x) = 6x,\ f'''(x) = 6 \Rightarrow f^{(n)}(x) = 0$ if $n \ge 4$; $f(2) = 8,\ f'(2) = 10,$
$f''(2) = 12,\ f'''(2) = 6,\ f^{(n)}(2) = 0$ if $n \ge 4 \Rightarrow x^3 - 2x + 4 = 8 + 10(x-2) + \frac{12}{2!}(x-2)^2 + \frac{6}{3!}(x-2)^3$
$= 8 + 10(x-2) + 6(x-2)^2 + (x-2)^3$

24. $f(x) = 2x^3 + x^2 + 3x - 8 \Rightarrow f'(x) = 6x^2 + 2x + 3,\ f''(x) = 12x + 2,\ f'''(x) = 12 \Rightarrow f^{(n)}(x) = 0$ if $n \ge 4$; $f(1) = -2,$
$f'(1) = 11,\ f''(1) = 14,\ f'''(1) = 12,\ f^{(n)}(1) = 0$ if $n \ge 4 \Rightarrow 2x^3 + x^2 + 3x - 8$
$= -2 + 11(x-1) + \frac{14}{2!}(x-1)^2 + \frac{12}{3!}(x-1)^3 = -2 + 11(x-1) + 7(x-1)^2 + 2(x-1)^3$

25. $f(x) = x^4 + x^2 + 1 \Rightarrow f'(x) = 4x^3 + 2x,\ f''(x) = 12x^2 + 2,\ f'''(x) = 24x,\ f^{(4)}(x) = 24,\ f^{(n)}(x) = 0$ if $n \ge 5$;
$f(-2) = 21,\ f'(-2) = -36,\ f''(-2) = 50,\ f'''(-2) = -48,\ f^{(4)}(-2) = 24,\ f^{(n)}(-2) = 0$ if $n \ge 5 \Rightarrow x^4 + x^2 + 1$
$= 21 - 36(x+2) + \frac{50}{2!}(x+2)^2 - \frac{48}{3!}(x+2)^3 + \frac{24}{4!}(x+2)^4 = 21 - 36(x+2) + 25(x+2)^2 - 8(x+2)^3 + (x+2)^4$

26. $f(x) = 3x^5 - x^4 + 2x^3 + x^2 - 2 \Rightarrow f'(x) = 15x^4 - 4x^3 + 6x^2 + 2x,\ f''(x) = 60x^3 - 12x^2 + 12x + 2,$
$f'''(x) = 180x^2 - 24x + 12,\ f^{(4)}(x) = 360x - 24,\ f^{(5)}(x) = 360,\ f^{(n)}(x) = 0$ if $n \ge 6$; $f(-1) = -7,$
$f'(-1) = 23,\ f''(-1) = -82,\ f'''(-1) = 216,\ f^{(4)}(-1) = -384,\ f^{(5)}(-1) = 360,\ f^{(n)}(-1) = 0$ if $n \ge 6$
$\Rightarrow 3x^5 - x^4 + 2x^3 + x^2 - 2 = -7 + 23(x+1) - \frac{82}{2!}(x+1)^2 + \frac{216}{3!}(x+1)^3 - \frac{384}{4!}(x+1)^4 + \frac{360}{5!}(x+1)^5$
$= -7 + 23(x+1) - 41(x+1)^2 + 36(x+1)^3 - 16(x+1)^4 + 3(x+1)^5$

27. $f(x) = x^{-2} \Rightarrow f'(x) = -2x^{-3},\ f''(x) = 3!\,x^{-4},\ f'''(x) = -4!\,x^{-5} \Rightarrow f^{(n)}(x) = (-1)^n (n+1)!\,x^{-n-2};$
$f(1) = 1,\ f'(1) = -2,\ f''(1) = 3!,\ f'''(1) = -4!,\ f^{(n)}(1) = (-1)^n (n+1)! \Rightarrow \frac{1}{x^2}$
$= 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 + \ldots = \sum_{n=0}^{\infty} (-1)^n (n+1)(x-1)^n$

28. $f(x) = \frac{1}{(1-x)^3} \Rightarrow f'(x) = 3(1-x)^{-4},\ f''(x) = 12(1-x)^{-5},\ f'''(x) = 60(1-x)^{-6} \Rightarrow f^{(n)}(x) = \frac{(n+2)!}{2}(1-x)^{-n-3};$
$f(0) = 1,\ f'(0) = 3,\ f''(0) = 12,\ f'''(0) = 60,\ \ldots,\ f^{(n)}(0) = \frac{(n+2)!}{2} \Rightarrow \frac{1}{(1-x)^3} = 1 + 3x + 6x^2 + 10x^3 + \ldots$
$= \sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2} x^n$

29. $f(x) = e^x \Rightarrow f'(x) = e^x, f''(x) = e^x \Rightarrow f^{(n)}(x) = e^x; f(2) = e^2, f'(2) = e^2, \ldots f^{(n)}(2) = e^2$
$\Rightarrow e^x = e^2 + e^2(x-2) + \frac{e^2}{2}(x-2)^2 + \frac{e^3}{3!}(x-2)^3 + \ldots = \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n$

30. $f(x) = 2^x \Rightarrow f'(x) = 2^x \ln 2, f''(x) = 2^x(\ln 2)^2, f'''(x) = 2^x(\ln 2)^3 \Rightarrow f^{(n)}(x) = 2^x(\ln 2)^n; f(1) = 2, f'(1) = 2\ln 2,$
$f''(1) = 2(\ln 2)^2, f'''(1) = 2(\ln 2)^3, \ldots, f^{(n)}(1) = 2(\ln 2)^n$
$\Rightarrow 2^x = 2 + (2\ln 2)(x-1) + \frac{2(\ln 2)^2}{2}(x-1)^2 + \frac{2(\ln 2)^3}{3!}(x-1)^3 + \ldots = \sum_{n=0}^{\infty} \frac{2(\ln 2)^n(x-1)^n}{n!}$

31. $f(x) = \cos\left(2x + \frac{\pi}{2}\right), f'(x) = -2\sin\left(2x + \frac{\pi}{2}\right), f''(x) = -4\cos\left(2x + \frac{\pi}{2}\right), f'''(x) = 8\sin\left(2x + \frac{\pi}{2}\right),$
$f^{(4)}(x) = 2^4\cos\left(2x + \frac{\pi}{2}\right), f^{(5)}(x) = -2^5\sin\left(2x + \frac{\pi}{2}\right), \ldots; f\left(\frac{\pi}{4}\right) = -1, f'\left(\frac{\pi}{4}\right) = 0, f''\left(\frac{\pi}{4}\right) = 4, f'''\left(\frac{\pi}{4}\right) = 0, f^{(4)}\left(\frac{\pi}{4}\right) = 2^4,$
$f^{(5)}\left(\frac{\pi}{4}\right) = 0, \ldots, f^{(2n)}\left(\frac{\pi}{4}\right) = (-1)^n 2^{2n} \Rightarrow \cos\left(2x + \frac{\pi}{2}\right) = -1 + 2\left(x - \frac{\pi}{4}\right)^2 - \frac{2}{3}\left(x - \frac{\pi}{4}\right)^4 + \ldots$
$= \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!}\left(x - \frac{\pi}{4}\right)^{2n}$

32. $f(x) = \sqrt{x+1}, f'(x) = \frac{1}{2}(x+1)^{-1/2}, f''(x) = -\frac{1}{4}(x+1)^{-3/2}, f'''(x) = \frac{3}{8}(x+1)^{-5/2}, f^{(4)}(x) = -\frac{15}{16}(x+1)^{-7/2}, \ldots;$
$f(0) = 1, f'(0) = \frac{1}{2}, f''(0) = -\frac{1}{4}, f'''(0) = \frac{3}{8}, f^{(4)}(0) = -\frac{15}{16}, \ldots \Rightarrow \sqrt{x+1} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \ldots$

33. The Maclaurin series generated by $\cos x$ is $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}$ which converges on $(-\infty, \infty)$ and the Maclaurin series generated by $\frac{2}{1-x}$ is $2\sum_{n=0}^{\infty} x^n$ which converges on $(-1, 1)$. Thus the Maclaurin series generated by $f(x) = \cos x - \frac{2}{1-x}$ is given by $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n} - 2\sum_{n=0}^{\infty} x^n = -1 - 2x - \frac{5}{2}x^2 - \ldots$. which converges on the intersection of $(-\infty, \infty)$ and $(-1, 1)$, so the interval of convergence is $(-1, 1)$.

34. The Maclaurin series generated by e^x is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ which converges on $(-\infty, \infty)$. The Maclaurin series generated by $f(x) = (1 - x + x^2)e^x$ is given by $(1 - x + x^2)\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{1}{2}x^2 + \frac{2}{3}x^3 \ldots$. which converges on $(-\infty, \infty)$.

35. The Maclaurin series generated by $\sin x$ is $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$ which converges on $(-\infty, \infty)$ and the Maclaurin series generated by $\ln(1+x)$ is $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}x^n$ which converges on $(-1, 1)$. Thus the Maclaurin series genereated by $f(x) = \sin x \cdot \ln(1+x)$ is given by $\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}\right)\left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}x^n\right) = x^2 - \frac{1}{2}x^3 + \frac{1}{6}x^4 - \ldots$. which converges on the intersection of $(-\infty, \infty)$ and $(-1, 1)$, so the interval of convergence is $(-1, 1)$.

36. The Maclaurin series generated by $\sin x$ is $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$ which converges on $(-\infty, \infty)$. The Maclaurin series genereated by $f(x) = x\sin^2 x$ is given by $x\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}\right)^2 = x\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}\right)\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}\right)$
$= x^3 - \frac{1}{3}x^5 + \frac{2}{45}x^7 + \ldots$ which converges on $(-\infty, \infty)$.

37. If $e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ and $f(x) = e^x$, we have $f^{(n)}(a) = e^a$ f or all $n = 0, 1, 2, 3, \ldots$
$\Rightarrow e^x = e^a\left[\frac{(x-a)^0}{0!} + \frac{(x-a)^1}{1!} + \frac{(x-a)^2}{2!} + \ldots\right] = e^a\left[1 + (x-a) + \frac{(x-a)^2}{2!} + \ldots\right]$ at $x = a$

38. $f(x) = e^x \Rightarrow f^{(n)}(x) = e^x$ for all n $\Rightarrow f^{(n)}(1) = e$ for all $n = 0, 1, 2, \ldots$
$\Rightarrow e^x = e + e(x-1) + \frac{e}{2!}(x-1)^2 + \frac{e}{3!}(x-1)^3 + \ldots = e\left[1 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \ldots\right]$

39. $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots \Rightarrow f'(x)$
$= f'(a) + f''(a)(x-a) + \frac{f'''(a)}{3!}3(x-a)^2 + \ldots \Rightarrow f''(x) = f''(a) + f'''(a)(x-a) + \frac{f^{(4)}(a)}{4!}4 \cdot 3(x-a)^2 + \ldots$
$\Rightarrow f^{(n)}(x) = f^{(n)}(a) + f^{(n+1)}(a)(x-a) + \frac{f^{(n+2)}(a)}{2}(x-a)^2 + \ldots$
$\Rightarrow f(a) = f(a) + 0, f'(a) = f'(a) + 0, \ldots, f^{(n)}(a) = f^{(n)}(a) + 0$

40. $E(x) = f(x) - b_0 - b_1(x-a) - b_2(x-a)^2 - b_3(x-a)^3 - \ldots - b_n(x-a)^n$
$\Rightarrow 0 = E(a) = f(a) - b_0 \Rightarrow b_0 = f(a)$; from condition (b),
$\lim_{x \to a} \frac{f(x) - f(a) - b_1(x-a) - b_2(x-a)^2 - b_3(x-a)^3 - \ldots - b_n(x-a)^n}{(x-a)^n} = 0$
$\Rightarrow \lim_{x \to a} \frac{f'(x) - b_1 - 2b_2(x-a) - 3b_3(x-a)^2 - \ldots - nb_n(x-a)^{n-1}}{n(x-a)^{n-1}} = 0$
$\Rightarrow b_1 = f'(a) \Rightarrow \lim_{x \to a} \frac{f''(x) - 2b_2 - 3!\,b_3(x-a) - \ldots - n(n-1)b_n(x-a)^{n-2}}{n(n-1)(x-a)^{n-2}} = 0$
$\Rightarrow b_2 = \frac{1}{2}f''(a) \Rightarrow \lim_{x \to a} \frac{f'''(x) - 3!\,b_3 - \ldots - n(n-1)(n-2)b_n(x-a)^{n-3}}{n(n-1)(n-2)(x-a)^{n-3}} = 0$
$= b_3 = \frac{1}{3!}f'''(a) \Rightarrow \lim_{x \to a} \frac{f^{(n)}(x) - n!\,b_n}{n!} = 0 \Rightarrow b_n = \frac{1}{n!}f^{(n)}(a)$; therefore,
$g(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n = P_n(x)$

41. $f(x) = \ln(\cos x) \Rightarrow f'(x) = -\tan x$ and $f''(x) = -\sec^2 x$; $f(0) = 0, f'(0) = 0, f''(0) = -1 \Rightarrow L(x) = 0$ and $Q(x) = -\frac{x^2}{2}$

42. $f(x) = e^{\sin x} \Rightarrow f'(x) = (\cos x)e^{\sin x}$ and $f''(x) = (-\sin x)e^{\sin x} + (\cos x)^2 e^{\sin x}$; $f(0) = 1, f'(0) = 1, f''(0) = 1$
$\Rightarrow L(x) = 1 + x$ and $Q(x) = 1 + x + \frac{x^2}{2}$

43. $f(x) = (1-x^2)^{-1/2} \Rightarrow f'(x) = x(1-x^2)^{-3/2}$ and $f''(x) = (1-x^2)^{-3/2} + 3x^2(1-x^2)^{-5/2}$; $f(0) = 1, f'(0) = 0,$
$f''(0) = 1 \Rightarrow L(x) = 1$ and $Q(x) = 1 + \frac{x^2}{2}$

44. $f(x) = \cosh x \Rightarrow f'(x) = \sinh x$ and $f''(x) = \cosh x$; $f(0) = 1, f'(0) = 0, f''(0) = 1 \Rightarrow L(x) = 1$ and $Q(x) = 1 + \frac{x^2}{2}$

45. $f(x) = \sin x \Rightarrow f'(x) = \cos x$ and $f''(x) = -\sin x$; $f(0) = 0, f'(0) = 1, f''(0) = 0 \Rightarrow L(x) = x$ and $Q(x) = x$

46. $f(x) = \tan x \Rightarrow f'(x) = \sec^2 x$ and $f''(x) = 2\sec^2 x \tan x$; $f(0) = 0, f'(0) = 1, f'' = 0 \Rightarrow L(x) = x$ and $Q(x) = x$

10.9 CONVERGENCE OF TAYLOR SERIES

1. $e^x = 1 + x + \frac{x^2}{2!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^{-5x} = 1 + (-5x) + \frac{(-5x)^2}{2!} + \ldots = 1 - 5x + \frac{5^2x^2}{2!} - \frac{5^3x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n 5^n x^n}{n!}$

2. $e^x = 1 + x + \frac{x^2}{2!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^{-x/2} = 1 + \left(\frac{-x}{2}\right) + \frac{\left(-\frac{x}{2}\right)^2}{2!} + \ldots = 1 - \frac{x}{2} + \frac{x^2}{2^2 2!} - \frac{x^3}{2^3 3!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^n n!}$

3. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \Rightarrow 5\sin(-x) = 5\left[(-x) - \frac{(-x)^3}{3!} + \frac{(-x)^5}{5!} - \ldots\right] = \sum_{n=0}^{\infty} \frac{5(-1)^{n+1}x^{2n+1}}{(2n+1)!}$

4. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \Rightarrow \sin\frac{\pi x}{2} = \frac{\pi x}{2} - \frac{\left(\frac{\pi x}{2}\right)^3}{3!} + \frac{\left(\frac{\pi x}{2}\right)^5}{5!} - \frac{\left(\frac{\pi x}{2}\right)^7}{7!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1} x^{2n+1}}{2^{2n+1}(2n+1)!}$

5. $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \Rightarrow \cos 5x^2 = \sum_{n=0}^{\infty} \frac{(-1)^n \left[5x^2\right]^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n} x^{4n}}{(2n)!} = 1 - \frac{25x^4}{2!} + \frac{625x^8}{4!} - \frac{15625x^{12}}{6!} + \dots$

6. $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \Rightarrow \cos\left(\frac{x^{3/2}}{\sqrt{2}}\right) = \cos\left(\left(\frac{x^3}{2}\right)^{1/2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\left(\frac{x^3}{2}\right)^{1/2}\right)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{2^n (2n)!}$
$= 1 - \frac{x^3}{2 \cdot 2!} + \frac{x^6}{2^2 \cdot 4!} - \frac{x^9}{2^3 \cdot 6!} + \dots$

7. $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \Rightarrow \ln(1+x^2) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (x^2)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n}}{n} = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \dots$

8. $\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \Rightarrow \tan^{-1}(3x^4) = \sum_{n=0}^{\infty} \frac{(-1)^n (3x^4)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1} x^{8n+4}}{n} = 3x^4 - 9x^{12} + \frac{243}{5}x^{20} - \frac{2187}{7}x^{28} + \dots$

9. $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \Rightarrow \frac{1}{1+\frac{3}{4}x^3} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{3}{4}x^3\right)^n = \sum_{n=0}^{\infty} (-1)^n \left(\frac{3}{4}\right)^n x^{3n} = 1 - \frac{3}{4}x^3 + \frac{9}{16}x^6 - \frac{27}{64}x^9 + \dots$

10. $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \Rightarrow \frac{1}{2-x} = \frac{1}{2} \frac{1}{1-\frac{1}{2}x} = \frac{1}{2}\sum_{n=0}^{\infty} \left(\frac{1}{2}x\right)^n = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} x^n = \frac{1}{2} + \frac{1}{4}x + \frac{1}{8}x^2 + \frac{1}{16}x^3 + \dots$

11. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow xe^x = x\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} = x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \frac{x^5}{4!} + \dots$

12. $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \Rightarrow x^2 \sin x = x^2 \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{(2n+1)!} = x^3 - \frac{x^5}{3!} + \frac{x^7}{5!} - \frac{x^9}{7!} + \dots$

13. $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \Rightarrow \frac{x^2}{2} - 1 + \cos x = \frac{x^2}{2} - 1 + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \frac{x^2}{2} - 1 + 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \dots$
$= \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \dots = \sum_{n=2}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

14. $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \Rightarrow \sin x - x + \frac{x^3}{3!} = \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}\right) - x + \frac{x^3}{3!}$
$= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \dots\right) - x + \frac{x^3}{3!} = \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \dots = \sum_{n=2}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

15. $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \Rightarrow x \cos \pi x = x \sum_{n=0}^{\infty} \frac{(-1)^n (\pi x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n} x^{2n+1}}{(2n)!} = x - \frac{\pi^2 x^3}{2!} + \frac{\pi^4 x^5}{4!} - \frac{\pi^6 x^7}{6!} + \dots$

16. $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \Rightarrow x^2 \cos(x^2) = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n)!} = x^2 - \frac{x^6}{2!} + \frac{x^{10}}{4!} - \frac{x^{14}}{6!} + \dots$

17. $\cos^2 x = \frac{1}{2} + \frac{\cos 2x}{2} = \frac{1}{2} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!} = \frac{1}{2} + \frac{1}{2}\left[1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \frac{(2x)^8}{8!} - \dots\right]$
$= 1 - \frac{(2x)^2}{2 \cdot 2!} + \frac{(2x)^4}{2 \cdot 4!} - \frac{(2x)^6}{2 \cdot 6!} + \frac{(2x)^8}{2 \cdot 8!} - \dots = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n (2x)^{2n}}{2 \cdot (2n)!} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n-1} x^{2n}}{(2n)!}$

18. $\sin^2 x = \left(\frac{1 - \cos 2x}{2}\right) = \frac{1}{2} - \frac{1}{2}\cos 2x = \frac{1}{2} - \frac{1}{2}\left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \dots\right) = \frac{(2x)^2}{2 \cdot 2!} - \frac{(2x)^4}{2 \cdot 4!} + \frac{(2x)^6}{2 \cdot 6!} - \dots$
$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (2x)^{2n}}{2 \cdot (2n)!} = \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n-1} x^{2n}}{(2n)!}$

19. $\frac{x^2}{1-2x} = x^2\left(\frac{1}{1-2x}\right) = x^2\sum_{n=0}^{\infty}(2x)^n = \sum_{n=0}^{\infty}2^n x^{n+2} = x^2 + 2x^3 + 2^2x^4 + 2^3x^5 + \ldots$

20. $x\ln(1+2x) = x\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2x)^n}{n} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}2^n x^{n+1}}{n} = 2x^2 - \frac{2^2x^3}{2} + \frac{2^3x^4}{4} - \frac{2^4x^5}{5} + \ldots$

21. $\frac{1}{1-x} = \sum_{n=0}^{\infty}x^n = 1 + x + x^2 + x^3 + \ldots \Rightarrow \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \ldots = \sum_{n=1}^{\infty}nx^{n-1} = \sum_{n=0}^{\infty}(n+1)x^n$

22. $\frac{2}{(1-x)^3} = \frac{d^2}{dx^2}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\left(\frac{1}{(1-x)^2}\right) = \frac{d}{dx}(1 + 2x + 3x^2 + \ldots) = 2 + 6x + 12x^2 + \ldots = \sum_{n=2}^{\infty}n(n-1)x^{n-2}$
$= \sum_{n=0}^{\infty}(n+2)(n+1)x^n$

23. $\tan^{-1}x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots \Rightarrow x\tan^{-1}x^2 = x\left(x^2 - \frac{1}{3}(x^2)^3 + \frac{1}{5}(x^2)^5 - \frac{1}{7}(x^2)^7 + \ldots\right)$
$= x^3 - \frac{1}{3}x^7 + \frac{1}{5}x^{11} - \frac{1}{7}x^{15} + \ldots = \sum_{n=1}^{\infty}\frac{(-1)^n x^{4n-1}}{2n-1}$

24. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \Rightarrow \sin x\cdot\cos x = \frac{1}{2}\sin 2x = \frac{1}{2}\left(2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \ldots\right)$
$= x - \frac{4x^3}{3!} + \frac{16x^5}{5!} - \frac{64x^7}{7!} + \ldots = x - \frac{2x^3}{3} + \frac{2x^5}{15} - \frac{4x^7}{315} + \ldots = \sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n}x^{2n+1}}{(2n+1)!}$

25. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ and $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots \Rightarrow e^x + \frac{1}{1+x}$
$= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right) + (1 - x + x^2 - x^3 + \ldots) = 2 + \frac{3}{2}x^2 - \frac{5}{6}x^3 + \frac{25}{24}x^4 + \ldots = \sum_{n=0}^{\infty}\left(\frac{1}{n!} + (-1)^n\right)x^n$

26. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$ and $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \Rightarrow \cos x - \sin x$
$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\right) - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right) = 1 - x - \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} - \frac{x^6}{6!} + \frac{x^7}{7!} + \ldots$
$= \sum_{n=0}^{\infty}\left(\frac{(-1)^n x^{2n}}{(2n)!} - \frac{(-1)^n x^{2n+1}}{(2n+1)!}\right)$

27. $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots \Rightarrow \frac{x}{3}\ln(1+x^2) = \frac{x}{3}\left(x^2 - \frac{1}{2}(x^2)^2 + \frac{1}{3}(x^2)^3 - \frac{1}{4}(x^2)^4 + \ldots\right)$
$= \frac{1}{3}x^3 - \frac{1}{6}x^5 + \frac{1}{9}x^7 - \frac{1}{12}x^9 + \ldots = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{3n}x^{2n+1}$

28. $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots$ and $\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots \Rightarrow \ln(1+x) - \ln(1-x)$
$= \left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots\right) - \left(-x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots\right) = 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \ldots = \sum_{n=0}^{\infty}\frac{2}{2n+1}x^{2n+1}$

29. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ and $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \Rightarrow e^x\cdot\sin x$
$= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right) = x + x^2 + \frac{1}{3}x^3 - \frac{1}{30}x^5 - \ldots.$

30. $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots$ and $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots \Rightarrow \frac{\ln(1+x)}{1-x} = \ln(1+x)\cdot\frac{1}{1-x}$
$= \left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots\right)(1 + x + x^2 + x^3 + \ldots) = x + \frac{1}{2}x^2 + \frac{5}{6}x^3 + \frac{7}{12}x^4 + \ldots.$

31. $\tan^{-1}x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots \Rightarrow (\tan^{-1}x)^2 = (\tan^{-1}x)(\tan^{-1}x)$
$= \left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots\right)\left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots\right) = x^2 - \frac{2}{3}x^4 - \frac{23}{45}x^6 - \frac{44}{105}x^8 + \ldots.$

32. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$ and $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \Rightarrow \cos^2 x \cdot \sin x = \cos x \cdot \cos x \cdot \sin x$
$= \cos x \cdot \frac{1}{2}\sin 2x = \frac{1}{2}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\right)\left(2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \ldots\right) = x - \frac{7}{6}x^3 + \frac{61}{120}x^5 - \frac{1247}{5040}x^7 + \ldots$

33. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$ and $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$
$\Rightarrow e^{\sin x} = 1 + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right) + \frac{1}{2}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right)^2 + \frac{1}{6}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right)^3 + \ldots$
$= 1 + x + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \ldots.$

34. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$ and $\tan^{-1}x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots \Rightarrow \sin(\tan^{-1}x) = \left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots\right)$
$- \frac{1}{6}\left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots\right)^3 + \frac{1}{120}\left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots\right)^5 - \frac{1}{5040}\left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots\right)^7 + \ldots$
$= x - \frac{1}{2}x^3 + \frac{3}{8}x^5 - \frac{5}{16}x^7 + \ldots$

35. Since $n = 3$, then $f^{(4)}(x) = \sin x$, $|f^{(4)}(x)| \le M$ on $[0, 0.1] \Rightarrow |\sin x| \le 1$ on $[0, 0.1] \Rightarrow M = 1$. Then $|R_3(0.1)| \le 1\frac{|0.1 - 0|^4}{4!}$
$= 4.2 \times 10^{-6} \Rightarrow$ error $\le 4.2 \times 10^{-6}$

36. Since $n = 4$, then $f^{(5)}(x) = e^x$, $|f^{(5)}(x)| \le M$ on $[0, 0.5] \Rightarrow |e^x| \le \sqrt{e}$ on $[0, 0.5] \Rightarrow M = 2.7$. Then
$|R_4(0.5)| \le 2.7\frac{|0.5 - 0|^5}{5!} = 7.03 \times 10^{-4} \Rightarrow$ error $\le 7.03 \times 10^{-4}$

37. By the Alternating Series Estimation Theorem, the error is less than $\frac{|x|^5}{5!} \Rightarrow |x|^5 < (5!)\,(5 \times 10^{-4}) \Rightarrow |x|^5 < 600 \times 10^{-4}$
$\Rightarrow |x| < \sqrt[5]{6 \times 10^{-2}} \approx 0.56968$

38. If $\cos x = 1 - \frac{x^2}{2}$ and $|x| < 0.5$, then the error is less than $\left|\frac{(.5)^4}{24}\right| = 0.0026$, by Alternating Series Estimation Theorem; since the next term in the series is positive, the approximation $1 - \frac{x^2}{2}$ is too small, by the Alternating Series Estimation Theorem

39. If $\sin x = x$ and $|x| < 10^{-3}$, then the error is less than $\frac{(10^{-3})^3}{3!} \approx 1.67 \times 10^{-10}$, by Alternating Series Estimation Theorem; The Alternating Series Estimation Theorem says $R_2(x)$ has the same sign as $-\frac{x^3}{3!}$. Moreover, $x < \sin x$
$\Rightarrow 0 < \sin x - x = R_2(x) \Rightarrow x < 0 \Rightarrow -10^{-3} < x < 0.$

40. $\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \ldots$. By the Alternating Series Estimation Theorem the $|\text{error}| < \left|\frac{-x^2}{8}\right| < \frac{(0.01)^2}{8}$
$= 1.25 \times 10^{-5}$

41. $|R_2(x)| = \left|\frac{e^c x^3}{3!}\right| < \frac{3^{(0.1)}(0.1)^3}{3!} < 1.87 \times 10^{-4}$, where c is between 0 and x

42. $|R_2(x)| = \left|\frac{e^c x^3}{3!}\right| < \frac{(0.1)^3}{3!} = 1.67 \times 10^{-4}$, where c is between 0 and x

43. $\sin^2 x = \left(\frac{1 - \cos 2x}{2}\right) = \frac{1}{2} - \frac{1}{2}\cos 2x = \frac{1}{2} - \frac{1}{2}\left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \ldots\right) = \frac{2x^2}{2!} - \frac{2^3x^4}{4!} + \frac{2^5x^6}{6!} - \ldots$
$\Rightarrow \frac{d}{dx}(\sin^2 x) = \frac{d}{dx}\left(\frac{2x^2}{2!} - \frac{2^3x^4}{4!} + \frac{2^5x^6}{6!} - \ldots\right) = 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \ldots \Rightarrow 2\sin x\cos x$
$= 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \ldots = \sin 2x$, which checks

44. $\cos^2 x = \cos 2x + \sin^2 x = \left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \frac{(2x)^8}{8!} + \ldots\right) + \left(\frac{2x^2}{2!} - \frac{2^3x^4}{4!} + \frac{2^5x^6}{6!} - \frac{2^7x^8}{8!} + \ldots\right)$
$= 1 - \frac{2x^2}{2!} + \frac{2^3x^4}{4!} - \frac{2^5x^6}{6!} + \ldots = 1 - x^2 + \frac{1}{3}x^4 - \frac{2}{45}x^6 + \frac{1}{315}x^8 - \ldots$

45. A special case of Taylor's Theorem is $f(b) = f(a) + f'(c)(b - a)$, where c is between a and b $\Rightarrow f(b) - f(a) = f'(c)(b - a)$, the Mean Value Theorem.

46. If f(x) is twice differentiable and at $x = a$ there is a point of inflection, then $f''(a) = 0$. Therefore, $L(x) = Q(x) = f(a) + f'(a)(x - a)$.

47. (a) $f'' \le 0$, $f'(a) = 0$ and $x = a$ interior to the interval I $\Rightarrow f(x) - f(a) = \frac{f''(c_2)}{2}(x - a)^2 \le 0$ throughout I $\Rightarrow f(x) \le f(a)$ throughout I $\Rightarrow$ f has a local maximum at $x = a$
(b) similar reasoning gives $f(x) - f(a) = \frac{f''(c_2)}{2}(x - a)^2 \ge 0$ throughout I $\Rightarrow f(x) \ge f(a)$ throughout I $\Rightarrow$ f has a local minimum at $x = a$

48. $f(x) = (1 - x)^{-1} \Rightarrow f'(x) = (1 - x)^{-2} \Rightarrow f''(x) = 2(1 - x)^{-3} \Rightarrow f^{(3)}(x) = 6(1 - x)^{-4}$
$\Rightarrow f^{(4)}(x) = 24(1 - x)^{-5}$; therefore $\frac{1}{1-x} \approx 1 + x + x^2 + x^3$. $|x| < 0.1 \Rightarrow \frac{10}{11} < \frac{1}{1-x} < \frac{10}{9} \Rightarrow \left|\frac{1}{(1-x)^5}\right| < \left(\frac{10}{9}\right)^5$
$\Rightarrow \left|\frac{x^4}{(1-x)^5}\right| < x^4\left(\frac{10}{9}\right)^5 \Rightarrow$ the error $e_3 \le \left|\frac{\max f^{(4)}(x)\, x^4}{4!}\right| < (0.1)^4\left(\frac{10}{9}\right)^5 = 0.00016935 < 0.00017$, since $\left|\frac{f^{(4)}(x)}{4!}\right| = \left|\frac{1}{(1-x)^5}\right|$.

49. (a) $f(x) = (1 + x)^k \Rightarrow f'(x) = k(1 + x)^{k-1} \Rightarrow f''(x) = k(k - 1)(1 + x)^{k-2}$; $f(0) = 1$, $f'(0) = k$, and $f''(0) = k(k - 1)$ $\Rightarrow Q(x) = 1 + kx + \frac{k(k-1)}{2}x^2$
(b) $|R_2(x)| = \left|\frac{3\cdot 2\cdot 1}{3!}x^3\right| < \frac{1}{100} \Rightarrow |x^3| < \frac{1}{100} \Rightarrow 0 < x < \frac{1}{100^{1/3}}$ or $0 < x < .21544$

50. (a) Let $P = x + \pi \Rightarrow |x| = |P - \pi| < .5 \times 10^{-n}$ since P approximates π accurate to n decimals. Then, $P + \sin P = (\pi + x) + \sin(\pi + x) = (\pi + x) - \sin x = \pi + (x - \sin x) \Rightarrow |(P + \sin P) - \pi|$
$= |\sin x - x| \le \frac{|x|^3}{3!} < \frac{0.125}{3!} \times 10^{-3n} < .5 \times 10^{-3n} \Rightarrow P + \sin P$ gives an approximation to π correct to 3n decimals.

51. If $f(x) = \sum_{n=0}^{\infty} a_n x^n$, then $f^{(k)}(x) = \sum_{n=k}^{\infty} n(n - 1)(n - 2)\cdots(n - k + 1)a_n x^{n-k}$ and $f^{(k)}(0) = k!\, a_k$
$\Rightarrow a_k = \frac{f^{(k)}(0)}{k!}$ for k a nonnegative integer. Therefore, the coefficients of f(x) are identical with the corresponding coefficients in the Maclaurin series of f(x) and the statement follows.

52. Note: f even $\Rightarrow f(-x) = f(x) \Rightarrow -f'(-x) = f'(x) \Rightarrow f'(-x) = -f'(x) \Rightarrow f'$ odd;
f odd $\Rightarrow f(-x) = -f(x) \Rightarrow -f'(-x) = -f'(x) \Rightarrow f'(-x) = f'(x) \Rightarrow f'$ even;
also, f odd $\Rightarrow f(-0) = f(0) \Rightarrow 2f(0) = 0 \Rightarrow f(0) = 0$
(a) If f(x) is even, then any odd-order derivative is odd and equal to 0 at $x = 0$. Therefore, $a_1 = a_3 = a_5 = \ldots = 0$; that is, the Maclaurin series for f contains only even powers.
(b) If f(x) is odd, then any even-order derivative is odd and equal to 0 at $x = 0$. Therefore, $a_0 = a_2 = a_4 = \ldots = 0$; that is, the Maclaurin series for f contains only odd powers.

53-58. Example CAS commands:
Maple:
```
f := x -> 1/sqrt(1+x);
x0 := -3/4;
x1 :=  3/4;
# Step 1:
plot( f(x), x=x0..x1, title="Step 1: #53 (Section 10.9)" );
```

```
# Step 2:
P1 := unapply( TaylorApproximation(f(x), x = 0, order=1), x );
P2 := unapply( TaylorApproximation(f(x), x = 0, order=2), x );
P3 := unapply( TaylorApproximation(f(x), x = 0, order=3), x );
# Step 3:
D2f := D(D(f));
D3f := D(D(D(f)));
D4f := D(D(D(D(f))));
plot( [D2f(x),D3f(x),D4f(x)], x=x0..x1, thickness=[0,2,4], color=[red,blue,green], title="Step 3: #57 (Section 9.9)" );
c1 := x0;
M1 := abs( D2f(c1) );
c2 := x0;
M2 := abs( D3f(c2) );
c3 := x0;
M3 := abs( D4f(c3) );
# Step 4:
R1 := unapply( abs(M1/2!*(x-0)^2), x );
R2 := unapply( abs(M2/3!*(x-0)^3), x );
R3 := unapply( abs(M3/4!*(x-0)^4), x );
plot( [R1(x),R2(x),R3(x)], x=x0..x1, thickness=[0,2,4], color=[red,blue,green], title="Step 4: #53 (Section 10.9)" );
# Step 5:
E1 := unapply( abs(f(x)-P1(x)), x );
E2 := unapply( abs(f(x)-P2(x)), x );
E3 := unapply( abs(f(x)-P3(x)), x );
plot( [E1(x),E2(x),E3(x),R1(x),R2(x),R3(x)], x=x0..x1, thickness=[0,2,4], color=[red,blue,green],
    linestyle=[1,1,1,3,3,3], title="Step 5: #53 (Section 10.9)" );
# Step 6:
TaylorApproximation( f(x), view=[x0..x1,DEFAULT], x=0, output=animation, order=1..3 );
L1 := fsolve( abs(f(x)-P1(x))=0.01, x=x0/2 );          # (a)
R1 := fsolve( abs(f(x)-P1(x))=0.01, x=x1/2 );
L2 := fsolve( abs(f(x)-P2(x))=0.01, x=x0/2 );
R2 := fsolve( abs(f(x)-P2(x))=0.01, x=x1/2 );
L3 := fsolve( abs(f(x)-P3(x))=0.01, x=x0/2 );
R3 := fsolve( abs(f(x)-P3(x))=0.01, x=x1/2 );
plot( [E1(x),E2(x),E3(x),0.01], x=min(L1,L2,L3)..max(R1,R2,R3), thickness=[0,2,4,0], linestyle=[0,0,0,2],
    color=[red,blue,green,black], view=[DEFAULT,0..0.01], title="#53(a) (Section 10.9)" );
abs(`f(x)`-`P`[1](x) ) <= evalf( E1(x0) );          # (b)
abs(`f(x)`-`P`[2](x) ) <= evalf( E2(x0) );
abs(`f(x)`-`P`[3](x) ) <= evalf( E3(x0) );
```

Mathematica: (assigned function and values for a, b, c, and n may vary)

```
Clear[x, f, c]
f[x_]= (1 + x)^(3/2)
{a, b}= {−1/2, 2};
pf=Plot[ f[x], {x, a, b}];
poly1[x_]=Series[f[x], {x,0,1}]//Normal
poly2[x_]=Series[f[x], {x,0,2}]//Normal
poly3[x_]=Series[f[x], {x,0,3}]//Normal
Plot[{f[x], poly1[x], poly2[x], poly3[x]}, {x, a, b},
    PlotStyle → {RGBColor[1,0,0], RGBColor[0,1,0], RGBColor[0,0,1], RGBColor[0,.5,.5]}];
```

The above defines the approximations. The following analyzes the derivatives to determine their maximum values.

```
f''[c]
Plot[f''[x], {x, a, b}];
f'''[c]
Plot[f'''[x], {x, a, b}];
f''''[c]
Plot[f''''[x], {x, a, b}];
```

Noting the upper bound for each of the above derivatives occurs at x = a, the upper bounds m1, m2, and m3 can be defined and bounds for remainders viewed as functions of x.

```
m1=f''[a]
m2=-f'''[a]
m3=f''''[a]
r1[x_]=m1 x^2 /2!
Plot[r1[x], {x, a, b}];
r2[x_]=m2 x^3 /3!
Plot[r2[x], {x, a, b}];
r3[x_]=m3 x^4 /4!
Plot[r3[x], {x, a, b}];
```

A three dimensional look at the error functions, allowing both c and x to vary can also be viewed. Recall that c must be a value between 0 and x, so some points on the surfaces where c is not in that interval are meaningless.

```
Plot3D[f''[c] x^2 /2!, {x, a, b}, {c, a, b}, PlotRange → All]
Plot3D[f'''[c] x^3 /3!, {x, a, b}, {c, a, b}, PlotRange → All]
Plot3D[f''''[c] x^4 /4!, {x, a, b}, {c, a, b}, PlotRange → All]
```

10.10 THE BINOMIAL SERIES

1. $(1+x)^{1/2} = 1 + \frac{1}{2}x + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)x^2}{2!} + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^3}{3!} + \ldots = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \ldots$

2. $(1+x)^{1/3} = 1 + \frac{1}{3}x + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)x^2}{2!} + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)x^3}{3!} + \ldots = 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 - \ldots$

3. $(1-x)^{-1/2} = 1 - \frac{1}{2}(-x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(-x)^2}{2!} + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)(-x)^3}{3!} + \ldots = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \ldots$

4. $(1-2x)^{1/2} = 1 + \frac{1}{2}(-2x) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)(-2x)^2}{2!} + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(-2x)^3}{3!} + \ldots = 1 - x - \frac{1}{2}x^2 - \frac{1}{2}x^3 - \ldots$

5. $\left(1+\frac{x}{2}\right)^{-2} = 1 - 2\left(\frac{x}{2}\right) + \frac{(-2)(-3)\left(\frac{x}{2}\right)^2}{2!} + \frac{(-2)(-3)(-4)\left(\frac{x}{2}\right)^3}{3!} + \ldots = 1 - x + \frac{3}{4}x^2 - \frac{1}{2}x^3$

6. $\left(1-\frac{x}{3}\right)^4 = 1 + 4\left(-\frac{x}{3}\right) + \frac{(4)(3)\left(-\frac{x}{3}\right)^2}{2!} + \frac{(4)(3)(2)\left(-\frac{x}{3}\right)^3}{3!} + \frac{(4)(3)(2)(1)\left(-\frac{x}{3}\right)^4}{4!} + 0 + \ldots = 1 - \frac{4}{3}x + \frac{2}{3}x^2 - \frac{4}{27}x^3 + \frac{1}{81}x^4$

7. $\left(1+x^3\right)^{-1/2} = 1 - \frac{1}{2}x^3 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(x^3\right)^2}{2!} + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(x^3\right)^3}{3!} + \ldots = 1 - \frac{1}{2}x^3 + \frac{3}{8}x^6 - \frac{5}{16}x^9 + \ldots$

8. $\left(1+x^2\right)^{-1/3} = 1 - \frac{1}{3}x^2 + \frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(x^2\right)^2}{2!} + \frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)\left(x^2\right)^3}{3!} + \ldots = 1 - \frac{1}{3}x^2 + \frac{2}{9}x^4 - \frac{14}{81}x^6 + \ldots$

9. $\left(1+\frac{1}{x}\right)^{1/2} = 1 + \frac{1}{2}\left(\frac{1}{x}\right) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{1}{x}\right)^2}{2!} + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{1}{x}\right)^3}{3!} + \ldots = 1 + \frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{16x^3} + \ldots$

10. $\frac{x}{\sqrt[3]{1+x}} = x(1+x)^{-1/3} = x\left(1 - \left(-\frac{1}{3}\right)x + \frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)x^2}{2!} + \frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)x^3}{3!} + \ldots\right) = x - \frac{1}{3}x^2 + \frac{2}{9}x^3 - \frac{14}{81}x^4 + \ldots$

11. $(1+x)^4 = 1 + 4x + \frac{(4)(3)x^2}{2!} + \frac{(4)(3)(2)x^3}{3!} + \frac{(4)(3)(2)x^4}{4!} = 1 + 4x + 6x^2 + 4x^3 + x^4$

12. $(1+x^2)^3 = 1 + 3x^2 + \frac{(3)(2)(x^2)^2}{2!} + \frac{(3)(2)(1)(x^2)^3}{3!} = 1 + 3x^2 + 3x^4 + x^6$

13. $(1-2x)^3 = 1 + 3(-2x) + \frac{(3)(2)(-2x)^2}{2!} + \frac{(3)(2)(1)(-2x)^3}{3!} = 1 - 6x + 12x^2 - 8x^3$

14. $\left(1 - \frac{x}{2}\right)^4 = 1 + 4\left(-\frac{x}{2}\right) + \frac{(4)(3)\left(-\frac{x}{2}\right)^2}{2!} + \frac{(4)(3)(2)\left(-\frac{x}{2}\right)^3}{3!} + \frac{(4)(3)(2)(1)\left(-\frac{x}{2}\right)^4}{4!} = 1 - 2x + \frac{3}{2}x^2 - \frac{1}{2}x^3 + \frac{1}{16}x^4$

15. $\int_0^{0.2} \sin x^2\, dx = \int_0^{0.2} \left(x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \ldots\right) dx = \left[\frac{x^3}{3} - \frac{x^7}{7\cdot 3!} + \ldots\right]_0^{0.2} \approx \left[\frac{x^3}{3}\right]_0^{0.2} \approx 0.00267$ with error $|E| \le \frac{(.2)^7}{7\cdot 3!} \approx 0.0000003$

16. $\int_0^{0.2} \frac{e^{-x}-1}{x}\, dx = \int_0^{0.2} \frac{1}{x}\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots - 1\right) dx = \int_0^{0.2} \left(-1 + \frac{x}{2} - \frac{x^2}{6} + \frac{x^3}{24} - \ldots\right) dx$
$= \left[-x + \frac{x^2}{4} - \frac{x^3}{18} + \ldots\right]_0^{0.2} \approx -0.19044$ with error $|E| \le \frac{(0.2)^4}{96} \approx 0.00002$

17. $\int_0^{0.1} \frac{1}{\sqrt{1+x^4}}\, dx = \int_0^{0.1} \left(1 - \frac{x^4}{2} + \frac{3x^8}{8} - \ldots\right) dx = \left[x - \frac{x^5}{10} + \ldots\right]_0^{0.1} \approx [x]_0^{0.1} \approx 0.1$ with error $|E| \le \frac{(0.1)^5}{10} = 0.000001$

18. $\int_0^{0.25} \sqrt[3]{1+x^2}\, dx = \int_0^{0.25} \left(1 + \frac{x^2}{3} - \frac{x^4}{9} + \ldots\right) dx = \left[x + \frac{x^3}{9} - \frac{x^5}{45} + \ldots\right]_0^{0.25} \approx \left[x + \frac{x^3}{9}\right]_0^{0.25} \approx 0.25174$ with error $|E| \le \frac{(0.25)^5}{45} \approx 0.0000217$

19. $\int_0^{0.1} \frac{\sin x}{x}\, dx = \int_0^{0.1} \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots\right) dx = \left[x - \frac{x^3}{3\cdot 3!} + \frac{x^5}{5\cdot 5!} - \frac{x^7}{7\cdot 7!} + \ldots\right]_0^{0.1} \approx \left[x - \frac{x^3}{3\cdot 3!} + \frac{x^5}{5\cdot 5!}\right]_0^{0.1}$
≈ 0.0999444611, $|E| \le \frac{(0.1)^7}{7\cdot 7!} \approx 2.8 \times 10^{-12}$

20. $\int_0^{0.1} \exp(-x^2)\, dx = \int_0^{0.1} \left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \ldots\right) dx = \left[x - \frac{x^3}{3} + \frac{x^5}{10} + \frac{x^7}{42} + \ldots\right]_0^{0.1} \approx \left[x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42}\right]_0^{0.1}$
≈ 0.0996676643, $|E| \le \frac{(0.1)^9}{216} \approx 4.6 \times 10^{-12}$

21. $(1+x^4)^{1/2} = (1)^{1/2} + \frac{\left(\frac{1}{2}\right)}{1}(1)^{-1/2}(x^4) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}(1)^{-3/2}(x^4)^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}(1)^{-5/2}(x^4)^3$
$+ \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{4!}(1)^{-7/2}(x^4)^4 + \ldots = 1 + \frac{x^4}{2} - \frac{x^8}{8} + \frac{x^{12}}{16} - \frac{5x^{16}}{128} + \ldots$
$\Rightarrow \int_0^{0.1} \left(1 + \frac{x^4}{2} - \frac{x^8}{8} + \frac{x^{12}}{16} - \frac{5x^{16}}{128} + \ldots\right) dx \approx \left[x + \frac{x^5}{10}\right]_0^{0.1} \approx 0.100001$, $|E| \le \frac{(0.1)^9}{72} \approx 1.39 \times 10^{-11}$

22. $\int_0^1 \left(\frac{1-\cos x}{x^2}\right) dx = \int_0^1 \left(\frac{1}{2} - \frac{x^2}{4!} + \frac{x^4}{6!} - \frac{x^6}{8!} + \frac{x^8}{10!} - \ldots\right) dx \approx \left[\frac{x}{2} - \frac{x^3}{3\cdot 4!} + \frac{x^5}{5\cdot 6!} - \frac{x^7}{7\cdot 8!} + \frac{x^9}{9\cdot 10!}\right]_0^1$
≈ 0.4863853764, $|E| \le \frac{1}{11\cdot 12!} \approx 1.9 \times 10^{-10}$

23. $\int_0^1 \cos t^2\, dt = \int_0^1 \left(1 - \frac{t^4}{2} + \frac{t^8}{4!} - \frac{t^{12}}{6!} + \ldots\right) dt = \left[t - \frac{t^5}{10} + \frac{t^9}{9\cdot 4!} - \frac{t^{13}}{13\cdot 6!} + \ldots\right]_0^1 \Rightarrow |\text{error}| < \frac{1}{13\cdot 6!} \approx .00011$

24. $\int_0^1 \cos\sqrt{t}\,dt = \int_0^1 \left(1 - \frac{t}{2} + \frac{t^2}{4!} - \frac{t^3}{6!} + \frac{t^4}{8!} - \ldots\right) dt = \left[t - \frac{t^2}{4} + \frac{t^3}{3\cdot 4!} - \frac{t^4}{4\cdot 6!} + \frac{t^5}{5\cdot 8!} - \ldots\right]_0^1$
$\Rightarrow\ |\text{error}| < \frac{1}{5\cdot 8!} \approx 0.000004960$

25. $F(x) = \int_0^x \left(t^2 - \frac{t^6}{3!} + \frac{t^{10}}{5!} - \frac{t^{14}}{7!} + \ldots\right) dt = \left[\frac{t^3}{3} - \frac{t^7}{7\cdot 3!} + \frac{t^{11}}{11\cdot 5!} - \frac{t^{15}}{15\cdot 7!} + \ldots\right]_0^x \approx \frac{x^3}{3} - \frac{x^7}{7\cdot 3!} + \frac{x^{11}}{11\cdot 5!}$
$\Rightarrow\ |\text{error}| < \frac{1}{15\cdot 7!} \approx 0.000013$

26. $F(x) = \int_0^x \left(t^2 - t^4 + \frac{t^6}{2!} - \frac{t^8}{3!} + \frac{t^{10}}{4!} - \frac{t^{12}}{5!} + \ldots\right) dt = \left[\frac{t^3}{3} - \frac{t^5}{5} + \frac{t^7}{7\cdot 2!} - \frac{t^9}{9\cdot 3!} + \frac{t^{11}}{11\cdot 4!} - \frac{t^{13}}{13\cdot 5!} + \ldots\right]_0^x$
$\approx \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7\cdot 2!} - \frac{x^9}{9\cdot 3!} + \frac{x^{11}}{11\cdot 4!}\ \Rightarrow\ |\text{error}| < \frac{1}{13\cdot 5!} \approx 0.00064$

27. (a) $F(x) = \int_0^x \left(t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \ldots\right) dt = \left[\frac{t^2}{2} - \frac{t^4}{12} + \frac{t^6}{30} - \ldots\right]_0^x \approx \frac{x^2}{2} - \frac{x^4}{12}\ \Rightarrow\ |\text{error}| < \frac{(0.5)^6}{30} \approx .00052$
(b) $|\text{error}| < \frac{1}{33\cdot 34} \approx .00089$ when $F(x) \approx \frac{x^2}{2} - \frac{x^4}{3\cdot 4} + \frac{x^6}{5\cdot 6} - \frac{x^8}{7\cdot 8} + \ldots + (-1)^{15}\,\frac{x^{32}}{31\cdot 32}$

28. (a) $F(x) = \int_0^x \left(1 - \frac{t}{2} + \frac{t^2}{3} - \frac{t^3}{4} + \ldots\right) dt = \left[t - \frac{t^2}{2\cdot 2} + \frac{t^3}{3\cdot 3} - \frac{t^4}{4\cdot 4} + \frac{t^5}{5\cdot 5} - \ldots\right]_0^x \approx x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + \frac{x^5}{5^2}$
$\Rightarrow\ |\text{error}| < \frac{(0.5)^6}{6^2} \approx .00043$
(b) $|\text{error}| < \frac{1}{32^2} \approx .00097$ when $F(x) \approx x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + \ldots + (-1)^{31}\,\frac{x^{31}}{31^2}$

29. $\frac{1}{x^2}\left(e^x - (1+x)\right) = \frac{1}{x^2}\left(\left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \ldots\right) - 1 - x\right) = \frac{1}{2} + \frac{x}{3!} + \frac{x^2}{4!} + \ldots\ \Rightarrow\ \lim_{x\to 0} \frac{e^x - (1+x)}{x^2}$
$= \lim_{x\to 0} \left(\frac{1}{2} + \frac{x}{3!} + \frac{x^2}{4!} + \ldots\right) = \frac{1}{2}$

30. $\frac{1}{x}\left(e^x - e^{-x}\right) = \frac{1}{x}\left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\right) - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots\right)\right] = \frac{1}{x}\left(2x + \frac{2x^3}{3!} + \frac{2x^5}{5!} + \frac{2x^7}{7!} + \ldots\right)$
$= 2 + \frac{2x^2}{3!} + \frac{2x^4}{5!} + \frac{2x^6}{7!} + \ldots\ \Rightarrow\ \lim_{x\to 0} \frac{e^x - e^{-x}}{x} = \lim_{x\to\infty} \left(2 + \frac{2x^2}{3!} + \frac{2x^4}{5!} + \frac{2x^6}{7!} + \ldots\right) = 2$

31. $\frac{1}{t^4}\left(1 - \cos t - \frac{t^2}{2}\right) = \frac{1}{t^4}\left[1 - \frac{t^2}{2} - \left(1 - \frac{t^2}{2} + \frac{t^4}{4!} - \frac{t^6}{6!} + \ldots\right)\right] = -\frac{1}{4!} + \frac{t^2}{6!} - \frac{t^4}{8!} + \ldots\ \Rightarrow\ \lim_{t\to 0} \frac{1 - \cos t - \left(\frac{t^2}{2}\right)}{t^4}$
$= \lim_{t\to 0} \left(-\frac{1}{4!} + \frac{t^2}{6!} - \frac{t^4}{8!} + \ldots\right) = -\frac{1}{24}$

32. $\frac{1}{\theta^5}\left(-\theta + \frac{\theta^3}{6} + \sin\theta\right) = \frac{1}{\theta^5}\left(-\theta + \frac{\theta^3}{6} + \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots\right) = \frac{1}{5!} - \frac{\theta^2}{7!} + \frac{\theta^4}{9!} - \ldots\ \Rightarrow\ \lim_{\theta\to 0} \frac{\sin\theta - \theta + \left(\frac{\theta^3}{6}\right)}{\theta^5}$
$= \lim_{\theta\to 0} \left(\frac{1}{5!} - \frac{\theta^2}{7!} + \frac{\theta^4}{9!} - \ldots\right) = \frac{1}{120}$

33. $\frac{1}{y^3}\left(y - \tan^{-1} y\right) = \frac{1}{y^3}\left[y - \left(y - \frac{y^3}{3} + \frac{y^5}{5} - \ldots\right)\right] = \frac{1}{3} - \frac{y^2}{5} + \frac{y^4}{7} - \ldots\ \Rightarrow\ \lim_{y\to 0} \frac{y - \tan^{-1} y}{y^3} = \lim_{y\to 0} \left(\frac{1}{3} - \frac{y^2}{5} + \frac{y^4}{7} - \ldots\right)$
$= \frac{1}{3}$

34. $\frac{\tan^{-1} y - \sin y}{y^3 \cos y} = \frac{\left(y - \frac{y^3}{3} + \frac{y^5}{5} - \ldots\right) - \left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \ldots\right)}{y^3 \cos y} = \frac{\left(-\frac{y^3}{6} + \frac{23y^5}{5!} - \ldots\right)}{y^3 \cos y} = \frac{\left(-\frac{1}{6} + \frac{23y^2}{5!} - \ldots\right)}{\cos y}$
$\Rightarrow\ \lim_{y\to 0} \frac{\tan^{-1} y - \sin y}{y^3 \cos y} = \lim_{y\to 0} \frac{\left(-\frac{1}{6} + \frac{23y^2}{5!} - \ldots\right)}{\cos y} = -\frac{1}{6}$

35. $x^2\left(-1 + e^{-1/x^2}\right) = x^2\left(-1 + 1 - \frac{1}{x^2} + \frac{1}{2x^4} - \frac{1}{6x^6} + \ldots\right) = -1 + \frac{1}{2x^2} - \frac{1}{6x^4} + \ldots\ \Rightarrow\ \lim_{x\to\infty} x^2\left(e^{-1/x^2} - 1\right)$
$= \lim_{x\to\infty} \left(-1 + \frac{1}{2x^2} - \frac{1}{6x^4} + \ldots\right) = -1$

36. $(x+1)\sin\left(\frac{1}{x+1}\right) = (x+1)\left(\frac{1}{x+1} - \frac{1}{3!(x+1)^3} + \frac{1}{5!(x+1)^5} - \ldots\right) = 1 - \frac{1}{3!(x+1)^2} + \frac{1}{5!(x+1)^4} - \ldots$
$\Rightarrow \lim_{x \to \infty} (x+1)\sin\left(\frac{1}{x+1}\right) = \lim_{x \to \infty} \left(1 - \frac{1}{3!(x+1)^2} + \frac{1}{5!(x+1)^4} - \ldots\right) = 1$

37. $\frac{\ln(1+x^2)}{1-\cos x} = \frac{\left(x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \ldots\right)}{1-\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right)} = \frac{\left(1 - \frac{x^2}{2} + \frac{x^4}{3} - \ldots\right)}{\left(\frac{1}{2!} - \frac{x^2}{4!} + \ldots\right)} \Rightarrow \lim_{x \to 0} \frac{\ln(1+x^2)}{1-\cos x} = \lim_{x \to 0} \frac{\left(1 - \frac{x^2}{2} + \frac{x^4}{3} - \ldots\right)}{\left(\frac{1}{2!} - \frac{x^2}{4!} + \ldots\right)} = 2! = 2$

38. $\frac{x^2-4}{\ln(x-1)} = \frac{(x-2)(x+2)}{\left[(x-2) - \frac{(x-2)^2}{2} + \frac{(x-2)^3}{3} - \ldots\right]} = \frac{x+2}{\left[1 - \frac{x-2}{2} + \frac{(x-2)^2}{3} - \ldots\right]} \Rightarrow \lim_{x \to 2} \frac{x^2-4}{\ln(x-1)}$
$= \lim_{x \to 2} \frac{x+2}{\left[1 - \frac{x-2}{2} + \frac{(x-2)^2}{3} - \ldots\right]} = 4$

39. $\sin 3x^2 = 3x^2 - \frac{9}{2}x^6 + \frac{81}{40}x^{10} - \ldots$ and $1 - \cos 2x = 2x^2 - \frac{2}{3}x^4 + \frac{4}{45}x^6 - \ldots \Rightarrow \lim_{x \to 0} \frac{\sin 3x^2}{1-\cos 2x}$
$= \lim_{x \to 0} \frac{3x^2 - \frac{9}{2}x^6 + \frac{81}{40}x^{10} - \ldots}{2x^2 - \frac{2}{3}x^4 + \frac{4}{45}x^6 - \ldots} = \lim_{x \to 0} \frac{3 - \frac{9}{2}x^4 + \frac{81}{40}x^8 - \ldots}{2 - \frac{2}{3}x^2 + \frac{4}{45}x^4 - \ldots} = \frac{3}{2}$

40. $\ln(1+x^3) = x^3 - \frac{x^6}{2} + \frac{x^9}{3} - \frac{x^{12}}{4} + \ldots$ and $x\sin x^2 = x^3 - \frac{1}{6}x^7 + \frac{1}{120}x^{11} - \frac{1}{5040}x^{15} + \ldots \Rightarrow \lim_{x \to 0} \frac{\ln(1+x^3)}{x\sin x^2}$
$= \lim_{x \to 0} \frac{x^3 - \frac{x^6}{2} + \frac{x^9}{3} - \frac{x^{12}}{4} + \ldots}{x^3 - \frac{1}{6}x^7 + \frac{1}{120}x^{11} - \frac{1}{5040}x^{15} + \ldots} = \lim_{x \to 0} \frac{1 - \frac{x^3}{2} + \frac{x^6}{3} - \frac{x^9}{4} + \ldots}{1 - \frac{1}{6}x^4 + \frac{1}{120}x^8 - \frac{1}{5040}x^{12} + \ldots} = 1$

41. $1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots = e^1 = e$

42. $\left(\frac{1}{4}\right)^3 + \left(\frac{1}{4}\right)^4 + \left(\frac{1}{4}\right)^5 + \ldots = \left(\frac{1}{4}\right)^3\left[1 + \left(\frac{1}{4}\right) + \left(\frac{1}{4}\right)^2 + \ldots\right] = \frac{1}{64}\frac{1}{1-1/4} = \frac{1}{64}\frac{4}{3} = \frac{1}{48}$

43. $1 - \frac{3^2}{4^2 2!} + \frac{3^4}{4^4 4!} - \frac{3^6}{4^6 6!} + \ldots = 1 - \frac{1}{2!}\left(\frac{3}{4}\right)^2 + \frac{1}{4!}\left(\frac{3}{4}\right)^4 - \frac{1}{6!}\left(\frac{3}{4}\right)^6 + \ldots = \cos\left(\frac{3}{4}\right)$

44. $\frac{1}{2} - \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} - \frac{1}{4\cdot 2^4} + \ldots = \left(\frac{1}{2}\right) - \frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{3}\left(\frac{1}{2}\right)^3 - \frac{1}{4}\left(\frac{1}{2}\right)^4 + \ldots = \ln\left(1 + \frac{1}{2}\right) = \ln\left(\frac{3}{2}\right)$

45. $\frac{\pi}{3} - \frac{\pi^3}{3^3 3!} + \frac{\pi^5}{3^5 5!} - \frac{\pi^7}{3^7 7!} + \ldots = \frac{\pi}{3} - \frac{1}{3!}\left(\frac{\pi}{3}\right)^3 + \frac{1}{5!}\left(\frac{\pi}{3}\right)^5 - \frac{1}{7!}\left(\frac{\pi}{3}\right)^7 + \ldots = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

46. $\frac{2}{3} - \frac{2^3}{3^3\cdot 3} + \frac{2^5}{3^5\cdot 5} - \frac{2^7}{3^7\cdot 7} + \ldots = \left(\frac{2}{3}\right) - \frac{1}{3}\left(\frac{2}{3}\right)^3 + \frac{1}{5}\left(\frac{2}{3}\right)^5 - \frac{1}{7}\left(\frac{2}{3}\right)^7 + \ldots = \tan^{-1}\left(\frac{2}{3}\right)$

47. $x^3 + x^4 + x^5 + x^6 + \ldots = x^3(1 + x + x^2 + x^3 + \ldots) = x^3\left(\frac{1}{1-x}\right) = \frac{x^3}{1-x}$

48. $1 - \frac{3^2x^2}{2!} + \frac{3^4x^4}{4!} - \frac{3^6x^6}{6!} + \ldots = 1 - \frac{1}{2!}(3x)^2 + \frac{1}{4!}(3x)^4 - \frac{1}{6!}(3x)^6 + \ldots = \cos(3x)$

49. $x^3 - x^5 + x^7 - x^9 + \ldots = x^3\left(1 - x^2 + (x^2)^2 - (x^2)^3 + \ldots\right) = x^3\left(\frac{1}{1+x^2}\right) = \frac{x^3}{1+x^2}$

50. $x^2 - 2x^3 + \frac{2^2x^4}{2!} - \frac{2^3x^5}{3!} + \frac{2^4x^6}{4!} - \ldots = x^2\left(1 - 2x + \frac{(2x)^2}{2!} - \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} - \ldots\right) = x^2e^{-2x}$

51. $-1 + 2x - 3x^2 + 4x^3 - 5x^4 + \ldots = \frac{d}{dx}(1 - x + x^2 - x^3 + x^4 - x^5 + \ldots) = \frac{d}{dx}\left(\frac{1}{1+x}\right) = \frac{-1}{(1+x)^2}$

52. $1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \frac{x^4}{5} + \ldots = -\frac{1}{x}\left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \ldots\right) = -\frac{1}{x}\ln(1-x) = -\frac{\ln(1-x)}{x}$

53. $\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) = \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\right)$

54. $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + \frac{(-1)^{n-1}x^n}{n} + \ldots \Rightarrow |\text{error}| = \left|\frac{(-1)^{n-1}x^n}{n}\right| = \frac{1}{n10^n}$ when $x = 0.1$;
$\frac{1}{n10^n} < \frac{1}{10^8} \Rightarrow n10^n > 10^8$ when $n \geq 8 \Rightarrow$ 7 terms

55. $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \ldots + \frac{(-1)^{n-1}x^{2n-1}}{2n-1} + \ldots \Rightarrow |\text{error}| = \left|\frac{(-1)^{n-1}x^{2n-1}}{2n-1}\right| = \frac{1}{2n-1}$ when $x = 1$;
$\frac{1}{2n-1} < \frac{1}{10^3} \Rightarrow n > \frac{1001}{2} = 500.5 \Rightarrow$ the first term not used is the 501st $\Rightarrow$ we must use 500 terms

56. $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \ldots + \frac{(-1)^{n-1}x^{2n-1}}{2n-1} + \ldots$ and $\lim\limits_{n\to\infty} \left|\frac{x^{2n+1}}{2n+1} \cdot \frac{2n-1}{x^{2n-1}}\right| = x^2 \lim\limits_{n\to\infty} \left|\frac{2n-1}{2n+1}\right| = x^2$
$\Rightarrow \tan^{-1} x$ converges for $|x| < 1$; when $x = -1$ we have $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{2n-1}$ which is a convergent series; when $x = 1$
we have $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$ which is a convergent series $\Rightarrow$ the series representing $\tan^{-1} x$ diverges for $|x| > 1$

57. $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \ldots + \frac{(-1)^{n-1}x^{2n-1}}{2n-1} + \ldots$ and when the series representing $48\tan^{-1}\left(\frac{1}{18}\right)$ has an error less than $\frac{1}{3} \cdot 10^{-6}$, then the series representing the sum
$48\tan^{-1}\left(\frac{1}{18}\right) + 32\tan^{-1}\left(\frac{1}{57}\right) - 20\tan^{-1}\left(\frac{1}{239}\right)$ also has an error of magnitude less than 10^{-6}; thus
$|\text{error}| = 48\frac{\left(\frac{1}{18}\right)^{2n-1}}{2n-1} < \frac{1}{3\cdot 10^6} \Rightarrow n \geq 4$ using a calculator $\Rightarrow$ 4 terms

58. $\ln(\sec x) = \int_0^x \tan t\, dt = \int_0^x \left(t + \frac{t^3}{3} + \frac{2t^5}{15} + \ldots\right) dt \approx \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \ldots$

59. (a) $(1-x^2)^{-1/2} \approx 1 + \frac{x^2}{2} + \frac{3x^4}{8} + \frac{5x^6}{16} \Rightarrow \sin^{-1} x \approx x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112}$; Using the Ratio Test:
$\lim\limits_{n\to\infty} \left|\frac{1\cdot3\cdot5\cdots(2n-1)(2n+1)x^{2n+3}}{2\cdot4\cdot6\cdots(2n)(2n+2)(2n+3)} \cdot \frac{2\cdot4\cdot6\cdots(2n)(2n+1)}{1\cdot3\cdot5\cdots(2n-1)x^{2n+1}}\right| < 1 \Rightarrow x^2 \lim\limits_{n\to\infty} \left|\frac{(2n+1)(2n+1)}{(2n+2)(2n+3)}\right| < 1$
$\Rightarrow |x| < 1 \Rightarrow$ the radius of convergence is 1. See Exercise 69.
(b) $\frac{d}{dx}(\cos^{-1} x) = -(1-x^2)^{-1/2} \Rightarrow \cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x \approx \frac{\pi}{2} - \left(x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112}\right) \approx \frac{\pi}{2} - x - \frac{x^3}{6} - \frac{3x^5}{40} - \frac{5x^7}{112}$

60. (a) $(1+t^2)^{-1/2} \approx (1)^{-1/2} + \left(-\frac{1}{2}\right)(1)^{-3/2}(t^2) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(1)^{-5/2}(t^2)^2}{2!} + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)(1)^{-7/2}(t^2)^3}{3!}$
$= 1 - \frac{t^2}{2} + \frac{3t^4}{2^2\cdot 2!} - \frac{3\cdot 5t^6}{2^3\cdot 3!} \Rightarrow \sinh^{-1} x \approx \int_0^x \left(1 - \frac{t^2}{2} + \frac{3t^4}{8} - \frac{5t^6}{16}\right) dt = x - \frac{x^3}{6} + \frac{3x^5}{40} - \frac{5x^7}{112}$
(b) $\sinh^{-1}\left(\frac{1}{4}\right) \approx \frac{1}{4} - \frac{1}{384} + \frac{3}{40{,}960} = 0.24746908$; the error is less than the absolute value of the first unused term, $\frac{5x^7}{112}$, evaluated at $t = \frac{1}{4}$ since the series is alternating $\Rightarrow |\text{error}| < \frac{5\left(\frac{1}{4}\right)^7}{112} \approx 2.725 \times 10^{-6}$

61. $\frac{-1}{1+x} = -\frac{1}{1-(-x)} = -1 + x - x^2 + x^3 - \ldots \Rightarrow \frac{d}{dx}\left(\frac{-1}{1+x}\right) = \frac{1}{1+x^2} = \frac{d}{dx}(-1 + x - x^2 + x^3 - \ldots)$
$= 1 - 2x + 3x^2 - 4x^3 + \ldots$

62. $\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + \ldots \Rightarrow \frac{d}{dx}\left(\frac{1}{1-x^2}\right) = \frac{2x}{(1-x^2)^2} = \frac{d}{dx}(1 + x^2 + x^4 + x^6 + \ldots) = 2x + 4x^3 + 6x^5 + \ldots$

63. Wallis' formula gives the approximation $\pi \approx 4\left[\frac{2\cdot4\cdot4\cdot6\cdot6\cdot8\cdots(2n-2)\cdot(2n)}{3\cdot3\cdot5\cdot5\cdot7\cdot7\cdots(2n-1)\cdot(2n-1)}\right]$ to produce the table

n	$\sim\pi$
10	3.221088998
20	3.181104886
30	3.167880758
80	3.151425420
90	3.150331383
93	3.150049112
94	3.149959030
95	3.149870848
100	3.149456425

At n = 1929 we obtain the first approximation accurate to 3 decimals: 3.141999845. At n = 30,000 we still do not obtain accuracy to 4 decimals: 3.141617732, so the convergence to π is very slow. Here is a Maple CAS procedure to produce these approximations:

```
pie :=
  proc(n)
  local i,j;
    a(2) := evalf(8/9);
    for i from 3 to n do a(i) := evalf(2*(2*i-2)*i/(2*i-1)^2*a(i-1)) od;
    [[j,4*a(j)] $ (j = n-5 .. n)]
  end
```

64. (a) $f(x)=1+\sum_{k=1}^{\infty}\binom{m}{k}x^k \Rightarrow f'(x)=\sum_{k=1}^{\infty}\binom{m}{k}k\,x^{k-1} \Rightarrow (1+x)\cdot f'(x)=(1+x)\sum_{k=1}^{\infty}\binom{m}{k}k\,x^{k-1}$

$$=\sum_{k=1}^{\infty}\binom{m}{k}k\,x^{k-1}+x\cdot\sum_{k=1}^{\infty}\binom{m}{k}k\,x^{k-1}=\sum_{k=1}^{\infty}\binom{m}{k}k\,x^{k-1}+\sum_{k=1}^{\infty}\binom{m}{k}k\,x^{k}=\binom{m}{1}(1)\,x^0+\sum_{k=2}^{\infty}\binom{m}{k}k\,x^{k-1}+\sum_{k=1}^{\infty}\binom{m}{k}k\,x^{k}$$

$=m+\sum_{k=2}^{\infty}\binom{m}{k}k\,x^{k-1}+\sum_{k=1}^{\infty}\binom{m}{k}k\,x^{k}$ Note that: $\sum_{k=2}^{\infty}\binom{m}{k}k\,x^{k-1}=\sum_{k=1}^{\infty}\binom{m}{k+1}(k+1)\,x^{k}$.

Thus, $(1+x)\cdot f'(x)=m+\sum_{k=2}^{\infty}\binom{m}{k}k\,x^{k-1}+\sum_{k=1}^{\infty}\binom{m}{k}k\,x^{k}=m+\sum_{k=1}^{\infty}\binom{m}{k+1}(k+1)\,x^{k}+\sum_{k=1}^{\infty}\binom{m}{k}k\,x^{k}$

$$=m+\sum_{k=1}^{\infty}\left[\binom{m}{k+1}(k+1)\,x^k+\binom{m}{k}k\,x^k\right]=m+\sum_{k=1}^{\infty}\left[\left(\binom{m}{k+1}(k+1)+\binom{m}{k}k\right)x^k\right].$$

Note that: $\binom{m}{k+1}(k+1)+\binom{m}{k}k=\frac{m\cdot(m-1)\cdots(m-(k+1)+1)}{(k+1)!}(k+1)+\frac{m\cdot(m-1)\cdots(m-k+1)}{k!}k$

$$=\frac{m\cdot(m-1)\cdots(m-k)}{k!}+\frac{m\cdot(m-1)\cdots(m-k+1)}{k!}k=\frac{m\cdot(m-1)\cdots(m-k+1)}{k!}((m-k)+k)=m\frac{m\cdot(m-1)\cdots(m-k+1)}{k!}=m\binom{m}{k}.$$

Thus, $(1+x)\cdot f'(x)=m+\sum_{k=1}^{\infty}\left[\left(\binom{m}{k+1}(k+1)+\binom{m}{k}k\right)x^k\right]=m+\sum_{k=1}^{\infty}\left[\left(m\binom{m}{k}\right)x^k\right]=m+m\sum_{k=1}^{\infty}\binom{m}{k}x^k$

$$=m\left(1+\sum_{k=1}^{\infty}\binom{m}{k}x^k\right)=m\cdot f(x)\Rightarrow f'(x)=\frac{m\cdot f(x)}{(1+x)} \text{ if } -1<x<1.$$

(b) Let $g(x)=(1+x)^{-m}f(x)\Rightarrow g'(x)=-m(1+x)^{-m-1}f(x)+(1+x)^{-m}f'(x)$

$$=-m(1+x)^{-m-1}f(x)+(1+x)^{-m}\cdot\frac{m\cdot f(x)}{(1+x)}=-m(1+x)^{-m-1}f(x)+(1+x)^{-m-1}\cdot m\cdot f(x)=0.$$

(c) $g'(x)=0\Rightarrow g(x)=c\Rightarrow(1+x)^{-m}f(x)=c\Rightarrow f(x)=\frac{c}{(1+x)^{-m}}=c(1+x)^m$. Since $f(x)=1+\sum_{k=1}^{\infty}\binom{m}{k}x^k$

$$\Rightarrow f(0)=1+\sum_{k=1}^{\infty}\binom{m}{k}(0)^k=1+0=1\Rightarrow c(1+0)^m=1\Rightarrow c=1\Rightarrow f(x)=(1+x)^m.$$

65. $(1-x^2)^{-1/2}=(1+(-x^2))^{-1/2}=(1)^{-1/2}+\left(-\frac{1}{2}\right)(1)^{-3/2}(-x^2)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(1)^{-5/2}(-x^2)^2}{2!}$

$$+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)(1)^{-7/2}(-x^2)^3}{3!}+\ldots=1+\frac{x^2}{2}+\frac{1\cdot3x^4}{2^2\cdot2!}+\frac{1\cdot3\cdot5x^6}{2^3\cdot3!}+\ldots=1+\sum_{n=1}^{\infty}\frac{1\cdot3\cdot5\cdots(2n-1)x^{2n}}{2^n\cdot n!}$$

$\Rightarrow \sin^{-1} x = \int_0^x (1-t^2)^{-1/2}\, dt = \int_0^x \left(1 + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)x^{2n}}{2^n\cdot n!}\right) dt = x + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)x^{2n+1}}{2\cdot 4\cdots(2n)(2n+1)}$, where $|x| < 1$

66. $\left[\tan^{-1} t\right]_x^{\infty} = \frac{\pi}{2} - \tan^{-1} x = \int_x^{\infty} \frac{dt}{1+t^2} = \int_x^{\infty} \left[\frac{\left(\frac{1}{t^2}\right)}{1+\left(\frac{1}{t^2}\right)}\right] dt = \int_x^{\infty} \frac{1}{t^2}\left(1 - \frac{1}{t^2} + \frac{1}{t^4} - \frac{1}{t^6} + \ldots\right) dt$
$= \int_x^{\infty} \left(\frac{1}{t^2} - \frac{1}{t^4} + \frac{1}{t^6} - \frac{1}{t^8} + \ldots\right) dt = \lim_{b\to\infty} \left[-\frac{1}{t} + \frac{1}{3t^3} - \frac{1}{5t^5} + \frac{1}{7t^7} - \ldots\right]_x^b = \frac{1}{x} - \frac{1}{3x^3} + \frac{1}{5x^5} - \frac{1}{7x^7} + \ldots$
$\Rightarrow \tan^{-1} x = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \ldots,\ x > 1$; $\left[\tan^{-1} t\right]_{-\infty}^{x} = \tan^{-1} x + \frac{\pi}{2} = \int_{-\infty}^{x} \frac{dt}{1+t^2}$
$= \lim_{b\to -\infty} \left[-\frac{1}{t} + \frac{1}{3t^3} - \frac{1}{5t^5} + \frac{1}{7t^7} - \ldots\right]_b^x = -\frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \frac{1}{7x^7} - \ldots \Rightarrow \tan^{-1} x = -\frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \ldots$,
$x < -1$

67. (a) $e^{-i\pi} = \cos(-\pi) + i\sin(-\pi) = -1 + i(0) = -1$
(b) $e^{i\pi/4} = \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} = \left(\frac{1}{\sqrt{2}}\right)(1+i)$
(c) $e^{-i\pi/2} = \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right) = 0 + i(-1) = -i$

68. $e^{i\theta} = \cos\theta + i\sin\theta \Rightarrow e^{-i\theta} = e^{i(-\theta)} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$;
$e^{i\theta} + e^{-i\theta} = \cos\theta + i\sin\theta + \cos\theta - i\sin\theta = 2\cos\theta \Rightarrow \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$;
$e^{i\theta} - e^{-i\theta} = \cos\theta + i\sin\theta - (\cos\theta - i\sin\theta) = 2i\sin\theta \Rightarrow \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$

69. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \Rightarrow e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \ldots$ and
$e^{-i\theta} = 1 - i\theta + \frac{(-i\theta)^2}{2!} + \frac{(-i\theta)^3}{3!} + \frac{(-i\theta)^4}{4!} + \ldots = 1 - i\theta + \frac{(i\theta)^2}{2!} - \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} - \ldots$
$\Rightarrow \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{\left(1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \ldots\right) + \left(1 - i\theta + \frac{(i\theta)^2}{2!} - \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} - \ldots\right)}{2}$
$= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots = \cos\theta$;
$\frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{\left(1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \ldots\right) - \left(1 - i\theta + \frac{(i\theta)^2}{2!} - \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} - \ldots\right)}{2i}$
$= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots = \sin\theta$

70. $e^{i\theta} = \cos\theta + i\sin\theta \Rightarrow e^{-i\theta} = e^{i(-\theta)} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$
(a) $e^{i\theta} + e^{-i\theta} = (\cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta) = 2\cos\theta \Rightarrow \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \cosh i\theta$
(b) $e^{i\theta} - e^{-i\theta} = (\cos\theta + i\sin\theta) - (\cos\theta - i\sin\theta) = 2i\sin\theta \Rightarrow i\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2} = \sinh i\theta$

71. $e^x \sin x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right)$
$= (1)x + (1)x^2 + \left(-\frac{1}{6} + \frac{1}{2}\right)x^3 + \left(-\frac{1}{6} + \frac{1}{6}\right)x^4 + \left(\frac{1}{120} - \frac{1}{12} + \frac{1}{24}\right)x^5 + \ldots = x + x^2 + \frac{1}{3}x^3 - \frac{1}{30}x^5 + \ldots$;
$e^x \cdot e^{ix} = e^{(1+i)x} = e^x(\cos x + i\sin x) = e^x\cos x + i(e^x\sin x) \Rightarrow e^x \sin x$ is the series of the imaginary part of $e^{(1+i)x}$ which we calculate next; $e^{(1+i)x} = \sum_{n=0}^{\infty} \frac{(x+ix)^n}{n!} = 1 + (x+ix) + \frac{(x+ix)^2}{2!} + \frac{(x+ix)^3}{3!} + \frac{(x+ix)^4}{4!} + \ldots$
$= 1 + x + ix + \frac{1}{2!}(2ix^2) + \frac{1}{3!}(2ix^3 - 2x^3) + \frac{1}{4!}(-4x^4) + \frac{1}{5!}(-4x^5 - 4ix^5) + \frac{1}{6!}(-8ix^6) + \ldots \Rightarrow$ the imaginary part of $e^{(1+i)x}$ is $x + \frac{2}{2!}x^2 + \frac{2}{3!}x^3 - \frac{4}{5!}x^5 - \frac{8}{6!}x^6 + \ldots = x + x^2 + \frac{1}{3}x^3 - \frac{1}{30}x^5 - \frac{1}{90}x^6 + \ldots$ in agreement with our product calculation. The series for $e^x \sin x$ converges for all values of x.

72. $\frac{d}{dx}\left(e^{(a+ib)}\right) = \frac{d}{dx}\left[e^{ax}(\cos bx + i\sin bx)\right] = ae^{ax}(\cos bx + i\sin bx) + e^{ax}(-b\sin bx + bi\cos bx)$
$= ae^{ax}(\cos bx + i\sin bx) + bie^{ax}(\cos bx + i\sin bx) = ae^{(a+ib)x} + ibe^{(a+ib)x} = (a+ib)e^{(a+ib)x}$

73. (a) $e^{i\theta_1}e^{i\theta_2} = (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1)$
$= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2) = e^{i(\theta_1+\theta_2)}$

(b) $e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta = (\cos\theta - i\sin\theta)\left(\frac{\cos\theta + i\sin\theta}{\cos\theta + i\sin\theta}\right) = \frac{1}{\cos\theta + i\sin\theta} = \frac{1}{e^{i\theta}}$

74. $\frac{a-bi}{a^2+b^2}e^{(a+bi)x} + C_1 + iC_2 = \left(\frac{a-bi}{a^2+b^2}\right)e^{ax}(\cos bx + i\sin bx) + C_1 + iC_2$
$= \frac{e^{ax}}{a^2+b^2}(a\cos bx + ia\sin bx - ib\cos bx + b\sin bx) + C_1 + iC_2$
$= \frac{e^{ax}}{a^2+b^2}[(a\cos bx + b\sin bx) + (a\sin bx - b\cos bx)i] + C_1 + iC_2$
$= \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2} + C_1 + \frac{ie^{ax}(a\sin bx - b\cos bx)}{a^2+b^2} + iC_2$;
$e^{(a+bi)x} = e^{ax}e^{ibx} = e^{ax}(\cos bx + i\sin bx) = e^{ax}\cos bx + ie^{ax}\sin bx$, so that given
$\int e^{(a+bi)x}\,dx = \frac{a-bi}{a^2+b^2}e^{(a+bi)x} + C_1 + iC_2$ we conclude that $\int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2} + C_1$
and $\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2} + C_2$

CHAPTER 10 PRACTICE EXERCISES

1. converges to 1, since $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1 + \frac{(-1)^n}{n}\right) = 1$

2. converges to 0, since $0 \le a_n \le \frac{2}{\sqrt{n}}$, $\lim_{n\to\infty} 0 = 0$, $\lim_{n\to\infty}\frac{2}{\sqrt{n}} = 0$ using the Sandwich Theorem for Sequences

3. converges to -1, since $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{1-2^n}{2^n}\right) = \lim_{n\to\infty}\left(\frac{1}{2^n} - 1\right) = -1$

4. converges to 1, since $\lim_{n\to\infty} a_n = \lim_{n\to\infty}[1 + (0.9)^n] = 1 + 0 = 1$

5. diverges, since $\left\{\sin\frac{n\pi}{2}\right\} = \{0, 1, 0, -1, 0, 1, \dots\}$

6. converges to 0, since $\{\sin n\pi\} = \{0, 0, 0, \dots\}$

7. converges to 0, since $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{\ln n^2}{n} = 2\lim_{n\to\infty}\frac{\left(\frac{1}{n}\right)}{1} = 0$

8. converges to 0, since $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{\ln(2n+1)}{n} = \lim_{n\to\infty}\frac{\left(\frac{2}{2n+1}\right)}{1} = 0$

9. converges to 1, since $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{n+\ln n}{n}\right) = \lim_{n\to\infty}\frac{1+\left(\frac{1}{n}\right)}{1} = 1$

10. converges to 0, since $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{\ln(2n^3+1)}{n} = \lim_{n\to\infty}\frac{\left(\frac{6n^2}{2n^3+1}\right)}{1} = \lim_{n\to\infty}\frac{12n}{6n^2} = \lim_{n\to\infty}\frac{2}{n} = 0$

11. converges to e^{-5}, since $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{n-5}{n}\right)^n = \lim_{n\to\infty}\left(1 + \frac{(-5)}{n}\right)^n = e^{-5}$ by Theorem 5

12. converges to $\frac{1}{e}$, since $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^{-n} = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e}$ by Theorem 5

13. converges to 3, since $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{3^n}{n}\right)^{1/n} = \lim_{n\to\infty}\frac{3}{n^{1/n}} = \frac{3}{1} = 3$ by Theorem 5

14. converges to 1, since $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{3}{n}\right)^{1/n} = \lim_{n\to\infty}\frac{3^{1/n}}{n^{1/n}} = \frac{1}{1} = 1$ by Theorem 5

15. converges to ln 2, since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} n\left(2^{1/n}-1\right) = \lim_{n\to\infty} \frac{2^{1/n}-1}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{\left[\frac{\left(-2^{1/n}\ln 2\right)}{n^2}\right]}{\left(\frac{-1}{n^2}\right)} = \lim_{n\to\infty} 2^{1/n}\ln 2$
$= 2^0 \cdot \ln 2 = \ln 2$

16. converges to 1, since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt[n]{2n+1} = \lim_{n\to\infty} \exp\left(\frac{\ln(2n+1)}{n}\right) = \lim_{n\to\infty} \exp\left(\frac{\frac{2}{2n+1}}{1}\right) = e^0 = 1$

17. diverges, since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty} (n+1) = \infty$

18. converges to 0, since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-4)^n}{n!} = 0$ by Theorem 5

19. $\frac{1}{(2n-3)(2n-1)} = \frac{\left(\frac{1}{2}\right)}{2n-3} - \frac{\left(\frac{1}{2}\right)}{2n-1} \Rightarrow s_n = \left[\frac{\left(\frac{1}{2}\right)}{3} - \frac{\left(\frac{1}{2}\right)}{5}\right] + \left[\frac{\left(\frac{1}{2}\right)}{5} - \frac{\left(\frac{1}{2}\right)}{7}\right] + \ldots + \left[\frac{\left(\frac{1}{2}\right)}{2n-3} - \frac{\left(\frac{1}{2}\right)}{2n-1}\right] = \frac{\left(\frac{1}{2}\right)}{3} - \frac{\left(\frac{1}{2}\right)}{2n-1}$
$\Rightarrow \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left[\frac{1}{6} - \frac{\left(\frac{1}{2}\right)}{2n-1}\right] = \frac{1}{6}$

20. $\frac{-2}{n(n+1)} = \frac{-2}{n} + \frac{2}{n+1} \Rightarrow s_n = \left(\frac{-2}{2} + \frac{2}{3}\right) + \left(\frac{-2}{3} + \frac{2}{4}\right) + \ldots + \left(\frac{-2}{n} + \frac{2}{n+1}\right) = -\frac{2}{2} + \frac{2}{n+1} \Rightarrow \lim_{n\to\infty} s_n$
$= \lim_{n\to\infty} \left(-1 + \frac{2}{n+1}\right) = -1$

21. $\frac{9}{(3n-1)(3n+2)} = \frac{3}{3n-1} - \frac{3}{3n+2} \Rightarrow s_n = \left(\frac{3}{2} - \frac{3}{5}\right) + \left(\frac{3}{5} - \frac{3}{8}\right) + \left(\frac{3}{8} - \frac{3}{11}\right) + \ldots + \left(\frac{3}{3n-1} - \frac{3}{3n+2}\right)$
$= \frac{3}{2} - \frac{3}{3n+2} \Rightarrow \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(\frac{3}{2} - \frac{3}{3n+2}\right) = \frac{3}{2}$

22. $\frac{-8}{(4n-3)(4n+1)} = \frac{-2}{4n-3} + \frac{2}{4n+1} \Rightarrow s_n = \left(\frac{-2}{9} + \frac{2}{13}\right) + \left(\frac{-2}{13} + \frac{2}{17}\right) + \left(\frac{-2}{17} + \frac{2}{21}\right) + \ldots + \left(\frac{-2}{4n-3} + \frac{2}{4n+1}\right)$
$= -\frac{2}{9} + \frac{2}{4n+1} \Rightarrow \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(-\frac{2}{9} + \frac{2}{4n+1}\right) = -\frac{2}{9}$

23. $\sum_{n=0}^{\infty} e^{-n} = \sum_{n=0}^{\infty} \frac{1}{e^n}$, a convergent geometric series with $r = \frac{1}{e}$ and $a = 1 \Rightarrow$ the sum is $\frac{1}{1-\left(\frac{1}{e}\right)} = \frac{e}{e-1}$

24. $\sum_{n=1}^{\infty} (-1)^n \frac{3}{4^n} = \sum_{n=0}^{\infty} \left(-\frac{3}{4}\right)\left(\frac{-1}{4}\right)^n$ a convergent geometric series with $r = -\frac{1}{4}$ and $a = \frac{-3}{4} \Rightarrow$ the sum is
$\frac{\left(-\frac{3}{4}\right)}{1-\left(\frac{-1}{4}\right)} = -\frac{3}{5}$

25. diverges, a p-series with $p = \frac{1}{2}$

26. $\sum_{n=1}^{\infty} \frac{-5}{n} = -5\sum_{n=1}^{\infty} \frac{1}{n}$, diverges since it is a nonzero multiple of the divergent harmonic series

27. Since $f(x) = \frac{1}{x^{1/2}} \Rightarrow f'(x) = -\frac{1}{2x^{3/2}} < 0 \Rightarrow f(x)$ is decreasing $\Rightarrow a_{n+1} < a_n$, and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges by the Alternating Series Test. Since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges, the given series converges conditionally.

28. converges absolutely by the Direct Comparison Test since $\frac{1}{2n^3} < \frac{1}{n^3}$ for $n \geq 1$, which is the nth term of a convergent p-series

29. The given series does not converge absolutely by the Direct Comparison Test since $\frac{1}{\ln(n+1)} > \frac{1}{n+1}$, which is the nth term of a divergent series. Since $f(x) = \frac{1}{\ln(x+1)} \Rightarrow f'(x) = -\frac{1}{(\ln(x+1))^2(x+1)} < 0 \Rightarrow f(x)$ is decreasing $\Rightarrow a_{n+1} < a_n$, and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\ln(n+1)} = 0$, the given series converges conditionally by the Alternating Series Test.

30. $\int_2^\infty \frac{1}{x(\ln x)^2}\,dx = \lim_{b\to\infty} \int_2^b \frac{1}{x(\ln x)^2}\,dx = \lim_{b\to\infty} \left[-(\ln x)^{-1}\right]_2^b = -\lim_{b\to\infty} \left(\frac{1}{\ln b} - \frac{1}{\ln 2}\right) = \frac{1}{\ln 2} \Rightarrow$ the series converges absolutely by the Integral Test

31. converges absolutely by the Direct Comparison Test since $\frac{\ln n}{n^3} < \frac{n}{n^3} = \frac{1}{n^2}$, the nth term of a convergent p-series

32. diverges by the Direct Comparison Test for $e^{n^n} > n \Rightarrow \ln\left(e^{n^n}\right) > \ln n \Rightarrow n^n > \ln n \Rightarrow \ln n^n > \ln(\ln n)$ $\Rightarrow n \ln n > \ln(\ln n) \Rightarrow \frac{\ln n}{\ln(\ln n)} > \frac{1}{n}$, the nth term of the divergent harmonic series

33. $\lim_{n\to\infty} \frac{\left(\frac{1}{n\sqrt{n^2+1}}\right)}{\left(\frac{1}{n^2}\right)} = \sqrt{\lim_{n\to\infty} \frac{n^2}{n^2+1}} = \sqrt{1} = 1 \Rightarrow$ converges absolutely by the Limit Comparison Test

34. Since $f(x) = \frac{3x^2}{x^3+1} \Rightarrow f'(x) = \frac{3x(2-x^3)}{(x^3+1)^2} < 0$ when $x \ge 2 \Rightarrow a_{n+1} < a_n$ for $n \ge 2$ and $\lim_{n\to\infty} \frac{3n^2}{n^3+1} = 0$, the series converges by the Alternating Series Test. The series does not converge absolutely: By the Limit Comparison Test, $\lim_{n\to\infty} \frac{\left(\frac{3n^2}{n^3+1}\right)}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{3n^3}{n^3+1} = 3$. Therefore the convergence is conditional.

35. converges absolutely by the Ratio Test since $\lim_{n\to\infty} \left[\frac{n+2}{(n+1)!} \cdot \frac{n!}{n+1}\right] = \lim_{n\to\infty} \frac{n+2}{(n+1)^2} = 0 < 1$

36. diverges since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^n(n^2+1)}{2n^2+n-1}$ does not exist

37. converges absolutely by the Ratio Test since $\lim_{n\to\infty} \left[\frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n}\right] = \lim_{n\to\infty} \frac{3}{n+1} = 0 < 1$

38. converges absolutely by the Root Test since $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{2^n 3^n}{n^n}} = \lim_{n\to\infty} \frac{6}{n} = 0 < 1$

39. converges absolutely by the Limit Comparison Test since $\lim_{n\to\infty} \frac{\left(\frac{1}{n^{3/2}}\right)}{\left(\frac{1}{\sqrt{n(n+1)(n+2)}}\right)} = \sqrt{\lim_{n\to\infty} \frac{n(n+1)(n+2)}{n^3}} = 1$

40. converges absolutely by the Limit Comparison Test since $\lim_{n\to\infty} \frac{\left(\frac{1}{n^2}\right)}{\left(\frac{1}{n\sqrt{n^2-1}}\right)} = \sqrt{\lim_{n\to\infty} \frac{n^2(n^2-1)}{n^4}} = 1$

41. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(x+4)^{n+1}}{(n+1)3^{n+1}} \cdot \frac{n3^n}{(x+4)^n}\right| < 1 \Rightarrow \frac{|x+4|}{3} \lim_{n\to\infty} \left(\frac{n}{n+1}\right) < 1 \Rightarrow \frac{|x+4|}{3} < 1$ $\Rightarrow |x+4| < 3 \Rightarrow -3 < x+4 < 3 \Rightarrow -7 < x < -1$; at $x = -7$ we have $\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, the alternating harmonic series, which converges conditionally; at $x = -1$ we have $\sum_{n=1}^{\infty} \frac{3^n}{n3^n} = \sum_{n=1}^{\infty} \frac{1}{n}$, the divergent harmonic series

(a) the radius is 3; the interval of convergence is $-7 \le x < -1$

(b) the interval of absolute convergence is $-7 < x < -1$

(c) the series converges conditionally at $x = -7$

42. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(x-1)^{2n}}{(2n+1)!} \cdot \frac{(2n-1)!}{(x-1)^{2n-2}}\right| < 1 \Rightarrow (x-1)^2 \lim_{n\to\infty} \frac{1}{(2n)(2n+1)} = 0 < 1$, which holds for all x

(a) the radius is ∞; the series converges for all x

(b) the series converges absolutely for all x

(c) there are no values for which the series converges conditionally

43. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(3x-1)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(3x-1)^n}\right| < 1 \Rightarrow |3x-1| \lim_{n\to\infty} \frac{n^2}{(n+1)^2} < 1 \Rightarrow |3x-1| < 1$ $\Rightarrow -1 < 3x-1 < 1 \Rightarrow 0 < 3x < 2 \Rightarrow 0 < x < \frac{2}{3}$; at $x = 0$ we have $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(-1)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^{2n-1}}{n^2}$ $= -\sum_{n=1}^{\infty} \frac{1}{n^2}$, a nonzero constant multiple of a convergent p-series, which is absolutely convergent; at $x = \frac{2}{3}$ we have $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(1)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$, which converges absolutely

(a) the radius is $\frac{1}{3}$; the interval of convergence is $0 \le x \le \frac{2}{3}$

(b) the interval of absolute convergence is $0 \le x \le \frac{2}{3}$

(c) there are no values for which the series converges conditionally

44. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{n+2}{2n+3} \cdot \frac{(2x+1)^{n+1}}{2^{n+1}} \cdot \frac{2n+1}{n+1} \cdot \frac{2^n}{(2x+1)^n}\right| < 1 \Rightarrow \frac{|2x+1|}{2} \lim_{n\to\infty} \left|\frac{n+2}{2n+3} \cdot \frac{2n+1}{n+1}\right| < 1$ $\Rightarrow \frac{|2x+1|}{2}(1) < 1 \Rightarrow |2x+1| < 2 \Rightarrow -2 < 2x+1 < 2 \Rightarrow -3 < 2x < 1 \Rightarrow -\frac{3}{2} < x < \frac{1}{2}$; at $x = -\frac{3}{2}$ we have $\sum_{n=1}^{\infty} \frac{n+1}{2n+1} \cdot \frac{(-2)^n}{2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n(n+1)}{2n+1}$ which diverges by the nth-Term Test for Divergence since $\lim_{n\to\infty} \left(\frac{n+1}{2n+1}\right) = \frac{1}{2} \neq 0$; at $x = \frac{1}{2}$ we have $\sum_{n=1}^{\infty} \frac{n+1}{2n+1} \cdot \frac{2^n}{2^n} = \sum_{n=1}^{\infty} \frac{n+1}{2n+1}$, which diverges by the nth-Term Test

(a) the radius is 1; the interval of convergence is $-\frac{3}{2} < x < \frac{1}{2}$

(b) the interval of absolute convergence is $-\frac{3}{2} < x < \frac{1}{2}$

(c) there are no values for which the series converges conditionally

45. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{x^n}\right| < 1 \Rightarrow |x| \lim_{n\to\infty} \left|\left(\frac{n}{n+1}\right)^n \left(\frac{1}{n+1}\right)\right| < 1 \Rightarrow \frac{|x|}{e} \lim_{n\to\infty} \left(\frac{1}{n+1}\right) < 1$ $\Rightarrow \frac{|x|}{e} \cdot 0 < 1$, which holds for all x

(a) the radius is ∞; the series converges for all x

(b) the series converges absolutely for all x

(c) there are no values for which the series converges conditionally

46. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{x^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{x^n}\right| < 1 \Rightarrow |x| \lim_{n\to\infty} \sqrt{\frac{n}{n+1}} < 1 \Rightarrow |x| < 1$; when $x = -1$ we have $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$, which converges by the Alternating Series Test; when $x = 1$ we have $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, a divergent p-series

(a) the radius is 1; the interval of convergence is $-1 \le x < 1$

(b) the interval of absolute convergence is $-1 < x < 1$

(c) the series converges conditionally at $x = -1$

47. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{(n+2)x^{2n+1}}{3^{n+1}} \cdot \frac{3^n}{(n+1)x^{2n-1}}\right| < 1 \Rightarrow \frac{x^2}{3} \lim_{n\to\infty} \left(\frac{n+2}{n+1}\right) < 1 \Rightarrow -\sqrt{3} < x < \sqrt{3}$; the series $\sum_{n=1}^{\infty} -\frac{n+1}{\sqrt{3}}$ and $\sum_{n=1}^{\infty} \frac{n+1}{\sqrt{3}}$, obtained with $x = \pm\sqrt{3}$, both diverge

(a) the radius is $\sqrt{3}$; the interval of convergence is $-\sqrt{3} < x < \sqrt{3}$

(b) the interval of absolute convergence is $-\sqrt{3} < x < \sqrt{3}$

(c) there are no values for which the series converges conditionally

48. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|<1 \Rightarrow \lim_{n\to\infty}\left|\frac{(x-1)^{2n+3}}{2n+3}\cdot\frac{2n+1}{(x-1)^{2n+1}}\right|<1 \Rightarrow (x-1)^2\lim_{n\to\infty}\left(\frac{2n+1}{2n+3}\right)<1 \Rightarrow (x-1)^2(1)<1$
$\Rightarrow (x-1)^2<1 \Rightarrow |x-1|<1 \Rightarrow -1<x-1<1 \Rightarrow 0<x<2$; at $x=0$ we have $\sum_{n=1}^{\infty}\frac{(-1)^n(-1)^{2n+1}}{2n+1}$
$=\sum_{n=1}^{\infty}\frac{(-1)^{3n+1}}{2n+1}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n+1}$ which converges conditionally by the Alternating Series Test and the fact that $\sum_{n=1}^{\infty}\frac{1}{2n+1}$ diverges; at $x=2$ we have $\sum_{n=1}^{\infty}\frac{(-1)^n(1)^{2n+1}}{2n+1}=\sum_{n=1}^{\infty}\frac{(-1)^n}{2n+1}$, which also converges conditionally
(a) the radius is 1; the interval of convergence is $0\le x\le 2$
(b) the interval of absolute convergence is $0<x<2$
(c) the series converges conditionally at $x=0$ and $x=2$

49. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|<1 \Rightarrow \lim_{n\to\infty}\left|\frac{\operatorname{csch}(n+1)x^{n+1}}{\operatorname{csch}(n)x^n}\right|<1 \Rightarrow |x|\lim_{n\to\infty}\left|\frac{\left(\frac{2}{e^{n+1}-e^{-n-1}}\right)}{\left(\frac{2}{e^n-e^{-n}}\right)}\right|<1$
$\Rightarrow |x|\lim_{n\to\infty}\left|\frac{e^{-1}-e^{-2n-1}}{1-e^{-2n-2}}\right|<1 \Rightarrow \frac{|x|}{e}<1 \Rightarrow -e<x<e$; the series $\sum_{n=1}^{\infty}(\pm e)^n\operatorname{csch} n$, obtained with $x=\pm e$, both diverge since $\lim_{n\to\infty}(\pm e)^n\operatorname{csch} n\neq 0$
(a) the radius is e; the interval of convergence is $-e<x<e$
(b) the interval of absolute convergence is $-e<x<e$
(c) there are no values for which the series converges conditionally

50. $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|<1 \Rightarrow \lim_{n\to\infty}\left|\frac{x^{n+1}\coth(n+1)}{x^n\coth(n)}\right|<1 \Rightarrow |x|\lim_{n\to\infty}\left|\frac{1+e^{-2n-2}}{1-e^{-2n-2}}\cdot\frac{1-e^{-2n}}{1+e^{-2n}}\right|<1 \Rightarrow |x|<1$
$\Rightarrow -1<x<1$; the series $\sum_{n=1}^{\infty}(\pm 1)^n\coth n$, obtained with $x=\pm 1$, both diverge since $\lim_{n\to\infty}(\pm 1)^n\coth n\neq 0$
(a) the radius is 1; the interval of convergence is $-1<x<1$
(b) the interval of absolute convergence is $-1<x<1$
(c) there are no values for which the series converges conditionally

51. The given series has the form $1-x+x^2-x^3+\ldots+(-x)^n+\ldots=\frac{1}{1+x}$, where $x=\frac{1}{4}$; the sum is $\frac{1}{1+\left(\frac{1}{4}\right)}=\frac{4}{5}$

52. The given series has the form $x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots+(-1)^{n-1}\frac{x^n}{n}+\ldots=\ln(1+x)$, where $x=\frac{2}{3}$; the sum is $\ln\left(\frac{5}{3}\right)\approx 0.510825624$

53. The given series has the form $x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots+(-1)^n\frac{x^{2n+1}}{(2n+1)!}+\ldots=\sin x$, where $x=\pi$; the sum is $\sin\pi=0$

54. The given series has the form $1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots+(-1)^n\frac{x^{2n}}{(2n)!}+\ldots=\cos x$, where $x=\frac{\pi}{3}$; the sum is $\cos\frac{\pi}{3}=\frac{1}{2}$

55. The given series has the form $1+x+\frac{x^2}{2!}+\frac{x^2}{3!}+\ldots+\frac{x^n}{n!}+\ldots=e^x$, where $x=\ln 2$; the sum is $e^{\ln(2)}=2$

56. The given series has the form $x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots+(-1)^n\frac{x^{2n-1}}{(2n-1)}+\ldots=\tan^{-1}x$, where $x=\frac{1}{\sqrt{3}}$; the sum is $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)=\frac{\pi}{6}$

57. Consider $\frac{1}{1-2x}$ as the sum of a convergent geometric series with $a=1$ and $r=2x \Rightarrow \frac{1}{1-2x}$
$=1+(2x)+(2x)^2+(2x)^3+\ldots=\sum_{n=0}^{\infty}(2x)^n=\sum_{n=0}^{\infty}2^nx^n$ where $|2x|<1 \Rightarrow |x|<\frac{1}{2}$

58. Consider $\frac{1}{1+x^3}$ as the sum of a convergent geometric series with a = 1 and r = $-x^3$ $\Rightarrow$ $\frac{1}{1+x^3} = \frac{1}{1-(-x^3)}$
$= 1 + (-x^3) + (-x^3)^2 + (-x^3)^3 + \ldots = \sum_{n=0}^{\infty} (-1)^n x^{3n}$ where $|-x^3| < 1 \Rightarrow |x^3| < 1 \Rightarrow |x| < 1$

59. $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \Rightarrow \sin \pi x = \sum_{n=0}^{\infty} \frac{(-1)^n (\pi x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1} x^{2n+1}}{(2n+1)!}$

60. $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \Rightarrow \sin \frac{2x}{3} = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{2x}{3}\right)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{3^{2n+1}(2n+1)!}$

61. $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \Rightarrow \cos\left(x^{5/3}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(x^{5/3}\right)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{10n/3}}{(2n)!}$

62. $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \Rightarrow \cos\left(\frac{x^3}{\sqrt{5}}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x^3}{\sqrt{5}}\right)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{5^n (2n)!}$

63. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^{(\pi x/2)} = \sum_{n=0}^{\infty} \frac{\left(\frac{\pi x}{2}\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{\pi^n x^n}{2^n n!}$

64. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$

65. $f(x) = \sqrt{3+x^2} = (3+x^2)^{1/2} \Rightarrow f'(x) = x(3+x^2)^{-1/2} \Rightarrow f''(x) = -x^2(3+x^2)^{-3/2} + (3+x^2)^{-1/2}$
$\Rightarrow f'''(x) = 3x^3(3+x^2)^{-5/2} - 3x(3+x^2)^{-3/2}$; $f(-1) = 2$, $f'(-1) = -\frac{1}{2}$, $f''(-1) = -\frac{1}{8} + \frac{1}{2} = \frac{3}{8}$,
$f'''(-1) = -\frac{3}{32} + \frac{3}{8} = \frac{9}{32} \Rightarrow \sqrt{3+x^2} = 2 - \frac{(x+1)}{2\cdot 1!} + \frac{3(x+1)^2}{2^3\cdot 2!} + \frac{9(x+1)^3}{2^5\cdot 3!} + \ldots$

66. $f(x) = \frac{1}{1-x} = (1-x)^{-1} \Rightarrow f'(x) = (1-x)^{-2} \Rightarrow f''(x) = 2(1-x)^{-3} \Rightarrow f'''(x) = 6(1-x)^{-4}$; $f(2) = -1$, $f'(2) = 1$,
$f''(2) = -2$, $f'''(2) = 6 \Rightarrow \frac{1}{1-x} = -1 + (x-2) - (x-2)^2 + (x-2)^3 - \ldots$

67. $f(x) = \frac{1}{x+1} = (x+1)^{-1} \Rightarrow f'(x) = -(x+1)^{-2} \Rightarrow f''(x) = 2(x+1)^{-3} \Rightarrow f'''(x) = -6(x+1)^{-4}$; $f(3) = \frac{1}{4}$,
$f'(3) = -\frac{1}{4^2}$, $f''(3) = \frac{2}{4^3}$, $f'''(2) = \frac{-6}{4^4} \Rightarrow \frac{1}{x+1} = \frac{1}{4} - \frac{1}{4^2}(x-3) + \frac{1}{4^3}(x-3)^2 - \frac{1}{4^4}(x-3)^3 + \ldots$

68. $f(x) = \frac{1}{x} = x^{-1} \Rightarrow f'(x) = -x^{-2} \Rightarrow f''(x) = 2x^{-3} \Rightarrow f'''(x) = -6x^{-4}$; $f(a) = \frac{1}{a}$, $f'(a) = -\frac{1}{a^2}$, $f''(a) = \frac{2}{a^3}$,
$f'''(a) = \frac{-6}{a^4} \Rightarrow \frac{1}{x} = \frac{1}{a} - \frac{1}{a^2}(x-a) + \frac{1}{a^3}(x-a)^2 - \frac{1}{a^4}(x-a)^3 + \ldots$

69. $\int_0^{1/2} \exp(-x^3)\,dx = \int_0^{1/2} \left(1 - x^3 + \frac{x^6}{2!} - \frac{x^9}{3!} + \frac{x^{12}}{4!} + \ldots\right) dx = \left[x - \frac{x^4}{4} + \frac{x^7}{7\cdot 2!} - \frac{x^{10}}{10\cdot 3!} + \frac{x^{13}}{13\cdot 4!} - \ldots\right]_0^{1/2}$
$\approx \frac{1}{2} - \frac{1}{2^4\cdot 4} + \frac{1}{2^7\cdot 7\cdot 2!} - \frac{1}{2^{10}\cdot 10\cdot 3!} + \frac{1}{2^{13}\cdot 13\cdot 4!} - \frac{1}{2^{16}\cdot 16\cdot 5!} \approx 0.484917143$

70. $\int_0^1 x\sin(x^3)\,dx = \int_0^1 x\left(x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \frac{x^{21}}{7!} + \frac{x^{27}}{9!} + \ldots\right) dx = \int_0^1 \left(x^4 - \frac{x^{10}}{3!} + \frac{x^{16}}{5!} - \frac{x^{22}}{7!} + \frac{x^{28}}{9!} - \ldots\right) dx$
$= \left[\frac{x^5}{5} - \frac{x^{11}}{11\cdot 3!} + \frac{x^{17}}{17\cdot 5!} - \frac{x^{23}}{23\cdot 7!} + \frac{x^{29}}{29\cdot 9!} - \ldots\right]_0^1 \approx 0.185330149$

71. $\int_1^{1/2} \frac{\tan^{-1} x}{x}\,dx = \int_1^{1/2} \left(1 - \frac{x^2}{3} + \frac{x^4}{5} - \frac{x^6}{7} + \frac{x^8}{9} - \frac{x^{10}}{11} + \ldots\right) dx = \left[x - \frac{x^3}{9} + \frac{x^5}{25} - \frac{x^7}{49} + \frac{x^9}{81} - \frac{x^{11}}{121} + \ldots\right]_0^{1/2}$
$\approx \frac{1}{2} - \frac{1}{9\cdot 2^3} + \frac{1}{5^2\cdot 2^5} - \frac{1}{7^2\cdot 2^7} + \frac{1}{9^2\cdot 2^9} - \frac{1}{11^2\cdot 2^{11}} + \frac{1}{13^2\cdot 2^{13}} - \frac{1}{15^2\cdot 2^{15}} + \frac{1}{17^2\cdot 2^{17}} - \frac{1}{19^2\cdot 2^{19}} + \frac{1}{21^2\cdot 2^{21}} \approx 0.4872223583$

72. $\int_0^{1/64} \frac{\tan^{-1} x}{\sqrt{x}}\,dx = \int_0^{1/64} \frac{1}{\sqrt{x}}\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\right) dx = \int_0^{1/64}\left(x^{1/2} - \frac{1}{3}x^{5/2} + \frac{1}{5}x^{9/2} - \frac{1}{7}x^{13/2} + \ldots\right) dx$
$= \left[\frac{2}{3}x^{3/2} - \frac{2}{21}x^{7/2} + \frac{2}{55}x^{11/2} - \frac{2}{105}x^{15/2} + \ldots\right]_0^{1/64} = \left(\frac{2}{3\cdot 8^3} - \frac{2}{21\cdot 8^7} + \frac{2}{55\cdot 8^{11}} - \frac{2}{105\cdot 8^{15}} + \ldots\right) \approx 0.0013020379$

73. $\lim_{x\to 0} \frac{7\sin x}{e^{2x}-1} = \lim_{x\to 0} \frac{7\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)}{\left(2x + \frac{2^2x^2}{2!} + \frac{2^3x^3}{3!} + \ldots\right)} = \lim_{x\to 0} \frac{7\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \ldots\right)}{\left(2 + \frac{2^2x}{2!} + \frac{2^3x^2}{3!} + \ldots\right)} = \frac{7}{2}$

74. $\lim_{\theta\to 0} \frac{e^\theta - e^{-\theta} - 2\theta}{\theta - \sin\theta} = \lim_{\theta\to 0} \frac{\left(1 + \theta + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \ldots\right) - \left(1 - \theta + \frac{\theta^2}{2!} - \frac{\theta^3}{3!} + \ldots\right) - 2\theta}{\theta - \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots\right)} = \lim_{\theta\to 0} \frac{2\left(\frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \ldots\right)}{\left(\frac{\theta^3}{3!} - \frac{\theta^5}{5!} + \ldots\right)}$
$= \lim_{\theta\to 0} \frac{2\left(\frac{1}{3!} + \frac{\theta^2}{5!} + \ldots\right)}{\left(\frac{1}{3!} - \frac{\theta^2}{5!} + \ldots\right)} = 2$

75. $\lim_{t\to 0} \left(\frac{1}{2 - 2\cos t} - \frac{1}{t^2}\right) = \lim_{t\to 0} \frac{t^2 - 2 + 2\cos t}{2t^2(1 - \cos t)} = \lim_{t\to 0} \frac{t^2 - 2 + 2\left(1 - \frac{t^2}{2} + \frac{t^4}{4!} - \ldots\right)}{2t^2\left(1 - 1 + \frac{t^2}{2} - \frac{t^4}{4!} + \ldots\right)} = \lim_{t\to 0} \frac{2\left(\frac{t^4}{4!} - \frac{t^6}{6!} + \ldots\right)}{\left(t^4 - \frac{2t^6}{4!} + \ldots\right)}$
$= \lim_{t\to 0} \frac{2\left(\frac{1}{4!} - \frac{t^2}{6!} + \ldots\right)}{\left(1 - \frac{2t^2}{4!} + \ldots\right)} = \frac{1}{12}$

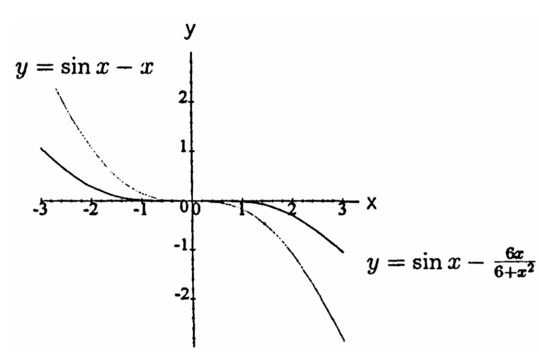
76. $\lim_{h\to 0} \frac{\left(\frac{\sin h}{h}\right) - \cos h}{h^2} = \lim_{h\to 0} \frac{\left(1 - \frac{h^2}{3!} + \frac{h^4}{5!} - \ldots\right) - \left(1 - \frac{h^2}{2!} + \frac{h^4}{4!} - \ldots\right)}{h^2}$
$= \lim_{h\to 0} \frac{\left(\frac{h^2}{2!} - \frac{h^2}{3!} + \frac{h^4}{5!} - \frac{h^4}{4!} + \frac{h^6}{6!} - \frac{h^6}{7!} + \ldots\right)}{h^2} = \lim_{h\to 0} \left(\frac{1}{2!} - \frac{1}{3!} + \frac{h^2}{5!} - \frac{h^2}{4!} + \frac{h^4}{6!} - \frac{h^4}{7!} + \ldots\right) = \frac{1}{3}$

77. $\lim_{z\to 0} \frac{1 - \cos^2 z}{\ln(1 - z) + \sin z} = \lim_{z\to 0} \frac{1 - \left(1 - z^2 + \frac{z^4}{3} - \ldots\right)}{\left(-z - \frac{z^2}{2} - \frac{z^3}{3} - \ldots\right) + \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots\right)} = \lim_{z\to 0} \frac{\left(z^2 - \frac{z^4}{3} + \ldots\right)}{\left(-\frac{z^2}{2} - \frac{2z^3}{3} - \frac{z^4}{4} - \ldots\right)}$
$= \lim_{z\to 0} \frac{\left(1 - \frac{z^2}{3} + \ldots\right)}{\left(-\frac{1}{2} - \frac{2z}{3} - \frac{z^2}{4} - \ldots\right)} = -2$

78. $\lim_{y\to 0} \frac{y^2}{\cos y - \cosh y} = \lim_{y\to 0} \frac{y^2}{\left(1 - \frac{y^2}{2} + \frac{y^4}{4!} - \frac{y^6}{6!} + \ldots\right) - \left(1 + \frac{y^2}{2!} + \frac{y^4}{4!} + \frac{y^6}{6!} + \ldots\right)} = \lim_{y\to 0} \frac{y^2}{\left(-\frac{2y^2}{2} - \frac{2y^6}{6!} - \ldots\right)}$
$= \lim_{y\to 0} \frac{1}{\left(-1 - \frac{2y^4}{6!} - \ldots\right)} = -1$

79. $\lim_{x\to 0} \left(\frac{\sin 3x}{x^3} + \frac{r}{x^2} + s\right) = \lim_{x\to 0} \left[\frac{\left(3x - \frac{(3x)^3}{6} + \frac{(3x)^5}{120} - \ldots\right)}{x^3} + \frac{r}{x^2} + s\right] = \lim_{x\to 0} \left(\frac{3}{x^2} - \frac{9}{2} + \frac{81x^2}{40} + \ldots + \frac{r}{x^2} + s\right) = 0$
$\Rightarrow \frac{r}{x^2} + \frac{3}{x^2} = 0$ and $s - \frac{9}{2} = 0 \Rightarrow r = -3$ and $s = \frac{9}{2}$

80. The approximation $\sin x \approx \frac{6x}{6 + x^2}$ is better than $\sin x \approx x$.



81. $\lim_{n\to\infty} \left|\frac{2\cdot5\cdot8\cdots(3n-1)(3n+2)x^{n+1}}{2\cdot4\cdot6\cdots(2n)(2n+2)} \cdot \frac{2\cdot4\cdot6\cdots(2n)}{2\cdot5\cdot8\cdots(3n-1)x^n}\right| < 1 \Rightarrow |x| \lim_{n\to\infty} \left|\frac{3n+2}{2n+2}\right| < 1 \Rightarrow |x| < \frac{2}{3}$
$\Rightarrow$ the radius of convergence is $\frac{2}{3}$

82. $\lim_{n\to\infty} \left|\frac{3\cdot5\cdot7\cdots(2n+1)(2n+3)(x-1)^{n+1}}{4\cdot9\cdot14\cdots(5n-1)(5n+4)} \cdot \frac{4\cdot9\cdot14\cdots(5n-1)}{3\cdot5\cdot7\cdots(2n+1)x^n}\right| < 1 \Rightarrow |x| \lim_{n\to\infty} \left|\frac{2n+3}{5n+4}\right| < 1 \Rightarrow |x| < \frac{5}{2}$
$\Rightarrow$ the radius of convergence is $\frac{5}{2}$

83. $\sum_{k=2}^{n} \ln\left(1-\frac{1}{k^2}\right) = \sum_{k=2}^{n} \left[\ln\left(1+\frac{1}{k}\right) + \ln\left(1-\frac{1}{k}\right)\right] = \sum_{k=2}^{n} [\ln(k+1) - \ln k + \ln(k-1) - \ln k]$
$= [\ln 3 - \ln 2 + \ln 1 - \ln 2] + [\ln 4 - \ln 3 + \ln 2 - \ln 3] + [\ln 5 - \ln 4 + \ln 3 - \ln 4] + [\ln 6 - \ln 5 + \ln 4 - \ln 5]$
$+ \ldots + [\ln(n+1) - \ln n + \ln(n-1) - \ln n] = [\ln 1 - \ln 2] + [\ln(n+1) - \ln n]$ after cancellation
$\Rightarrow \sum_{k=2}^{n} \ln\left(1-\frac{1}{k^2}\right) = \ln\left(\frac{n+1}{2n}\right) \Rightarrow \sum_{k=2}^{\infty} \ln\left(1-\frac{1}{k^2}\right) = \lim_{n\to\infty} \ln\left(\frac{n+1}{2n}\right) = \ln\frac{1}{2}$ is the sum

84. $\sum_{k=2}^{n} \frac{1}{k^2-1} = \frac{1}{2}\sum_{k=2}^{n}\left(\frac{1}{k-1} - \frac{1}{k+1}\right) = \frac{1}{2}\left[\left(\frac{1}{1}-\frac{1}{3}\right) + \left(\frac{1}{2}-\frac{1}{4}\right) + \left(\frac{1}{3}-\frac{1}{5}\right) + \left(\frac{1}{4}-\frac{1}{6}\right) + \ldots + \left(\frac{1}{n-2}-\frac{1}{n}\right)\right.$
$\left.+\left(\frac{1}{n-1}-\frac{1}{n+1}\right)\right] = \frac{1}{2}\left(\frac{1}{1}+\frac{1}{2}-\frac{1}{n}-\frac{1}{n+1}\right) = \frac{1}{2}\left(\frac{3}{2}-\frac{1}{n}-\frac{1}{n+1}\right) = \frac{1}{2}\left[\frac{3n(n+1)-2(n+1)-2n}{2n(n+1)}\right] = \frac{3n^2-n-2}{4n(n+1)}$
$\Rightarrow \sum_{k=2}^{\infty} \frac{1}{k^2-1} = \lim_{n\to\infty} \frac{1}{2}\left(\frac{3}{2}-\frac{1}{n}-\frac{1}{n+1}\right) = \frac{3}{4}$

85. (a) $\lim_{n\to\infty} \left|\frac{1\cdot4\cdot7\cdots(3n-2)(3n+1)x^{3n+3}}{(3n+3)!} \cdot \frac{(3n)!}{1\cdot4\cdot7\cdots(3n-2)x^{3n}}\right| < 1 \Rightarrow |x^3| \lim_{n\to\infty} \frac{(3n+1)}{(3n+1)(3n+2)(3n+3)}$
$= |x^3| \cdot 0 < 1 \Rightarrow$ the radius of convergence is ∞

(b) $y = 1 + \sum_{n=1}^{\infty} \frac{1\cdot4\cdot7\cdots(3n-2)}{(3n)!} x^{3n} \Rightarrow \frac{dy}{dx} = \sum_{n=1}^{\infty} \frac{1\cdot4\cdot7\cdots(3n-2)}{(3n-1)!} x^{3n-1}$
$\Rightarrow \frac{d^2y}{dx^2} = \sum_{n=1}^{\infty} \frac{1\cdot4\cdot7\cdots(3n-2)}{(3n-2)!} x^{3n-2} = x + \sum_{n=2}^{\infty} \frac{1\cdot4\cdot7\cdots(3n-5)}{(3n-3)!} x^{3n-2}$
$= x\left(1 + \sum_{n=1}^{\infty} \frac{1\cdot4\cdot7\cdots(3n-2)}{(3n)!} x^{3n}\right) = xy + 0 \Rightarrow a = 1$ and $b = 0$

86. (a) $\frac{x^2}{1+x} = \frac{x^2}{1-(-x)} = x^2 + x^2(-x) + x^2(-x)^2 + x^2(-x)^3 + \ldots = x^2 - x^3 + x^4 - x^5 + \ldots = \sum_{n=2}^{\infty} (-1)^n x^n$ which converges absolutely for $|x| < 1$

(b) $x = 1 \Rightarrow \sum_{n=2}^{\infty} (-1)^n x^n = \sum_{n=2}^{\infty} (-1)^n$ which diverges

87. Yes, the series $\sum_{n=1}^{\infty} a_n b_n$ converges as we now show. Since $\sum_{n=1}^{\infty} a_n$ converges it follows that $a_n \to 0 \Rightarrow a_n < 1$ for $n >$ some index $N \Rightarrow a_n b_n < b_n$ for $n > N \Rightarrow \sum_{n=1}^{\infty} a_n b_n$ converges by the Direct Comparison Test with $\sum_{n=1}^{\infty} b_n$

88. No, the series $\sum_{n=1}^{\infty} a_n b_n$ might diverge (as it would if a_n and b_n both equaled n) or it might converge (as it would if a_n and b_n both equaled $\frac{1}{n}$).

89. $\sum_{n=1}^{\infty} (x_{n+1} - x_n) = \lim_{n\to\infty} \sum_{k=1}^{\infty} (x_{k+1} - x_k) = \lim_{n\to\infty} (x_{n+1} - x_1) = \lim_{n\to\infty} (x_{n+1}) - x_1 \Rightarrow$ both the series and sequence must either converge or diverge.

90. It converges by the Limit Comparison Test since $\lim_{n\to\infty} \frac{\left(\frac{a_n}{1+a_n}\right)}{a_n} = \lim_{n\to\infty} \frac{1}{1+a_n} = 1$ because $\sum_{n=1}^{\infty} a_n$ converges and so $a_n \to 0$.

91. $\sum_{n=1}^{\infty} \frac{a_n}{n} = a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \frac{a_4}{4} + \ldots \geq a_1 + \left(\frac{1}{2}\right) a_2 + \left(\frac{1}{3} + \frac{1}{4}\right) a_4 + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) a_8$
$+ \left(\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \ldots + \frac{1}{16}\right) a_{16} + \ldots \geq \frac{1}{2}(a_2 + a_4 + a_8 + a_{16} + \ldots)$ which is a divergent series

92. $a_n = \frac{1}{\ln n}$ for $n \geq 2 \Rightarrow a_2 \geq a_3 \geq a_4 \geq \ldots$, and $\frac{1}{\ln 2} + \frac{1}{\ln 4} + \frac{1}{\ln 8} + \ldots = \frac{1}{\ln 2} + \frac{1}{2 \ln 2} + \frac{1}{3 \ln 2} + \ldots$
$= \frac{1}{\ln 2}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots\right)$ which diverges so that $1 + \sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges by the Integral Test.

CHAPTER 10 ADDITIONAL AND ADVANCED EXERCISES

1. converges since $\frac{1}{(3n-2)^{(2n+1)/2}} < \frac{1}{(3n-2)^{3/2}}$ and $\sum_{n=1}^{\infty} \frac{1}{(3n-2)^{3/2}}$ converges by the Limit Comparison Test:
$\lim_{n\to\infty} \frac{\left(\frac{1}{n^{3/2}}\right)}{\left(\frac{1}{(3n-2)^{3/2}}\right)} = \lim_{n\to\infty} \left(\frac{3n-2}{n}\right)^{3/2} = 3^{3/2}$

2. converges by the Integral Test: $\int_1^{\infty} (\tan^{-1} x)^2 \frac{dx}{x^2+1} = \lim_{b\to\infty} \left[\frac{(\tan^{-1} x)^3}{3}\right]_1^b = \lim_{b\to\infty} \left[\frac{(\tan^{-1} b)^3}{3} - \frac{\pi^3}{192}\right]$
$= \left(\frac{\pi^3}{24} - \frac{\pi^3}{192}\right) = \frac{7\pi^3}{192}$

3. diverges by the nth-Term Test since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^n \tanh n = \lim_{b\to\infty} (-1)^n \left(\frac{1-e^{-2n}}{1+e^{-2n}}\right) = \lim_{n\to\infty} (-1)^n$ does not exist

4. converges by the Direct Comparison Test: $n! < n^n \Rightarrow \ln(n!) < n \ln(n) \Rightarrow \frac{\ln(n!)}{\ln(n)} < n$
$\Rightarrow \log_n(n!) < n \Rightarrow \frac{\log_n(n!)}{n^3} < \frac{1}{n^2}$, which is the nth-term of a convergent p-series

5. converges by the Direct Comparison Test: $a_1 = 1 = \frac{12}{(1)(3)(2)^2}$, $a_2 = \frac{1 \cdot 2}{3 \cdot 4} = \frac{12}{(2)(4)(3)^2}$, $a_3 = \left(\frac{2 \cdot 3}{4 \cdot 5}\right)\left(\frac{1 \cdot 2}{3 \cdot 4}\right)$
$= \frac{12}{(3)(5)(4)^2}$, $a_4 = \left(\frac{3 \cdot 4}{5 \cdot 6}\right)\left(\frac{2 \cdot 3}{4 \cdot 5}\right)\left(\frac{1 \cdot 2}{3 \cdot 4}\right) = \frac{12}{(4)(6)(5)^2}, \ldots \Rightarrow 1 + \sum_{n=1}^{\infty} \frac{12}{(n+1)(n+3)(n+2)^2}$ represents the given series and $\frac{12}{(n+1)(n+3)(n+2)^2} < \frac{12}{n^4}$, which is the nth-term of a convergent p-series

6. converges by the Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n}{(n-1)(n+1)} = 0 < 1$

7. diverges by the nth-Term Test since if $a_n \to L$ as $n \to \infty$, then $L = \frac{1}{1+L} \Rightarrow L^2 + L - 1 = 0 \Rightarrow L = \frac{-1 \pm \sqrt{5}}{2} \neq 0$

8. Split the given series into $\sum_{n=1}^{\infty} \frac{1}{3^{2n+1}}$ and $\sum_{n=1}^{\infty} \frac{2n}{3^{2n}}$; the first subseries is a convergent geometric series and the second converges by the Root Test: $\lim_{n\to\infty} \sqrt[n]{\frac{2n}{3^{2n}}} = \lim_{n\to\infty} \frac{\sqrt[n]{2}\sqrt[n]{n}}{9} = \frac{1 \cdot 1}{9} = \frac{1}{9} < 1$

9. $f(x) = \cos x$ with $a = \frac{\pi}{3} \Rightarrow f\left(\frac{\pi}{3}\right) = 0.5$, $f'\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$, $f''\left(\frac{\pi}{3}\right) = -0.5$, $f'''\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$, $f^{(4)}\left(\frac{\pi}{3}\right) = 0.5$;
$\cos x = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{4}\left(x - \frac{\pi}{3}\right)^2 + \frac{\sqrt{3}}{12}\left(x - \frac{\pi}{3}\right)^3 + \ldots$

10. $f(x) = \sin x$ with $a = 2\pi \Rightarrow f(2\pi) = 0, f'(2\pi) = 1, f''(2\pi) = 0, f'''(2\pi) = -1, f^{(4)}(2\pi) = 0, f^{(5)}(2\pi) = 1,$
$f^{(6)}(2\pi) = 0, f^{(7)}(2\pi) = -1$; $\sin x = (x - 2\pi) - \frac{(x-2\pi)^3}{3!} + \frac{(x-2\pi)^5}{5!} - \frac{(x-2\pi)^7}{7!} + \ldots$

11. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ with $a = 0$

12. $f(x) = \ln x$ with $a = 1 \Rightarrow f(1) = 0, f'(1) = 1, f''(1) = -1, f'''(1) = 2, f^{(4)}(1) = -6$;
$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots$

13. $f(x) = \cos x$ with $a = 22\pi \Rightarrow f(22\pi) = 1, f'(22\pi) = 0, f''(22\pi) = -1, f'''(22\pi) = 0, f^{(4)}(22\pi) = 1,$
$f^{(5)}(22\pi) = 0, f^{(6)}(22\pi) = -1$; $\cos x = 1 - \frac{1}{2}(x - 22\pi)^2 + \frac{1}{4!}(x - 22\pi)^4 - \frac{1}{6!}(x - 22\pi)^6 + \ldots$

14. $f(x) = \tan^{-1} x$ with $a = 1 \Rightarrow f(1) = \frac{\pi}{4}, f'(1) = \frac{1}{2}, f''(1) = -\frac{1}{2}, f'''(1) = \frac{1}{2}$;
$\tan^{-1} x = \frac{\pi}{4} + \frac{(x-1)}{2} - \frac{(x-1)^2}{4} + \frac{(x-1)^3}{12} + \ldots$

15. Yes, the sequence converges: $c_n = (a^n + b^n)^{1/n} \Rightarrow c_n = b\left(\left(\frac{a}{b}\right)^n + 1\right)^{1/n} \Rightarrow \lim_{n\to\infty} c_n = \ln b + \lim_{n\to\infty} \frac{\ln\left(\left(\frac{a}{b}\right)^n + 1\right)}{n}$
$= \ln b + \lim_{n\to\infty} \frac{\left(\frac{a}{b}\right)^n \ln\left(\frac{a}{b}\right)}{\left(\frac{a}{b}\right)^n + 1} = \ln b + \frac{0 \cdot \ln\left(\frac{a}{b}\right)}{0+1} = \ln b$ since $0 < a < b$. Thus, $\lim_{n\to\infty} c_n = e^{\ln b} = b$.

16. $1 + \frac{2}{10} + \frac{3}{10^2} + \frac{7}{10^3} + \frac{2}{10^4} + \frac{3}{10^5} + \frac{7}{10^6} + \ldots = 1 + \sum_{n=1}^{\infty} \frac{2}{10^{3n-2}} + \sum_{n=1}^{\infty} \frac{3}{10^{3n-1}} + \sum_{n=1}^{\infty} \frac{7}{10^{3n}}$
$= 1 + \sum_{n=0}^{\infty} \frac{2}{10^{3n+1}} + \sum_{n=0}^{\infty} \frac{3}{10^{3n+2}} + \sum_{n=0}^{\infty} \frac{7}{10^{3n+3}} = 1 + \frac{\left(\frac{2}{10}\right)}{1 - \left(\frac{1}{10}\right)^3} + \frac{\left(\frac{3}{10^2}\right)}{1 - \left(\frac{1}{10}\right)^3} + \frac{\left(\frac{7}{10^3}\right)}{1 - \left(\frac{1}{10}\right)^3}$
$= 1 + \frac{200}{999} + \frac{30}{999} + \frac{7}{999} = \frac{999+237}{999} = \frac{412}{333}$

17. $s_n = \sum_{k=0}^{n-1} \int_k^{k+1} \frac{dx}{1+x^2} \Rightarrow s_n = \int_0^1 \frac{dx}{1+x^2} + \int_1^2 \frac{dx}{1+x^2} + \ldots + \int_{n-1}^{n} \frac{dx}{1+x^2} \Rightarrow s_n = \int_0^n \frac{dx}{1+x^2}$
$\Rightarrow \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(\tan^{-1} n - \tan^{-1} 0\right) = \frac{\pi}{2}$

18. $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)x^{n+1}}{(n+2)(2x+1)^{n+1}} \cdot \frac{(n+1)(2x+1)^n}{nx^n}\right| = \lim_{n\to\infty} \left|\frac{x}{2x+1} \cdot \frac{(n+1)^2}{n(n+2)}\right| = \left|\frac{x}{2x+1}\right| < 1$
$\Rightarrow |x| < |2x+1|$; if $x > 0$, $|x| < |2x+1| \Rightarrow x < 2x+1 \Rightarrow x > -1$; if $-\frac{1}{2} < x < 0$, $|x| < |2x+1|$
$\Rightarrow -x < 2x+1 \Rightarrow 3x > -1 \Rightarrow x > -\frac{1}{3}$; if $x < -\frac{1}{2}$, $|x| < |2x+1| \Rightarrow -x < -2x-1 \Rightarrow x < -1$. Therefore, the series converges absolutely for $x < -1$ and $x > -\frac{1}{3}$.

19. (a) No, the limit does not appear to depend on the value of the constant a
(b) Yes, the limit depends on the value of b
(c) $s = \left(1 - \frac{\cos\left(\frac{a}{n}\right)}{n}\right)^n \Rightarrow \ln s = \frac{\ln\left(1 - \frac{\cos\left(\frac{a}{n}\right)}{n}\right)}{\left(\frac{1}{n}\right)} \Rightarrow \lim_{n\to\infty} \ln s = \frac{\left(\frac{1}{1 - \frac{\cos\left(\frac{a}{n}\right)}{n}}\right)\left(\frac{-\frac{a}{n}\sin\left(\frac{a}{n}\right) + \cos\left(\frac{a}{n}\right)}{n^2}\right)}{\left(-\frac{1}{n^2}\right)}$
$= \lim_{n\to\infty} \frac{\frac{a}{n}\sin\left(\frac{a}{n}\right) - \cos\left(\frac{a}{n}\right)}{1 - \frac{\cos\left(\frac{a}{n}\right)}{n}} = \frac{0-1}{1-0} = -1 \Rightarrow \lim_{n\to\infty} s = e^{-1} \approx 0.3678794412$; similarly,
$\lim_{n\to\infty} \left(1 - \frac{\cos\left(\frac{a}{n}\right)}{bn}\right)^n = e^{-1/b}$

20. $\sum_{n=1}^{\infty} a_n$ converges $\Rightarrow \lim_{n\to\infty} a_n = 0$; $\lim_{n\to\infty} \left[\left(\frac{1+\sin a_n}{2}\right)^n\right]^{1/n} = \lim_{n\to\infty} \left(\frac{1+\sin a_n}{2}\right) = \frac{1+\sin\left(\lim_{n\to\infty} a_n\right)}{2} = \frac{1+\sin 0}{2}$
$= \frac{1}{2} \Rightarrow$ the series converges by the nth-Root Test

21. $\lim_{n \to \infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \lim_{n \to \infty} \left|\frac{b^{n+1}x^{n+1}}{\ln(n+1)} \cdot \frac{\ln n}{b^n x^n}\right| < 1 \Rightarrow |bx| < 1 \Rightarrow -\frac{1}{b} < x < \frac{1}{b} = 5 \Rightarrow b = \pm\frac{1}{5}$

22. A polynomial has only a finite number of nonzero terms in its Taylor series, but the functions sin x, ln x and e^x have infinitely many nonzero terms in their Taylor expansions.

23. $\lim_{x \to 0} \frac{\sin(ax) - \sin x - x}{x^3} = \lim_{x \to 0} \frac{\left(ax - \frac{a^3x^3}{3!} + \ldots\right) - \left(x - \frac{x^3}{3!} + \ldots\right) - x}{x^3} = \lim_{x \to 0} \left[\frac{a-2}{x^2} - \frac{a^3}{3!} + \frac{1}{3!} - \left(\frac{a^5}{5!} - \frac{1}{5!}\right)x^2 + \ldots\right]$ is finite if $a - 2 = 0 \Rightarrow a = 2$; $\lim_{x \to 0} \frac{\sin 2x - \sin x - x}{x^3} = -\frac{2^3}{3!} + \frac{1}{3!} = -\frac{7}{6}$

24. $\lim_{x \to 0} \frac{\cos ax - b}{2x^2} = -1 \Rightarrow \lim_{x \to 0} \frac{\left(1 - \frac{a^2x^2}{2} + \frac{a^4x^4}{4!} - \ldots\right) - b}{2x^2} = -1 \Rightarrow \lim_{x \to 0} \left(\frac{1-b}{2x^2} - \frac{a^2}{4} + \frac{a^2x^2}{48} - \ldots\right) = -1$ $\Rightarrow b = 1$ and $a = \pm 2$

25. (a) $\frac{u_n}{u_{n+1}} = \frac{(n+1)^2}{n^2} = 1 + \frac{2}{n} + \frac{1}{n^2} \Rightarrow C = 2 > 1$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

(b) $\frac{u_n}{u_{n+1}} = \frac{n+1}{n} = 1 + \frac{1}{n} + \frac{0}{n^2} \Rightarrow C = 1 \le 1$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

26. $\frac{u_n}{u_{n+1}} = \frac{2n(2n+1)}{(2n-1)^2} = \frac{4n^2+2n}{4n^2-4n+1} = 1 + \frac{\left(\frac{6}{4}\right)}{n} + \frac{5}{4n^2-4n+1} = 1 + \frac{\left(\frac{3}{2}\right)}{n} + \frac{\left[\frac{5n^2}{(4n^2-4n+1)}\right]}{n^2}$ after long division $\Rightarrow C = \frac{3}{2} > 1$ and $|f(n)| = \frac{5n^2}{4n^2-4n+1} = \frac{5}{\left(4 - \frac{4}{n} + \frac{1}{n^2}\right)} \le 5 \Rightarrow \sum_{n=1}^{\infty} u_n$ converges by Raabe's Test

27. (a) $\sum_{n=1}^{\infty} a_n = L \Rightarrow a_n^2 \le a_n \sum_{n=1}^{\infty} a_n = a_n L \Rightarrow \sum_{n=1}^{\infty} a_n^2$ converges by the Direct Comparison Test

(b) converges by the Limit Comparison Test: $\lim_{n \to \infty} \frac{\left(\frac{a_n}{1-a_n}\right)}{a_n} = \lim_{n \to \infty} \frac{1}{1-a_n} = 1$ since $\sum_{n=1}^{\infty} a_n$ converges and therefore $\lim_{x \to \infty} a_n = 0$

28. If $0 < a_n < 1$ then $|\ln(1-a_n)| = -\ln(1-a_n) = a_n + \frac{a_n^2}{2} + \frac{a_n^3}{3} + \ldots < a_n + a_n^2 + a_n^3 + \ldots = \frac{a_n}{1-a_n}$, a positive term of a convergent series, by the Limit Comparison Test and Exercise 27b

29. $(1-x)^{-1} = 1 + \sum_{n=1}^{\infty} x^n$ where $|x| < 1 \Rightarrow \frac{1}{(1-x)^2} = \frac{d}{dx}(1-x)^{-1} = \sum_{n=1}^{\infty} nx^{n-1}$ and when $x = \frac{1}{2}$ we have $4 = 1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \ldots + n\left(\frac{1}{2}\right)^{n-1} + \ldots$

30. (a) $\sum_{n=1}^{\infty} x^{n+1} = \frac{x^2}{1-x} \Rightarrow \sum_{n=1}^{\infty} (n+1)x^n = \frac{2x-x^2}{(1-x)^2} \Rightarrow \sum_{n=1}^{\infty} n(n+1)x^{n-1} = \frac{2}{(1-x)^3} \Rightarrow \sum_{n=1}^{\infty} n(n+1)x^n = \frac{2x}{(1-x)^3}$ $\Rightarrow \sum_{n=1}^{\infty} \frac{n(n+1)}{x^n} = \frac{\frac{2}{x}}{\left(1-\frac{1}{x}\right)^3} = \frac{2x^2}{(x-1)^3}$, $|x| > 1$

(b) $x = \sum_{n=1}^{\infty} \frac{n(n+1)}{x^n} \Rightarrow x = \frac{2x^2}{(x-1)^3} \Rightarrow x^3 - 3x^2 + x - 1 = 0 \Rightarrow x = 1 + \left(1 + \frac{\sqrt{57}}{9}\right)^{1/3} + \left(1 - \frac{\sqrt{57}}{9}\right)^{1/3}$ ≈ 2.769292, using a CAS or calculator

31. (a) $\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}(1 + x + x^2 + x^3 + \ldots) = 1 + 2x + 3x^2 + 4x^3 + \ldots = \sum_{n=1}^{\infty} nx^{n-1}$

(b) from part (a) we have $\sum_{n=1}^{\infty} n\left(\frac{5}{6}\right)^{n-1}\left(\frac{1}{6}\right) = \left(\frac{1}{6}\right)\left[\frac{1}{1-\left(\frac{5}{6}\right)}\right]^2 = 6$

(c) from part (a) we have $\sum_{n=1}^{\infty} np^{n-1}q = \frac{q}{(1-p)^2} = \frac{q}{q^2} = \frac{1}{q}$

32. (a) $\sum_{k=1}^{\infty} p_k = \sum_{k=1}^{\infty} 2^{-k} = \frac{\left(\frac{1}{2}\right)}{1-\left(\frac{1}{2}\right)} = 1$ and $E(x) = \sum_{k=1}^{\infty} kp_k = \sum_{k=1}^{\infty} k2^{-k} = \frac{1}{2}\sum_{k=1}^{\infty} k2^{1-k} = \left(\frac{1}{2}\right)\frac{1}{\left[1-\left(\frac{1}{2}\right)\right]^2} = 2$ by Exercise 31(a)

(b) $\sum_{k=1}^{\infty} p_k = \sum_{k=1}^{\infty} \frac{5^{k-1}}{6^k} = \frac{1}{5}\sum_{k=1}^{\infty}\left(\frac{5}{6}\right)^k = \left(\frac{1}{5}\right)\left[\frac{\left(\frac{5}{6}\right)}{1-\left(\frac{5}{6}\right)}\right] = 1$ and $E(x) = \sum_{k=1}^{\infty} kp_k = \sum_{k=1}^{\infty} k\,\frac{5^{k-1}}{6^k} = \frac{1}{6}\sum_{k=1}^{\infty} k\left(\frac{5}{6}\right)^{k-1}$
$= \left(\frac{1}{6}\right)\frac{1}{\left[1-\left(\frac{5}{6}\right)\right]^2} = 6$

(c) $\sum_{k=1}^{\infty} p_k = \sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+1}\right) = \lim_{k\to\infty}\left(1 - \frac{1}{k+1}\right) = 1$ and $E(x) = \sum_{k=1}^{\infty} kp_k = \sum_{k=1}^{\infty} k\left(\frac{1}{k(k+1)}\right)$
$= \sum_{k=1}^{\infty} \frac{1}{k+1}$, a divergent series so that E(x) does not exist

33. (a) $R_n = C_0e^{-kt_0} + C_0e^{-2kt_0} + \ldots + C_0e^{-nkt_0} = \frac{C_0e^{-kt_0}\left(1-e^{-nkt_0}\right)}{1-e^{-kt_0}} \Rightarrow R = \lim_{n\to\infty} R_n = \frac{C_0e^{-kt_0}}{1-e^{-kt_0}} = \frac{C_0}{e^{kt_0}-1}$

(b) $R_n = \frac{e^{-1}\left(1-e^{-n}\right)}{1-e^{-1}} \Rightarrow R_1 = e^{-1} \approx 0.36787944$ and $R_{10} = \frac{e^{-1}\left(1-e^{-10}\right)}{1-e^{-1}} \approx 0.58195028$;
$R = \frac{1}{e-1} \approx 0.58197671$; $R - R_{10} \approx 0.00002643 \Rightarrow \frac{R-R_{10}}{R} < 0.0001$

(c) $R_n = \frac{e^{-.1}\left(1-e^{-.1n}\right)}{1-e^{-.1}}$, $\frac{R}{2} = \frac{1}{2}\left(\frac{1}{e^{.1}-1}\right) \approx 4.7541659$; $R_n > \frac{R}{2} \Rightarrow \frac{1-e^{-.1n}}{e^{.1}-1} > \left(\frac{1}{2}\right)\left(\frac{1}{e^{.1}-1}\right)$
$\Rightarrow 1 - e^{-n/10} > \frac{1}{2} \Rightarrow e^{-n/10} < \frac{1}{2} \Rightarrow -\frac{n}{10} < \ln\left(\frac{1}{2}\right) \Rightarrow \frac{n}{10} > -\ln\left(\frac{1}{2}\right) \Rightarrow n > 6.93 \Rightarrow n = 7$

34. (a) $R = \frac{C_0}{e^{kt_0}-1} \Rightarrow Re^{kt_0} = R + C_0 = C_H \Rightarrow e^{kt_0} = \frac{C_H}{C_L} \Rightarrow t_0 = \frac{1}{k}\ln\left(\frac{C_H}{C_L}\right)$

(b) $t_0 = \frac{1}{0.05}\ln e = 20$ hrs

(c) Give an initial dose that produces a concentration of 2 mg/ml followed every $t_0 = \frac{1}{0.02}\ln\left(\frac{2}{0.5}\right) \approx 69.31$ hrs by a dose that raises the concentration by 1.5 mg/ml

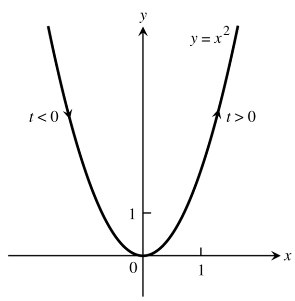
(d) $t_0 = \frac{1}{0.2}\ln\left(\frac{0.1}{0.03}\right) = 5\ln\left(\frac{10}{3}\right) \approx 6$ hrs

NOTES:

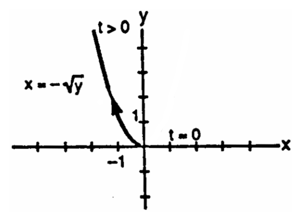
CHAPTER 11 PARAMETRIC EQUATIONS AND POLAR COORDINATES

11.1 PARAMETRIZATIONS OF PLANE CURVES

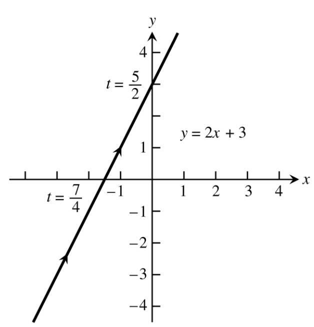
1. $x = 3t,\ y = 9t^2,\ -\infty < t < \infty \ \Rightarrow\ y = x^2$



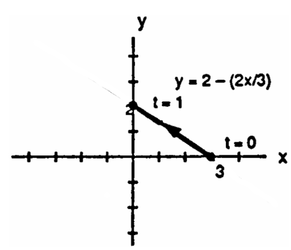
2. $x = -\sqrt{t},\ y = t,\ t \ge 0 \ \Rightarrow\ x = -\sqrt{y}$ or $y = x^2,\ x \le 0$



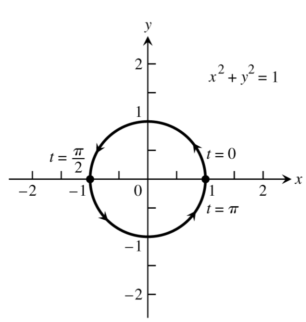
3. $x = 2t - 5,\ y = 4t - 7,\ -\infty < t < \infty$
$\Rightarrow\ x + 5 = 2t \ \Rightarrow\ 2(x + 5) = 4t$
$\Rightarrow\ y = 2(x + 5) - 7 \ \Rightarrow\ y = 2x + 3$



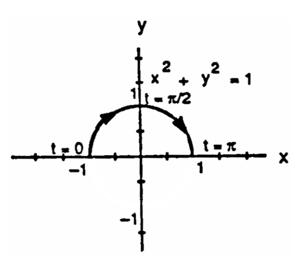
4. $x = 3 - 3t,\ y = 2t,\ 0 \le t \le 1 \ \Rightarrow\ \frac{y}{2} = t$
$\Rightarrow\ x = 3 - 3\left(\frac{y}{2}\right) \ \Rightarrow\ 2x = 6 - 3y$
$\Rightarrow\ y = 2 - \frac{2}{3}x,\ 0 \le x \le 3$



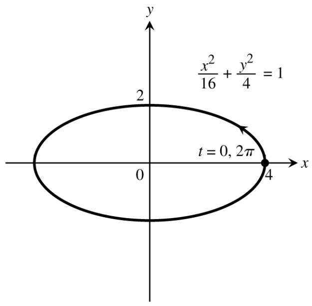
5. $x = \cos 2t,\ y = \sin 2t,\ 0 \le t \le \pi$
$\Rightarrow\ \cos^2 2t + \sin^2 2t = 1 \ \Rightarrow\ x^2 + y^2 = 1$



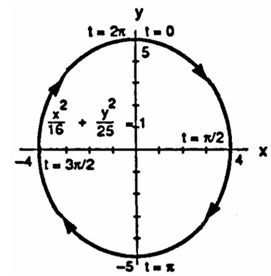
6. $x = \cos(\pi - t),\ y = \sin(\pi - t),\ 0 \le t \le \pi$
$\Rightarrow\ \cos^2(\pi - t) + \sin^2(\pi - t) = 1$
$\Rightarrow\ x^2 + y^2 = 1,\ y \ge 0$



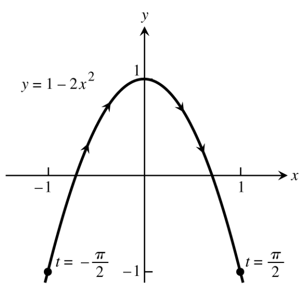
7. $x = 4\cos t,\ y = 2\sin t,\ 0 \le t \le 2\pi$
$\Rightarrow \frac{16\cos^2 t}{16} + \frac{4\sin^2 t}{4} = 1 \Rightarrow \frac{x^2}{16} + \frac{y^2}{4} = 1$



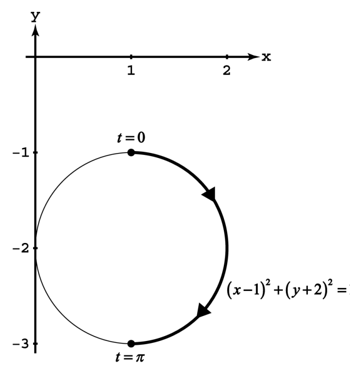
8. $x = 4\sin t,\ y = 5\cos t,\ 0 \le t \le 2\pi$
$\Rightarrow \frac{16\sin^2 t}{16} + \frac{25\cos^2 t}{25} = 1 \Rightarrow \frac{x^2}{16} + \frac{y^2}{25} = 1$



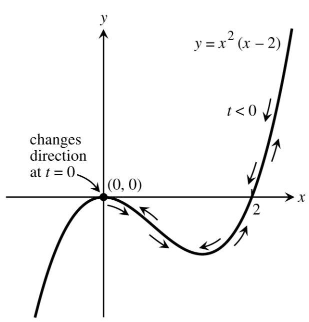
9. $x = \sin t,\ y = \cos 2t,\ -\frac{\pi}{2} \le t \le \frac{\pi}{2}$
$\Rightarrow y = \cos 2t = 1 - 2\sin^2 t \Rightarrow y = 1 - 2x^2$



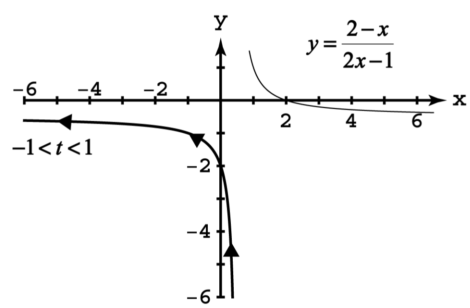
10. $x = 1 + \sin t,\ y = \cos t - 2,\ 0 \le t \le \pi$
$\Rightarrow \sin^2 t + \cos^2 t = 1 \Rightarrow (x - 1)^2 + (y + 2)^2 = 1$



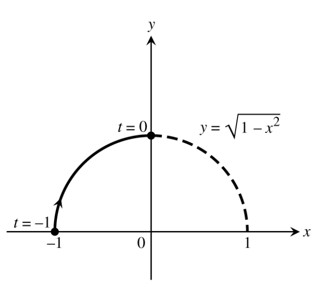
11. $x = t^2,\ y = t^6 - 2t^4,\ -\infty < t < \infty$
$\Rightarrow y = (t^2)^3 - 2(t^2)^2 \Rightarrow y = x^3 - 2x^2$



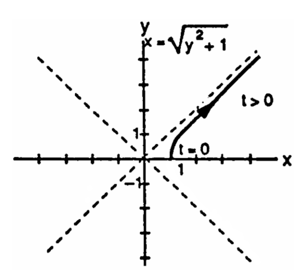
12. $x = \frac{t}{t-1},\ y = \frac{t-2}{t+1},\ -1 < t < 1$
$\Rightarrow t = \frac{x}{x-1} \Rightarrow y = \frac{2-x}{2x-1}$



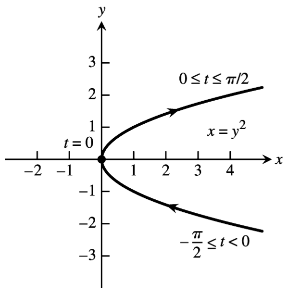
13. $x = t,\ y = \sqrt{1 - t^2},\ -1 \le t \le 0$
$\Rightarrow y = \sqrt{1 - x^2}$



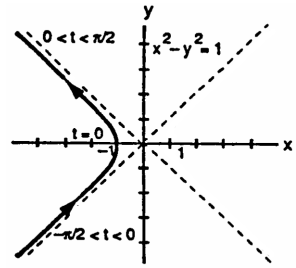
14. $x = \sqrt{t + 1},\ y = \sqrt{t},\ t \ge 0$
$\Rightarrow y^2 = t \Rightarrow x = \sqrt{y^2 + 1},\ y \ge 0$



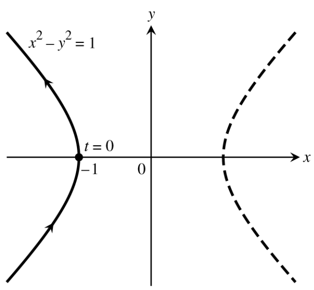
15. $x = \sec^2 t - 1,\ y = \tan t,\ -\frac{\pi}{2} < t < \frac{\pi}{2}$
$\Rightarrow\ \sec^2 t - 1 = \tan^2 t\ \Rightarrow\ x = y^2$



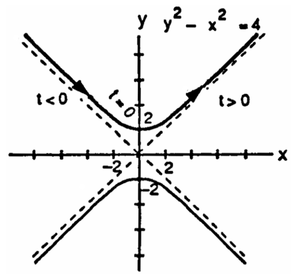
16. $x = -\sec t,\ y = \tan t,\ -\frac{\pi}{2} < t < \frac{\pi}{2}$
$\Rightarrow\ \sec^2 t - \tan^2 t = 1\ \Rightarrow\ x^2 - y^2 = 1$



17. $x = -\cosh t,\ y = \sinh t,\ -\infty < 1 < \infty$
$\Rightarrow\ \cosh^2 t - \sinh^2 t = 1\ \Rightarrow\ x^2 - y^2 = 1$



18. $x = 2\sinh t,\ y = 2\cosh t,\ -\infty < t < \infty$
$\Rightarrow\ 4\cosh^2 t - 4\sinh^2 t = 4\ \Rightarrow\ y^2 - x^2 = 4$



19. (a) $x = a\cos t,\ y = -a\sin t,\ 0 \le t \le 2\pi$
(b) $x = a\cos t,\ y = a\sin t,\ 0 \le t \le 2\pi$
(c) $x = a\cos t,\ y = -a\sin t,\ 0 \le t \le 4\pi$
(d) $x = a\cos t,\ y = a\sin t,\ 0 \le t \le 4\pi$

20. (a) $x = a\sin t,\ y = b\cos t,\ \frac{\pi}{2} \le t \le \frac{5\pi}{2}$
(b) $x = a\cos t,\ y = b\sin t,\ 0 \le t \le 2\pi$
(c) $x = a\sin t,\ y = b\cos t,\ \frac{\pi}{2} \le t \le \frac{9\pi}{2}$
(d) $x = a\cos t,\ y = b\sin t,\ 0 \le t \le 4\pi$

21. Using $(-1, -3)$ we create the parametric equations $x = -1 + at$ and $y = -3 + bt$, representing a line which goes through $(-1, -3)$ at $t = 0$. We determine a and b so that the line goes through $(4, 1)$ when $t = 1$. Since $4 = -1 + a \Rightarrow a = 5$. Since $1 = -3 + b\ \Rightarrow b = 4$. Therefore, one possible parameterization is $x = -1 + 5t$, $y = -3 + 4t,\ 0 \le t \le 1$.

22. Using $(-1, 3)$ we create the parametric equations $x = -1 + at$ and $y = 3 + bt$, representing a line which goes through $(-1, 3)$ at $t = 0$. We determine a and b so that the line goes through $(3, -2)$ when $t = 1$. Since $3 = -1 + a \Rightarrow a = 4$. Since $-2 = 3 + b\ \Rightarrow b = -5$. Therefore, one possible parameterization is $x = -1 + 4t,\ y = 3 - 5t,\ 0 \le t \le 1$.

23. The lower half of the parabola is given by $x = y^2 + 1$ for $y \le 0$. Substituting t for y, we obtain one possible parameterization $x = t^2 + 1,\ y = t,\ t \le 0$.

24. The vertex of the parabola is at $(-1, -1)$, so the left half of the parabola is given by $y = x^2 + 2x$ for $x \le -1$. Substituting t for x, we obtain one possible parametrization: $x = t,\ y = t^2 + 2t,\ t \le -1$.

25. For simplicity, we assume that x and y are linear functions of t and that the point(x, y) starts at $(2, 3)$ for $t = 0$ and passes through $(-1, -1)$ at $t = 1$. Then $x = f(t)$, where $f(0) = 2$ and $f(1) = -1$.
Since slope $= \frac{\Delta x}{\Delta t} = \frac{-1-2}{1-0} = -3$, $x = f(t) = -3t + 2 = 2 - 3t$. Also, $y = g(t)$, where $g(0) = 3$ and $g(1) = -1$.
Since slope $= \frac{\Delta y}{\Delta t} = \frac{-1-3}{1-0} = -4$. $y = g(t) = -4t + 3 = 3 - 4t$.
One possible parameterization is: $x = 2 - 3t,\ y = 3 - 4t,\ t \ge 0$.

26. For simplicity, we assume that x and y are linear functions of t and that the point(x, y) starts at $(-1, 2)$ for $t = 0$ and passes through $(0, 0)$ at $t = 1$. Then $x = f(t)$, where $f(0) = -1$ and $f(1) = 0$.
Since slope $= \frac{\Delta x}{\Delta t} = \frac{0-(-1)}{1-0} = 1$, $x = f(t) = 1t + (-1) = -1 + t$. Also, $y = g(t)$, where $g(0) = 2$ and $g(1) = 0$.
Since slope $= \frac{\Delta y}{\Delta t} = \frac{0-2}{1-0} = -2$. $y = g(t) = -2t + 2 = 2 - 2t$.
One possible parameterization is: $x = -1 + t$, $y = 2 - 2t$, $t \geq 0$.

27. Since we only want the top half of a circle, $y \geq 0$, so let $x = 2\cos t$, $y = 2|\sin t|$, $0 \leq t \leq 4\pi$

28. Since we want x to stay between -3 and 3, let $x = 3\sin t$, then $y = (3\sin t)^2 = 9\sin^2 t$, thus $x = 3\sin t$, $y = 9\sin^2 t$, $0 \leq t < \infty$

29. $x^2 + y^2 = a^2 \Rightarrow 2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$; let $t = \frac{dy}{dx} \Rightarrow -\frac{x}{y} = t \Rightarrow x = -yt$. Substitution yields $y^2t^2 + y^2 = a^2 \Rightarrow y = \frac{a}{\sqrt{1+t^2}}$ and $x = \frac{-at}{\sqrt{1+t}}$, $-\infty < t < \infty$

30. In terms of θ, parametric equations for the circle are $x = a\cos\theta$, $y = a\sin\theta$, $0 \leq \theta < 2\pi$. Since $\theta = \frac{s}{a}$, the arc length parametrizations are: $x = a\cos\frac{s}{a}$, $y = a\sin\frac{s}{a}$, and $0 \leq \frac{s}{a} < 2\pi \Rightarrow 0 \leq s \leq 2\pi a$ is the interval for s.

31. Drop a vertical line from the point (x, y) to the x-axis, then θ is an angle in a right triangle, and from trigonometry we know that $\tan\theta = \frac{y}{x} \Rightarrow y = x\tan\theta$. The equation of the line through (0, 2) and (4, 0) is given by $y = -\frac{1}{2}x + 2$. Thus $x\tan\theta = -\frac{1}{2}x + 2 \Rightarrow x = \frac{4}{2\tan\theta + 1}$ and $y = \frac{4\tan\theta}{2\tan\theta + 1}$ where $0 \leq \theta < \frac{\pi}{2}$.

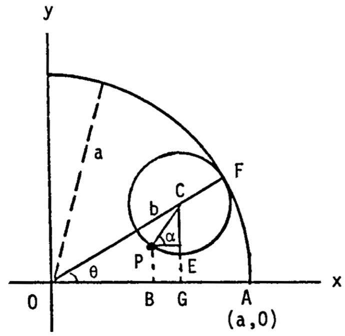
32. Drop a vertical line from the point (x, y) to the x-axis, then θ is an angle in a right triangle, and from trigonometry we know that $\tan\theta = \frac{y}{x} \Rightarrow y = x\tan\theta$. Since $y = \sqrt{x} \Rightarrow y^2 = x \Rightarrow (x\tan\theta)^2 = x \Rightarrow x = \cot^2\theta \Rightarrow y = \cot\theta$ where $0 < \theta \leq \frac{\pi}{2}$.

33. The equation of the circle is given by $(x-2)^2 + y^2 = 1$. Drop a vertical line from the point (x, y) on the circle to the x-axis, then θ is an angle in a right triangle. So that we can start at (1, 0) and rotate in a clockwise direction, let $x = 2 - \cos\theta$, $y = \sin\theta$, $0 \leq \theta \leq 2\pi$.

34. Drop a vertical line from the point (x, y) to the x-axis, then θ is an angle in a right triangle, whose height is y and whose base is $x + 2$. By trigonometry we have $\tan\theta = \frac{y}{x+2} \Rightarrow y = (x+2)\tan\theta$. The equation of the circle is given by $x^2 + y^2 = 1 \Rightarrow x^2 + ((x+2)\tan\theta)^2 = 1 \Rightarrow x^2\sec^2\theta + 4x\tan^2\theta + 4\tan^2\theta - 1 = 0$. Solving for x we obtain
$x = \frac{-4\tan^2\theta \pm \sqrt{(4\tan^2\theta)^2 - 4\sec^2\theta\,(4\tan^2\theta - 1)}}{2\sec^2\theta} = \frac{-4\tan^2\theta \pm 2\sqrt{1 - 3\tan^2\theta}}{2\sec^2\theta} = -2\sin^2\theta \pm \cos\theta\sqrt{\cos^2\theta - 3\sin^2\theta}$
$= -2 + 2\cos^2\theta \pm \cos\theta\sqrt{4\cos^2\theta - 3}$ and $y = \left(-2 + 2\cos^2\theta \pm \cos\theta\sqrt{4\cos^2\theta - 3} + 2\right)\tan\theta$
$= 2\sin\theta\cos\theta \pm \sin\theta\sqrt{4\cos^2\theta - 3}$. Since we only need to go from (1, 0) to (0, 1), let
$x = -2 + 2\cos^2\theta + \cos\theta\sqrt{4\cos^2\theta - 3}$, $y = 2\sin\theta\cos\theta + \sin\theta\sqrt{4\cos^2\theta - 3}$, $0 \leq \theta \leq \tan^{-1}\left(\frac{1}{2}\right)$.
To obtain the upper limit for θ, note that $x = 0$ and $y = 1$, using $y = (x+2)\tan\theta \Rightarrow 1 = 2\tan\theta \Rightarrow \theta = \tan^{-1}\left(\frac{1}{2}\right)$.

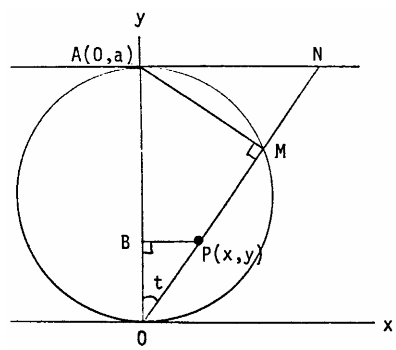
35. Extend the vertical line through A to the x-axis and let C be the point of intersection. Then $OC = AQ = x$ and $\tan t = \frac{2}{OC} = \frac{2}{x} \Rightarrow x = \frac{2}{\tan t} = 2\cot t$; $\sin t = \frac{2}{OA} \Rightarrow OA = \frac{2}{\sin t}$; and $(AB)(OA) = (AQ)^2 \Rightarrow AB\left(\frac{2}{\sin t}\right) = x^2$ $\Rightarrow AB\left(\frac{2}{\sin t}\right) = \left(\frac{2}{\tan t}\right)^2 \Rightarrow AB = \frac{2\sin t}{\tan^2 t}$. Next $y = 2 - AB\sin t \Rightarrow y = 2 - \left(\frac{2\sin t}{\tan^2 t}\right)\sin t =$ $2 - \frac{2\sin^2 t}{\tan^2 t} = 2 - 2\cos^2 t = 2\sin^2 t$. Therefore let $x = 2\cot t$ and $y = 2\sin^2 t$, $0 < t < \pi$.

36. Arc PF = Arc AF since each is the distance rolled and $\frac{\text{Arc PF}}{b} = \angle FCP \Rightarrow \text{Arc PF} = b(\angle FCP)$; $\frac{\text{Arc AF}}{a} = \theta$
$\Rightarrow \text{Arc AF} = a\theta \Rightarrow a\theta = b(\angle FCP) \Rightarrow \angle FCP = \frac{a}{b}\theta$;
$\angle OCG = \frac{\pi}{2} - \theta$; $\angle OCG = \angle OCP + \angle PCE$
$= \angle OCP + \left(\frac{\pi}{2} - \alpha\right)$. Now $\angle OCP = \pi - \angle FCP$
$= \pi - \frac{a}{b}\theta$. Thus $\angle OCG = \pi - \frac{a}{b}\theta + \frac{\pi}{2} - \alpha \Rightarrow \frac{\pi}{2} - \theta$
$= \pi - \frac{a}{b}\theta + \frac{\pi}{2} - \alpha \Rightarrow \alpha = \pi - \frac{a}{b}\theta + \theta = \pi - \left(\frac{a-b}{b}\theta\right)$.

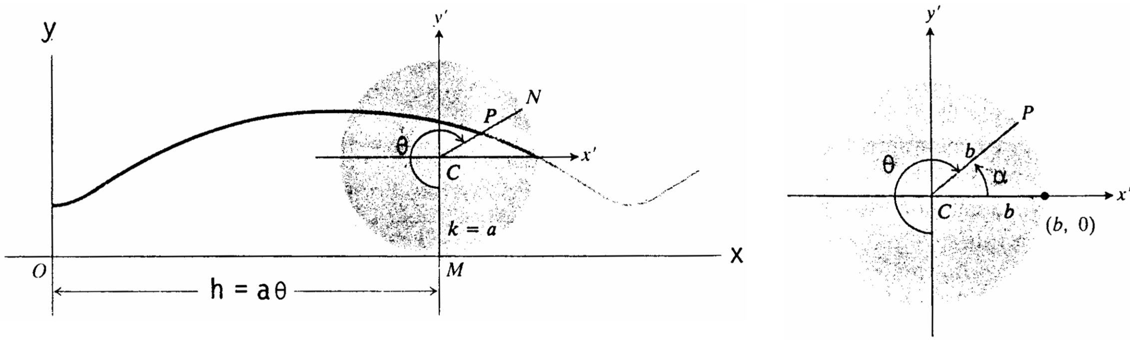


Then $x = OG - BG = OG - PE = (a-b)\cos\theta - b\cos\alpha = (a-b)\cos\theta - b\cos\left(\pi - \frac{a-b}{b}\theta\right)$
$= (a-b)\cos\theta + b\cos\left(\frac{a-b}{b}\theta\right)$. Also $y = EG = CG - CE = (a-b)\sin\theta - b\sin\alpha$
$= (a-b)\sin\theta - b\sin\left(\pi - \frac{a-b}{b}\theta\right) = (a-b)\sin\theta - b\sin\left(\frac{a-b}{b}\theta\right)$. Therefore
$x = (a-b)\cos\theta + b\cos\left(\frac{a-b}{b}\theta\right)$ and $y = (a-b)\sin\theta - b\sin\left(\frac{a-b}{b}\theta\right)$.
If $b = \frac{a}{4}$, then $x = \left(a - \frac{a}{4}\right)\cos\theta + \frac{a}{4}\cos\left(\frac{a - \left(\frac{a}{4}\right)}{\left(\frac{a}{4}\right)}\theta\right)$
$= \frac{3a}{4}\cos\theta + \frac{a}{4}\cos 3\theta = \frac{3a}{4}\cos\theta + \frac{a}{4}(\cos\theta\cos 2\theta - \sin\theta\sin 2\theta)$
$= \frac{3a}{4}\cos\theta + \frac{a}{4}((\cos\theta)(\cos^2\theta - \sin^2\theta) - (\sin\theta)(2\sin\theta\cos\theta))$
$= \frac{3a}{4}\cos\theta + \frac{a}{4}\cos^3\theta - \frac{a}{4}\cos\theta\sin^2\theta - \frac{2a}{4}\sin^2\theta\cos\theta$
$= \frac{3a}{4}\cos\theta + \frac{a}{4}\cos^3\theta - \frac{3a}{4}(\cos\theta)(1 - \cos^2\theta) = a\cos^3\theta$;
$y = \left(a - \frac{a}{4}\right)\sin\theta - \frac{a}{4}\sin\left(\frac{a - \left(\frac{a}{4}\right)}{\left(\frac{a}{4}\right)}\theta\right) = \frac{3a}{4}\sin\theta - \frac{a}{4}\sin 3\theta = \frac{3a}{4}\sin\theta - \frac{a}{4}(\sin\theta\cos 2\theta + \cos\theta\sin 2\theta)$
$= \frac{3a}{4}\sin\theta - \frac{a}{4}((\sin\theta)(\cos^2\theta - \sin^2\theta) + (\cos\theta)(2\sin\theta\cos\theta))$
$= \frac{3a}{4}\sin\theta - \frac{a}{4}\sin\theta\cos^2\theta + \frac{a}{4}\sin^3\theta - \frac{2a}{4}\cos^2\theta\sin\theta$
$= \frac{3a}{4}\sin\theta - \frac{3a}{4}\sin\theta\cos^2\theta + \frac{a}{4}\sin^3\theta$
$= \frac{3a}{4}\sin\theta - \frac{3a}{4}(\sin\theta)(1 - \sin^2\theta) + \frac{a}{4}\sin^3\theta = a\sin^3\theta$.

37. Draw line AM in the figure and note that $\angle AMO$ is a right angle since it is an inscribed angle which spans the diameter of a circle. Then $AN^2 = MN^2 + AM^2$. Now, $OA = a$, $\frac{AN}{a} = \tan t$, and $\frac{AM}{a} = \sin t$. Next $MN = OP$
$\Rightarrow OP^2 = AN^2 - AM^2 = a^2\tan^2 t - a^2\sin^2 t$
$\Rightarrow OP = \sqrt{a^2\tan^2 t - a^2\sin^2 t}$
$= (a\sin t)\sqrt{\sec^2 t - 1} = \frac{a\sin^2 t}{\cos t}$. In triangle BPO,
$x = OP\sin t = \frac{a\sin^3 t}{\cos t} = a\sin^2 t\tan t$ and
$y = OP\cos t = a\sin^2 t \Rightarrow x = a\sin^2 t\tan t$ and $y = a\sin^2 t$.



38. Let the x-axis be the line the wheel rolls along with the y-axis through a low point of the trochoid (see the accompanying figure).

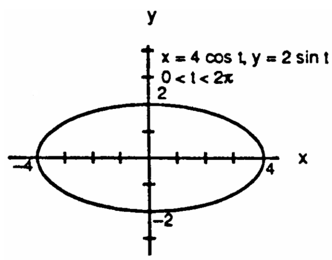


Let θ denote the angle through which the wheel turns. Then $h = a\theta$ and $k = a$. Next introduce $x'y'$-axes parallel to the xy-axes and having their origin at the center C of the wheel. Then $x' = b\cos\alpha$ and $y' = b\sin\alpha$, where $\alpha = \frac{3\pi}{2} - \theta$. It follows that $x' = b\cos\left(\frac{3\pi}{2} - \theta\right) = -b\sin\theta$ and $y' = b\sin\left(\frac{3\pi}{2} - \theta\right)$ $= -b\cos\theta \Rightarrow x = h + x' = a\theta - b\sin\theta$ and $y = k + y' = a - b\cos\theta$ are parametric equations of the trochoid.

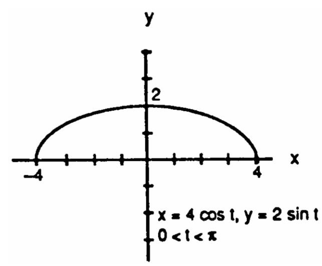
39. $D = \sqrt{(x-2)^2 + \left(y - \frac{1}{2}\right)^2} \Rightarrow D^2 = (x-2)^2 + \left(y - \frac{1}{2}\right)^2 = (t-2)^2 + \left(t^2 - \frac{1}{2}\right)^2 \Rightarrow D^2 = t^4 - 4t + \frac{17}{4}$ $\Rightarrow \frac{d\,(D^2)}{dt} = 4t^3 - 4 = 0 \Rightarrow t = 1$. The second derivative is always positive for $t \neq 0 \Rightarrow t = 1$ gives a local minimum for D^2 (and hence D) which is an absolute minimum since it is the only extremum $\Rightarrow$ the closest point on the parabola is $(1, 1)$.

40. $D = \sqrt{\left(2\cos t - \frac{3}{4}\right)^2 + (\sin t - 0)^2} \Rightarrow D^2 = \left(2\cos t - \frac{3}{4}\right)^2 + \sin^2 t \Rightarrow \frac{d\,(D^2)}{dt}$ $= 2\left(2\cos t - \frac{3}{4}\right)(-2\sin t) + 2\sin t\cos t = (-2\sin t)\left(3\cos t - \frac{3}{2}\right) = 0 \Rightarrow -2\sin t = 0$ or $3\cos t - \frac{3}{2} = 0$ $\Rightarrow t = 0, \pi$ or $t = \frac{\pi}{3}, \frac{5\pi}{3}$. Now $\frac{d^2\,(D^2)}{dt^2} = -6\cos^2 t + 3\cos t + 6\sin^2 t$ so that $\frac{d^2\,(D^2)}{dt^2}(0) = -3 \Rightarrow$ relative maximum, $\frac{d^2\,(D^2)}{dt^2}(\pi) = -9 \Rightarrow$ relative maximum, $\frac{d^2\,(D^2)}{dt^2}\left(\frac{\pi}{3}\right) = \frac{9}{2} \Rightarrow$ relative minimum, and $\frac{d^2\,(D^2)}{dt^2}\left(\frac{5\pi}{3}\right) = \frac{9}{2} \Rightarrow$ relative minimum. Therefore both $t = \frac{\pi}{3}$ and $t = \frac{5\pi}{3}$ give points on the ellipse closest to the point $\left(\frac{3}{4}, 0\right) \Rightarrow \left(1, \frac{\sqrt{3}}{2}\right)$ and $\left(1, -\frac{\sqrt{3}}{2}\right)$ are the desired points.

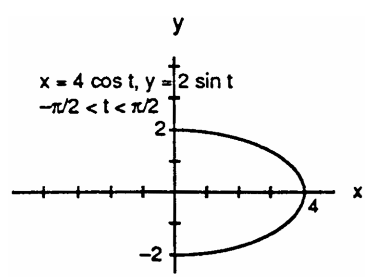
41. (a)



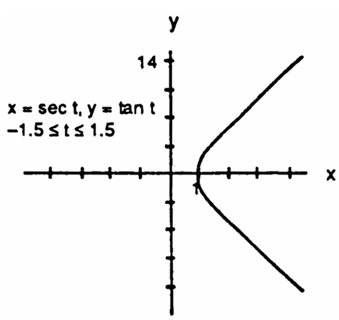
(b)



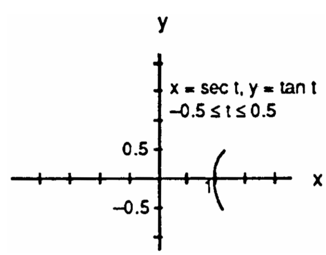
(c)



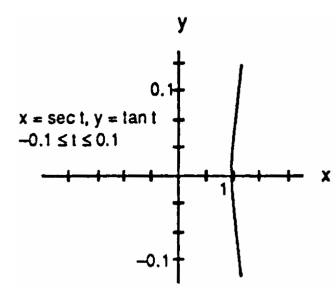
42. (a)



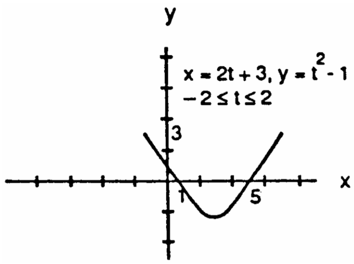
(b)



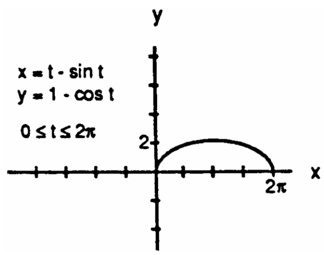
(c)



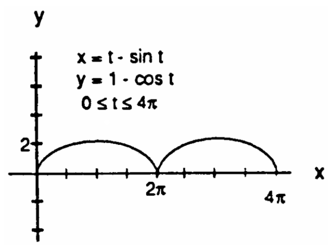
43.



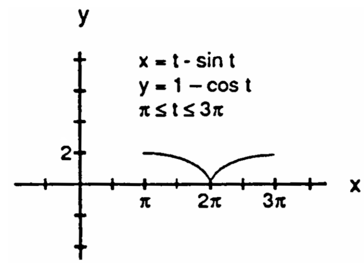
44. (a)



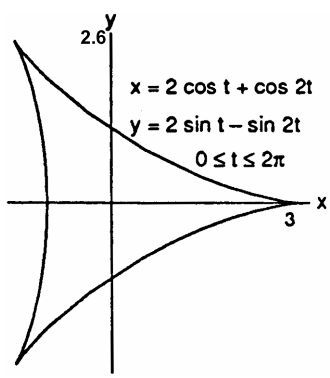
(b)



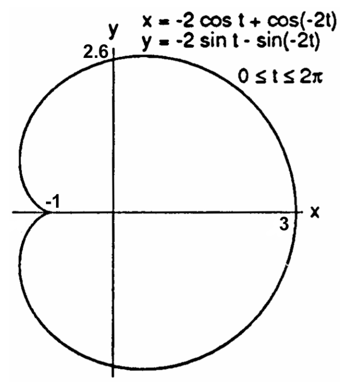
(c)



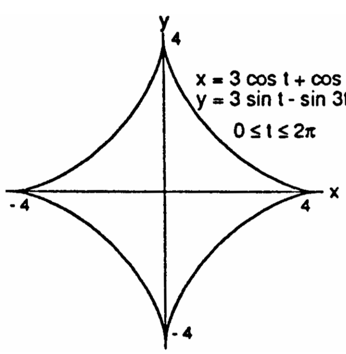
45. (a)



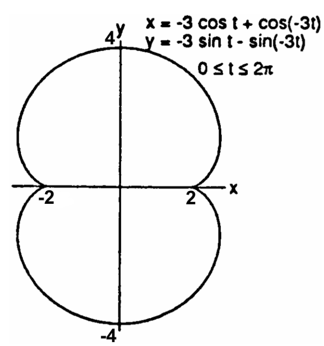
(b)



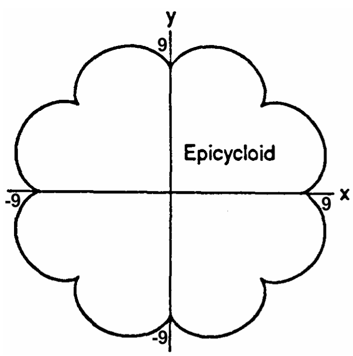
46. (a)



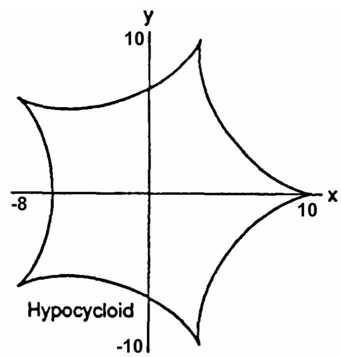
(b)



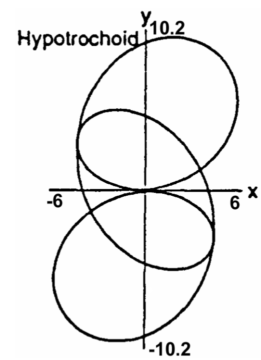
47. (a)



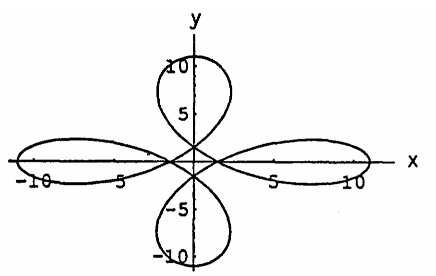
(b)



(c)

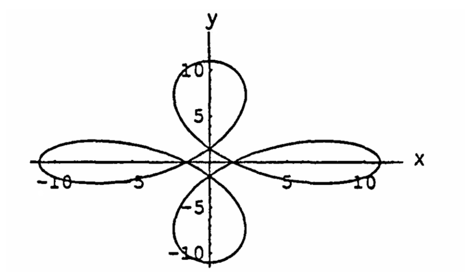


48. (a)



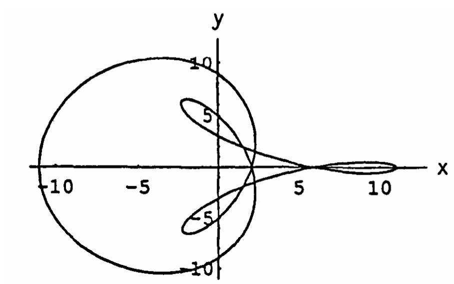
$x = 6\cos t + 5\cos 3t,\quad y = 6\sin t - 5\sin 3t,$
$0 \le t \le 2\pi$

(b)



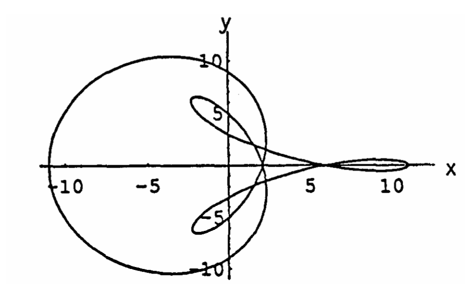
$x = 6\cos 2t + 5\cos 6t,\quad y = 6\sin 2t - 5\sin 6t,$
$0 \le t \le \pi$

(c)



$x = 6\cos t + 5\cos 3t,\quad y = 6\sin 2t - 5\sin 3t,$
$0 \le t \le 2\pi$

(d)



$x = 6\cos 2t + 5\cos 6t,\quad y = 6\sin 4t - 5\sin 6t,$
$0 \le t \le \pi$

11.2 CALCULUS WITH PARAMETRIC CURVES

1. $t = \frac{\pi}{4} \Rightarrow x = 2\cos\frac{\pi}{4} = \sqrt{2},\ y = 2\sin\frac{\pi}{4} = \sqrt{2};\ \frac{dx}{dt} = -2\sin t,\ \frac{dy}{dt} = 2\cos t \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\cos t}{-2\sin t} = -\cot t$
$\Rightarrow \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{4}} = -\cot\frac{\pi}{4} = -1$; tangent line is $y - \sqrt{2} = -1\left(x - \sqrt{2}\right)$ or $y = -x + 2\sqrt{2}$; $\frac{dy'}{dt} = \csc^2 t$
$\Rightarrow \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{\csc^2 t}{-2\sin t} = -\frac{1}{2\sin^3 t} \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=\frac{\pi}{4}} = -\sqrt{2}$

2. $t = -\frac{1}{6} \Rightarrow x = \sin\left(2\pi\left(-\frac{1}{6}\right)\right) = \sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2},\ y = \cos\left(2\pi\left(-\frac{1}{6}\right)\right) = \cos\left(-\frac{\pi}{3}\right) = \frac{1}{2};\ \frac{dx}{dt} = 2\pi\cos 2\pi t,$
$\frac{dy}{dt} = -2\pi\sin 2\pi t \Rightarrow \frac{dy}{dx} = \frac{-2\pi\sin 2\pi t}{2\pi\cos 2\pi t} = -\tan 2\pi t \Rightarrow \left.\frac{dy}{dx}\right|_{t=-\frac{1}{6}} = -\tan\left(2\pi\left(-\frac{1}{6}\right)\right) = -\tan\left(-\frac{\pi}{3}\right) = \sqrt{3};$
tangent line is $y - \frac{1}{2} = \sqrt{3}\left[x - \left(-\frac{\sqrt{3}}{2}\right)\right]$ or $y = \sqrt{3}x + 2$; $\frac{dy'}{dt} = -2\pi\sec^2 2\pi t \Rightarrow \frac{d^2y}{dx^2} = \frac{-2\pi\sec^2 2\pi t}{2\pi\cos 2\pi t}$
$= -\frac{1}{\cos^3 2\pi t} \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=-\frac{1}{6}} = -8$

3. $t = \frac{\pi}{4} \Rightarrow x = 4 \sin \frac{\pi}{4} = 2\sqrt{2}, y = 2 \cos \frac{\pi}{4} = \sqrt{2}; \frac{dx}{dt} = 4 \cos t, \frac{dy}{dt} = -2 \sin t \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-2 \sin t}{4 \cos t}$
$= -\frac{1}{2} \tan t \Rightarrow \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{4}} = -\frac{1}{2} \tan \frac{\pi}{4} = -\frac{1}{2}$; tangent line is $y - \sqrt{2} = -\frac{1}{2}\left(x - 2\sqrt{2}\right)$ or $y = -\frac{1}{2}x + 2\sqrt{2}$;
$\frac{dy'}{dt} = -\frac{1}{2} \sec^2 t \Rightarrow \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{-\frac{1}{2}\sec^2 t}{4 \cos t} = -\frac{1}{8 \cos^3 t} \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=\frac{\pi}{4}} = -\frac{\sqrt{2}}{4}$

4. $t = \frac{2\pi}{3} \Rightarrow x = \cos \frac{2\pi}{3} = -\frac{1}{2}, y = \sqrt{3} \cos \frac{2\pi}{3} = -\frac{\sqrt{3}}{2}; \frac{dx}{dt} = -\sin t, \frac{dy}{dt} = -\sqrt{3} \sin t \Rightarrow \frac{dy}{dx} = \frac{-\sqrt{3}\sin t}{-\sin t} = \sqrt{3}$
$\Rightarrow \left.\frac{dy}{dx}\right|_{t=\frac{2\pi}{3}} = \sqrt{3}$; tangent line is $y - \left(-\frac{\sqrt{3}}{2}\right) = \sqrt{3}\left[x - \left(-\frac{1}{2}\right)\right]$ or $y = \sqrt{3}x$; $\frac{dy'}{dt} = 0 \Rightarrow \frac{d^2y}{dx^2} = \frac{0}{-\sin t} = 0$
$\Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=\frac{2\pi}{3}} = 0$

5. $t = \frac{1}{4} \Rightarrow x = \frac{1}{4}, y = \frac{1}{2}; \frac{dx}{dt} = 1, \frac{dy}{dt} = \frac{1}{2\sqrt{t}} \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1}{2\sqrt{t}} \Rightarrow \left.\frac{dy}{dx}\right|_{t=\frac{1}{4}} = \frac{1}{2\sqrt{\frac{1}{4}}} = 1$; tangent line is
$y - \frac{1}{2} = 1 \cdot \left(x - \frac{1}{4}\right)$ or $y = x + \frac{1}{4}$; $\frac{dy'}{dt} = -\frac{1}{4}t^{-3/2} \Rightarrow \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = -\frac{1}{4}t^{-3/2} \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=\frac{1}{4}} = -2$

6. $t = -\frac{\pi}{4} \Rightarrow x = \sec^2\left(-\frac{\pi}{4}\right) - 1 = 1, y = \tan\left(-\frac{\pi}{4}\right) = -1; \frac{dx}{dt} = 2\sec^2 t \tan t, \frac{dy}{dt} = \sec^2 t$
$\Rightarrow \frac{dy}{dx} = \frac{\sec^2 t}{2\sec^2 t \tan t} = \frac{1}{2 \tan t} = \frac{1}{2}\cot t \Rightarrow \left.\frac{dy}{dx}\right|_{t=-\frac{\pi}{4}} = \frac{1}{2}\cot\left(-\frac{\pi}{4}\right) = -\frac{1}{2}$; tangent line is
$y - (-1) = -\frac{1}{2}(x - 1)$ or $y = -\frac{1}{2}x - \frac{1}{2}$; $\frac{dy'}{dt} = -\frac{1}{2}\csc^2 t \Rightarrow \frac{d^2y}{dx^2} = \frac{-\frac{1}{2}\csc^2 t}{2\sec^2 t \tan t} = -\frac{1}{4}\cot^3 t$
$\Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=-\frac{\pi}{4}} = \frac{1}{4}$

7. $t = \frac{\pi}{6} \Rightarrow x = \sec \frac{\pi}{6} = \frac{2}{\sqrt{3}}, y = \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}; \frac{dx}{dt} = \sec t \tan t, \frac{dy}{dt} = \sec^2 t \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$
$= \frac{\sec^2 t}{\sec t \tan t} = \csc t \Rightarrow \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{6}} = \csc \frac{\pi}{6} = 2$; tangent line is $y - \frac{1}{\sqrt{3}} = 2\left(x - \frac{2}{\sqrt{3}}\right)$ or $y = 2x - \sqrt{3}$;
$\frac{dy'}{dt} = -\csc t \cot t \Rightarrow \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{-\csc t \cot t}{\sec t \tan t} = -\cot^3 t \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=\frac{\pi}{6}} = -3\sqrt{3}$

8. $t = 3 \Rightarrow x = -\sqrt{3+1} = -2, y = \sqrt{3(3)} = 3; \frac{dx}{dt} = -\frac{1}{2}(t+1)^{-1/2}, \frac{dy}{dt} = \frac{3}{2}(3t)^{-1/2} \Rightarrow \frac{dy}{dx} = \frac{\left(\frac{3}{2}\right)(3t)^{-1/2}}{\left(-\frac{1}{2}\right)(t+1)^{-1/2}}$
$= -\frac{3\sqrt{t+1}}{\sqrt{3t}} = \left.\frac{dy}{dx}\right|_{t=3} = \frac{-3\sqrt{3+1}}{\sqrt{3(3)}} = -2$; tangent line is $y - 3 = -2[x - (-2)]$ or $y = -2x - 1$;
$\frac{dy'}{dt} = \frac{\sqrt{3t}\left[-\frac{3}{2}(t+1)^{-1/2}\right] + 3\sqrt{t+1}\left[\frac{3}{2}(3t)^{-1/2}\right]}{3t} = \frac{3}{2t\sqrt{3t}\sqrt{t+1}} \Rightarrow \frac{d^2y}{dx^2} = \frac{\left(\frac{3}{2t\sqrt{3t}\sqrt{t+1}}\right)}{\left(\frac{-1}{2\sqrt{t+1}}\right)} = -\frac{3}{t\sqrt{3t}}$
$\Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=3} = -\frac{1}{3}$

9. $t = -1 \Rightarrow x = 5, y = 1; \frac{dx}{dt} = 4t, \frac{dy}{dt} = 4t^3 \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4t^3}{4t} = t^2 \Rightarrow \left.\frac{dy}{dx}\right|_{t=-1} = (-1)^2 = 1$; tangent line is
$y - 1 = 1 \cdot (x - 5)$ or $y = x - 4$; $\frac{dy'}{dt} = 2t \Rightarrow \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{2t}{4t} = \frac{1}{2} \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=-1} = \frac{1}{2}$

10. $t = 1 \Rightarrow x = 1, y = -2; \frac{dx}{dt} = -\frac{1}{t^2}, \frac{dy}{dt} = \frac{1}{t} \Rightarrow \frac{dy}{dx} = \frac{\left(\frac{1}{t}\right)}{\left(-\frac{1}{t^2}\right)} = -t \Rightarrow \left.\frac{dy}{dx}\right|_{t=1} = -1$; tangent line is
$y - (-2) = -1(x - 1)$ or $y = -x - 1$; $\frac{dy'}{dt} = -1 \Rightarrow \frac{d^2y}{dx^2} = \frac{-1}{\left(-\frac{1}{t^2}\right)} = t^2 \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=1} = 1$

11. $t = \frac{\pi}{3} \Rightarrow x = \frac{\pi}{3} - \sin \frac{\pi}{3} = \frac{\pi}{3} - \frac{\sqrt{3}}{2}, y = 1 - \cos \frac{\pi}{3} = 1 - \frac{1}{2} = \frac{1}{2}; \frac{dx}{dt} = 1 - \cos t, \frac{dy}{dt} = \sin t \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$
$= \frac{\sin t}{1 - \cos t} \Rightarrow \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{3}} = \frac{\sin\left(\frac{\pi}{3}\right)}{1 - \cos\left(\frac{\pi}{3}\right)} = \frac{\left(\frac{\sqrt{3}}{2}\right)}{\left(\frac{1}{2}\right)} = \sqrt{3}$; tangent line is $y - \frac{1}{2} = \sqrt{3}\left(x - \frac{\pi}{3} + \frac{\sqrt{3}}{2}\right)$

$\Rightarrow\ y = \sqrt{3}x - \frac{\pi\sqrt{3}}{3} + 2;\ \frac{dy'}{dt} = \frac{(1-\cos t)(\cos t) - (\sin t)(\sin t)}{(1-\cos t)^2} = \frac{-1}{1-\cos t}\ \Rightarrow\ \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{\left(\frac{-1}{1-\cos t}\right)}{1-\cos t}$

$= \frac{-1}{(1-\cos t)^2}\ \Rightarrow\ \left.\frac{d^2y}{dx^2}\right|_{t=\frac{\pi}{3}} = -4$

12. $t = \frac{\pi}{2}\ \Rightarrow\ x = \cos\frac{\pi}{2} = 0,\ y = 1 + \sin\frac{\pi}{2} = 2;\ \frac{dx}{dt} = -\sin t,\ \frac{dy}{dt} = \cos t\ \Rightarrow\ \frac{dy}{dx} = \frac{\cos t}{-\sin t} = -\cot t$

$\Rightarrow\ \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{2}} = -\cot\frac{\pi}{2} = 0$; tangent line is $y = 2$; $\frac{dy'}{dt} = \csc^2 t\ \Rightarrow\ \frac{d^2y}{dx^2} = \frac{\csc^2 t}{-\sin t} = -\csc^3 t\ \Rightarrow\ \left.\frac{d^2y}{dx^2}\right|_{t=\frac{\pi}{2}} = -1$

13. $t = 2 \Rightarrow x = \frac{1}{2+1} = \frac{1}{3},\ y = \frac{2}{2-1} = 2;\ \frac{dx}{dt} = \frac{-1}{(t+1)^2},\ \frac{dy}{dt} = \frac{-1}{(t-1)^2} \Rightarrow \frac{dy}{dx} = \frac{(t+1)^2}{(t-1)^2} \Rightarrow \left.\frac{dy}{dx}\right|_{t=2} = \frac{(2+1)^2}{(2-1)^2} = 9;$

tangent line is $y = 9x - 1$; $\frac{dy'}{dt} = -\frac{4(t+1)}{(t-1)^3} \Rightarrow \frac{d^2y}{dx^2} = \frac{4(t+1)^3}{(t-1)^3} \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=2} = \frac{4(2+1)^3}{(2-1)^3} = 108$

14. $t = 0 \Rightarrow x = 0 + e^0 = 1,\ y = 1 - e^0 = 0;\ \frac{dx}{dt} = 1 + e^t,\ \frac{dy}{dt} = -e^t \Rightarrow \frac{dy}{dx} = \frac{-e^t}{1+e^t} \Rightarrow \left.\frac{dy}{dx}\right|_{t=0} = \frac{-e^0}{1+e^0} = -\frac{1}{2};$

tangent line is $y = -\frac{1}{2}x + \frac{1}{2}$; $\frac{dy'}{dt} = \frac{-e^t}{(1+e^t)^2} \Rightarrow \frac{d^2y}{dx^2} = \frac{-e^t}{(1+e^t)^3} \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=0} = \frac{-e^0}{(1+e^0)^3} = -\frac{1}{8}$

15. $x^3 + 2t^2 = 9\ \Rightarrow\ 3x^2\frac{dx}{dt} + 4t = 0\ \Rightarrow\ 3x^2\frac{dx}{dt} = -4t\ \Rightarrow\ \frac{dx}{dt} = \frac{-4t}{3x^2};$

$2y^3 - 3t^2 = 4\ \Rightarrow\ 6y^2\frac{dy}{dt} - 6t = 0\ \Rightarrow\ \frac{dy}{dt} = \frac{6t}{6y^2} = \frac{t}{y^2}$; thus $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\left(\frac{t}{y^2}\right)}{\left(\frac{-4t}{3x^2}\right)} = \frac{t(3x^2)}{y^2(-4t)} = \frac{3x^2}{-4y^2}$; $t = 2$

$\Rightarrow\ x^3 + 2(2)^2 = 9\ \Rightarrow\ x^3 + 8 = 9\ \Rightarrow\ x^3 = 1\ \Rightarrow\ x = 1;\ t = 2\ \Rightarrow\ 2y^3 - 3(2)^2 = 4$

$\Rightarrow\ 2y^3 = 16\ \Rightarrow\ y^3 = 8\ \Rightarrow\ y = 2$; therefore $\left.\frac{dy}{dx}\right|_{t=2} = \frac{3(1)^2}{-4(2)^2} = -\frac{3}{16}$

16. $x = \sqrt{5-\sqrt{t}}\ \Rightarrow\ \frac{dx}{dt} = \frac{1}{2}\left(5-\sqrt{t}\right)^{-1/2}\left(-\frac{1}{2}t^{-1/2}\right) = -\frac{1}{4\sqrt{t}\sqrt{5-\sqrt{t}}}$; $y(t-1) = \sqrt{t}\ \Rightarrow y + (t-1)\frac{dy}{dt} = \frac{1}{2}t^{-1/2}$

$\Rightarrow\ (t-1)\frac{dy}{dt} = \frac{1}{2\sqrt{t}} - y\ \Rightarrow\ \frac{dy}{dt} = \frac{\frac{1}{2\sqrt{t}} - y}{(t-1)} = \frac{1-2y\sqrt{t}}{2t\sqrt{t}-2\sqrt{t}}$; thus $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{1-2y\sqrt{t}}{2t\sqrt{t}-2\sqrt{t}}}{\frac{-1}{4\sqrt{t}\sqrt{5-\sqrt{t}}}} = \frac{1-2y\sqrt{t}}{2\sqrt{t}(t-1)}\cdot\frac{4\sqrt{t}\sqrt{5-\sqrt{t}}}{-1}$

$= \frac{2\left(1-2y\sqrt{t}\right)\sqrt{5-\sqrt{t}}}{1-t}$; $t = 4\ \Rightarrow\ x = \sqrt{5-\sqrt{4}} = \sqrt{3}$; $t = 4\ \Rightarrow y\cdot 3 = \sqrt{4}\ \Rightarrow y = \frac{2}{3}$

therefore, $\left.\frac{dy}{dx}\right|_{t=4} = \frac{2\left(1-2\left(\frac{2}{3}\right)\sqrt{4}\right)\sqrt{5-\sqrt{4}}}{1-4} = \frac{10\sqrt{3}}{9}$

17. $x + 2x^{3/2} = t^2 + t\ \Rightarrow\ \frac{dx}{dt} + 3x^{1/2}\frac{dx}{dt} = 2t + 1\ \Rightarrow\ \left(1 + 3x^{1/2}\right)\frac{dx}{dt} = 2t + 1\ \Rightarrow\ \frac{dx}{dt} = \frac{2t+1}{1+3x^{1/2}}$; $y\sqrt{t+1} + 2t\sqrt{y} = 4$

$\Rightarrow\ \frac{dy}{dt}\sqrt{t+1} + y\left(\frac{1}{2}\right)(t+1)^{-1/2} + 2\sqrt{y} + 2t\left(\frac{1}{2}y^{-1/2}\right)\frac{dy}{dt} = 0\ \Rightarrow\ \frac{dy}{dt}\sqrt{t+1} + \frac{y}{2\sqrt{t+1}} + 2\sqrt{y} + \left(\frac{t}{\sqrt{y}}\right)\frac{dy}{dt} = 0$

$\Rightarrow\ \left(\sqrt{t+1} + \frac{t}{\sqrt{y}}\right)\frac{dy}{dt} = \frac{-y}{2\sqrt{t+1}} - 2\sqrt{y}\ \Rightarrow\ \frac{dy}{dt} = \frac{\left(\frac{-y}{2\sqrt{t+1}} - 2\sqrt{y}\right)}{\left(\sqrt{t+1} + \frac{t}{\sqrt{y}}\right)} = \frac{-y\sqrt{y} - 4y\sqrt{t+1}}{2\sqrt{y}(t+1) + 2t\sqrt{t+1}}$; thus

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\left(\frac{-y\sqrt{y}-4y\sqrt{t+1}}{2\sqrt{y}(t+1)+2t\sqrt{t+1}}\right)}{\left(\frac{2t+1}{1+3x^{1/2}}\right)}$; $t = 0\ \Rightarrow\ x + 2x^{3/2} = 0\ \Rightarrow\ x\left(1 + 2x^{1/2}\right) = 0\ \Rightarrow\ x = 0;\ t = 0$

$\Rightarrow\ y\sqrt{0+1} + 2(0)\sqrt{y} = 4\ \Rightarrow\ y = 4$; therefore $\left.\frac{dy}{dx}\right|_{t=0} = \frac{\left(\frac{-4\sqrt{4}-4(4)\sqrt{0+1}}{2\sqrt{4}(0+1)+2(0)\sqrt{0+1}}\right)}{\left(\frac{2(0)+1}{1+3(0)^{1/2}}\right)} = -6$

18. $x\sin t + 2x = t\ \Rightarrow\ \frac{dx}{dt}\sin t + x\cos t + 2\frac{dx}{dt} = 1\ \Rightarrow\ (\sin t + 2)\frac{dx}{dt} = 1 - x\cos t\ \Rightarrow\ \frac{dx}{dt} = \frac{1-x\cos t}{\sin t+2};$

$t\sin t - 2t = y\ \Rightarrow\ \sin t + t\cos t - 2 = \frac{dy}{dt}$; thus $\frac{dy}{dx} = \frac{\sin t + t\cos t - 2}{\left(\frac{1-x\cos t}{\sin t+2}\right)}$; $t = \pi\ \Rightarrow\ x\sin\pi + 2x = \pi$

$\Rightarrow\ x = \frac{\pi}{2}$; therefore $\left.\frac{dy}{dx}\right|_{t=\pi} = \frac{\sin\pi + \pi\cos\pi - 2}{\left[\frac{1-\left(\frac{\pi}{2}\right)\cos\pi}{\sin\pi+2}\right]} = \frac{-4\pi-8}{2+\pi} = -4$

19. $x = t^3 + t,\ y + 2t^3 = 2x + t^2 \Rightarrow \frac{dx}{dt} = 3t^2 + 1,\ \frac{dy}{dt} + 6t^2 = 2\frac{dx}{dt} + 2t \Rightarrow \frac{dy}{dt} = 2(3t^2+1) + 2t - 6t^2 = 2t + 2$
$\Rightarrow \frac{dy}{dx} = \frac{2t+2}{3t^2+1} \Rightarrow \left.\frac{dy}{dx}\right|_{t=1} = \frac{2(1)+2}{3(1)^2+1} = 1$

20. $t = \ln(x - t),\ y = te^t \Rightarrow 1 = \frac{1}{x-t}\left(\frac{dx}{dt} - 1\right) \Rightarrow x - t = \frac{dx}{dt} - 1 \Rightarrow \frac{dx}{dt} = x - t + 1,\ \frac{dy}{dt} = te^t + e^t;$
$\Rightarrow \frac{dy}{dx} = \frac{te^t + e^t}{x-t+1};\ t = 0 \Rightarrow 0 = \ln(x - 0) \Rightarrow x = 1 \Rightarrow \left.\frac{dy}{dx}\right|_{t=0} = \frac{(0)e^0 + e^0}{1-0+1} = \frac{1}{2}$

21. $A = \int_0^{2\pi} y\,dx = \int_0^{2\pi} a(1-\cos t)a(1-\cos t)dt = a^2\int_0^{2\pi}(1-\cos t)^2dt = a^2\int_0^{2\pi}(1 - 2\cos t + \cos^2 t)dt$
$= a^2\int_0^{2\pi}\left(1 - 2\cos t + \frac{1+\cos 2t}{2}\right)dt = a^2\int_0^{2\pi}\left(\frac{3}{2} - 2\cos t + \frac{1}{2}\cos 2t\right)dt = a^2\left[\frac{3}{2}t - 2\sin t + \frac{1}{4}\sin 2t\right]_0^{2\pi}$
$= a^2(3\pi - 0 + 0) - 0 = 3\pi a^2$

22. $A = \int_0^1 x\,dy = \int_0^1 (t - t^2)(-e^{-t})dt \quad \left[u = t - t^2 \Rightarrow du = (1-2t)dt;\ dv = (-e^{-t})dt \Rightarrow v = e^{-t}\right]$
$= e^{-t}(t - t^2)\Big|_0^1 - \int_0^1 e^{-t}(1-2t)dt \quad \left[u = 1 - 2t \Rightarrow du = -2dt;\ dv = e^{-t}dt \Rightarrow v = -e^{-t}\right]$
$= e^{-t}(t - t^2)\Big|_0^1 - \left[-e^{-t}(1-2t)\Big|_0^1 - \int_0^1 2e^{-t}dt\right] = \left[e^{-t}(t-t^2) + e^{-t}(1-2t) - 2e^{-t}\right]\Big|_0^1$
$= (e^{-1}(0) + e^{-1}(-1) - 2e^{-1}) - (e^0(0) + e^0(1) - 2e^0) = 1 - 3e^{-1} = 1 - \frac{3}{e}$

23. $A = 2\int_\pi^0 y\,dx = 2\int_\pi^0 (b\sin t)(-a\sin t)dt = 2ab\int_0^\pi \sin^2 t\,dt = 2ab\int_0^\pi \frac{1-\cos 2t}{2}\,dt = ab\int_0^\pi (1 - \cos 2t)\,dt$
$= ab\left[t - \frac{1}{2}\sin 2t\right]_0^\pi = ab((\pi - 0) - 0) = \pi ab$

24. (a) $x = t^2,\ y = t^6,\ 0 \le t \le 1 \Rightarrow A = \int_0^1 y\,dx = \int_0^1 (t^6)2t\,dt = \int_0^1 2t^7\,dt = \left[\frac{1}{4}t^8\right]_0^1 = \frac{1}{4} - 0 = \frac{1}{4}$
(b) $x = t^3,\ y = t^9,\ 0 \le t \le 1 \Rightarrow A = \int_0^1 y\,dx = \int_0^1 (t^9)3t^2\,dt = \int_0^1 3t^{11}\,dt = \left[\frac{1}{4}t^{12}\right]_0^1 = \frac{1}{4} - 0 = \frac{1}{4}$

25. $\frac{dx}{dt} = -\sin t$ and $\frac{dy}{dt} = 1 + \cos t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(-\sin t)^2 + (1+\cos t)^2} = \sqrt{2 + 2\cos t}$
$\Rightarrow$ Length $= \int_0^\pi \sqrt{2 + 2\cos t}\,dt = \sqrt{2}\int_0^\pi \sqrt{\left(\frac{1-\cos t}{1-\cos t}\right)(1+\cos t)}\,dt = \sqrt{2}\int_0^\pi \sqrt{\frac{\sin^2 t}{1-\cos t}}\,dt$
$= \sqrt{2}\int_0^\pi \frac{\sin t}{\sqrt{1-\cos t}}\,dt$ (since $\sin t \ge 0$ on $[0,\pi]$); $[u = 1 - \cos t \Rightarrow du = \sin t\,dt;\ t = 0 \Rightarrow u = 0,$
$t = \pi \Rightarrow u = 2] \rightarrow \sqrt{2}\int_0^2 u^{-1/2}\,du = \sqrt{2}\left[2u^{1/2}\right]_0^2 = 4$

26. $\frac{dx}{dt} = 3t^2$ and $\frac{dy}{dt} = 3t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(3t^2)^2 + (3t)^2} = \sqrt{9t^4 + 9t^2} = 3t\sqrt{t^2+1}$ (since $t \ge 0$ on $[0, \sqrt{3}]$)
$\Rightarrow$ Length $= \int_0^{\sqrt{3}} 3t\sqrt{t^2+1}\,dt;\ \left[u = t^2 + 1 \Rightarrow \frac{3}{2}du = 3t\,dt;\ t = 0 \Rightarrow u = 1,\ t = \sqrt{3} \Rightarrow u = 4\right]$
$\rightarrow \int_1^4 \frac{3}{2}u^{1/2}\,du = \left[u^{3/2}\right]_1^4 = (8 - 1) = 7$

27. $\frac{dx}{dt} = t$ and $\frac{dy}{dt} = (2t+1)^{1/2} \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{t^2 + (2t+1)} = \sqrt{(t+1)^2} = |t + 1| = t + 1$ since $0 \le t \le 4$
$\Rightarrow$ Length $= \int_0^4 (t+1)\,dt = \left[\frac{t^2}{2} + t\right]_0^4 = (8 + 4) = 12$

28. $\frac{dx}{dt} = (2t+3)^{1/2}$ and $\frac{dy}{dt} = 1+t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(2t+3)+(1+t)^2} = \sqrt{t^2+4t+4} = |t+2| = t+2$ since $0 \le t \le 3 \Rightarrow \text{Length} = \int_0^3 (t+2)\,dt = \left[\frac{t^2}{2}+2t\right]_0^3 = \frac{21}{2}$

29. $\frac{dx}{dt} = 8t\cos t$ and $\frac{dy}{dt} = 8t\sin t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(8t\cos t)^2 + (8t\sin t)^2} = \sqrt{64t^2\cos^2 t + 64t^2\sin^2 t}$ $= |8t| = 8t$ since $0 \le t \le \frac{\pi}{2} \Rightarrow \text{Length} = \int_0^{\pi/2} 8t\,dt = [4t^2]_0^{\pi/2} = \pi^2$

30. $\frac{dx}{dt} = \left(\frac{1}{\sec t + \tan t}\right)(\sec t\tan t + \sec^2 t) - \cos t = \sec t - \cos t$ and $\frac{dy}{dt} = -\sin t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$
$= \sqrt{(\sec t - \cos t)^2 + (-\sin t)^2} = \sqrt{\sec^2 t - 1} = \sqrt{\tan^2 t} = |\tan t| = \tan t$ since $0 \le t \le \frac{\pi}{3}$
$\Rightarrow \text{Length} = \int_0^{\pi/3} \tan t\,dt = \int_0^{\pi/3} \frac{\sin t}{\cos t}\,dt = [-\ln|\cos t|]_0^{\pi/3} = -\ln\frac{1}{2} + \ln 1 = \ln 2$

31. $\frac{dx}{dt} = -\sin t$ and $\frac{dy}{dt} = \cos t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1 \Rightarrow \text{Area} = \int 2\pi y\,ds$
$= \int_0^{2\pi} 2\pi(2+\sin t)(1)dt = 2\pi\,[2t - \cos t]_0^{2\pi} = 2\pi[(4\pi - 1) - (0-1)] = 8\pi^2$

32. $\frac{dx}{dt} = t^{1/2}$ and $\frac{dy}{dt} = t^{-1/2} \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{t + t^{-1}} = \sqrt{\frac{t^2+1}{t}} \Rightarrow \text{Area} = \int 2\pi x\,ds$
$= \int_0^{\sqrt{3}} 2\pi\left(\frac{2}{3}t^{3/2}\right)\sqrt{\frac{t^2+1}{t}}\,dt = \frac{4\pi}{3}\int_0^{\sqrt{3}} t\sqrt{t^2+1}\,dt$; $[u = t^2+1 \Rightarrow du = 2t\,dt;\ t = 0 \Rightarrow u = 1,$
$\left.t = \sqrt{3} \Rightarrow u = 4\right] \rightarrow \int_1^4 \frac{2\pi}{3}\sqrt{u}\,du = \left[\frac{4\pi}{9}u^{3/2}\right]_1^4 = \frac{28\pi}{9}$
Note: $\int_0^{\sqrt{3}} 2\pi\left(\frac{2}{3}t^{3/2}\right)\sqrt{\frac{t^2+1}{t}}\,dt$ is an improper integral but $\lim_{t\to 0^+} f(t)$ exists and is equal to 0, where $f(t) = 2\pi\left(\frac{2}{3}t^{3/2}\right)\sqrt{\frac{t^2+1}{t}}$. Thus the discontinuity is removable: define $F(t) = f(t)$ for $t > 0$ and $F(0) = 0$
$\Rightarrow \int_0^{\sqrt{3}} F(t)\,dt = \frac{28\pi}{9}$.

33. $\frac{dx}{dt} = 1$ and $\frac{dy}{dt} = t + \sqrt{2} \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{1^2 + \left(t+\sqrt{2}\right)^2} = \sqrt{t^2 + 2\sqrt{2}t + 3} \Rightarrow \text{Area} = \int 2\pi x\,ds$
$= \int_{-\sqrt{2}}^{\sqrt{2}} 2\pi\left(t+\sqrt{2}\right)\sqrt{t^2+2\sqrt{2}t+3}\,dt$; $\left[u = t^2 + 2\sqrt{2}t + 3 \Rightarrow du = \left(2t + 2\sqrt{2}\right)dt;\ t = -\sqrt{2} \Rightarrow u = 1,\right.$
$\left.t = \sqrt{2} \Rightarrow u = 9\right] \rightarrow \int_1^9 \pi\sqrt{u}\,du = \left[\frac{2}{3}\pi u^{3/2}\right]_1^9 = \frac{2\pi}{3}(27-1) = \frac{52\pi}{3}$

34. From Exercise 30, $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \tan t \Rightarrow \text{Area} = \int 2\pi y\,ds = \int_0^{\pi/3} 2\pi\cos t\tan t\,dt = 2\pi\int_0^{\pi/3}\sin t\,dt$
$= 2\pi\,[-\cos t]_0^{\pi/3} = 2\pi\left[-\frac{1}{2} - (-1)\right] = \pi$

35. $\frac{dx}{dt} = 2$ and $\frac{dy}{dt} = 1 \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{2^2+1^2} = \sqrt{5} \Rightarrow \text{Area} = \int 2\pi y\,ds = \int_0^1 2\pi(t+1)\sqrt{5}\,dt$
$= 2\pi\sqrt{5}\left[\frac{t^2}{2} + t\right]_0^1 = 3\pi\sqrt{5}$. Check: slant height is $\sqrt{5} \Rightarrow$ Area is $\pi(1+2)\sqrt{5} = 3\pi\sqrt{5}$.

36. $\frac{dx}{dt} = h$ and $\frac{dy}{dt} = r \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{h^2 + r^2} \Rightarrow \text{Area} = \int 2\pi y\, ds = \int_0^1 2\pi r t\sqrt{h^2 + r^2}\, dt$
$= 2\pi r\sqrt{h^2 + r^2} \int_0^1 t\, dt = 2\pi r\sqrt{h^2 + r^2} \left[\frac{t^2}{2}\right]_0^1 = \pi r\sqrt{h^2 + r^2}$. Check: slant height is $\sqrt{h^2 + r^2} \Rightarrow$ Area is $\pi r\sqrt{h^2 + r^2}$.

37. Let the density be $\delta = 1$. Then $x = \cos t + t \sin t \Rightarrow \frac{dx}{dt} = t \cos t$, and $y = \sin t - t \cos t \Rightarrow \frac{dy}{dt} = t \sin t$
$\Rightarrow dm = 1 \cdot ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \sqrt{(t \cos t)^2 + (t \sin t)^2} = |t|\, dt = t\, dt$ since $0 \le t \le \frac{\pi}{2}$. The curve's mass is $M = \int dm = \int_0^{\pi/2} t\, dt = \frac{\pi^2}{8}$. Also $M_x = \int \tilde{y}\, dm = \int_0^{\pi/2} (\sin t - t \cos t)\, t\, dt = \int_0^{\pi/2} t \sin t\, dt - \int_0^{\pi/2} t^2 \cos t\, dt$
$= [\sin t - t \cos t]_0^{\pi/2} - [t^2 \sin t - 2 \sin t + 2t \cos t]_0^{\pi/2} = 3 - \frac{\pi^2}{4}$, where we integrated by parts. Therefore,
$\overline{y} = \frac{M_x}{M} = \frac{\left(3 - \frac{\pi^2}{4}\right)}{\left(\frac{\pi^2}{8}\right)} = \frac{24}{\pi^2} - 2$. Next, $M_y = \int \tilde{x}\, dm = \int_0^{\pi/2} (\cos t + t \sin t)\, t\, dt = \int_0^{\pi/2} t \cos t\, dt + \int_0^{\pi/2} t^2 \sin t\, dt$
$= [\cos t + t \sin t]_0^{\pi/2} + [-t^2 \cos t + 2 \cos t + 2t \sin t]_0^{\pi/2} = \frac{3\pi}{2} - 3$, again integrating by parts. Hence $\overline{x} = \frac{M_y}{M} = \frac{\left(\frac{3\pi}{2} - 3\right)}{\left(\frac{\pi^2}{8}\right)} = \frac{12}{\pi} - \frac{24}{\pi^2}$. Therefore $(\overline{x}, \overline{y}) = \left(\frac{12}{\pi} - \frac{24}{\pi^2}, \frac{24}{\pi^2} - 2\right)$.

38. Let the density be $\delta = 1$. Then $x = e^t \cos t \Rightarrow \frac{dx}{dt} = e^t \cos t - e^t \sin t$, and $y = e^t \sin t \Rightarrow \frac{dy}{dt} = e^t \sin t + e^t \cos t$
$\Rightarrow dm = 1 \cdot ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \sqrt{\left(e^t \cos t - e^t \sin t\right)^2 + \left(e^t \sin t + e^t \cos t\right)^2}\, dt = \sqrt{2e^{2t}}\, dt = \sqrt{2}\, e^t\, dt$.
The curve's mass is $M = \int dm = \int_0^{\pi} \sqrt{2}\, e^t\, dt = \sqrt{2}\, e^{\pi} - \sqrt{2}$. Also $M_x = \int \tilde{y}\, dm = \int_0^{\pi} (e^t \sin t)\left(\sqrt{2}\, e^t\right) dt$
$= \int_0^{\pi} \sqrt{2}\, e^{2t} \sin t\, dt = \sqrt{2}\left[\frac{e^{2t}}{5}(2 \sin t - \cos t)\right]_0^{\pi} = \sqrt{2}\left(\frac{e^{2\pi}}{5} + \frac{1}{5}\right) \Rightarrow \overline{y} = \frac{M_x}{M} = \frac{\sqrt{2}\left(\frac{e^{2\pi}}{5} + \frac{1}{5}\right)}{\sqrt{2}\, e^{\pi} - \sqrt{2}} = \frac{e^{2\pi} + 1}{5(e^{\pi} - 1)}$.
Next $M_y = \int \tilde{x}\, dm = \int_0^{\pi} (e^t \cos t)\left(\sqrt{2}\, e^t\right) dt = \int_0^{\pi} \sqrt{2}\, e^{2t} \cos t\, dt = \sqrt{2}\left[\frac{e^{2t}}{5}(2 \cos t + \sin t)\right]_0^{\pi} = -\sqrt{2}\left(\frac{2e^{2\pi}}{5} + \frac{2}{5}\right)$
$\Rightarrow \overline{x} = \frac{M_y}{M} = \frac{-\sqrt{2}\left(\frac{2e^{2\pi}}{5} + \frac{2}{5}\right)}{\sqrt{2}\, e^{\pi} - \sqrt{2}} = -\frac{2e^{2\pi} + 2}{5(e^{\pi} - 1)}$. Therefore $(\overline{x}, \overline{y}) = \left(-\frac{2e^{2\pi} + 2}{5(e^{\pi} - 1)}, \frac{e^{2\pi} + 1}{5(e^{\pi} - 1)}\right)$.

39. Let the density be $\delta = 1$. Then $x = \cos t \Rightarrow \frac{dx}{dt} = -\sin t$, and $y = t + \sin t \Rightarrow \frac{dy}{dt} = 1 + \cos t$
$\Rightarrow dm = 1 \cdot ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \sqrt{(-\sin t)^2 + (1 + \cos t)^2}\, dt = \sqrt{2 + 2 \cos t}\, dt$. The curve's mass is $M = \int dm = \int_0^{\pi} \sqrt{2 + 2 \cos t}\, dt = \sqrt{2}\int_0^{\pi} \sqrt{1 + \cos t}\, dt = \sqrt{2}\int_0^{\pi} \sqrt{2 \cos^2\left(\frac{t}{2}\right)}\, dt = 2\int_0^{\pi} \left|\cos\left(\frac{t}{2}\right)\right| dt$
$= 2\int_0^{\pi} \cos\left(\frac{t}{2}\right) dt$ $\left(\text{since } 0 \le t \le \pi \Rightarrow 0 \le \frac{t}{2} \le \frac{\pi}{2}\right) = 2\left[2 \sin\left(\frac{t}{2}\right)\right]_0^{\pi} = 4$. Also $M_x = \int \tilde{y}\, dm$
$= \int_0^{\pi} (t + \sin t)\left(2 \cos \frac{t}{2}\right) dt = \int_0^{\pi} 2t \cos\left(\frac{t}{2}\right) dt + \int_0^{\pi} 2 \sin t \cos\left(\frac{t}{2}\right) dt$
$= 2\left[4 \cos\left(\frac{t}{2}\right) + 2t \sin\left(\frac{t}{2}\right)\right]_0^{\pi} + 2\left[-\frac{1}{3} \cos\left(\frac{3}{2}t\right) - \cos\left(\frac{1}{2}t\right)\right]_0^{\pi} = 4\pi - \frac{16}{3} \Rightarrow \overline{y} = \frac{M_x}{M} = \frac{\left(4\pi - \frac{16}{3}\right)}{4} = \pi - \frac{4}{3}$.
Next $M_y = \int \tilde{x}\, dm = \int_0^{\pi} (\cos t)\left(2 \cos \frac{t}{2}\right) dt = \int_0^{\pi} \cos t \cos\left(\frac{t}{2}\right) dt = 2\left[\sin\left(\frac{t}{2}\right) + \frac{\sin\left(\frac{3}{2}t\right)}{3}\right]_0^{\pi} = 2 - \frac{2}{3}$
$= \frac{4}{3} \Rightarrow \overline{x} = \frac{M_y}{M} = \frac{\left(\frac{4}{3}\right)}{4} = \frac{1}{3}$. Therefore $(\overline{x}, \overline{y}) = \left(\frac{1}{3}, \pi - \frac{4}{3}\right)$.

40. Let the density be $\delta = 1$. Then $x = t^3 \Rightarrow \frac{dx}{dt} = 3t^2$, and $y = \frac{3t^2}{2} \Rightarrow \frac{dy}{dt} = 3t \Rightarrow dm = 1 \cdot ds$
$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \sqrt{\left(3t^2\right)^2 + (3t)^2}\, dt = 3|t|\sqrt{t^2 + 1}\, dt = 3t\sqrt{t^2 + 1}\, dt$ since $0 \le t \le \sqrt{3}$. The curve's mass is $M = \int dm = \int_0^{\sqrt{3}} 3t\sqrt{t^2 + 1}\, dt = \left[\left(t^2 + 1\right)^{3/2}\right]_0^{\sqrt{3}} = 7$. Also $M_x = \int \tilde{y}\, dm = \int_0^{\sqrt{3}} \frac{3t^2}{2}\left(3t\sqrt{t^2 + 1}\right) dt$
$= \frac{9}{2}\int_0^{\sqrt{3}} t^3\sqrt{t^2 + 1}\, dt = \frac{87}{5} = 17.4$ (by computer) $\Rightarrow \overline{y} = \frac{M_x}{M} = \frac{17.4}{7} \approx 2.49$. Next $M_y = \int \tilde{x}\, dm$

$= \int_0^{\sqrt{3}} t^3 \cdot 3t\sqrt{(t^2+1)}\,dt = 3\int_0^{\sqrt{3}} t^4\sqrt{t^2+1}\,dt \approx 16.4849$ (by computer) $\Rightarrow \overline{x} = \frac{M_y}{M} = \frac{16.4849}{7} \approx 2.35$.
Therefore, $(\overline{x}, \overline{y}) \approx (2.35, 2.49)$.

41. (a) $\frac{dx}{dt} = -2\sin 2t$ and $\frac{dy}{dt} = 2\cos 2t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(-2\sin 2t)^2 + (2\cos 2t)^2} = 2$

$\Rightarrow$ Length $= \int_0^{\pi/2} 2\,dt = [2t]_0^{\pi/2} = \pi$

(b) $\frac{dx}{dt} = \pi\cos\pi t$ and $\frac{dy}{dt} = -\pi\sin\pi t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(\pi\cos\pi t)^2 + (-\pi\sin\pi t)^2} = \pi$

$\Rightarrow$ Length $= \int_{-1/2}^{1/2} \pi\,dt = [\pi t]_{-1/2}^{1/2} = \pi$

42. (a) $x = g(y)$ has the parametrization $x = g(y)$ and $y = y$ for $c \le y \le d \Rightarrow \frac{dx}{dy} = g'(y)$ and $\frac{dy}{dy} = 1$; then

Length $= \int_c^d \sqrt{\left(\frac{dy}{dy}\right)^2 + \left(\frac{dx}{dy}\right)^2}\,dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy = \int_c^d \sqrt{1 + [g'(y)]^2}\,dy$

(b) $x = y^{3/2},\ 0 \le y \le \frac{4}{3} \Rightarrow \frac{dx}{dy} = \frac{3}{2}y^{1/2} \Rightarrow L = \int_0^{4/3} \sqrt{1 + \left(\frac{3}{2}y^{1/2}\right)^2}\,dy = \int_0^{4/3} \sqrt{1 + \frac{9}{4}y}\,dy = \left[\frac{4}{9}\cdot\frac{2}{3}\left(1 + \frac{9}{4}y\right)^{3/2}\right]_0^{4/3}$

$= \frac{8}{27}(4)^{3/2} - \frac{8}{27}(1)^{3/2} = \frac{56}{27}$

(c) $x = \frac{3}{2}y^{2/3},\ 0 \le y \le 1 \Rightarrow \frac{dx}{dy} = y^{-1/3} \Rightarrow L = \int_0^1 \sqrt{1 + (y^{-1/3})^2}\,dy = \int_0^1 \sqrt{1 + \frac{1}{y^{2/3}}}\,dy = \lim_{a\to 0^+} \int_a^1 \sqrt{\frac{y^{2/3}+1}{y^{2/3}}}\,dy$

$= \lim_{a\to 0^+} \frac{3}{2}\int_a^1 \left(y^{2/3}+1\right)^{1/2}\left(\frac{2}{3}y^{-1/3}\right)dy = \lim_{a\to 0^+}\left[\frac{3}{2}\cdot\frac{2}{3}\left(y^{2/3}+1\right)^{3/2}\right]_a^1 = \lim_{a\to 0^+}\left((2)^{3/2} - \left(a^{2/3}+1\right)^{3/2}\right) = 2\sqrt{2} - 1$

43. $x = (1 + 2\sin\theta)\cos\theta,\ y = (1 + 2\sin\theta)\sin\theta \Rightarrow \frac{dx}{d\theta} = 2\cos^2\theta - \sin\theta(1 + 2\sin\theta),\ \frac{dy}{d\theta} = 2\cos\theta\sin\theta + \cos\theta(1 + 2\sin\theta)$

$\Rightarrow \frac{dy}{dx} = \frac{2\cos\theta\sin\theta + \cos\theta(1+2\sin\theta)}{2\cos^2\theta - \sin\theta(1+2\sin\theta)} = \frac{4\cos\theta\sin\theta + \cos\theta}{2\cos^2\theta - 2\sin^2\theta - \sin\theta} = \frac{2\sin 2\theta + \cos\theta}{2\cos 2\theta - \sin\theta}$

(a) $x = (1 + 2\sin(0))\cos(0) = 1,\ y = (1 + 2\sin(0))\sin(0) = 0;\ \left.\frac{dy}{dx}\right|_{\theta=0} = \frac{2\sin(2(0)) + \cos(0)}{2\cos(2(0)) - \sin(0)} = \frac{0+1}{2-0} = \frac{1}{2}$

(b) $x = \left(1 + 2\sin\left(\frac{\pi}{2}\right)\right)\cos\left(\frac{\pi}{2}\right) = 0,\ y = \left(1 + 2\sin\left(\frac{\pi}{2}\right)\right)\sin\left(\frac{\pi}{2}\right) = 3;\ \left.\frac{dy}{dx}\right|_{\theta=\pi/2} = \frac{2\sin\left(2\left(\frac{\pi}{2}\right)\right) + \cos\left(\frac{\pi}{2}\right)}{2\cos\left(2\left(\frac{\pi}{2}\right)\right) - \sin\left(\frac{\pi}{2}\right)} = \frac{0+0}{-2-1} = 0$

(c) $x = \left(1 + 2\sin\left(\frac{4\pi}{3}\right)\right)\cos\left(\frac{4\pi}{3}\right) = \frac{\sqrt{3}-1}{2},\ y = \left(1 + 2\sin\left(\frac{4\pi}{3}\right)\right)\sin\left(\frac{4\pi}{3}\right) = \frac{3-\sqrt{3}}{2};\ \left.\frac{dy}{dx}\right|_{\theta=4\pi/3} = \frac{2\sin\left(2\left(\frac{4\pi}{3}\right)\right) + \cos\left(\frac{4\pi}{3}\right)}{2\cos\left(2\left(\frac{4\pi}{3}\right)\right) - \sin\left(\frac{4\pi}{3}\right)}$

$= \frac{\sqrt{3} - \frac{1}{2}}{-1 + \frac{\sqrt{3}}{2}} = \frac{2\sqrt{3}-1}{\sqrt{3}-2} = -\left(4 + 3\sqrt{3}\right)$

44. $x = t,\ y = 1 - \cos t,\ 0 \le t \le 2\pi \Rightarrow \frac{dx}{dt} = 1,\ \frac{dy}{dt} = \sin t \Rightarrow \frac{dy}{dx} = \frac{\sin t}{1} = \sin t \Rightarrow \frac{d}{dt}\left(\frac{dy}{dx}\right) = \cos t \Rightarrow \frac{d^2y}{dx^2} = \frac{\cos t}{1} = \cos t$. The maximum and minimum slope will occur at points that maximize/minimize $\frac{dy}{dx}$, in other words, points where $\frac{d^2y}{dx^2} = 0$

$\Rightarrow \cos t = 0 \Rightarrow t = \frac{\pi}{2}$ or $t = \frac{3\pi}{2} \Rightarrow \frac{d^2y}{dx^2} = +++ \underset{\pi/2}{|} - - - \underset{3\pi/2}{|} +++$

(a) the maximum slope is $\left.\frac{dy}{dx}\right|_{t=\pi/2} = \sin\left(\frac{\pi}{2}\right) = 1$, which occurs at $x = \frac{\pi}{2},\ y = 1 - \cos\left(\frac{\pi}{2}\right) = 1$

(a) the minimum slope is $\left.\frac{dy}{dx}\right|_{t=3\pi/2} = \sin\left(\frac{3\pi}{2}\right) = -1$, which occurs at $x = \frac{3\pi}{2},\ y = 1 - \cos\left(\frac{3\pi}{2}\right) = 1$

45. $\frac{dx}{dt} = \cos t$ and $\frac{dy}{dt} = 2\cos 2t \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\cos 2t}{\cos t} = \frac{2\left(2\cos^2 t - 1\right)}{\cos t}$; then $\frac{dy}{dx} = 0 \Rightarrow \frac{2\left(2\cos^2 t - 1\right)}{\cos t} = 0$

$\Rightarrow 2\cos^2 t - 1 = 0 \Rightarrow \cos t = \pm\frac{1}{\sqrt{2}} \Rightarrow t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$. In the 1st quadrant: $t = \frac{\pi}{4} \Rightarrow x = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ and $y = \sin 2\left(\frac{\pi}{4}\right) = 1 \Rightarrow \left(\frac{\sqrt{2}}{2}, 1\right)$ is the point where the tangent line is horizontal. At the origin: $x = 0$ and $y = 0$

$\Rightarrow \sin t = 0 \Rightarrow t = 0$ or $t = \pi$ and $\sin 2t = 0 \Rightarrow t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$; thus $t = 0$ and $t = \pi$ give the tangent lines at the origin. Tangents at origin: $\left.\frac{dy}{dx}\right|_{t=0} = 2 \Rightarrow y = 2x$ and $\left.\frac{dy}{dx}\right|_{t=\pi} = -2 \Rightarrow y = -2x$

46. $\frac{dx}{dt} = 2\cos 2t$ and $\frac{dy}{dt} = 3\cos 3t \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3\cos 3t}{2\cos 2t} = \frac{3(\cos 2t\cos t - \sin 2t\sin t)}{2(2\cos^2 t - 1)}$
$= \frac{3[(2\cos^2 t - 1)(\cos t) - 2\sin t\cos t\sin t]}{2(2\cos^2 t - 1)} = \frac{(3\cos t)(2\cos^2 t - 1 - 2\sin^2 t)}{2(2\cos^2 t - 1)} = \frac{(3\cos t)(4\cos^2 t - 3)}{2(2\cos^2 t - 1)}$; then
$\frac{dy}{dx} = 0 \Rightarrow \frac{(3\cos t)(4\cos^2 t - 3)}{2(2\cos^2 t - 1)} = 0 \Rightarrow 3\cos t = 0$ or $4\cos^2 t - 3 = 0$: $3\cos t = 0 \Rightarrow t = \frac{\pi}{2}, \frac{3\pi}{2}$ and $4\cos^2 t - 3 = 0 \Rightarrow \cos t = \pm\frac{\sqrt{3}}{2} \Rightarrow t = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$. In the 1st quadrant: $t = \frac{\pi}{6} \Rightarrow x = \sin 2\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ and $y = \sin 3\left(\frac{\pi}{6}\right) = 1 \Rightarrow \left(\frac{\sqrt{3}}{2}, 1\right)$ is the point where the graph has a horizontal tangent. At the origin: $x = 0$ and $y = 0 \Rightarrow \sin 2t = 0$ and $\sin 3t = 0 \Rightarrow t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ and $t = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3} \Rightarrow t = 0$ and $t = \pi$ give the tangent lines at the origin. Tangents at the origin: $\left.\frac{dy}{dx}\right|_{t=0} = \frac{3\cos 0}{2\cos 0} = \frac{3}{2} \Rightarrow y = \frac{3}{2}x$, and $\left.\frac{dy}{dx}\right|_{t=\pi}$
$= \frac{3\cos(3\pi)}{2\cos(2\pi)} = -\frac{3}{2} \Rightarrow y = -\frac{3}{2}x$

47. (a) $x = a(t - \sin t), y = a(1 - \cos t), 0 \le t \le 2\pi \Rightarrow \frac{dx}{dt} = a(1 - \cos t), \frac{dy}{dt} = a\sin t \Rightarrow$ Length
$= \int_0^{2\pi}\sqrt{(a(1-\cos t))^2 + (a\sin t)^2}\,dt = \int_0^{2\pi}\sqrt{a^2 - 2a^2\cos t + a^2\cos^2 t + a^2\sin^2 t}\,dt$
$= a\sqrt{2}\int_0^{2\pi}\sqrt{1 - \cos t}\,dt = a\sqrt{2}\int_0^{2\pi}\sqrt{2\sin^2\left(\frac{t}{2}\right)}\,dt = 2a\int_0^{2\pi}\sin\left(\frac{t}{2}\right)dt = \left[-4a\cos\left(\frac{t}{2}\right)\right]_0^{2\pi}$
$= -4a\cos\pi + 4a\cos(0) = 8a$

(b) $a = 1 \Rightarrow x = t - \sin t, y = 1 - \cos t, 0 \le t \le 2\pi \Rightarrow \frac{dx}{dt} = 1 - \cos t, \frac{dy}{dt} = \sin t \Rightarrow$ Surface area $=$
$= \int_0^{2\pi} 2\pi(1 - \cos t)\sqrt{(1-\cos t)^2 + (\sin t)^2}\,dt = \int_0^{2\pi} 2\pi(1-\cos t)\sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t}\,dt$
$= 2\pi\int_0^{2\pi}(1 - \cos t)\sqrt{2 - 2\cos t}\,dt = 2\sqrt{2}\pi\int_0^{2\pi}(1-\cos t)^{3/2}\,dt = 2\sqrt{2}\pi\int_0^{2\pi}\left(1 - \cos\left(2\cdot\frac{t}{2}\right)\right)^{3/2}dt$
$= 2\sqrt{2}\pi\int_0^{2\pi}\left(2\sin^2\left(\frac{t}{2}\right)\right)^{3/2}dt = 8\pi\int_0^{2\pi}\sin^3\left(\frac{t}{2}\right)dt$
$\left[u = \frac{t}{2} \Rightarrow du = \frac{1}{2}dt \Rightarrow dt = 2\,du;\ t = 0 \Rightarrow u = 0,\ t = 2\pi \Rightarrow u = \pi\right]$
$= 16\pi\int_0^{\pi}\sin^3 u\,du = 16\pi\int_0^{\pi}\sin^2 u\sin u\,du = 16\pi\int_0^{\pi}(1 - \cos^2 u)\sin u\,du = 16\pi\int_0^{\pi}\sin u\,du - 16\pi\int_0^{\pi}\cos^2 u\sin u\,du$
$= \left[-16\pi\cos u + \frac{16\pi}{3}\cos^3 u\right]_0^{\pi} = \left(16\pi - \frac{16\pi}{3}\right) - \left(-16\pi + \frac{16\pi}{3}\right) = \frac{64\pi}{3}$

48. $x = t - \sin t, y = 1 - \cos t, 0 \le t \le 2\pi$; Volume $= \int_0^{2\pi}\pi y^2 dx = \int_0^{2\pi}\pi(1-\cos t)^2(1-\cos t)dt$
$= \pi\int_0^{2\pi}(1 - 3\cos t + 3\cos^2 t - \cos^3 t)dt = \pi\int_0^{2\pi}\left(1 - 3\cos t + 3\left(\frac{1+\cos 2t}{2}\right) - \cos^2 t\cos t\right)dt$
$= \pi\int_0^{2\pi}\left(\frac{5}{2} - 3\cos t + \frac{3}{2}\cos 2t - (1 - \sin^2 t)\cos t\right)dt = \pi\int_0^{2\pi}\left(\frac{5}{2} - 4\cos t + \frac{3}{2}\cos 2t + \sin^2 t\cos t\right)dt$
$= \pi\left[\frac{5}{2}t - 4\sin t + \frac{3}{4}\sin 2t + \frac{1}{3}\sin^3 t\right]_0^{2\pi} = \pi(5\pi - 0 + 0 + 0) - 0 = 5\pi^2$

47-50. Example CAS commands:

Maple:

```
with( plots );
with( student );
x := t -> t^3/3;
y := t -> t^2/2;
a := 0;
b := 1;
N := [2, 4, 8 ];
for n in N do
```

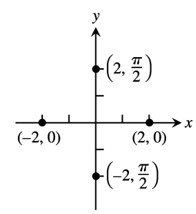
```
  tt := [seq( a+i*(b-a)/n, i=0..n )];
  pts := [seq([x(t),y(t)],t=tt)];
  L := simplify(add( student[distance](pts[i+1],pts[i]), i=1..n ));      # (b)
  T := sprintf("#47(a) (Section 11.2)\nn=%3d  L=%8.5f\n", n, L );
  P[n] := plot( [[x(t),y(t),t=a..b],pts], title=T ):                     # (a)
end do:
display( [seq(P[n],n=N)], insequence=true );
ds := t ->sqrt( simplify(D(x)(t)^2 + D(y)(t)^2) ):                       # (c)
L := Int( ds(t), t=a..b ):
L = evalf(L);
```

11.3 POLAR COORDINATES

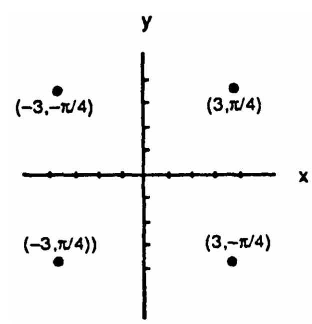
1. a, e; b, g; c, h; d, f

2. a, f; b, h; c, g; d, e

3. (a) $\left(2, \frac{\pi}{2} + 2n\pi\right)$ and $\left(-2, \frac{\pi}{2} + (2n+1)\pi\right)$, n an integer
   (b) $(2, 2n\pi)$ and $(-2, (2n+1)\pi)$, n an integer
   (c) $\left(2, \frac{3\pi}{2} + 2n\pi\right)$ and $\left(-2, \frac{3\pi}{2} + (2n+1)\pi\right)$, n an integer
   (d) $(2, (2n+1)\pi)$ and $(-2, 2n\pi)$, n an integer



4. (a) $\left(3, \frac{\pi}{4} + 2n\pi\right)$ and $\left(-3, \frac{5\pi}{4} + 2n\pi\right)$, n an integer
   (b) $\left(-3, \frac{\pi}{4} + 2n\pi\right)$ and $\left(3, \frac{5\pi}{4} + 2n\pi\right)$, n an integer
   (c) $\left(3, -\frac{\pi}{4} + 2n\pi\right)$ and $\left(-3, \frac{3\pi}{4} + 2n\pi\right)$, n an integer
   (d) $\left(-3, -\frac{\pi}{4} + 2n\pi\right)$ and $\left(3, \frac{3\pi}{4} + 2n\pi\right)$, n an integer



5. (a) $x = r\cos\theta = 3\cos 0 = 3$, $y = r\sin\theta = 3\sin 0 = 0 \Rightarrow$ Cartesian coordinates are $(3, 0)$
   (b) $x = r\cos\theta = -3\cos 0 = -3$, $y = r\sin\theta = -3\sin 0 = 0 \Rightarrow$ Cartesian coordinates are $(-3, 0)$
   (c) $x = r\cos\theta = 2\cos\frac{2\pi}{3} = -1$, $y = r\sin\theta = 2\sin\frac{2\pi}{3} = \sqrt{3} \Rightarrow$ Cartesian coordinates are $\left(-1, \sqrt{3}\right)$
   (d) $x = r\cos\theta = 2\cos\frac{7\pi}{3} = 1$, $y = r\sin\theta = 2\sin\frac{7\pi}{3} = \sqrt{3} \Rightarrow$ Cartesian coordinates are $\left(1, \sqrt{3}\right)$
   (e) $x = r\cos\theta = -3\cos\pi = 3$, $y = r\sin\theta = -3\sin\pi = 0 \Rightarrow$ Cartesian coordinates are $(3, 0)$
   (f) $x = r\cos\theta = 2\cos\frac{\pi}{3} = 1$, $y = r\sin\theta = 2\sin\frac{\pi}{3} = \sqrt{3} \Rightarrow$ Cartesian coordinates are $\left(1, \sqrt{3}\right)$
   (g) $x = r\cos\theta = -3\cos 2\pi = -3$, $y = r\sin\theta = -3\sin 2\pi = 0 \Rightarrow$ Cartesian coordinates are $(-3, 0)$
   (h) $x = r\cos\theta = -2\cos\left(-\frac{\pi}{3}\right) = -1$, $y = r\sin\theta = -2\sin\left(-\frac{\pi}{3}\right) = \sqrt{3} \Rightarrow$ Cartesian coordinates are $\left(-1, \sqrt{3}\right)$

6. (a) $x = \sqrt{2}\cos\frac{\pi}{4} = 1$, $y = \sqrt{2}\sin\frac{\pi}{4} = 1 \Rightarrow$ Cartesian coordinates are $(1, 1)$
   (b) $x = 1\cos 0 = 1$, $y = 1\sin 0 = 0 \Rightarrow$ Cartesian coordinates are $(1, 0)$
   (c) $x = 0\cos\frac{\pi}{2} = 0$, $y = 0\sin\frac{\pi}{2} = 0 \Rightarrow$ Cartesian coordinates are $(0, 0)$
   (d) $x = -\sqrt{2}\cos\left(\frac{\pi}{4}\right) = -1$, $y = -\sqrt{2}\sin\left(\frac{\pi}{4}\right) = -1 \Rightarrow$ Cartesian coordinates are $(-1, -1)$
   (e) $x = -3\cos\frac{5\pi}{6} = \frac{3\sqrt{3}}{2}$, $y = -3\sin\frac{5\pi}{6} = -\frac{3}{2} \Rightarrow$ Cartesian coordinates are $\left(\frac{3\sqrt{3}}{2}, -\frac{3}{2}\right)$
   (f) $x = 5\cos\left(\tan^{-1}\frac{4}{3}\right) = 3$, $y = 5\sin\left(\tan^{-1}\frac{4}{3}\right) = 4 \Rightarrow$ Cartesian coordinates are $(3, 4)$

(g) $x = -1 \cos 7\pi = 1$, $y = -1 \sin 7\pi = 0 \Rightarrow$ Cartesian coordinates are $(1, 0)$

(h) $x = 2\sqrt{3} \cos \frac{2\pi}{3} = -\sqrt{3}$, $y = 2\sqrt{3} \sin \frac{2\pi}{3} = 3 \Rightarrow$ Cartesian coordinates are $\left(-\sqrt{3}, 3\right)$

7. (a) $(1, 1) \Rightarrow r = \sqrt{1^2 + 1^2} = \sqrt{2}$, $\sin\theta = \frac{1}{\sqrt{2}}$ and $\cos\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4} \Rightarrow$ Polar coordinates are $\left(\sqrt{2}, \frac{\pi}{4}\right)$

(b) $(-3, 0) \Rightarrow r = \sqrt{(-3)^2 + 0^2} = 3$, $\sin\theta = 0$ and $\cos\theta = -1 \Rightarrow \theta = \pi \Rightarrow$ Polar coordinates are $(3, \pi)$

(c) $\left(\sqrt{3}, -1\right) \Rightarrow r = \sqrt{\left(\sqrt{3}\right)^2 + (-1)^2} = 2$, $\sin\theta = -\frac{1}{2}$ and $\cos\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{11\pi}{6} \Rightarrow$ Polar coordinates are $\left(2, \frac{11\pi}{6}\right)$

(d) $(-3, 4) \Rightarrow r = \sqrt{(-3)^2 + 4^2} = 5$, $\sin\theta = \frac{4}{5}$ and $\cos\theta = -\frac{3}{5} \Rightarrow \theta = \pi - \arctan\left(\frac{4}{3}\right) \Rightarrow$ Polar coordinates are $\left(5, \pi - \arctan\left(\frac{4}{3}\right)\right)$

8. (a) $(-2, -2) \Rightarrow r = \sqrt{(-2)^2 + (-2)^2} = 2\sqrt{2}$, $\sin\theta = -\frac{1}{\sqrt{2}}$ and $\cos\theta = -\frac{1}{\sqrt{2}} \Rightarrow \theta = -\frac{3\pi}{4} \Rightarrow$ Polar coordinates are $\left(2\sqrt{2}, -\frac{3\pi}{4}\right)$

(b) $(0, 3) \Rightarrow r = \sqrt{0^2 + 3^2} = 3$, $\sin\theta = 1$ and $\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2} \Rightarrow$ Polar coordinates are $\left(3, \frac{\pi}{2}\right)$

(c) $\left(-\sqrt{3}, 1\right) \Rightarrow r = \sqrt{\left(-\sqrt{3}\right)^2 + 1^2} = 2$, $\sin\theta = \frac{1}{2}$ and $\cos\theta = -\frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{5\pi}{6} \Rightarrow$ Polar coordinates are $\left(2, \frac{5\pi}{6}\right)$

(d) $(5, -12) \Rightarrow r = \sqrt{5^2 + (-12)^2} = 13$, $\sin\theta = -\frac{12}{13}$ and $\cos\theta = \frac{5}{12} \Rightarrow \theta = -\arctan\left(\frac{12}{5}\right) \Rightarrow$ Polar coordinates are $\left(13, -\arctan\left(\frac{12}{5}\right)\right)$

9. (a) $(3, 3) \Rightarrow r = -\sqrt{3^2 + 3^2} = -3\sqrt{2}$, $\sin\theta = -\frac{1}{\sqrt{2}}$ and $\cos\theta = -\frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{5\pi}{4} \Rightarrow$ Polar coordinates are $\left(-3\sqrt{2}, \frac{5\pi}{4}\right)$

(b) $(-1, 0) \Rightarrow r = -\sqrt{(-1)^2 + 0^2} = -1$, $\sin\theta = 0$ and $\cos\theta = 1 \Rightarrow \theta = 0 \Rightarrow$ Polar coordinates are $(-1, 0)$

(c) $\left(-1, \sqrt{3}\right) \Rightarrow r = -\sqrt{(-1)^2 + \left(\sqrt{3}\right)^2} = -2$, $\sin\theta = -\frac{\sqrt{3}}{2}$ and $\cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{5\pi}{3} \Rightarrow$ Polar coordinates are $\left(-2, \frac{5\pi}{3}\right)$

(d) $(4, -3) \Rightarrow r = -\sqrt{4^2 + (-3)^2} = -5$, $\sin\theta = \frac{3}{5}$ and $\cos\theta = -\frac{4}{5} \Rightarrow \theta = \pi - \arctan\left(\frac{3}{4}\right) \Rightarrow$ Polar coordinates are $\left(-5, \pi - \arctan\left(\frac{4}{3}\right)\right)$

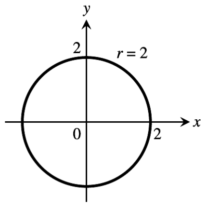
10. (a) $(-2, 0) \Rightarrow r = -\sqrt{(-2)^2 + 0^2} = -2$, $\sin\theta = 0$ and $\cos\theta = 1 \Rightarrow \theta = 0 \Rightarrow$ Polar coordinates are $(-2, 0)$

(b) $(1, 0) \Rightarrow r = -\sqrt{1^2 + 0^2} = -1$, $\sin\theta = 0$ and $\cos\theta = -1 \Rightarrow \theta = \pi$ or $\theta = -\pi \Rightarrow$ Polar coordinates are $(-1, \pi)$ or $(-1, -\pi)$

(c) $(0, -3) \Rightarrow r = -\sqrt{0^2 + (-3)^2} = -3$, $\sin\theta = 1$ and $\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2} \Rightarrow$ Polar coordinates are $\left(-3, \frac{\pi}{2}\right)$

(d) $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) \Rightarrow r = -\sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = -1$, $\sin\theta = -\frac{1}{2}$ and $\cos\theta = -\frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{7\pi}{6}$ or $\theta = -\frac{5\pi}{6} \Rightarrow$ Polar coordinates are $\left(-1, \frac{7\pi}{6}\right)$ or $\left(-1, -\frac{5\pi}{6}\right)$

11.



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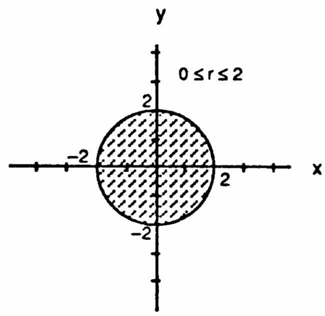

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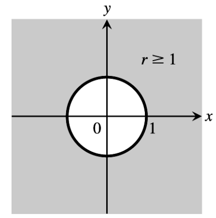

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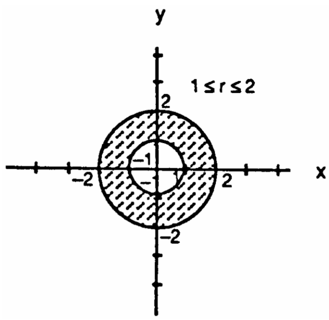

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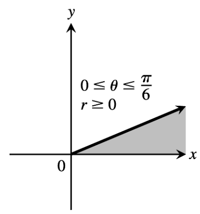

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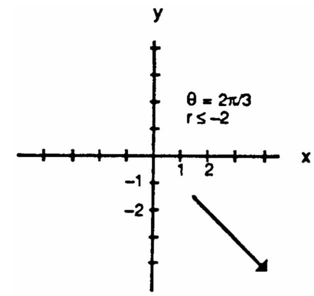

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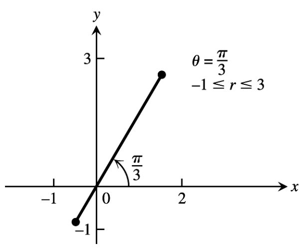

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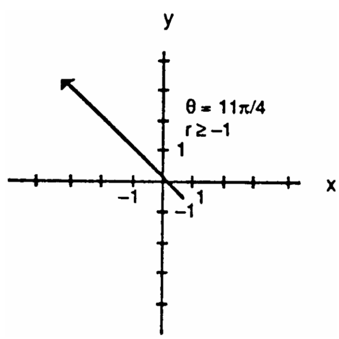

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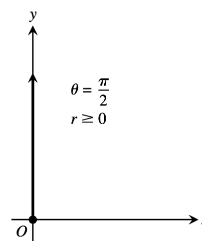

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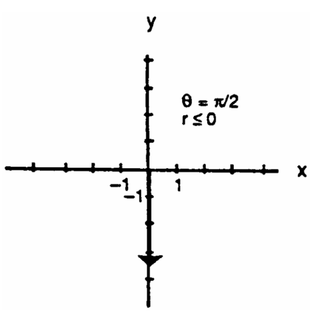

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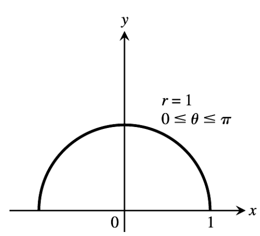

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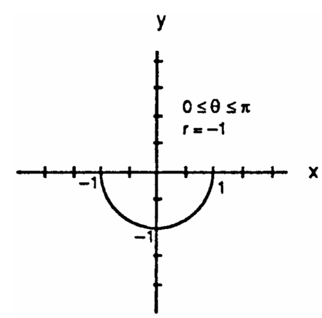

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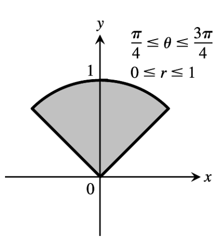

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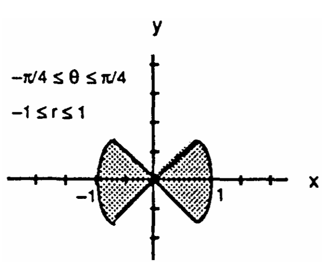

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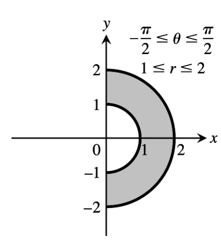
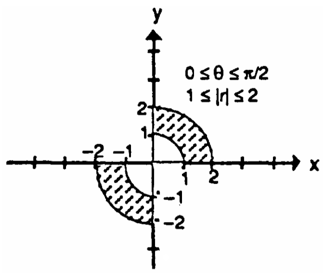


26. 

27. $r \cos\theta = 2 \Rightarrow x = 2$, vertical line through $(2, 0)$

28. $r \sin\theta = -1 \Rightarrow y = -1$, horizontal line through $(0, -1)$

29. $r \sin\theta = 0 \Rightarrow y = 0$, the x-axis

30. $r \cos\theta = 0 \Rightarrow x = 0$, the y-axis

31. $r = 4 \csc\theta \Rightarrow r = \frac{4}{\sin\theta} \Rightarrow r \sin\theta = 4 \Rightarrow y = 4$, a horizontal line through $(0, 4)$

32. $r = -3 \sec\theta \Rightarrow r = \frac{-3}{\cos\theta} \Rightarrow r \cos\theta = -3 \Rightarrow x = -3$, a vertical line through $(-3, 0)$

33. $r \cos\theta + r \sin\theta = 1 \Rightarrow x + y = 1$, line with slope $m = -1$ and intercept $b = 1$

34. $r \sin\theta = r \cos\theta \Rightarrow y = x$, line with slope $m = 1$ and intercept $b = 0$

35. $r^2 = 1 \Rightarrow x^2 + y^2 = 1$, circle with center $C = (0, 0)$ and radius 1

36. $r^2 = 4r \sin\theta \Rightarrow x^2 + y^2 = 4y \Rightarrow x^2 + y^2 - 4y + 4 = 4 \Rightarrow x^2 + (y - 2)^2 = 4$, circle with center $C = (0, 2)$ and radius 2

37. $r = \frac{5}{\sin\theta - 2\cos\theta} \Rightarrow r \sin\theta - 2r \cos\theta = 5 \Rightarrow y - 2x = 5$, line with slope $m = 2$ and intercept $b = 5$

38. $r^2 \sin 2\theta = 2 \Rightarrow 2r^2 \sin\theta \cos\theta = 2 \Rightarrow (r \sin\theta)(r \cos\theta) = 1 \Rightarrow xy = 1$, hyperbola with focal axis $y = x$

39. $r = \cot\theta \csc\theta = \left(\frac{\cos\theta}{\sin\theta}\right)\left(\frac{1}{\sin\theta}\right) \Rightarrow r \sin^2\theta = \cos\theta \Rightarrow r^2 \sin^2\theta = r \cos\theta \Rightarrow y^2 = x$, parabola with vertex $(0, 0)$ which opens to the right

40. $r = 4 \tan\theta \sec\theta \Rightarrow r = 4\left(\frac{\sin\theta}{\cos^2\theta}\right) \Rightarrow r \cos^2\theta = 4 \sin\theta \Rightarrow r^2 \cos^2\theta = 4r \sin\theta \Rightarrow x^2 = 4y$, parabola with vertex $= (0, 0)$ which opens upward

41. $r = (\csc\theta)\, e^{r\cos\theta} \Rightarrow r \sin\theta = e^{r\cos\theta} \Rightarrow y = e^x$, graph of the natural exponential function

42. $r \sin\theta = \ln r + \ln\cos\theta = \ln(r \cos\theta) \Rightarrow y = \ln x$, graph of the natural logarithm function

43. $r^2 + 2r^2 \cos\theta \sin\theta = 1 \Rightarrow x^2 + y^2 + 2xy = 1 \Rightarrow x^2 + 2xy + y^2 = 1 \Rightarrow (x + y)^2 = 1 \Rightarrow x + y = \pm 1$, two parallel straight lines of slope -1 and y-intercepts $b = \pm 1$

44. $\cos^2\theta = \sin^2\theta \Rightarrow r^2 \cos^2\theta = r^2 \sin^2\theta \Rightarrow x^2 = y^2 \Rightarrow |x| = |y| \Rightarrow \pm x = y$, two perpendicular lines through the origin with slopes 1 and -1, respectively.

45. $r^2 = -4r \cos\theta \Rightarrow x^2 + y^2 = -4x \Rightarrow x^2 + 4x + y^2 = 0 \Rightarrow x^2 + 4x + 4 + y^2 = 4 \Rightarrow (x + 2)^2 + y^2 = 4$, a circle with center $C(-2, 0)$ and radius 2

46. $r^2 = -6r \sin\theta \Rightarrow x^2 + y^2 = -6y \Rightarrow x^2 + y^2 + 6y = 0 \Rightarrow x^2 + y^2 + 6y + 9 = 9 \Rightarrow x^2 + (y+3)^2 = 9$, a circle with center $C(0, -3)$ and radius 3

47. $r = 8\sin\theta \Rightarrow r^2 = 8r\sin\theta \Rightarrow x^2 + y^2 = 8y \Rightarrow x^2 + y^2 - 8y = 0 \Rightarrow x^2 + y^2 - 8y + 16 = 16 \Rightarrow x^2 + (y-4)^2 = 16$, a circle with center $C(0, 4)$ and radius 4

48. $r = 3\cos\theta \Rightarrow r^2 = 3r\cos\theta \Rightarrow x^2 + y^2 = 3x \Rightarrow x^2 + y^2 - 3x = 0 \Rightarrow x^2 - 3x + \frac{9}{4} + y^2 = \frac{9}{4}$
$\Rightarrow \left(x - \frac{3}{2}\right)^2 + y^2 = \frac{9}{4}$, a circle with center $C\left(\frac{3}{2}, 0\right)$ and radius $\frac{3}{2}$

49. $r = 2\cos\theta + 2\sin\theta \Rightarrow r^2 = 2r\cos\theta + 2r\sin\theta \Rightarrow x^2 + y^2 = 2x + 2y \Rightarrow x^2 - 2x + y^2 - 2y = 0$
$\Rightarrow (x-1)^2 + (y-1)^2 = 2$, a circle with center $C(1, 1)$ and radius $\sqrt{2}$

50. $r = 2\cos\theta - \sin\theta \Rightarrow r^2 = 2r\cos\theta - r\sin\theta \Rightarrow x^2 + y^2 = 2x - y \Rightarrow x^2 - 2x + y^2 + y = 0$
$\Rightarrow (x-1)^2 + \left(y + \frac{1}{2}\right)^2 = \frac{5}{4}$, a circle with center $C\left(1, -\frac{1}{2}\right)$ and radius $\frac{\sqrt{5}}{2}$

51. $r\sin\left(\theta + \frac{\pi}{6}\right) = 2 \Rightarrow r\left(\sin\theta\cos\frac{\pi}{6} + \cos\theta\sin\frac{\pi}{6}\right) = 2 \Rightarrow \frac{\sqrt{3}}{2} r\sin\theta + \frac{1}{2} r\cos\theta = 2 \Rightarrow \frac{\sqrt{3}}{2} y + \frac{1}{2} x = 2$
$\Rightarrow \sqrt{3}\,y + x = 4$, line with slope $m = -\frac{1}{\sqrt{3}}$ and intercept $b = \frac{4}{\sqrt{3}}$

52. $r\sin\left(\frac{2\pi}{3} - \theta\right) = 5 \Rightarrow r\left(\sin\frac{2\pi}{3}\cos\theta - \cos\frac{2\pi}{3}\sin\theta\right) = 5 \Rightarrow \frac{\sqrt{3}}{2} r\cos\theta + \frac{1}{2} r\sin\theta = 5 \Rightarrow \frac{\sqrt{3}}{2} x + \frac{1}{2} y = 5$
$\Rightarrow \sqrt{3}\,x + y = 10$, line with slope $m = -\sqrt{3}$ and intercept $b = 10$

53. $x = 7 \Rightarrow r\cos\theta = 7$

54. $y = 1 \Rightarrow r\sin\theta = 1$

55. $x = y \Rightarrow r\cos\theta = r\sin\theta \Rightarrow \theta = \frac{\pi}{4}$

56. $x - y = 3 \Rightarrow r\cos\theta - r\sin\theta = 3$

57. $x^2 + y^2 = 4 \Rightarrow r^2 = 4 \Rightarrow r = 2$ or $r = -2$

58. $x^2 - y^2 = 1 \Rightarrow r^2\cos^2\theta - r^2\sin^2\theta = 1 \Rightarrow r^2(\cos^2\theta - \sin^2\theta) = 1 \Rightarrow r^2\cos 2\theta = 1$

59. $\frac{x^2}{9} + \frac{y^2}{4} = 1 \Rightarrow 4x^2 + 9y^2 = 36 \Rightarrow 4r^2\cos^2\theta + 9r^2\sin^2\theta = 36$

60. $xy = 2 \Rightarrow (r\cos\theta)(r\sin\theta) = 2 \Rightarrow r^2\cos\theta\sin\theta = 2 \Rightarrow 2r^2\cos\theta\sin\theta = 4 \Rightarrow r^2\sin 2\theta = 4$

61. $y^2 = 4x \Rightarrow r^2\sin^2\theta = 4r\cos\theta \Rightarrow r\sin^2\theta = 4\cos\theta$

62. $x^2 + xy + y^2 = 1 \Rightarrow x^2 + y^2 + xy = 1 \Rightarrow r^2 + r^2\sin\theta\cos\theta = 1 \Rightarrow r^2(1 + \sin\theta\cos\theta) = 1$

63. $x^2 + (y-2)^2 = 4 \Rightarrow x^2 + y^2 - 4y + 4 = 4 \Rightarrow x^2 + y^2 = 4y \Rightarrow r^2 = 4r\sin\theta \Rightarrow r = 4\sin\theta$

64. $(x-5)^2 + y^2 = 25 \Rightarrow x^2 - 10x + 25 + y^2 = 25 \Rightarrow x^2 + y^2 = 10x \Rightarrow r^2 = 10r\cos\theta \Rightarrow r = 10\cos\theta$

65. $(x-3)^2 + (y+1)^2 = 4 \Rightarrow x^2 - 6x + 9 + y^2 + 2y + 1 = 4 \Rightarrow x^2 + y^2 = 6x - 2y - 6 \Rightarrow r^2 = 6r\cos\theta - 2r\sin\theta - 6$

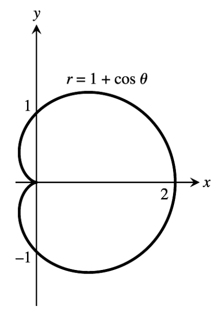
66. $(x+2)^2 + (y-5)^2 = 16 \Rightarrow x^2 + 4x + 4 + y^2 - 10y + 25 = 16 \Rightarrow x^2 + y^2 = -4x + 10y - 13$
$\Rightarrow r^2 = -4r\cos\theta + 10r\sin\theta - 13$

67. $(0, \theta)$ where θ is any angle

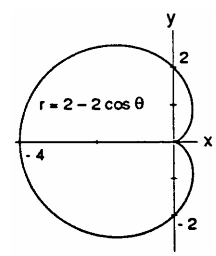
68. (a) $x = a \Rightarrow r\cos\theta = a \Rightarrow r = \frac{a}{\cos\theta} \Rightarrow r = a\sec\theta$
(b) $y = b \Rightarrow r\sin\theta = b \Rightarrow r = \frac{b}{\sin\theta} \Rightarrow r = b\csc\theta$

11.4 GRAPHING IN POLAR COORDINATES

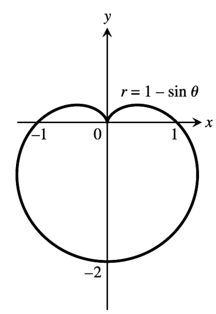
1. $1 + \cos(-\theta) = 1 + \cos\theta = r \Rightarrow$ symmetric about the x-axis; $1 + \cos(-\theta) \neq -r$ and $1 + \cos(\pi - \theta) = 1 - \cos\theta \neq r \Rightarrow$ not symmetric about the y-axis; therefore not symmetric about the origin



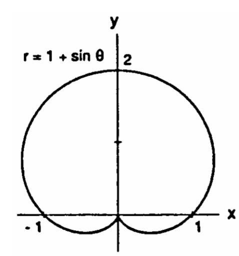
2. $2 - 2\cos(-\theta) = 2 - 2\cos\theta = r \Rightarrow$ symmetric about the x-axis; $2 - 2\cos(-\theta) \neq -r$ and $2 - 2\cos(\pi - \theta) = 2 + 2\cos\theta \neq r \Rightarrow$ not symmetric about the y-axis; therefore not symmetric about the origin



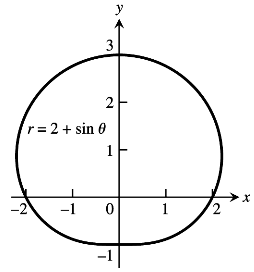
3. $1 - \sin(-\theta) = 1 + \sin\theta \neq r$ and $1 - \sin(\pi - \theta) = 1 - \sin\theta \neq -r \Rightarrow$ not symmetric about the x-axis; $1 - \sin(\pi - \theta) = 1 - \sin\theta = r \Rightarrow$ symmetric about the y-axis; therefore not symmetric about the origin



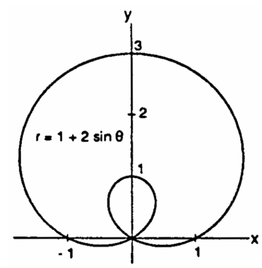
4. $1 + \sin(-\theta) = 1 - \sin\theta \neq r$ and $1 + \sin(\pi - \theta) = 1 + \sin\theta \neq -r \Rightarrow$ not symmetric about the x-axis; $1 + \sin(\pi - \theta) = 1 + \sin\theta = r \Rightarrow$ symmetric about the y-axis; therefore not symmetric about the origin



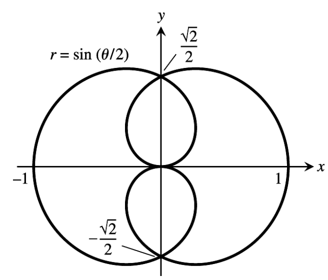
5. $2 + \sin(-\theta) = 2 - \sin\theta \neq \text{r}$ and $2 + \sin(\pi - \theta)$ $= 2 + \sin\theta \neq -\text{r} \Rightarrow$ not symmetric about the x-axis; $2 + \sin(\pi - \theta) = 2 + \sin\theta = \text{r} \Rightarrow$ symmetric about the y-axis; therefore not symmetric about the origin



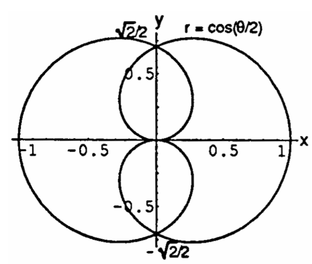
6. $1 + 2\sin(-\theta) = 1 - 2\sin\theta \neq \text{r}$ and $1 + 2\sin(\pi - \theta)$ $= 1 + 2\sin\theta \neq -\text{r} \Rightarrow$ not symmetric about the x-axis; $1 + 2\sin(\pi - \theta) = 1 + 2\sin\theta = \text{r} \Rightarrow$ symmetric about the y-axis; therefore not symmetric about the origin



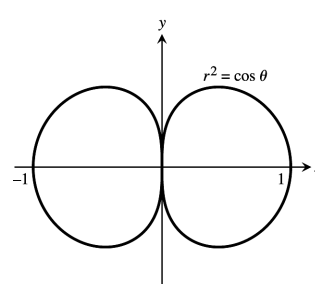
7. $\sin\left(-\frac{\theta}{2}\right) = -\sin\left(\frac{\theta}{2}\right) = -\text{r} \Rightarrow$ symmetric about the y-axis; $\sin\left(\frac{2\pi - \theta}{2}\right) = \sin\left(\frac{\theta}{2}\right)$, so the graph is symmetric about the x-axis, and hence the origin.



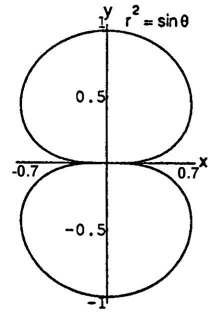
8. $\cos\left(-\frac{\theta}{2}\right) = \cos\left(\frac{\theta}{2}\right) = \text{r} \Rightarrow$ symmetric about the x-axis; $\cos\left(\frac{2\pi - \theta}{2}\right) = \cos\left(\frac{\theta}{2}\right)$, so the graph is symmetric about the y-axis, and hence the origin.



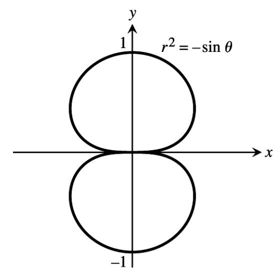
9. $\cos(-\theta) = \cos\theta = \text{r}^2 \Rightarrow (\text{r}, -\theta)$ and $(-\text{r}, -\theta)$ are on the graph when (r, θ) is on the graph $\Rightarrow$ symmetric about the x-axis and the y-axis; therefore symmetric about the origin



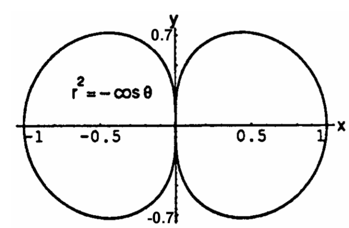
10. $\sin(\pi - \theta) = \sin\theta = r^2 \Rightarrow (r, \pi - \theta)$ and $(-r, \pi - \theta)$ are on the graph when (r, θ) is on the graph $\Rightarrow$ symmetric about the y-axis and the x-axis; therefore symmetric about the origin



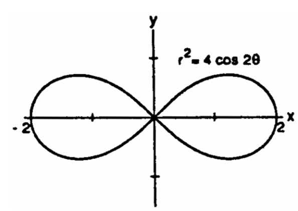
11. $-\sin(\pi - \theta) = -\sin\theta = r^2 \Rightarrow (r, \pi - \theta)$ and $(-r, \pi - \theta)$ are on the graph when (r, θ) is on the graph $\Rightarrow$ symmetric about the y-axis and the x-axis; therefore symmetric about the origin



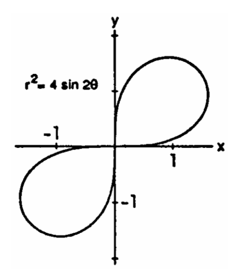
12. $-\cos(-\theta) = -\cos\theta = r^2 \Rightarrow (r, -\theta)$ and $(-r, -\theta)$ are on the graph when (r, θ) is on the graph $\Rightarrow$ symmetric about the x-axis and the y-axis; therefore symmetric about the origin



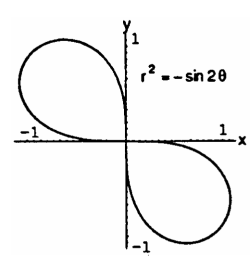
13. Since $(\pm r, -\theta)$ are on the graph when (r, θ) is on the graph $\left((\pm r)^2 = 4\cos 2(-\theta) \Rightarrow r^2 = 4\cos 2\theta\right)$, the graph is symmetric about the x-axis and the y-axis $\Rightarrow$ the graph is symmetric about the origin



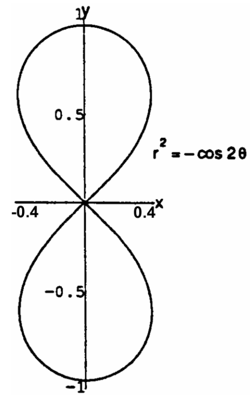
14. Since (r, θ) on the graph $\Rightarrow (-r, \theta)$ is on the graph $\left((\pm r)^2 = 4\sin 2\theta \Rightarrow r^2 = 4\sin 2\theta\right)$, the graph is symmetric about the origin. But $4\sin 2(-\theta) = -4\sin 2\theta \neq r^2$ and $4\sin 2(\pi - \theta) = 4\sin(2\pi - 2\theta) = 4\sin(-2\theta) = -4\sin 2\theta \neq r^2 \Rightarrow$ the graph is not symmetric about the x-axis; therefore the graph is not symmetric about the y-axis



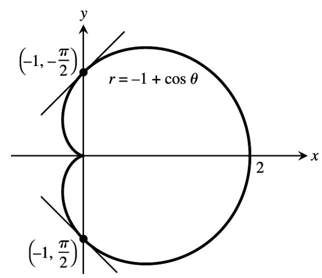
15. Since (r, θ) on the graph $\Rightarrow (-r, \theta)$ is on the graph $\left((\pm r)^2 = -\sin 2\theta \Rightarrow r^2 = -\sin 2\theta\right)$, the graph is symmetric about the origin. But $-\sin 2(-\theta) = -(-\sin 2\theta)$ $\sin 2\theta \neq r^2$ and $-\sin 2(\pi - \theta) = -\sin(2\pi - 2\theta) = -\sin(-2\theta) = -(-\sin 2\theta) = \sin 2\theta \neq r^2 \Rightarrow$ the graph is not symmetric about the x-axis; therefore the graph is not symmetric about the y-axis



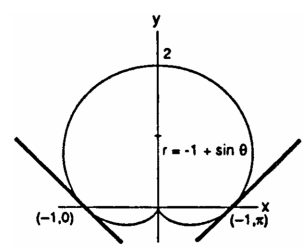
16. Since$(\pm \mathrm{r}, -\theta)$ are on the graph when (r, θ) is on the graph $\left((\pm \mathrm{r})^2 = -\cos 2(-\theta) \Rightarrow \mathrm{r}^2 = -\cos 2\theta\right)$, the graph is symmetric about the x-axis and the y-axis $\Rightarrow$ the graph is symmetric about the origin.



17. $\theta = \frac{\pi}{2} \Rightarrow \mathrm{r} = -1 \Rightarrow \left(-1, \frac{\pi}{2}\right)$, and $\theta = -\frac{\pi}{2} \Rightarrow \mathrm{r} = -1 \Rightarrow \left(-1, -\frac{\pi}{2}\right)$; $\mathrm{r}' = \frac{d\mathrm{r}}{d\theta} = -\sin\theta$; Slope $= \frac{\mathrm{r}'\sin\theta + \mathrm{r}\cos\theta}{\mathrm{r}'\cos\theta - \mathrm{r}\sin\theta} = \frac{-\sin^2\theta + \mathrm{r}\cos\theta}{-\sin\theta\cos\theta - \mathrm{r}\sin\theta} \Rightarrow$ Slope at $\left(-1, \frac{\pi}{2}\right)$ is $\frac{-\sin^2\left(\frac{\pi}{2}\right) + (-1)\cos\frac{\pi}{2}}{-\sin\frac{\pi}{2}\cos\frac{\pi}{2} - (-1)\sin\frac{\pi}{2}} = -1$; Slope at $\left(-1, -\frac{\pi}{2}\right)$ is $\frac{-\sin^2\left(-\frac{\pi}{2}\right) + (-1)\cos\left(-\frac{\pi}{2}\right)}{-\sin\left(-\frac{\pi}{2}\right)\cos\left(-\frac{\pi}{2}\right) - (-1)\sin\left(-\frac{\pi}{2}\right)} = 1$



18. $\theta = 0 \Rightarrow \mathrm{r} = -1 \Rightarrow (-1, 0)$, and $\theta = \pi \Rightarrow \mathrm{r} = -1 \Rightarrow (-1, \pi)$; $\mathrm{r}' = \frac{d\mathrm{r}}{d\theta} = \cos\theta$;
Slope $= \frac{\mathrm{r}'\sin\theta + \mathrm{r}\cos\theta}{\mathrm{r}'\cos\theta - \mathrm{r}\sin\theta} = \frac{\cos\theta\sin\theta + \mathrm{r}\cos\theta}{\cos\theta\cos\theta - \mathrm{r}\sin\theta} = \frac{\cos\theta\sin\theta + \mathrm{r}\cos\theta}{\cos^2\theta - \mathrm{r}\sin\theta} \Rightarrow$ Slope at $(-1, 0)$ is $\frac{\cos 0\sin 0 + (-1)\cos 0}{\cos^2 0 - (-1)\sin 0} = -1$; Slope at $(-1, \pi)$ is $\frac{\cos\pi\sin\pi + (-1)\cos\pi}{\cos^2\pi - (-1)\sin\pi} = 1$

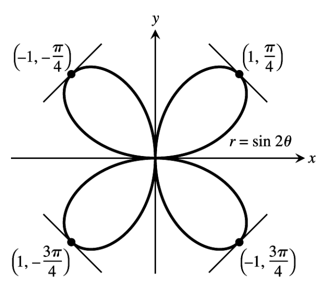


19. $\theta = \frac{\pi}{4} \Rightarrow \mathrm{r} = 1 \Rightarrow \left(1, \frac{\pi}{4}\right)$; $\theta = -\frac{\pi}{4} \Rightarrow \mathrm{r} = -1 \Rightarrow \left(-1, -\frac{\pi}{4}\right)$; $\theta = \frac{3\pi}{4} \Rightarrow \mathrm{r} = -1 \Rightarrow \left(-1, \frac{3\pi}{4}\right)$; $\theta = -\frac{3\pi}{4} \Rightarrow \mathrm{r} = 1 \Rightarrow \left(1, -\frac{3\pi}{4}\right)$;
$\mathrm{r}' = \frac{d\mathrm{r}}{d\theta} = 2\cos 2\theta$;
Slope $= \frac{\mathrm{r}'\sin\theta + \mathrm{r}\cos\theta}{\mathrm{r}'\cos\theta - \mathrm{r}\sin\theta} = \frac{2\cos 2\theta\sin\theta + \mathrm{r}\cos\theta}{2\cos 2\theta\cos\theta - \mathrm{r}\sin\theta}$
$\Rightarrow$ Slope at $\left(1, \frac{\pi}{4}\right)$ is $\frac{2\cos\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{4}\right) + (1)\cos\left(\frac{\pi}{4}\right)}{2\cos\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{4}\right) - (1)\sin\left(\frac{\pi}{4}\right)} = -1$;

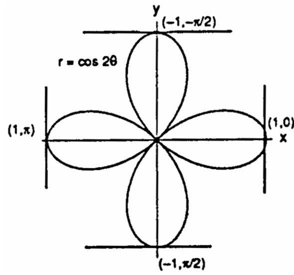
Slope at $\left(-1, -\frac{\pi}{4}\right)$ is $\frac{2\cos\left(-\frac{\pi}{2}\right)\sin\left(-\frac{\pi}{4}\right) + (-1)\cos\left(-\frac{\pi}{4}\right)}{2\cos\left(-\frac{\pi}{2}\right)\cos\left(-\frac{\pi}{4}\right) - (-1)\sin\left(-\frac{\pi}{4}\right)} = 1$;

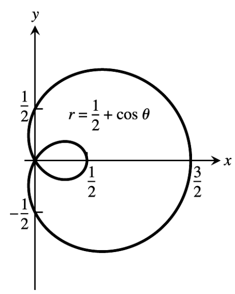
Slope at $\left(-1, \frac{3\pi}{4}\right)$ is $\frac{2\cos\left(\frac{3\pi}{2}\right)\sin\left(\frac{3\pi}{4}\right) + (-1)\cos\left(\frac{3\pi}{4}\right)}{2\cos\left(\frac{3\pi}{2}\right)\cos\left(\frac{3\pi}{4}\right) - (-1)\sin\left(\frac{3\pi}{4}\right)} = 1$;

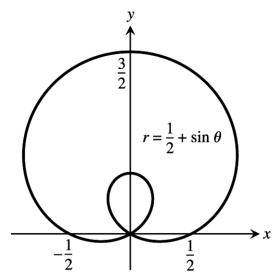
Slope at $\left(1, -\frac{3\pi}{4}\right)$ is $\frac{2\cos\left(-\frac{3\pi}{2}\right)\sin\left(-\frac{3\pi}{4}\right) + (1)\cos\left(-\frac{3\pi}{4}\right)}{2\cos\left(-\frac{3\pi}{2}\right)\cos\left(-\frac{3\pi}{4}\right) - (1)\sin\left(-\frac{3\pi}{4}\right)} = -1$

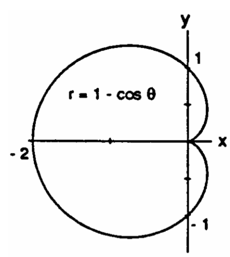


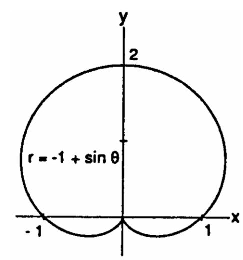
20. $\theta = 0 \Rightarrow r = 1 \Rightarrow (1, 0)$; $\theta = \frac{\pi}{2} \Rightarrow r = -1 \Rightarrow \left(-1, \frac{\pi}{2}\right)$;
$\theta = -\frac{\pi}{2} \Rightarrow r = -1 \Rightarrow \left(-1, -\frac{\pi}{2}\right)$; $\theta = \pi \Rightarrow r = 1$
$\Rightarrow (1, \pi)$; $r' = \frac{dr}{d\theta} = -2 \sin 2\theta$;
Slope $= \frac{r' \sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta} = \frac{-2\sin 2\theta \sin\theta + r\cos\theta}{-2\sin 2\theta\cos\theta - r\sin\theta}$
$\Rightarrow$ Slope at $(1, 0)$ is $\frac{-2\sin 0 \sin 0 + \cos 0}{-2\sin 0\cos 0 - \sin 0}$, which is undefined;
Slope at $\left(-1, \frac{\pi}{2}\right)$ is $\frac{-2\sin 2\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right) + (-1)\cos\left(\frac{\pi}{2}\right)}{-2\sin 2\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right) - (-1)\sin\left(\frac{\pi}{2}\right)} = 0$;
Slope at $\left(-1, -\frac{\pi}{2}\right)$ is $\frac{-2\sin 2\left(-\frac{\pi}{2}\right)\sin\left(-\frac{\pi}{2}\right) + (-1)\cos\left(-\frac{\pi}{2}\right)}{-2\sin 2\left(-\frac{\pi}{2}\right)\cos\left(-\frac{\pi}{2}\right) - (-1)\sin\left(-\frac{\pi}{2}\right)} = 0$;
Slope at $(1, \pi)$ is $\frac{-2\sin 2\pi \sin\pi + \cos\pi}{-2\sin 2\pi\cos\pi - \sin\pi}$, which is undefined

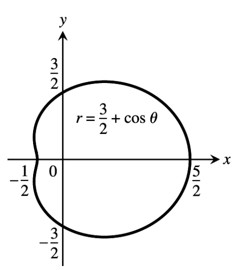


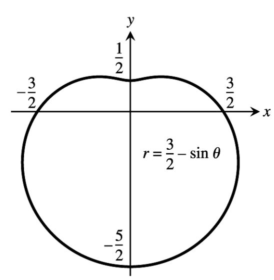
21. (a)


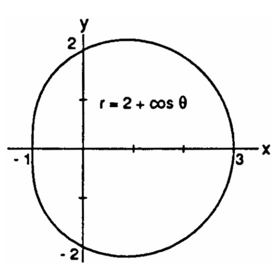
(b)


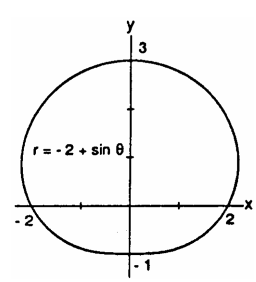
22. (a)


(b)


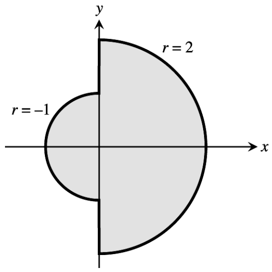
23. (a)


(b)


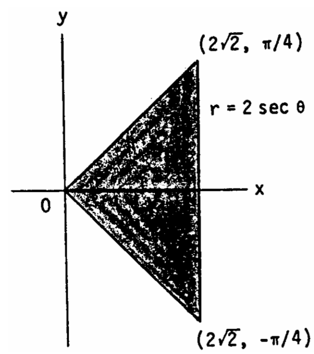
24. (a)


(b)


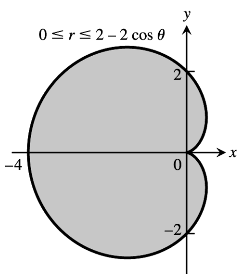
25.



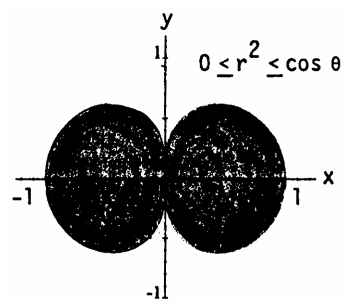
26. $r = 2\sec\theta \Rightarrow r = \frac{2}{\cos\theta} \Rightarrow r\cos\theta = 2 \Rightarrow x = 2$



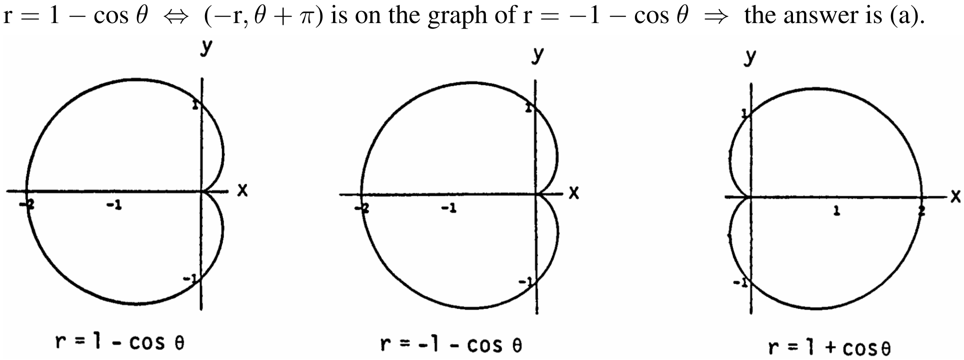
27.



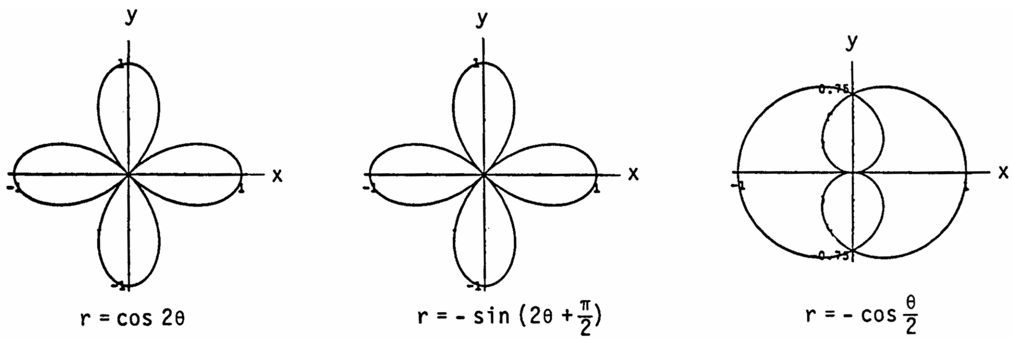
28.



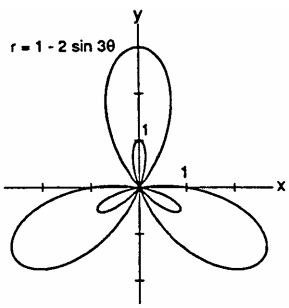
29. Note that (r, θ) and $(-r, \theta + \pi)$ describe the same point in the plane. Then $r = 1 - \cos\theta \Leftrightarrow -1 - \cos(\theta + \pi)$ $= -1 - (\cos\theta\cos\pi - \sin\theta\sin\pi) = -1 + \cos\theta = -(1 - \cos\theta) = -r$; therefore (r, θ) is on the graph of $r = 1 - \cos\theta \Leftrightarrow (-r, \theta + \pi)$ is on the graph of $r = -1 - \cos\theta \Rightarrow$ the answer is (a).



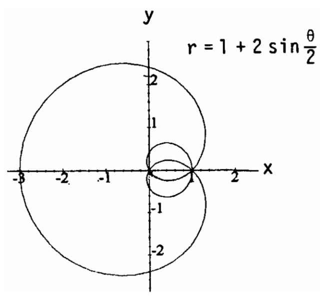
30. Note that (r, θ) and $(-r, \theta + \pi)$ describe the same point in the plane. Then $r = \cos 2\theta \Leftrightarrow -\sin\left(2(\theta + \pi)) + \frac{\pi}{2}\right)$ $= -\sin\left(2\theta + \frac{5\pi}{2}\right) = -\sin(2\theta)\cos\left(\frac{5\pi}{2}\right) - \cos(2\theta)\sin\left(\frac{5\pi}{2}\right) = -\cos 2\theta = -r$; therefore (r, θ) is on the graph of $r = -\sin\left(2\theta + \frac{\pi}{2}\right) \Rightarrow$ the answer is (a).



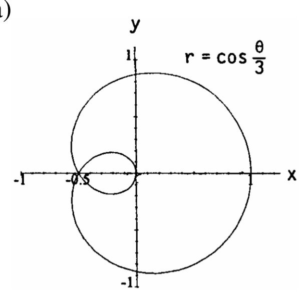
31.



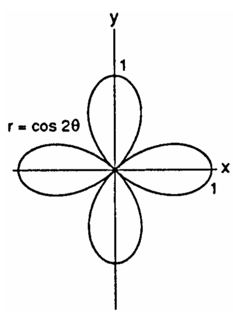
32.



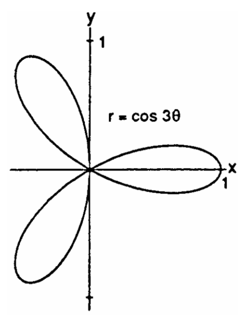
33. (a)



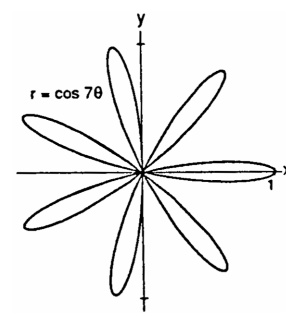
(b)



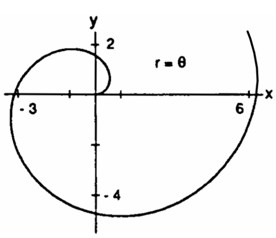
(c)



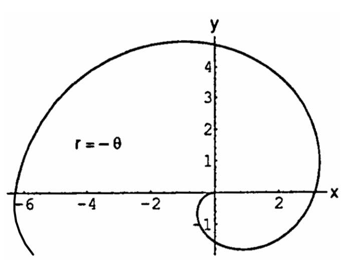
(d)



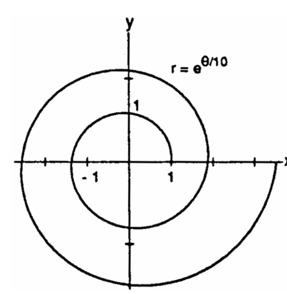
34. (a)



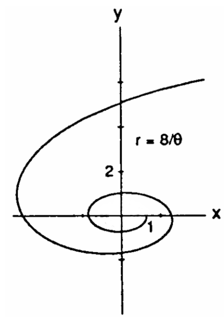
(b)



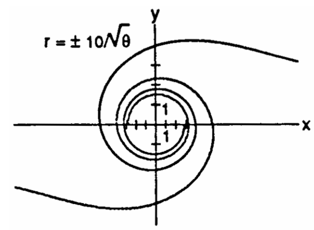
(c)



(d)



(e)



11.5 AREA AND LENGTHS IN POLAR COORDINATES

1. $A = \int_0^{\pi} \frac{1}{2}\theta^2\, d\theta = \left[\frac{1}{6}\theta^3\right]_0^{\pi} = \frac{\pi^3}{6}$

2. $A = \int_{\pi/4}^{\pi/2} \frac{1}{2}(2\sin\theta)^2\, d\theta = 2\int_{\pi/4}^{\pi/2} \sin^2\theta\, d\theta = 2\int_{\pi/4}^{\pi/2} \frac{1-\cos 2\theta}{2}\, d\theta = \int_{\pi/4}^{\pi/2} (1-\cos 2\theta) d\theta = \left[\theta - \frac{1}{2}\sin 2\theta\right]_{\pi/4}^{\pi/2}$
$= \left(\frac{\pi}{2} - 0\right) - \left(\frac{\pi}{4} - \frac{1}{2}\right) = \frac{\pi}{4} + \frac{1}{2}$

3. $A = \int_0^{2\pi} \frac{1}{2}(4 + 2\cos\theta)^2\, d\theta = \int_0^{2\pi} \frac{1}{2}(16 + 16\cos\theta + 4\cos^2\theta)\, d\theta = \int_0^{2\pi} \left[8 + 8\cos\theta + 2\left(\frac{1+\cos 2\theta}{2}\right)\right] d\theta$
$= \int_0^{2\pi} (9 + 8\cos\theta + \cos 2\theta)\, d\theta = \left[9\theta + 8\sin\theta + \frac{1}{2}\sin 2\theta\right]_0^{2\pi} = 18\pi$

4. $A = \int_0^{2\pi} \frac{1}{2}[a(1+\cos\theta)]^2\, d\theta = \int_0^{2\pi} \frac{1}{2} a^2 (1 + 2\cos\theta + \cos^2\theta)\, d\theta = \frac{1}{2}a^2 \int_0^{2\pi} \left(1 + 2\cos\theta + \frac{1+\cos 2\theta}{2}\right) d\theta$
$= \frac{1}{2}a^2 \int_0^{2\pi} \left(\frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos 2\theta\right) d\theta = \frac{1}{2}a^2 \left[\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \frac{3}{2}\pi a^2$

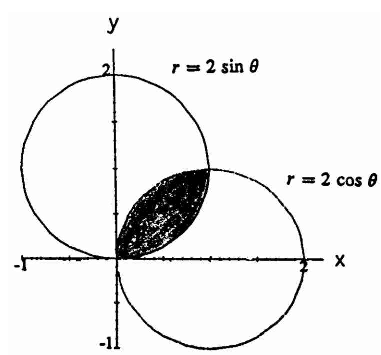
5. $A = 2\int_0^{\pi/4} \frac{1}{2}\cos^2 2\theta\, d\theta = \int_0^{\pi/4} \frac{1+\cos 4\theta}{2}\, d\theta = \frac{1}{2}\left[\theta + \frac{\sin 4\theta}{4}\right]_0^{\pi/4} = \frac{\pi}{8}$

6. $A = \int_{-\pi/6}^{\pi/6} \frac{1}{2}(\cos 3\theta)^2 d\theta = \frac{1}{2}\int_{-\pi/6}^{\pi/6} \cos^2 3\theta\, d\theta = \frac{1}{2}\int_{-\pi/6}^{\pi/6} \frac{1+\cos 6\theta}{2}\, d\theta = \frac{1}{4}\int_{-\pi/6}^{\pi/6} (1+\cos 6\theta)\, d\theta$
$= \frac{1}{4}\left[\theta + \frac{1}{6}\sin 6\theta\right]_{-\pi/6}^{\pi/6} = \frac{1}{4}\left(\frac{\pi}{6} + 0\right) - \frac{1}{4}\left(-\frac{\pi}{6} + 0\right) = \frac{\pi}{12}$

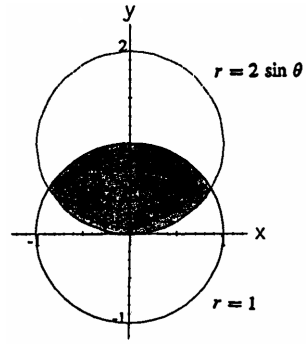
7. $A = \int_0^{\pi/2} \frac{1}{2}(4\sin 2\theta)\, d\theta = \int_0^{\pi/2} 2\sin 2\theta\, d\theta = [-\cos 2\theta]_0^{\pi/2} = 2$

8. $A = (6)(2)\int_0^{\pi/6} \frac{1}{2}(2\sin 3\theta)\, d\theta = 12\int_0^{\pi/6} \sin 3\theta\, d\theta = 12\left[-\frac{\cos 3\theta}{3}\right]_0^{\pi/6} = 4$

9. $r = 2\cos\theta$ and $r = 2\sin\theta \Rightarrow 2\cos\theta = 2\sin\theta$
$\Rightarrow \cos\theta = \sin\theta \Rightarrow \theta = \frac{\pi}{4}$; therefore
$A = 2\int_0^{\pi/4} \frac{1}{2}(2\sin\theta)^2\, d\theta = \int_0^{\pi/4} 4\sin^2\theta\, d\theta$
$= \int_0^{\pi/4} 4\left(\frac{1-\cos 2\theta}{2}\right) d\theta = \int_0^{\pi/4} (2 - 2\cos 2\theta)\, d\theta$
$= [2\theta - \sin 2\theta]_0^{\pi/4} = \frac{\pi}{2} - 1$

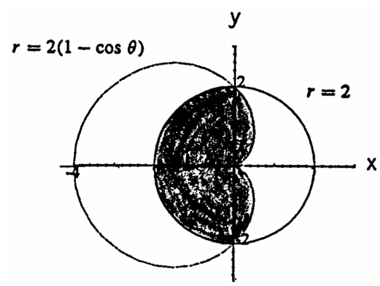


10. $r = 1$ and $r = 2\sin\theta \Rightarrow 2\sin\theta = 1 \Rightarrow \sin\theta = \frac{1}{2}$
$\Rightarrow \theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$; therefore



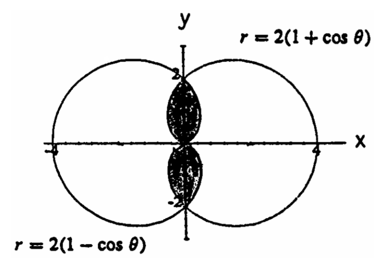
$A = \pi(1)^2 - \int_{\pi/6}^{5\pi/6} \frac{1}{2}\left[(2\sin\theta)^2 - 1^2\right] d\theta$
$= \pi - \int_{\pi/6}^{5\pi/6} \left(2\sin^2\theta - \frac{1}{2}\right) d\theta$
$= \pi - \int_{\pi/6}^{5\pi/6} \left(1 - \cos 2\theta - \frac{1}{2}\right) d\theta$
$= \pi - \int_{\pi/6}^{5\pi/6} \left(\frac{1}{2} - \cos 2\theta\right) d\theta = \pi - \left[\frac{1}{2}\theta - \frac{\sin 2\theta}{2}\right]_{\pi/6}^{5\pi/6}$
$= \pi - \left(\frac{5\pi}{12} - \frac{1}{2}\sin\frac{5\pi}{3}\right) + \left(\frac{\pi}{12} - \frac{1}{2}\sin\frac{\pi}{3}\right) = \frac{4\pi - 3\sqrt{3}}{6}$

11. $r = 2$ and $r = 2(1 - \cos\theta) \Rightarrow 2 = 2(1 - \cos\theta)$
$\Rightarrow \cos\theta = 0 \Rightarrow \theta = \pm\frac{\pi}{2}$; therefore



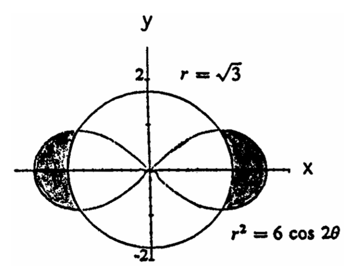
$A = 2\int_0^{\pi/2} \frac{1}{2}\left[2(1 - \cos\theta)\right]^2 d\theta + \frac{1}{2}$area of the circle
$= \int_0^{\pi/2} 4\left(1 - 2\cos\theta + \cos^2\theta\right) d\theta + \left(\frac{1}{2}\pi\right)(2)^2$
$= \int_0^{\pi/2} 4\left(1 - 2\cos\theta + \frac{1 + \cos 2\theta}{2}\right) d\theta + 2\pi$
$= \int_0^{\pi/2} (4 - 8\cos\theta + 2 + 2\cos 2\theta)\, d\theta + 2\pi$
$= \left[6\theta - 8\sin\theta + \sin 2\theta\right]_0^{\pi/2} + 2\pi = 5\pi - 8$

12. $r = 2(1 - \cos\theta)$ and $r = 2(1 + \cos\theta) \Rightarrow 1 - \cos\theta$
$= 1 + \cos\theta \Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$; the graph also gives the point of intersection $(0, 0)$; therefore



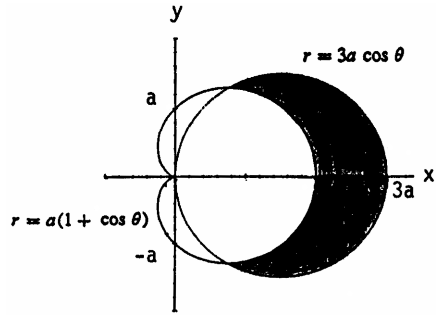
$A = 2\int_0^{\pi/2} \frac{1}{2}\left[2(1 - \cos\theta)\right]^2 d\theta + 2\int_{\pi/2}^{\pi} \frac{1}{2}\left[2(1 + \cos\theta)\right]^2 d\theta$
$= \int_0^{\pi/2} 4(1 - 2\cos\theta + \cos^2\theta) d\theta$
$+ \int_{\pi/2}^{\pi} 4\left(1 + 2\cos\theta + \cos^2\theta\right) d\theta$
$= \int_0^{\pi/2} 4\left(1 - 2\cos\theta + \frac{1 + \cos 2\theta}{2}\right) d\theta + \int_{\pi/2}^{\pi} 4\left(1 + 2\cos\theta + \frac{1 + \cos 2\theta}{2}\right) d\theta$
$= \int_0^{\pi/2} (6 - 8\cos\theta + 2\cos 2\theta)\, d\theta + \int_{\pi/2}^{\pi} (6 + 8\cos\theta + 2\cos 2\theta)\, d\theta$
$= \left[6\theta - 8\sin\theta + \sin 2\theta\right]_0^{\pi/2} + \left[6\theta + 8\sin\theta + \sin 2\theta\right]_{\pi/2}^{\pi} = 6\pi - 16$

13. $r = \sqrt{3}$ and $r^2 = 6\cos 2\theta \Rightarrow 3 = 6\cos 2\theta \Rightarrow \cos 2\theta = \frac{1}{2}$
$\Rightarrow \theta = \frac{\pi}{6}$ (in the 1st quadrant); we use symmetry of the graph to find the area, so

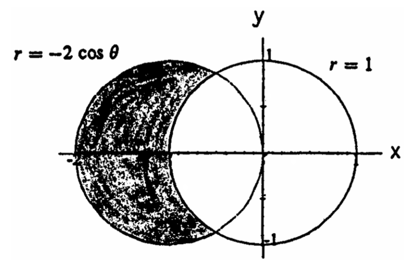


$A = 4\int_0^{\pi/6} \left[\frac{1}{2}(6\cos 2\theta) - \frac{1}{2}\left(\sqrt{3}\right)^2\right] d\theta$
$= 2\int_0^{\pi/6} (6\cos 2\theta - 3)\, d\theta = 2\left[3\sin 2\theta - 3\theta\right]_0^{\pi/6}$
$= 3\sqrt{3} - \pi$

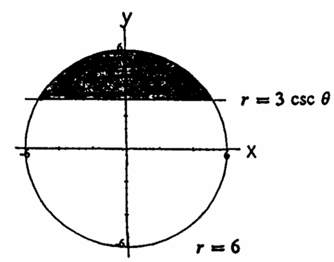
14. $r = 3a\cos\theta$ and $r = a(1+\cos\theta) \Rightarrow 3a\cos\theta = a(1+\cos\theta)$
$\Rightarrow\ 3\cos\theta = 1+\cos\theta \ \Rightarrow\ \cos\theta = \frac{1}{2} \ \Rightarrow\ \theta = \frac{\pi}{3}$ or $-\frac{\pi}{3}$;
the graph also gives the point of intersection $(0,0)$; therefore
$A = 2\int_0^{\pi/3} \frac{1}{2}\left[(3a\cos\theta)^2 - a^2(1+\cos\theta)^2\right] d\theta$
$= \int_0^{\pi/3} (9a^2\cos^2\theta - a^2 - 2a^2\cos\theta - a^2\cos^2\theta)\, d\theta$
$= \int_0^{\pi/3} (8a^2\cos^2\theta - 2a^2\cos\theta - a^2)\, d\theta$
$= \int_0^{\pi/3} [4a^2(1+\cos 2\theta) - 2a^2\cos\theta - a^2]\, d\theta$
$= \int_0^{\pi/3} (3a^2 + 4a^2\cos 2\theta - 2a^2\cos\theta)\, d\theta$
$= \left[3a^2\theta + 2a^2\sin 2\theta - 2a^2\sin\theta\right]_0^{\pi/3} = \pi a^2 + 2a^2\left(\frac{1}{2}\right) - 2a^2\left(\frac{\sqrt{3}}{2}\right) = a^2\left(\pi + 1 - \sqrt{3}\right)$



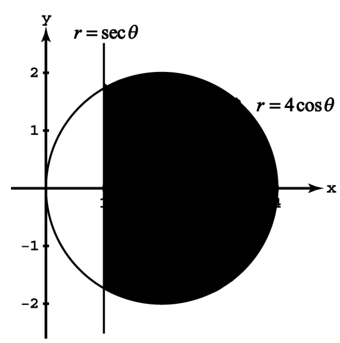
15. $r = 1$ and $r = -2\cos\theta \ \Rightarrow\ 1 = -2\cos\theta \ \Rightarrow\ \cos\theta = -\frac{1}{2}$
$\Rightarrow\ \theta = \frac{2\pi}{3}$ in quadrant II; therefore
$A = 2\int_{2\pi/3}^{\pi} \frac{1}{2}\left[(-2\cos\theta)^2 - 1^2\right] d\theta = \int_{2\pi/3}^{\pi} (4\cos^2\theta - 1)\, d\theta$
$= \int_{2\pi/3}^{\pi} [2(1+\cos 2\theta) - 1]\, d\theta = \int_{2\pi/3}^{\pi} (1 + 2\cos 2\theta)\, d\theta$
$= [\theta + \sin 2\theta]_{2\pi/3}^{\pi} = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$



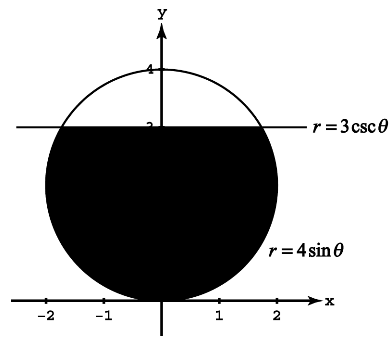
16. $r = 6$ and $r = 3\csc\theta \ \Rightarrow\ 6\sin\theta = 3 \ \Rightarrow\ \sin\theta = \frac{1}{2}$
$\Rightarrow\ \theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$; therefore $A = \int_{\pi/6}^{5\pi/6} \frac{1}{2}\left(6^2 - 9\csc^2\theta\right) d\theta$
$= \int_{\pi/6}^{5\pi/6} \left(18 - \frac{9}{2}\csc^2\theta\right) d\theta = \left[18\theta + \frac{9}{2}\cot\theta\right]_{\pi/6}^{5\pi/6}$
$= \left(15\pi - \frac{9}{2}\sqrt{3}\right) - \left(3\pi + \frac{9}{2}\sqrt{3}\right) = 12\pi - 9\sqrt{3}$



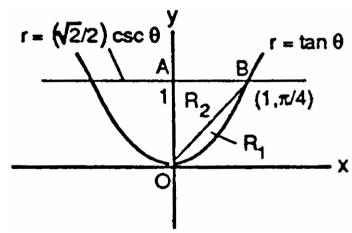
17. $r = \sec\theta$ and $r = 4\cos\theta \ \Rightarrow 4\cos\theta = \sec\theta \ \Rightarrow\ \cos^2\theta = \frac{1}{4}$
$\Rightarrow\ \theta = \frac{\pi}{3},\ \frac{2\pi}{3},\ \frac{4\pi}{3}$, or $\frac{5\pi}{3}$; therefore
$A = 2\int_0^{\pi/3} \frac{1}{2}\left(16\cos^2\theta - \sec^2\theta\right) d\theta$
$= \int_0^{\pi/3} (8 + 8\cos 2\theta - \sec^2\theta)\, d\theta$
$= [8\theta + 4\sin 2\theta - \tan\theta]_0^{\pi/3}$
$= \left(\frac{8\pi}{3} + 2\sqrt{3} - \sqrt{3}\right) - (0 + 0 - 0) = \frac{8\pi}{3} + \sqrt{3}$



18. $r = 3\csc\theta$ and $r = 4\sin\theta \Rightarrow 4\sin\theta = 3\csc\theta \Rightarrow \sin^2\theta = \frac{3}{4}$
$\Rightarrow \theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$, or $\frac{5\pi}{3}$; therefore
$A = 4\pi - 2\int_{\pi/3}^{\pi/2} \frac{1}{2}\left(16\sin^2\theta - 9\csc^2\theta\right) d\theta$
$= 4\pi - \int_{\pi/3}^{\pi/2} \left(8 - 8\cos 2\theta - 9\csc^2\theta\right) d\theta$
$= 4\pi - \left[8\theta - 4\sin 2\theta + 9\cot\theta\right]_{\pi/3}^{\pi/2}$
$= 4\pi - \left[(4\pi - 0 + 0) - \left(\frac{8\pi}{3} - 2\sqrt{3} + 3\sqrt{3}\right)\right]$
$= \frac{8\pi}{3} + \sqrt{3}$



19. (a) $r = \tan\theta$ and $r = \left(\frac{\sqrt{2}}{2}\right)\csc\theta \Rightarrow \tan\theta = \left(\frac{\sqrt{2}}{2}\right)\csc\theta$
$\Rightarrow \sin^2\theta = \left(\frac{\sqrt{2}}{2}\right)\cos\theta \Rightarrow 1 - \cos^2\theta = \left(\frac{\sqrt{2}}{2}\right)\cos\theta$
$\Rightarrow \cos^2\theta + \left(\frac{\sqrt{2}}{2}\right)\cos\theta - 1 = 0 \Rightarrow \cos\theta = -\sqrt{2}$ or
$\frac{\sqrt{2}}{2}$ (use the quadratic formula) $\Rightarrow \theta = \frac{\pi}{4}$ (the solution in the first quadrant); therefore the area of R_1 is
$A_1 = \int_0^{\pi/4} \frac{1}{2}\tan^2\theta\, d\theta = \frac{1}{2}\int_0^{\pi/4} (\sec^2\theta - 1)\, d\theta = \frac{1}{2}[\tan\theta - \theta]_0^{\pi/4} = \frac{1}{2}\left(\tan\frac{\pi}{4} - \frac{\pi}{4}\right) = \frac{1}{2} - \frac{\pi}{8}$; $AO = \left(\frac{\sqrt{2}}{2}\right)\csc\frac{\pi}{2}$
$= \frac{\sqrt{2}}{2}$ and $OB = \left(\frac{\sqrt{2}}{2}\right)\csc\frac{\pi}{4} = 1 \Rightarrow AB = \sqrt{1^2 - \left(\frac{\sqrt{2}}{2}\right)^2} = \frac{\sqrt{2}}{2} \Rightarrow$ the area of R_2 is $A_2 = \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \frac{1}{4}$;
therefore the area of the region shaded in the text is $2\left(\frac{1}{2} - \frac{\pi}{8} + \frac{1}{4}\right) = \frac{3}{2} - \frac{\pi}{4}$. Note: The area must be found this way since no common interval generates the region. For example, the interval $0 \le \theta \le \frac{\pi}{4}$ generates the arc OB of $r = \tan\theta$ but does not generate the segment AB of the line $r = \frac{\sqrt{2}}{2}\csc\theta$. Instead the interval generates the half-line from B to $+\infty$ on the line $r = \frac{\sqrt{2}}{2}\csc\theta$.



(b) $\lim_{\theta \to \pi/2^-} \tan\theta = \infty$ and the line $x = 1$ is $r = \sec\theta$ in polar coordinates; then $\lim_{\theta \to \pi/2^-} (\tan\theta - \sec\theta)$
$= \lim_{\theta \to \pi/2^-} \left(\frac{\sin\theta}{\cos\theta} - \frac{1}{\cos\theta}\right) = \lim_{\theta \to \pi/2^-} \left(\frac{\sin\theta - 1}{\cos\theta}\right) = \lim_{\theta \to \pi/2^-} \left(\frac{\cos\theta}{-\sin\theta}\right) = 0 \Rightarrow r = \tan\theta$ approaches

$r = \sec\theta$ as $\theta \to \frac{\pi^-}{2} \Rightarrow r = \sec\theta$ (or $x = 1$) is a vertical asymptote of $r = \tan\theta$. Similarly, $r = -\sec\theta$ (or $x = -1$) is a vertical asymptote of $r = \tan\theta$.

20. It is not because the circle is generated twice from $\theta = 0$ to 2π. The area of the cardioid is
$A = 2\int_0^{\pi} \frac{1}{2}(\cos\theta + 1)^2\, d\theta = \int_0^{\pi} (\cos^2\theta + 2\cos\theta + 1)\, d\theta = \int_0^{\pi} \left(\frac{1 + \cos 2\theta}{2} + 2\cos\theta + 1\right) d\theta$
$= \left[\frac{3\theta}{2} + \frac{\sin 2\theta}{4} + 2\sin\theta\right]_0^{\pi} = \frac{3\pi}{2}$. The area of the circle is $A = \pi\left(\frac{1}{2}\right)^2 = \frac{\pi}{4} \Rightarrow$ the area requested is actually $\frac{3\pi}{2} - \frac{\pi}{4} = \frac{5\pi}{4}$

21. $r = \theta^2, 0 \le \theta \le \sqrt{5} \Rightarrow \frac{dr}{d\theta} = 2\theta$; therefore Length $= \int_0^{\sqrt{5}} \sqrt{(\theta^2)^2 + (2\theta)^2}\, d\theta = \int_0^{\sqrt{5}} \sqrt{\theta^4 + 4\theta^2}\, d\theta$
$= \int_0^{\sqrt{5}} |\theta| \sqrt{\theta^2 + 4}\, d\theta =$ (since $\theta \ge 0$) $\int_0^{\sqrt{5}} \theta\sqrt{\theta^2 + 4}\, d\theta$; $\left[u = \theta^2 + 4 \Rightarrow \frac{1}{2}\, du = \theta\, d\theta; \theta = 0 \Rightarrow u = 4,\right.$
$\left.\theta = \sqrt{5} \Rightarrow u = 9\right] \to \int_4^9 \frac{1}{2}\sqrt{u}\, du = \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_4^9 = \frac{19}{3}$

22. $r = \frac{e^\theta}{\sqrt{2}}, 0 \le \theta \le \pi \Rightarrow \frac{dr}{d\theta} = \frac{e^\theta}{\sqrt{2}}$; therefore Length $= \int_0^{\pi} \sqrt{\left(\frac{e^\theta}{\sqrt{2}}\right)^2 + \left(\frac{e^\theta}{\sqrt{2}}\right)^2}\, d\theta = \int_0^{\pi} \sqrt{2\left(\frac{e^{2\theta}}{2}\right)}\, d\theta$
$= \int_0^{\pi} e^\theta\, d\theta = \left[e^\theta\right]_0^{\pi} = e^\pi - 1$

23. $r = 1 + \cos\theta \Rightarrow \frac{dr}{d\theta} = -\sin\theta$; therefore Length $= \int_0^{2\pi} \sqrt{(1+\cos\theta)^2 + (-\sin\theta)^2}\, d\theta$
$= 2\int_0^{\pi} \sqrt{2 + 2\cos\theta}\, d\theta = 2\int_0^{\pi} \sqrt{\frac{4(1+\cos\theta)}{2}}\, d\theta = 4\int_0^{\pi} \sqrt{\frac{1+\cos\theta}{2}}\, d\theta = 4\int_0^{\pi} \cos\left(\frac{\theta}{2}\right) d\theta = 4\left[2\sin\frac{\theta}{2}\right]_0^{\pi} = 8$

24. $r = a\sin^2\frac{\theta}{2}, 0 \le \theta \le \pi, a > 0 \Rightarrow \frac{dr}{d\theta} = a\sin\frac{\theta}{2}\cos\frac{\theta}{2}$; therefore Length $= \int_0^{\pi} \sqrt{\left(a\sin^2\frac{\theta}{2}\right)^2 + \left(a\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^2}\, d\theta$
$= \int_0^{\pi} \sqrt{a^2\sin^4\frac{\theta}{2} + a^2\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}}\, d\theta = \int_0^{\pi} a\left|\sin\frac{\theta}{2}\right| \sqrt{\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}}\, d\theta =$ (since $0 \le \theta \le \pi$) $a\int_0^{\pi} \sin\left(\frac{\theta}{2}\right) d\theta$
$= \left[-2a\cos\frac{\theta}{2}\right]_0^{\pi} = 2a$

25. $r = \frac{6}{1+\cos\theta}, 0 \le \theta \le \frac{\pi}{2} \Rightarrow \frac{dr}{d\theta} = \frac{6\sin\theta}{(1+\cos\theta)^2}$; therefore Length $= \int_0^{\pi/2} \sqrt{\left(\frac{6}{1+\cos\theta}\right)^2 + \left(\frac{6\sin\theta}{(1+\cos\theta)^2}\right)^2}\, d\theta$
$= \int_0^{\pi/2} \sqrt{\frac{36}{(1+\cos\theta)^2} + \frac{36\sin^2\theta}{(1+\cos\theta)^4}}\, d\theta = 6\int_0^{\pi/2} \left|\frac{1}{1+\cos\theta}\right| \sqrt{1 + \frac{\sin^2\theta}{(1+\cos\theta)^2}}\, d\theta$
$= \left(\text{since } \frac{1}{1+\cos\theta} > 0 \text{ on } 0 \le \theta \le \frac{\pi}{2}\right)\ 6\int_0^{\pi/2} \left(\frac{1}{1+\cos\theta}\right) \sqrt{\frac{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta}{(1+\cos\theta)^2}}\, d\theta$
$= 6\int_0^{\pi/2} \left(\frac{1}{1+\cos\theta}\right) \sqrt{\frac{2+2\cos\theta}{(1+\cos\theta)^2}}\, d\theta = 6\sqrt{2}\int_0^{\pi/2} \frac{d\theta}{(1+\cos\theta)^{3/2}} = 6\sqrt{2}\int_0^{\pi/2} \frac{d\theta}{\left(2\cos^2\frac{\theta}{2}\right)^{3/2}} = 3\int_0^{\pi/2} \left|\sec^3\frac{\theta}{2}\right| d\theta$
$= 3\int_0^{\pi/2} \sec^3\frac{\theta}{2}\, d\theta = 6\int_0^{\pi/4} \sec^3 u\, du =$ (use tables) $6\left(\left[\frac{\sec u \tan u}{2}\right]_0^{\pi/4} + \frac{1}{2}\int_0^{\pi/4} \sec u\, du\right)$
$= 6\left(\frac{1}{\sqrt{2}} + \left[\frac{1}{2}\ln|\sec u + \tan u|\right]_0^{\pi/4}\right) = 3\left[\sqrt{2} + \ln\left(1 + \sqrt{2}\right)\right]$

26. $r = \frac{2}{1-\cos\theta}, \frac{\pi}{2} \le \theta \le \pi \Rightarrow \frac{dr}{d\theta} = \frac{-2\sin\theta}{(1-\cos\theta)^2}$; therefore Length $= \int_{\pi/2}^{\pi} \sqrt{\left(\frac{2}{1-\cos\theta}\right)^2 + \left(\frac{-2\sin\theta}{(1-\cos\theta)^2}\right)^2}\, d\theta$
$= \int_{\pi/2}^{\pi} \sqrt{\frac{4}{(1-\cos\theta)^2}\left(1 + \frac{\sin^2\theta}{(1-\cos\theta)^2}\right)}\, d\theta = \int_{\pi/2}^{\pi} \left|\frac{2}{1-\cos\theta}\right| \sqrt{\frac{(1-\cos\theta)^2 + \sin^2\theta}{(1-\cos\theta)^2}}\, d\theta$
$= \left(\text{since } 1 - \cos\theta \ge 0 \text{ on } \frac{\pi}{2} \le \theta \le \pi\right)\ 2\int_{\pi/2}^{\pi} \left(\frac{1}{1-\cos\theta}\right) \sqrt{\frac{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}{(1-\cos\theta)^2}}\, d\theta$
$= 2\int_{\pi/2}^{\pi} \left(\frac{1}{1-\cos\theta}\right) \sqrt{\frac{2-2\cos\theta}{(1-\cos\theta)^2}}\, d\theta = 2\sqrt{2}\int_{\pi/2}^{\pi} \frac{d\theta}{(1-\cos\theta)^{3/2}} = 2\sqrt{2}\int_{\pi/2}^{\pi} \frac{d\theta}{\left(2\sin^2\frac{\theta}{2}\right)^{3/2}} = \int_{\pi/2}^{\pi} \left|\csc^3\frac{\theta}{2}\right| d\theta$
$= \int_{\pi/2}^{\pi} \csc^3\left(\frac{\theta}{2}\right) d\theta = \left(\text{since } \csc\frac{\theta}{2} \ge 0 \text{ on } \frac{\pi}{2} \le \theta \le \pi\right)\ 2\int_{\pi/4}^{\pi/2} \csc^3 u\, du =$ (use tables)
$2\left(\left[-\frac{\csc u \cot u}{2}\right]_{\pi/4}^{\pi/2} + \frac{1}{2}\int_{\pi/4}^{\pi/2} \csc u\, du\right) = 2\left(\frac{1}{\sqrt{2}} - \left[\frac{1}{2}\ln|\csc u + \cot u|\right]_{\pi/4}^{\pi/2}\right) = 2\left[\frac{1}{\sqrt{2}} + \frac{1}{2}\ln\left(\sqrt{2} + 1\right)\right]$
$= \sqrt{2} + \ln\left(1 + \sqrt{2}\right)$

27. $r = \cos^3\frac{\theta}{3} \Rightarrow \frac{dr}{d\theta} = -\sin\frac{\theta}{3}\cos^2\frac{\theta}{3}$; therefore Length $= \int_0^{\pi/4} \sqrt{\left(\cos^3\frac{\theta}{3}\right)^2 + \left(-\sin\frac{\theta}{3}\cos^2\frac{\theta}{3}\right)^2}\, d\theta$
$= \int_0^{\pi/4} \sqrt{\cos^6\left(\frac{\theta}{3}\right) + \sin^2\left(\frac{\theta}{3}\right)\cos^4\left(\frac{\theta}{3}\right)}\, d\theta = \int_0^{\pi/4} \left(\cos^2\frac{\theta}{3}\right) \sqrt{\cos^2\left(\frac{\theta}{3}\right) + \sin^2\left(\frac{\theta}{3}\right)}\, d\theta = \int_0^{\pi/4} \cos^2\left(\frac{\theta}{3}\right) d\theta$
$= \int_0^{\pi/4} \frac{1 + \cos\left(\frac{2\theta}{3}\right)}{2}\, d\theta = \frac{1}{2}\left[\theta + \frac{3}{2}\sin\frac{2\theta}{3}\right]_0^{\pi/4} = \frac{\pi}{8} + \frac{3}{8}$

28. $r = \sqrt{1 + \sin 2\theta}, 0 \le \theta \le \pi\sqrt{2} \Rightarrow \frac{dr}{d\theta} = \frac{1}{2}(1 + \sin 2\theta)^{-1/2}(2\cos 2\theta) = (\cos 2\theta)(1 + \sin 2\theta)^{-1/2}$; therefore
Length $= \int_0^{\pi\sqrt{2}} \sqrt{(1 + \sin 2\theta) + \frac{\cos^2 2\theta}{(1 + \sin 2\theta)}}\, d\theta = \int_0^{\pi\sqrt{2}} \sqrt{\frac{1 + 2\sin 2\theta + \sin^2 2\theta + \cos^2 2\theta}{1 + \sin 2\theta}}\, d\theta$
$= \int_0^{\pi\sqrt{2}} \sqrt{\frac{2 + 2\sin 2\theta}{1 + \sin 2\theta}}\, d\theta = \int_0^{\pi\sqrt{2}} \sqrt{2}\, d\theta = \left[\sqrt{2}\,\theta\right]_0^{\pi\sqrt{2}} = 2\pi$

29. Let $r = f(\theta)$. Then $x = f(\theta)\cos\theta \Rightarrow \frac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta \Rightarrow \left(\frac{dx}{d\theta}\right)^2 = [f'(\theta)\cos\theta - f(\theta)\sin\theta]^2$
$= [f'(\theta)]^2\cos^2\theta - 2f'(\theta)\,f(\theta)\sin\theta\cos\theta + [f(\theta)]^2\sin^2\theta$; $y = f(\theta)\sin\theta \Rightarrow \frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta$
$\Rightarrow \left(\frac{dy}{d\theta}\right)^2 = [f'(\theta)\sin\theta + f(\theta)\cos\theta]^2 = [f'(\theta)]^2\sin^2\theta + 2f'(\theta)f(\theta)\sin\theta\cos\theta + [f(\theta)]^2\cos^2\theta$. Therefore
$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = [f'(\theta)]^2(\cos^2\theta + \sin^2\theta) + [f(\theta)]^2(\cos^2\theta + \sin^2\theta) = [f'(\theta)]^2 + [f(\theta)]^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2$.
Thus, $L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$.

30. (a) $r = a \Rightarrow \frac{dr}{d\theta} = 0$; Length $= \int_0^{2\pi}\sqrt{a^2 + 0^2}\,d\theta = \int_0^{2\pi}|a|\ d\theta = [a\theta]_0^{2\pi} = 2\pi a$
(b) $r = a\cos\theta \Rightarrow \frac{dr}{d\theta} = -a\sin\theta$; Length $= \int_0^\pi \sqrt{(a\cos\theta)^2 + (-a\sin\theta)^2}\,d\theta = \int_0^\pi \sqrt{a^2(\cos^2\theta + \sin^2\theta)}\,d\theta$
$= \int_0^\pi |a|\ d\theta = [a\theta]_0^\pi = \pi a$
(c) $r = a\sin\theta \Rightarrow \frac{dr}{d\theta} = a\cos\theta$; Length $= \int_0^\pi \sqrt{(a\cos\theta)^2 + (a\sin\theta)^2}\,d\theta = \int_0^\pi \sqrt{a^2(\cos^2\theta + \sin^2\theta)}\,d\theta$
$= \int_0^\pi |a|\ d\theta = [a\theta]_0^\pi = \pi a$

31. (a) $r_{av} = \frac{1}{2\pi - 0}\int_0^{2\pi} a(1 - \cos\theta)\,d\theta = \frac{a}{2\pi}[\theta - \sin\theta]_0^{2\pi} = a$
(b) $r_{av} = \frac{1}{2\pi - 0}\int_0^{2\pi} a\,d\theta = \frac{1}{2\pi}[a\theta]_0^{2\pi} = a$
(c) $r_{av} = \frac{1}{\left(\frac{\pi}{2}\right) - \left(-\frac{\pi}{2}\right)}\int_{-\pi/2}^{\pi/2} a\cos\theta\,d\theta = \frac{1}{\pi}[a\sin\theta]_{-\pi/2}^{\pi/2} = \frac{2a}{\pi}$

32. $r = 2f(\theta),\ \alpha \le \theta \le \beta \Rightarrow \frac{dr}{d\theta} = 2f'(\theta) \Rightarrow r^2 + \left(\frac{dr}{d\theta}\right)^2 = [2f(\theta)]^2 + [2f'(\theta)]^2 \Rightarrow$ Length $= \int_\alpha^\beta \sqrt{4[f(\theta)]^2 + 4[f'(\theta)]^2}\,d\theta$
$= 2\int_\alpha^\beta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\,d\theta$ which is twice the length of the curve $r = f(\theta)$ for $\alpha \le \theta \le \beta$.

11.6 CONIC SECTIONS

1. $x = \frac{y^2}{8} \Rightarrow 4p = 8 \Rightarrow p = 2$; focus is $(2, 0)$, directrix is $x = -2$

2. $x = -\frac{y^2}{4} \Rightarrow 4p = 4 \Rightarrow p = 1$; focus is $(-1, 0)$, directrix is $x = 1$

3. $y = -\frac{x^2}{6} \Rightarrow 4p = 6 \Rightarrow p = \frac{3}{2}$; focus is $\left(0, -\frac{3}{2}\right)$, directrix is $y = \frac{3}{2}$

4. $y = \frac{x^2}{2} \Rightarrow 4p = 2 \Rightarrow p = \frac{1}{2}$; focus is $\left(0, \frac{1}{2}\right)$, directrix is $y = -\frac{1}{2}$

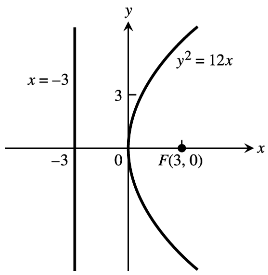
5. $\frac{x^2}{4} - \frac{y^2}{9} = 1 \Rightarrow c = \sqrt{4 + 9} = \sqrt{13} \Rightarrow$ foci are $\left(\pm\sqrt{13}, 0\right)$; vertices are $(\pm 2, 0)$; asymptotes are $y = \pm\frac{3}{2}x$

6. $\frac{x^2}{4} + \frac{y^2}{9} = 1 \Rightarrow c = \sqrt{9 - 4} = \sqrt{5} \Rightarrow$ foci are $\left(0, \pm\sqrt{5}\right)$; vertices are $(0, \pm 3)$

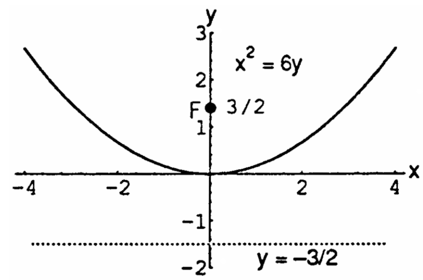
7. $\frac{x^2}{2} + y^2 = 1 \Rightarrow c = \sqrt{2 - 1} = 1 \Rightarrow$ foci are $(\pm 1, 0)$; vertices are $\left(\pm\sqrt{2}, 0\right)$

8. $\frac{y^2}{4} - x^2 = 1 \Rightarrow c = \sqrt{4 + 1} = \sqrt{5} \Rightarrow$ foci are $\left(0, \pm\sqrt{5}\right)$; vertices are $(0, \pm 2)$; asymptotes are $y = \pm 2x$

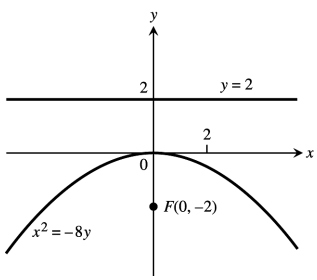
9. $y^2 = 12x \Rightarrow x = \frac{y^2}{12} \Rightarrow 4p = 12 \Rightarrow p = 3$;
focus is $(3, 0)$, directrix is $x = -3$



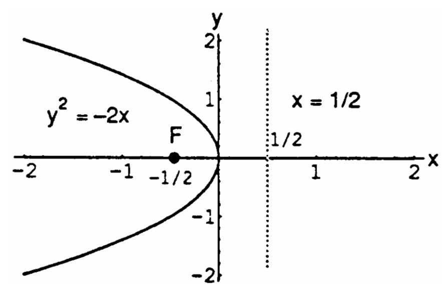
10. $x^2 = 6y \Rightarrow y = \frac{x^2}{6} \Rightarrow 4p = 6 \Rightarrow p = \frac{3}{2}$;
focus is $\left(0, \frac{3}{2}\right)$, directrix is $y = -\frac{3}{2}$



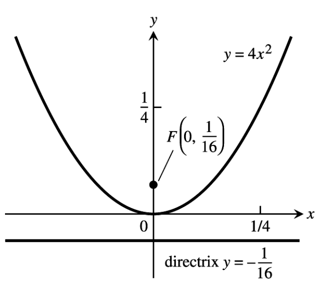
11. $x^2 = -8y \Rightarrow y = \frac{x^2}{-8} \Rightarrow 4p = 8 \Rightarrow p = 2$;
focus is $(0, -2)$, directrix is $y = 2$



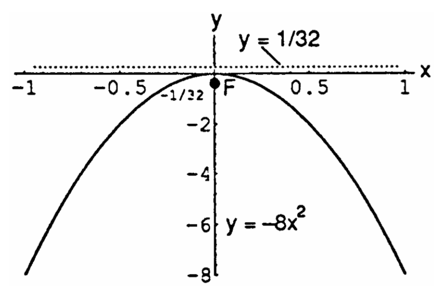
12. $y^2 = -2x \Rightarrow x = \frac{y^2}{-2} \Rightarrow 4p = 2 \Rightarrow p = \frac{1}{2}$;
focus is $\left(-\frac{1}{2}, 0\right)$, directrix is $x = \frac{1}{2}$



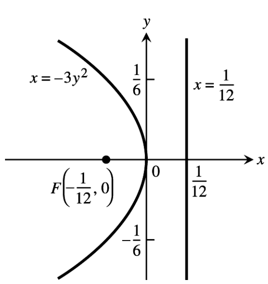
13. $y = 4x^2 \Rightarrow y = \frac{x^2}{\left(\frac{1}{4}\right)} \Rightarrow 4p = \frac{1}{4} \Rightarrow p = \frac{1}{16}$;
focus is $\left(0, \frac{1}{16}\right)$, directrix is $y = -\frac{1}{16}$



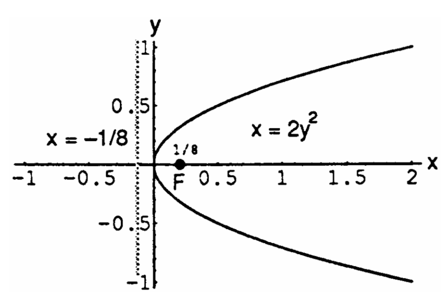
14. $y = -8x^2 \Rightarrow y = -\frac{x^2}{\left(\frac{1}{8}\right)} \Rightarrow 4p = \frac{1}{8} \Rightarrow p = \frac{1}{32}$;
focus is $\left(0, -\frac{1}{32}\right)$, directrix is $y = \frac{1}{32}$



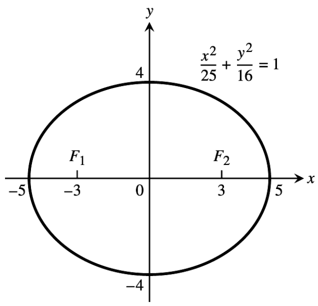
15. $x = -3y^2 \Rightarrow x = -\frac{y^2}{\left(\frac{1}{3}\right)} \Rightarrow 4p = \frac{1}{3} \Rightarrow p = \frac{1}{12}$;
focus is $\left(-\frac{1}{12}, 0\right)$, directrix is $x = \frac{1}{12}$



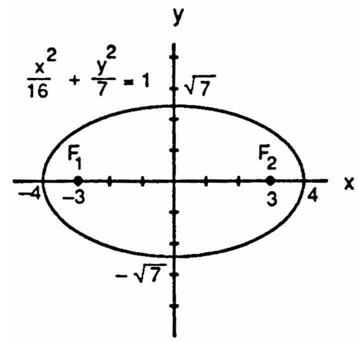
16. $x = 2y^2 \Rightarrow x = \frac{y^2}{\left(\frac{1}{2}\right)} \Rightarrow 4p = \frac{1}{2} \Rightarrow p = \frac{1}{8}$;
focus is $\left(\frac{1}{8}, 0\right)$, directrix is $x = -\frac{1}{8}$



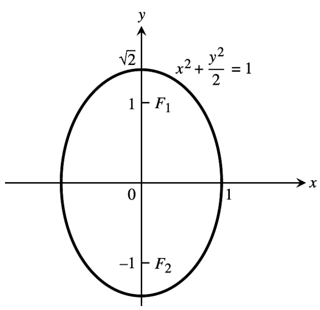
17. $16x^2 + 25y^2 = 400 \Rightarrow \frac{x^2}{25} + \frac{y^2}{16} = 1$
$\Rightarrow c = \sqrt{a^2 - b^2} = \sqrt{25 - 16} = 3$



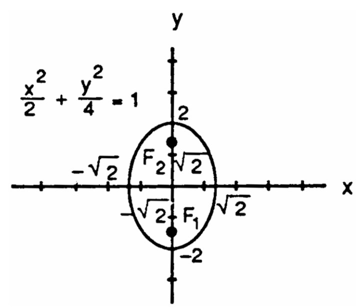
18. $7x^2 + 16y^2 = 112 \Rightarrow \frac{x^2}{16} + \frac{y^2}{7} = 1$
$\Rightarrow c = \sqrt{a^2 - b^2} = \sqrt{16 - 7} = 3$



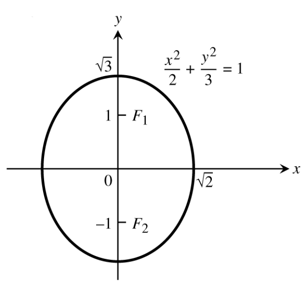
19. $2x^2 + y^2 = 2 \Rightarrow x^2 + \frac{y^2}{2} = 1$
$\Rightarrow c = \sqrt{a^2 - b^2} = \sqrt{2 - 1} = 1$



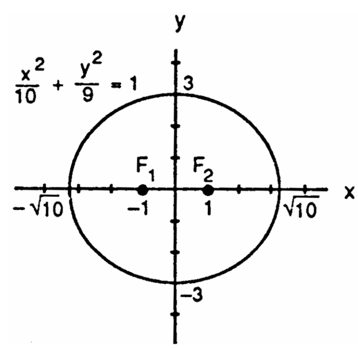
20. $2x^2 + y^2 = 4 \Rightarrow \frac{x^2}{2} + \frac{y^2}{4} = 1$
$\Rightarrow c = \sqrt{a^2 - b^2} = \sqrt{4 - 2} = \sqrt{2}$



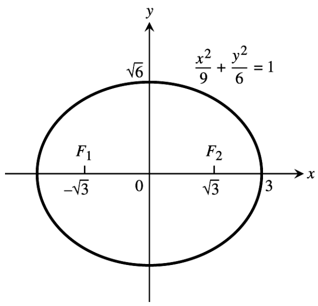
21. $3x^2 + 2y^2 = 6 \Rightarrow \frac{x^2}{2} + \frac{y^2}{3} = 1$
$\Rightarrow c = \sqrt{a^2 - b^2} = \sqrt{3 - 2} = 1$



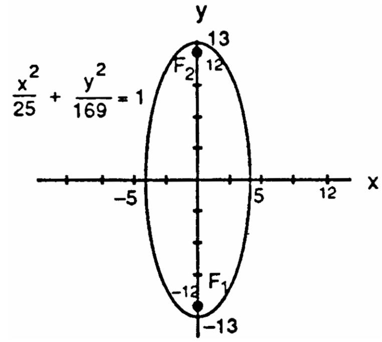
22. $9x^2 + 10y^2 = 90 \Rightarrow \frac{x^2}{10} + \frac{y^2}{9} = 1$
$\Rightarrow c = \sqrt{a^2 - b^2} = \sqrt{10 - 9} = 1$



23. $6x^2 + 9y^2 = 54 \Rightarrow \frac{x^2}{9} + \frac{y^2}{6} = 1$
$\Rightarrow c = \sqrt{a^2 - b^2} = \sqrt{9-6} = \sqrt{3}$



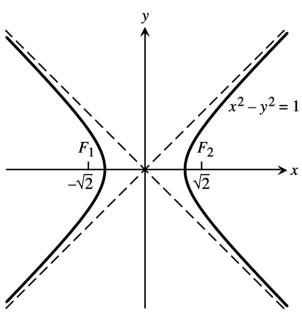
24. $169x^2 + 25y^2 = 4225 \Rightarrow \frac{x^2}{25} + \frac{y^2}{169} = 1$
$\Rightarrow c = \sqrt{a^2 - b^2} = \sqrt{169 - 25} = 12$



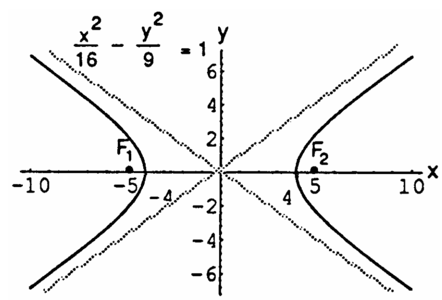
25. Foci: $\left(\pm\sqrt{2}, 0\right)$, Vertices: $(\pm 2, 0) \Rightarrow a = 2, c = \sqrt{2} \Rightarrow b^2 = a^2 - c^2 = 4 - \left(\sqrt{2}\right)^2 = 2 \Rightarrow \frac{x^2}{4} + \frac{y^2}{2} = 1$

26. Foci: $(0, \pm 4)$, Vertices: $(0, \pm 5) \Rightarrow a = 5, c = 4 \Rightarrow b^2 = 25 - 16 = 9 \Rightarrow \frac{x^2}{9} + \frac{y^2}{25} = 1$

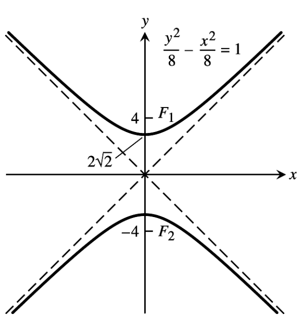
27. $x^2 - y^2 = 1 \Rightarrow c = \sqrt{a^2 + b^2} = \sqrt{1+1} = \sqrt{2}$; asymptotes are $y = \pm x$



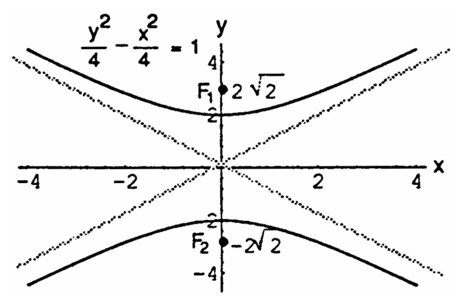
28. $9x^2 - 16y^2 = 144 \Rightarrow \frac{x^2}{16} - \frac{y^2}{9} = 1$
$\Rightarrow c = \sqrt{a^2 + b^2} = \sqrt{16 + 9} = 5$;
asymptotes are $y = \pm \frac{3}{4} x$



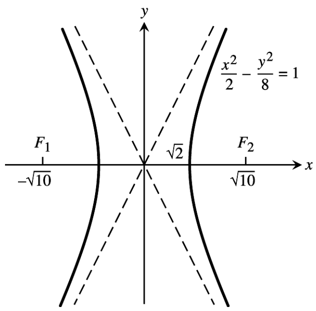
29. $y^2 - x^2 = 8 \Rightarrow \frac{y^2}{8} - \frac{x^2}{8} = 1 \Rightarrow c = \sqrt{a^2 + b^2}$
$= \sqrt{8+8} = 4$; asymptotes are $y = \pm x$



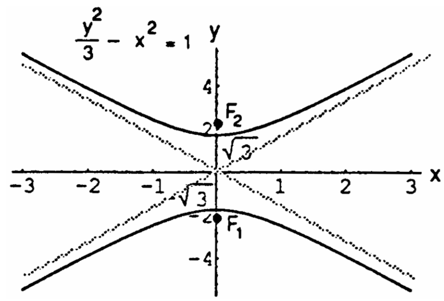
30. $y^2 - x^2 = 4 \Rightarrow \frac{y^2}{4} - \frac{x^2}{4} = 1 \Rightarrow c = \sqrt{a^2 + b^2}$
$= \sqrt{4+4} = 2\sqrt{2}$; asymptotes are $y = \pm x$



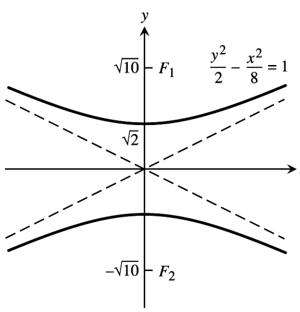
31. $8x^2 - 2y^2 = 16 \Rightarrow \frac{x^2}{2} - \frac{y^2}{8} = 1 \Rightarrow c = \sqrt{a^2 + b^2} = \sqrt{2+8} = \sqrt{10}$; asymptotes are $y = \pm 2x$



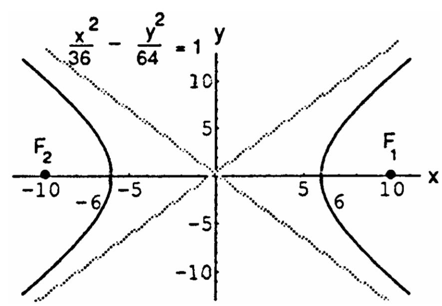
32. $y^2 - 3x^2 = 3 \Rightarrow \frac{y^2}{3} - x^2 = 1 \Rightarrow c = \sqrt{a^2 + b^2} = \sqrt{3+1} = 2$; asymptotes are $y = \pm \sqrt{3}x$



33. $8y^2 - 2x^2 = 16 \Rightarrow \frac{y^2}{2} - \frac{x^2}{8} = 1 \Rightarrow c = \sqrt{a^2 + b^2} = \sqrt{2+8} = \sqrt{10}$; asymptotes are $y = \pm \frac{x}{2}$



34. $64x^2 - 36y^2 = 2304 \Rightarrow \frac{x^2}{36} - \frac{y^2}{64} = 1 \Rightarrow c = \sqrt{a^2 + b^2} = \sqrt{36+64} = 10$; asymptotes are $y = \pm \frac{4}{3}$



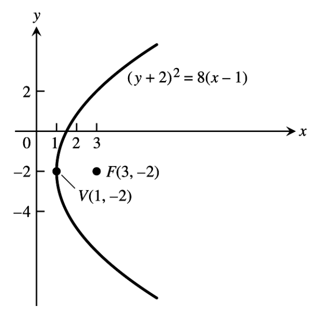
35. Foci: $\left(0, \pm\sqrt{2}\right)$, Asymptotes: $y = \pm x \Rightarrow c = \sqrt{2}$ and $\frac{a}{b} = 1 \Rightarrow a = b \Rightarrow c^2 = a^2 + b^2 = 2a^2 \Rightarrow 2 = 2a^2$ $\Rightarrow a = 1 \Rightarrow b = 1 \Rightarrow y^2 - x^2 = 1$

36. Foci: $(\pm 2, 0)$, Asymptotes: $y = \pm \frac{1}{\sqrt{3}}x \Rightarrow c = 2$ and $\frac{b}{a} = \frac{1}{\sqrt{3}} \Rightarrow b = \frac{a}{\sqrt{3}} \Rightarrow c^2 = a^2 + b^2 = a^2 + \frac{a^2}{3} = \frac{4a^2}{3}$ $\Rightarrow 4 = \frac{4a^2}{3} \Rightarrow a^2 = 3 \Rightarrow a = \sqrt{3} \Rightarrow b = 1 \Rightarrow \frac{x^2}{3} - y^2 = 1$

37. Vertices: $(\pm 3, 0)$, Asymptotes: $y = \pm \frac{4}{3}x \Rightarrow a = 3$ and $\frac{b}{a} = \frac{4}{3} \Rightarrow b = \frac{4}{3}(3) = 4 \Rightarrow \frac{x^2}{9} - \frac{y^2}{16} = 1$

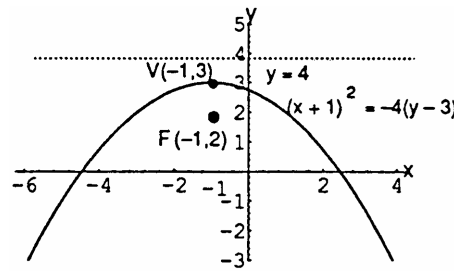
38. Vertices: $(0, \pm 2)$, Asymptotes: $y = \pm \frac{1}{2}x \Rightarrow a = 2$ and $\frac{a}{b} = \frac{1}{2} \Rightarrow b = 2(2) = 4 \Rightarrow \frac{y^2}{4} - \frac{x^2}{16} = 1$

39. (a) $y^2 = 8x \Rightarrow 4p = 8 \Rightarrow p = 2 \Rightarrow$ directrix is $x = -2$, focus is $(2, 0)$, and vertex is $(0, 0)$; therefore the new directrix is $x = -1$, the new focus is $(3, -2)$, and the new vertex is $(1, -2)$



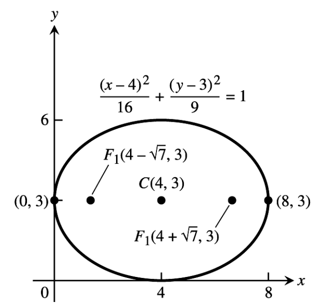
40. (a) $x^2 = -4y \Rightarrow 4p = 4 \Rightarrow p = 1 \Rightarrow$ directrix is $y = 1$, focus is $(0, -1)$, and vertex is $(0, 0)$; therefore the new directrix is $y = 4$, the new focus is $(-1, 2)$, and the new vertex is $(-1, 3)$

(b)



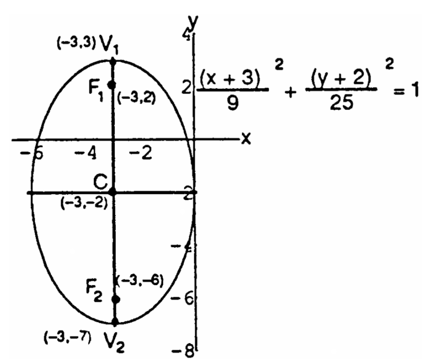
41. (a) $\frac{x^2}{16} + \frac{y^2}{9} = 1 \Rightarrow$ center is $(0, 0)$, vertices are $(-4, 0)$ and $(4, 0)$; $c = \sqrt{a^2 - b^2} = \sqrt{7} \Rightarrow$ foci are $\left(\sqrt{7}, 0\right)$ and $\left(-\sqrt{7}, 0\right)$; therefore the new center is $(4, 3)$, the new vertices are $(0, 3)$ and $(8, 3)$, and the new foci are $\left(4 \pm \sqrt{7}, 3\right)$

(b)



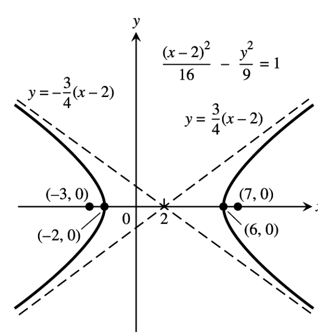
42. (a) $\frac{x^2}{9} + \frac{y^2}{25} = 1 \Rightarrow$ center is $(0, 0)$, vertices are $(0, 5)$ and $(0, -5)$; $c = \sqrt{a^2 - b^2} = \sqrt{16} = 4 \Rightarrow$ foci are $(0, 4)$ and $(0, -4)$; therefore the new center is $(-3, -2)$, the new vertices are $(-3, 3)$ and $(-3, -7)$, and the new foci are $(-3, 2)$ and $(-3, -6)$

(b)



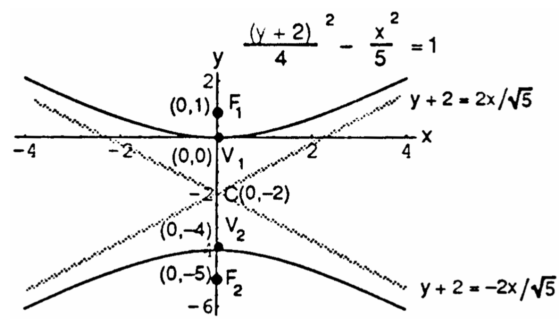
43. (a) $\frac{x^2}{16} - \frac{y^2}{9} = 1 \Rightarrow$ center is $(0, 0)$, vertices are $(-4, 0)$ and $(4, 0)$, and the asymptotes are $\frac{x}{4} = \pm \frac{y}{3}$ or $y = \pm \frac{3x}{4}$; $c = \sqrt{a^2 + b^2} = \sqrt{25} = 5 \Rightarrow$ foci are $(-5, 0)$ and $(5, 0)$; therefore the new center is $(2, 0)$, the new vertices are $(-2, 0)$ and $(6, 0)$, the new foci are $(-3, 0)$ and $(7, 0)$, and the new asymptotes are $y = \pm \frac{3(x-2)}{4}$

(b)



44. (a) $\frac{y^2}{4} - \frac{x^2}{5} = 1 \Rightarrow$ center is $(0, 0)$, vertices are $(0, -2)$ and $(0, 2)$, and the asymptotes are $\frac{y}{2} = \pm \frac{x}{\sqrt{5}}$ or $y = \pm \frac{2x}{\sqrt{5}}$; $c = \sqrt{a^2 + b^2} = \sqrt{9} = 3 \Rightarrow$ foci are $(0, 3)$ and $(0, -3)$; therefore the new center is $(0, -2)$, the new vertices are $(0, -4)$ and $(0, 0)$, the new foci are $(0, 1)$ and $(0, -5)$, and the new asymptotes are $y + 2 = \pm \frac{2x}{\sqrt{5}}$

(b)



45. $y^2 = 4x \Rightarrow 4p = 4 \Rightarrow p = 1 \Rightarrow$ focus is $(1, 0)$, directrix is $x = -1$, and vertex is $(0, 0)$; therefore the new vertex is $(-2, -3)$, the new focus is $(-1, -3)$, and the new directrix is $x = -3$; the new equation is $(y + 3)^2 = 4(x + 2)$

46. $y^2 = -12x \Rightarrow 4p = 12 \Rightarrow p = 3 \Rightarrow$ focus is $(-3, 0)$, directrix is $x = 3$, and vertex is $(0, 0)$; therefore the new vertex is $(4, 3)$, the new focus is $(1, 3)$, and the new directrix is $x = 7$; the new equation is $(y - 3)^2 = -12(x - 4)$

47. $x^2 = 8y \Rightarrow 4p = 8 \Rightarrow p = 2 \Rightarrow$ focus is $(0, 2)$, directrix is $y = -2$, and vertex is $(0, 0)$; therefore the new vertex is $(1, -7)$, the new focus is $(1, -5)$, and the new directrix is $y = -9$; the new equation is $(x - 1)^2 = 8(y + 7)$

48. $x^2 = 6y \Rightarrow 4p = 6 \Rightarrow p = \frac{3}{2} \Rightarrow$ focus is $\left(0, \frac{3}{2}\right)$, directrix is $y = -\frac{3}{2}$, and vertex is $(0, 0)$; therefore the new vertex is $(-3, -2)$, the new focus is $\left(-3, -\frac{1}{2}\right)$, and the new directrix is $y = -\frac{7}{2}$; the new equation is $(x + 3)^2 = 6(y + 2)$

49. $\frac{x^2}{6} + \frac{y^2}{9} = 1 \Rightarrow$ center is $(0, 0)$, vertices are $(0, 3)$ and $(0, -3)$; $c = \sqrt{a^2 - b^2} = \sqrt{9 - 6} = \sqrt{3} \Rightarrow$ foci are $\left(0, \sqrt{3}\right)$ and $\left(0, -\sqrt{3}\right)$; therefore the new center is $(-2, -1)$, the new vertices are $(-2, 2)$ and $(-2, -4)$, and the new foci are $\left(-2, -1 \pm \sqrt{3}\right)$; the new equation is $\frac{(x+2)^2}{6} + \frac{(y+1)^2}{9} = 1$

50. $\frac{x^2}{2} + y^2 = 1 \Rightarrow$ center is $(0, 0)$, vertices are $\left(\sqrt{2}, 0\right)$ and $\left(-\sqrt{2}, 0\right)$; $c = \sqrt{a^2 - b^2} = \sqrt{2 - 1} = 1 \Rightarrow$ foci are $(-1, 0)$ and $(1, 0)$; therefore the new center is $(3, 4)$, the new vertices are $\left(3 \pm \sqrt{2}, 4\right)$, and the new foci are $(2, 4)$ and $(4, 4)$; the new equation is $\frac{(x-3)^2}{2} + (y - 4)^2 = 1$

51. $\frac{x^2}{3} + \frac{y^2}{2} = 1 \Rightarrow$ center is $(0, 0)$, vertices are $\left(\sqrt{3}, 0\right)$ and $\left(-\sqrt{3}, 0\right)$; $c = \sqrt{a^2 - b^2} = \sqrt{3 - 2} = 1 \Rightarrow$ foci are $(-1, 0)$ and $(1, 0)$; therefore the new center is $(2, 3)$, the new vertices are $\left(2 \pm \sqrt{3}, 3\right)$, and the new foci are $(1, 3)$ and $(3, 3)$; the new equation is $\frac{(x-2)^2}{3} + \frac{(y-3)^2}{2} = 1$

52. $\frac{x^2}{16} + \frac{y^2}{25} = 1 \Rightarrow$ center is $(0, 0)$, vertices are $(0, 5)$ and $(0, -5)$; $c = \sqrt{a^2 - b^2} = \sqrt{25 - 16} = 3 \Rightarrow$ foci are $(0, 3)$ and $(0, -3)$; therefore the new center is $(-4, -5)$, the new vertices are $(-4, 0)$ and $(-4, -10)$, and the new foci are $(-4, -2)$ and $(-4, -8)$; the new equation is $\frac{(x+4)^2}{16} + \frac{(y+5)^2}{25} = 1$

53. $\frac{x^2}{4} - \frac{y^2}{5} = 1 \Rightarrow$ center is $(0, 0)$, vertices are $(2, 0)$ and $(-2, 0)$; $c = \sqrt{a^2 + b^2} = \sqrt{4 + 5} = 3 \Rightarrow$ foci are $(3, 0)$ and $(-3, 0)$; the asymptotes are $\pm \frac{x}{2} = \frac{y}{\sqrt{5}} \Rightarrow y = \pm \frac{\sqrt{5}x}{2}$; therefore the new center is $(2, 2)$, the new vertices are

$(4, 2)$ and $(0, 2)$, and the new foci are $(5, 2)$ and $(-1, 2)$; the new asymptotes are $y - 2 = \pm \frac{\sqrt{5}(x-2)}{2}$; the new equation is $\frac{(x-2)^2}{4} - \frac{(y-2)^2}{5} = 1$

54. $\frac{x^2}{16} - \frac{y^2}{9} = 1 \Rightarrow$ center is $(0, 0)$, vertices are $(4, 0)$ and $(-4, 0)$; $c = \sqrt{a^2 + b^2} = \sqrt{16 + 9} = 5 \Rightarrow$ foci are $(-5, 0)$ and $(5, 0)$; the asymptotes are $\pm \frac{x}{4} = \frac{y}{3} \Rightarrow y = \pm \frac{3x}{4}$; therefore the new center is $(-5, -1)$, the new vertices are $(-1, -1)$ and $(-9, -1)$, and the new foci are $(-10, -1)$ and $(0, -1)$; the new asymptotes are $y + 1 = \pm \frac{3(x+5)}{4}$; the new equation is $\frac{(x+5)^2}{16} - \frac{(y+1)^2}{9} = 1$

55. $y^2 - x^2 = 1 \Rightarrow$ center is $(0, 0)$, vertices are $(0, 1)$ and $(0, -1)$; $c = \sqrt{a^2 + b^2} = \sqrt{1 + 1} = \sqrt{2} \Rightarrow$ foci are $\left(0, \pm\sqrt{2}\right)$; the asymptotes are $y = \pm x$; therefore the new center is $(-1, -1)$, the new vertices are $(-1, 0)$ and $(-1, -2)$, and the new foci are $\left(-1, -1 \pm \sqrt{2}\right)$; the new asymptotes are $y + 1 = \pm (x + 1)$; the new equation is $(y + 1)^2 - (x + 1)^2 = 1$

56. $\frac{y^2}{3} - x^2 = 1 \Rightarrow$ center is $(0, 0)$, vertices are $\left(0, \sqrt{3}\right)$ and $\left(0, -\sqrt{3}\right)$; $c = \sqrt{a^2 + b^2} = \sqrt{3 + 1} = 2 \Rightarrow$ foci are $(0, 2)$ and $(0, -2)$; the asymptotes are $\pm x = \frac{y}{\sqrt{3}} \Rightarrow y = \pm \sqrt{3}x$; therefore the new center is $(1, 3)$, the new vertices are $\left(1, 3 \pm \sqrt{3}\right)$, and the new foci are $(1, 5)$ and $(1, 1)$; the new asymptotes are $y - 3 = \pm \sqrt{3}(x - 1)$; the new equation is $\frac{(y-3)^2}{3} - (x - 1)^2 = 1$

57. $x^2 + 4x + y^2 = 12 \Rightarrow x^2 + 4x + 4 + y^2 = 12 + 4 \Rightarrow (x + 2)^2 + y^2 = 16$; this is a circle: center at $C(-2, 0)$, $a = 4$

58. $2x^2 + 2y^2 - 28x + 12y + 114 = 0 \Rightarrow x^2 - 14x + 49 + y^2 + 6y + 9 = -57 + 49 + 9 \Rightarrow (x - 7)^2 + (y + 3)^2 = 1$; this is a circle: center at $C(7, -3)$, $a = 1$

59. $x^2 + 2x + 4y - 3 = 0 \Rightarrow x^2 + 2x + 1 = -4y + 3 + 1 \Rightarrow (x + 1)^2 = -4(y - 1)$; this is a parabola: $V(-1, 1)$, $F(-1, 0)$

60. $y^2 - 4y - 8x - 12 = 0 \Rightarrow y^2 - 4y + 4 = 8x + 12 + 4 \Rightarrow (y - 2)^2 = 8(x + 2)$; this is a parabola: $V(-2, 2)$, $F(0, 2)$

61. $x^2 + 5y^2 + 4x = 1 \Rightarrow x^2 + 4x + 4 + 5y^2 = 5 \Rightarrow (x + 2)^2 + 5y^2 = 5 \Rightarrow \frac{(x+2)^2}{5} + y^2 = 1$; this is an ellipse: the center is $(-2, 0)$, the vertices are $\left(-2 \pm \sqrt{5}, 0\right)$; $c = \sqrt{a^2 - b^2} = \sqrt{5 - 1} = 2 \Rightarrow$ the foci are $(-4, 0)$ and $(0, 0)$

62. $9x^2 + 6y^2 + 36y = 0 \Rightarrow 9x^2 + 6\left(y^2 + 6y + 9\right) = 54 \Rightarrow 9x^2 + 6(y + 3)^2 = 54 \Rightarrow \frac{x^2}{6} + \frac{(y+3)^2}{9} = 1$; this is an ellipse: the center is $(0, -3)$, the vertices are $(0, 0)$ and $(0, -6)$; $c = \sqrt{a^2 - b^2} = \sqrt{9 - 6} = \sqrt{3} \Rightarrow$ the foci are $\left(0, -3 \pm \sqrt{3}\right)$

63. $x^2 + 2y^2 - 2x - 4y = -1 \Rightarrow x^2 - 2x + 1 + 2\left(y^2 - 2y + 1\right) = 2 \Rightarrow (x - 1)^2 + 2(y - 1)^2 = 2$ $\Rightarrow \frac{(x-1)^2}{2} + (y - 1)^2 = 1$; this is an ellipse: the center is $(1, 1)$, the vertices are $\left(1 \pm \sqrt{2}, 1\right)$; $c = \sqrt{a^2 - b^2} = \sqrt{2 - 1} = 1 \Rightarrow$ the foci are $(2, 1)$ and $(0, 1)$

64. $4x^2 + y^2 + 8x - 2y = -1 \Rightarrow 4\left(x^2 + 2x + 1\right) + y^2 - 2y + 1 = 4 \Rightarrow 4(x + 1)^2 + (y - 1)^2 = 4$ $\Rightarrow (x + 1)^2 + \frac{(y-1)^2}{4} = 1$; this is an ellipse: the center is $(-1, 1)$, the vertices are $(-1, 3)$ and $(-1, -1)$; $c = \sqrt{a^2 - b^2} = \sqrt{4 - 1} = \sqrt{3} \Rightarrow$ the foci are $\left(-1, 1 \pm \sqrt{3}\right)$

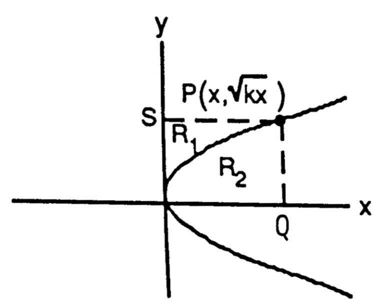
65. $x^2 - y^2 - 2x + 4y = 4 \Rightarrow x^2 - 2x + 1 - (y^2 - 4y + 4) = 1 \Rightarrow (x-1)^2 - (y-2)^2 = 1$; this is a hyperbola: the center is $(1, 2)$, the vertices are $(2, 2)$ and $(0, 2)$; $c = \sqrt{a^2 + b^2} = \sqrt{1+1} = \sqrt{2} \Rightarrow$ the foci are $\left(1 \pm \sqrt{2}, 2\right)$; the asymptotes are $y - 2 = \pm(x-1)$

66. $x^2 - y^2 + 4x - 6y = 6 \Rightarrow x^2 + 4x + 4 - (y^2 + 6y + 9) = 1 \Rightarrow (x+2)^2 - (y+3)^2 = 1$; this is a hyperbola: the center is $(-2, -3)$, the vertices are $(-1, -3)$ and $(-3, -3)$; $c = \sqrt{a^2 + b^2} = \sqrt{1+1} = \sqrt{2} \Rightarrow$ the foci are $\left(-2 \pm \sqrt{2}, -3\right)$; the asymptotes are $y + 3 = \pm(x+2)$

67. $2x^2 - y^2 + 6y = 3 \Rightarrow 2x^2 - (y^2 - 6y + 9) = -6 \Rightarrow \frac{(y-3)^2}{6} - \frac{x^2}{3} = 1$; this is a hyperbola: the center is $(0, 3)$, the vertices are $\left(0, 3 \pm \sqrt{6}\right)$; $c = \sqrt{a^2 + b^2} = \sqrt{6+3} = 3 \Rightarrow$ the foci are $(0, 6)$ and $(0, 0)$; the asymptotes are $\frac{y-3}{\sqrt{6}} = \pm \frac{x}{\sqrt{3}} \Rightarrow y = \pm\sqrt{2}x + 3$

68. $y^2 - 4x^2 + 16x = 24 \Rightarrow y^2 - 4(x^2 - 4x + 4) = 8 \Rightarrow \frac{y^2}{8} - \frac{(x-2)^2}{2} = 1$; this is a hyperbola: the center is $(2, 0)$, the vertices are $\left(2, \pm\sqrt{8}\right)$; $c = \sqrt{a^2 + b^2} = \sqrt{8+2} = \sqrt{10} \Rightarrow$ the foci are $\left(2, \pm\sqrt{10}\right)$; the asymptotes are $\frac{y}{\sqrt{8}} = \pm\frac{x-2}{\sqrt{2}} \Rightarrow y = \pm 2(x-2)$

69. (a) $y^2 = kx \Rightarrow x = \frac{y^2}{k}$; the volume of the solid formed by revolving R_1 about the y-axis is $V_1 = \int_0^{\sqrt{kx}} \pi \left(\frac{y^2}{k}\right)^2 dy = \frac{\pi}{k^2}\int_0^{\sqrt{kx}} y^4\, dy = \frac{\pi x^2\sqrt{kx}}{5}$; the volume of the right circular cylinder formed by revolving PQ about the y-axis is $V_2 = \pi x^2\sqrt{kx} \Rightarrow$ the volume of the solid formed by revolving R_2 about the y-axis is $V_3 = V_2 - V_1 = \frac{4\pi x^2\sqrt{kx}}{5}$. Therefore we can see the ratio of V_3 to V_1 is 4:1.



(b) The volume of the solid formed by revolving R_2 about the x-axis is $V_1 = \int_0^x \pi\left(\sqrt{kt}\right)^2 dt = \pi k\int_0^x t\, dt = \frac{\pi k x^2}{2}$. The volume of the right circular cylinder formed by revolving PS about the x-axis is $V_2 = \pi\left(\sqrt{kx}\right)^2 x = \pi k x^2 \Rightarrow$ the volume of the solid formed by revolving R_1 about the x-axis is $V_3 = V_2 - V_1 = \pi k x^2 - \frac{\pi k x^2}{2} = \frac{\pi k x^2}{2}$. Therefore the ratio of V_3 to V_1 is 1:1.

70. $y = \int \frac{w}{H} x\, dx = \frac{w}{H}\left(\frac{x^2}{2}\right) + C = \frac{wx^2}{2H} + C$; $y = 0$ when $x = 0 \Rightarrow 0 = \frac{w(0)^2}{2H} + C \Rightarrow C = 0$; therefore $y = \frac{wx^2}{2H}$ is the equation of the cable's curve

71. $x^2 = 4py$ and $y = p \Rightarrow x^2 = 4p^2 \Rightarrow x = \pm 2p$. Therefore the line $y = p$ cuts the parabola at points $(-2p, p)$ and $(2p, p)$, and these points are $\sqrt{[2p - (-2p)]^2 + (p-p)^2} = 4p$ units apart.

72. $\lim\limits_{x\to\infty}\left(\frac{b}{a}x - \frac{b}{a}\sqrt{x^2 - a^2}\right) = \frac{b}{a}\lim\limits_{x\to\infty}\left(x - \sqrt{x^2 - a^2}\right) = \frac{b}{a}\lim\limits_{x\to\infty}\left[\frac{\left(x - \sqrt{x^2 - a^2}\right)\left(x + \sqrt{x^2 - a^2}\right)}{x + \sqrt{x^2 - a^2}}\right]$
$= \frac{b}{a}\lim\limits_{x\to\infty}\left[\frac{x^2 - (x^2 - a^2)}{x + \sqrt{x^2 - a^2}}\right] = \frac{b}{a}\lim\limits_{x\to\infty}\left[\frac{a^2}{x + \sqrt{x^2 - a^2}}\right] = 0$

73. Let $y = \sqrt{1 - \frac{x^2}{4}}$ on the interval $0 \le x \le 2$. The area of the inscribed rectangle is given by
$A(x) = 2x\left(2\sqrt{1 - \frac{x^2}{4}}\right) = 4x\sqrt{1 - \frac{x^2}{4}}$ (since the length is 2x and the height is 2y)
$\Rightarrow A'(x) = 4\sqrt{1 - \frac{x^2}{4}} - \frac{x^2}{\sqrt{1-\frac{x^2}{4}}}$. Thus $A'(x) = 0 \Rightarrow 4\sqrt{1 - \frac{x^2}{4}} - \frac{x^2}{\sqrt{1-\frac{x^2}{4}}} = 0 \Rightarrow 4\left(1 - \frac{x^2}{4}\right) - x^2 = 0 \Rightarrow x^2 = 2$
$\Rightarrow x = \sqrt{2}$ (only the positive square root lies in the interval). Since $A(0) = A(2) = 0$ we have that $A\left(\sqrt{2}\right) = 4$ is the maximum area when the length is $2\sqrt{2}$ and the height is $\sqrt{2}$.

74. (a) Around the x-axis: $9x^2 + 4y^2 = 36 \Rightarrow y^2 = 9 - \frac{9}{4}x^2 \Rightarrow y = \pm\sqrt{9 - \frac{9}{4}x^2}$ and we use the positive root
$\Rightarrow V = 2\int_0^2 \pi\left(\sqrt{9 - \frac{9}{4}x^2}\right)^2 dx = 2\int_0^2 \pi\left(9 - \frac{9}{4}x^2\right) dx = 2\pi\left[9x - \frac{3}{4}x^3\right]_0^2 = 24\pi$
(b) Around the y-axis: $9x^2 + 4y^2 = 36 \Rightarrow x^2 = 4 - \frac{4}{9}y^2 \Rightarrow x = \pm\sqrt{4 - \frac{4}{9}y^2}$ and we use the positive root
$\Rightarrow V = 2\int_0^3 \pi\left(\sqrt{4 - \frac{4}{9}y^2}\right)^2 dy = 2\int_0^3 \pi\left(4 - \frac{4}{9}y^2\right) dy = 2\pi\left[4y - \frac{4}{27}y^3\right]_0^3 = 16\pi$

75. $9x^2 - 4y^2 = 36 \Rightarrow y^2 = \frac{9x^2 - 36}{4} \Rightarrow y = \pm\frac{3}{2}\sqrt{x^2 - 4}$ on the interval $2 \le x \le 4 \Rightarrow V = \int_2^4 \pi\left(\frac{3}{2}\sqrt{x^2 - 4}\right)^2 dx$
$= \frac{9\pi}{4}\int_2^4 (x^2 - 4)\, dx = \frac{9\pi}{4}\left[\frac{x^3}{3} - 4x\right]_2^4 = \frac{9\pi}{4}\left[\left(\frac{64}{3} - 16\right) - \left(\frac{8}{3} - 8\right)\right] = \frac{9\pi}{4}\left(\frac{56}{3} - 8\right) = \frac{3\pi}{4}(56 - 24) = 24\pi$

76. Let $P_1(-p, y_1)$ be any point on $x = -p$, and let $P(x, y)$ be a point where a tangent intersects $y^2 = 4px$. Now
$y^2 = 4px \Rightarrow 2y\frac{dy}{dx} = 4p \Rightarrow \frac{dy}{dx} = \frac{2p}{y}$; then the slope of a tangent line from P_1 is $\frac{y - y_1}{x - (-p)} = \frac{dy}{dx} = \frac{2p}{y}$
$\Rightarrow y^2 - yy_1 = 2px + 2p^2$. Since $x = \frac{y^2}{4p}$, we have $y^2 - yy_1 = 2p\left(\frac{y^2}{4p}\right) + 2p^2 \Rightarrow y^2 - yy_1 = \frac{1}{2}y^2 + 2p^2$
$\Rightarrow \frac{1}{2}y^2 - yy_1 - 2p^2 = 0 \Rightarrow y = \frac{2y_1 \pm \sqrt{4y_1^2 + 16p^2}}{2} = y_1 \pm \sqrt{y_1^2 + 4p^2}$. Therefore the slopes of the two tangents from P_1 are $m_1 = \frac{2p}{y_1 + \sqrt{y_1^2 + 4p^2}}$ and $m_2 = \frac{2p}{y_1 - \sqrt{y_1^2 + 4p^2}} \Rightarrow m_1m_2 = \frac{4p^2}{y_1^2 - (y_1^2 + 4p^2)} = -1$
$\Rightarrow$ the lines are perpendicular

77. $(x - 2)^2 + (y - 1)^2 = 5 \Rightarrow 2(x - 2) + 2(y - 1)\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x-2}{y-1}$; $y = 0 \Rightarrow (x - 2)^2 + (0 - 1)^2 = 5$
$\Rightarrow (x - 2)^2 = 4 \Rightarrow x = 4$ or $x = 0 \Rightarrow$ the circle crosses the x-axis at $(4, 0)$ and $(0, 0)$; $x = 0$
$\Rightarrow (0 - 2)^2 + (y - 1)^2 = 5 \Rightarrow (y - 1)^2 = 1 \Rightarrow y = 2$ or $y = 0 \Rightarrow$ the circle crosses the y-axis at $(0, 2)$ and $(0, 0)$.
At $(4, 0)$: $\frac{dy}{dx} = -\frac{4-2}{0-1} = 2 \Rightarrow$ the tangent line is $y = 2(x - 4)$ or $y = 2x - 8$
At $(0, 0)$: $\frac{dy}{dx} = -\frac{0-2}{0-1} = -2 \Rightarrow$ the tangent line is $y = -2x$
At $(0, 2)$: $\frac{dy}{dx} = -\frac{0-2}{2-1} = 2 \Rightarrow$ the tangent line is $y - 2 = 2x$ or $y = 2x + 2$

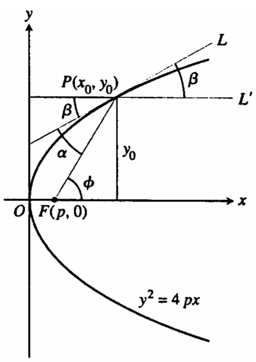
78. $x^2 - y^2 = 1 \Rightarrow x = \pm\sqrt{1 + y^2}$ on the interval $-3 \le y \le 3 \Rightarrow V = \int_{-3}^3 \pi\left(\sqrt{1 + y^2}\right)^2 dy = 2\int_0^3 \pi\left(\sqrt{1 + y^2}\right)^2 dy$
$= 2\pi\int_0^3 (1 + y^2)\, dy = 2\pi\left[y + \frac{y^3}{3}\right]_0^3 = 24\pi$

79. Let $y = \sqrt{16 - \frac{16}{9}x^2}$ on the interval $-3 \le x \le 3$. Since the plate is symmetric about the y-axis, $\overline{x} = 0$. For a vertical strip: $(\tilde{x}, \tilde{y}) = \left(x, \frac{\sqrt{16 - \frac{16}{9}x^2}}{2}\right)$, length $= \sqrt{16 - \frac{16}{9}x^2}$, width $= dx \Rightarrow$ area $= dA = \sqrt{16 - \frac{16}{9}x^2}\, dx$
$\Rightarrow$ mass $= dm = \delta\, dA = \delta\sqrt{16 - \frac{16}{9}x^2}\, dx$. Moment of the strip about the x-axis:
$\tilde{y}\, dm = \frac{\sqrt{16 - \frac{16}{9}x^2}}{2}\left(\delta\sqrt{16 - \frac{16}{9}x^2}\right) dx = \delta\left(8 - \frac{8}{9}x^2\right) dx$ so the moment of the plate about the x-axis is

$M_x = \int \tilde{y}\, dm = \int_{-3}^{3} \delta\left(8 - \frac{8}{9}x^2\right) dx = \delta\left[8x - \frac{8}{27}x^3\right]_{-3}^{3} = 32\delta$; also the mass of the plate is
$M = \int_{-3}^{3}\delta\sqrt{16 - \frac{16}{9}x^2}\,dx = \int_{-3}^{3} 4\delta\sqrt{1-\left(\frac{1}{3}x\right)^2}\,dx = 4\delta\int_{-1}^{1} 3\sqrt{1-u^2}\,du$ where $u = \frac{x}{3} \Rightarrow 3\,du = dx$; $x = -3$
$\Rightarrow u = -1$ and $x = 3 \Rightarrow u = 1$. Hence, $4\delta\int_{-1}^{1} 3\sqrt{1-u^2}\,du = 12\delta\int_{-1}^{1}\sqrt{1-u^2}\,du$
$= 12\delta\left[\frac{1}{2}\left(u\sqrt{1-u^2} + \sin^{-1}u\right)\right]_{-1}^{1} = 6\pi\delta \Rightarrow \bar{y} = \frac{M_x}{M} = \frac{32\delta}{6\pi\delta} = \frac{16}{3\pi}$. Therefore the center of mass is $\left(0, \frac{16}{3\pi}\right)$.

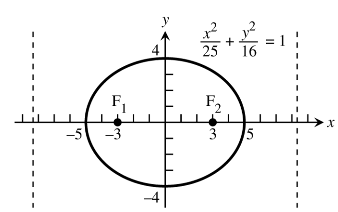
80. $y = \sqrt{x^2+1} \Rightarrow \frac{dy}{dx} = \frac{1}{2}(x^2+1)^{-1/2}(2x) = \frac{x}{\sqrt{x^2+1}} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{x^2}{x^2+1} \Rightarrow \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{x^2}{x^2+1}}$
$= \sqrt{\frac{2x^2+1}{x^2+1}} \Rightarrow S = \int_0^{\sqrt{2}} 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^{\sqrt{2}} 2\pi\sqrt{x^2+1}\sqrt{\frac{2x^2+1}{x^2+1}}\,dx = \int_0^{\sqrt{2}} 2\pi\sqrt{2x^2+1}\,dx$;
$\begin{bmatrix} u = \sqrt{2}x \\ du = \sqrt{2}\,dx \end{bmatrix} \rightarrow \frac{2\pi}{\sqrt{2}}\int_0^2\sqrt{u^2+1}\,du = \frac{2\pi}{\sqrt{2}}\left[\frac{1}{2}\left(u\sqrt{u^2+1} + \ln\left(u+\sqrt{u^2+1}\right)\right)\right]_0^2 = \frac{\pi}{\sqrt{2}}\left[2\sqrt{5} + \ln\left(2+\sqrt{5}\right)\right]$

81. (a) $\tan\beta = m_L \Rightarrow \tan\beta = f'(x_0)$ where $f(x) = \sqrt{4px}$;
$f'(x) = \frac{1}{2}(4px)^{-1/2}(4p) = \frac{2p}{\sqrt{4px}} \Rightarrow f'(x_0) = \frac{2p}{\sqrt{4px_0}}$
$= \frac{2p}{y_0} \Rightarrow \tan\beta = \frac{2p}{y_0}$.
(b) $\tan\phi = m_{FP} = \frac{y_0 - 0}{x_0 - p} = \frac{y_0}{x_0 - p}$
(c) $\tan\alpha = \frac{\tan\phi - \tan\beta}{1 + \tan\phi\tan\beta} = \frac{\left(\frac{y_0}{x_0-p} - \frac{2p}{y_0}\right)}{1 + \left(\frac{y_0}{x_0-p}\right)\left(\frac{2p}{y_0}\right)}$
$= \frac{y_0^2 - 2p(x_0-p)}{y_0(x_0-p+2p)} = \frac{4px_0 - 2px_0 + 2p^2}{y_0(x_0+p)} = \frac{2p(x_0+p)}{y_0(x_0+p)} = \frac{2p}{y_0}$

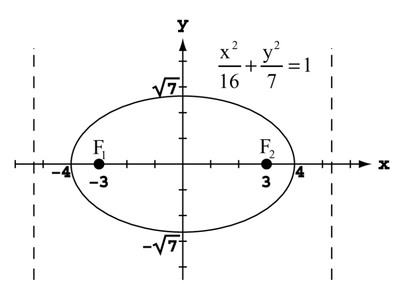


11.7 CONICS IN POLAR COORDINATES

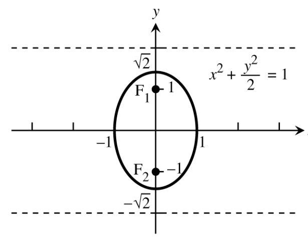
1. $16x^2 + 25y^2 = 400 \Rightarrow \frac{x^2}{25} + \frac{y^2}{16} = 1 \Rightarrow c = \sqrt{a^2 - b^2}$
$= \sqrt{25-16} = 3 \Rightarrow e = \frac{c}{a} = \frac{3}{5}$; $F(\pm 3, 0)$;
directrices are $x = 0 \pm \frac{a}{e} = \pm\frac{5}{\left(\frac{3}{5}\right)} = \pm\frac{25}{3}$



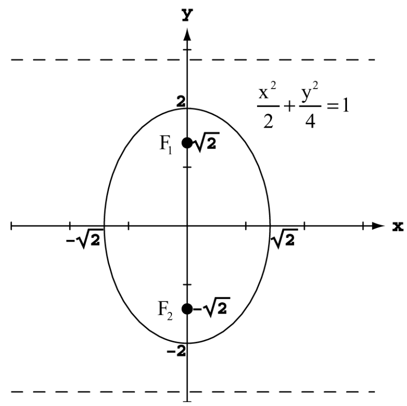
2. $7x^2 + 16y^2 = 112 \Rightarrow \frac{x^2}{16} + \frac{y^2}{7} = 1 \Rightarrow c = \sqrt{a^2 - b^2}$
$= \sqrt{16-7} = 3 \Rightarrow e = \frac{c}{a} = \frac{3}{4}$; $F(\pm 3, 0)$;
directrices are $x = 0 \pm \frac{a}{e} = \pm\frac{4}{\left(\frac{3}{4}\right)} = \pm\frac{16}{3}$



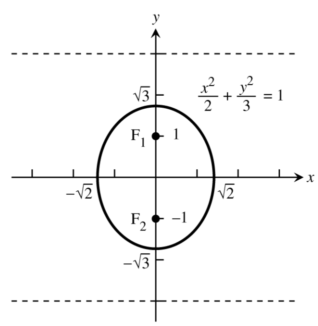
3. $2x^2 + y^2 = 2 \Rightarrow x^2 + \frac{y^2}{2} = 1 \Rightarrow c = \sqrt{a^2 - b^2} = \sqrt{2-1} = 1 \Rightarrow e = \frac{c}{a} = \frac{1}{\sqrt{2}}$; $F(0, \pm 1)$; directrices are $y = 0 \pm \frac{a}{e} = \pm \frac{\sqrt{2}}{\left(\frac{1}{\sqrt{2}}\right)} = \pm 2$



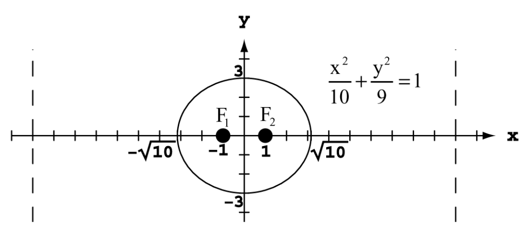
4. $2x^2 + y^2 = 4 \Rightarrow \frac{x^2}{2} + \frac{y^2}{4} = 1 \Rightarrow c = \sqrt{a^2 - b^2} = \sqrt{4-2} = \sqrt{2} \Rightarrow e = \frac{c}{a} = \frac{\sqrt{2}}{2}$; $F\left(0, \pm\sqrt{2}\right)$; directrices are $y = 0 \pm \frac{a}{e} = \pm \frac{2}{\left(\frac{\sqrt{2}}{2}\right)} = \pm 2\sqrt{2}$



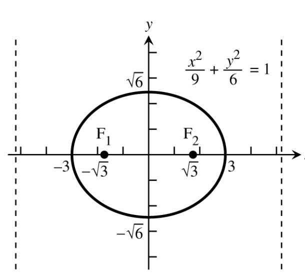
5. $3x^2 + 2y^2 = 6 \Rightarrow \frac{x^2}{2} + \frac{y^2}{3} = 1 \Rightarrow c = \sqrt{a^2 - b^2} = \sqrt{3-2} = 1 \Rightarrow e = \frac{c}{a} = \frac{1}{\sqrt{3}}$; $F(0, \pm 1)$; directrices are $y = 0 \pm \frac{a}{e} = \pm \frac{\sqrt{3}}{\left(\frac{1}{\sqrt{3}}\right)} = \pm 3$



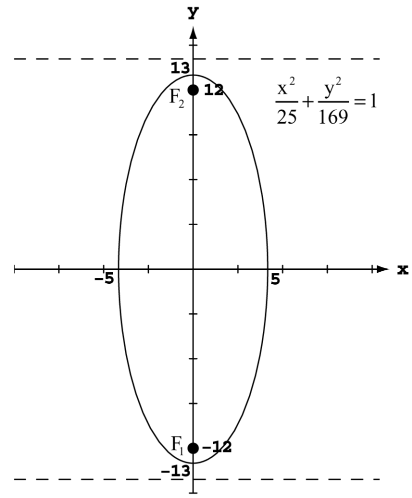
6. $9x^2 + 10y^2 = 90 \Rightarrow \frac{x^2}{10} + \frac{y^2}{9} = 1 \Rightarrow c = \sqrt{a^2 - b^2} = \sqrt{10-9} = 1 \Rightarrow e = \frac{c}{a} = \frac{1}{\sqrt{10}}$; $F(\pm 1, 0)$; directrices are $x = 0 \pm \frac{a}{e} = \pm \frac{\sqrt{10}}{\left(\frac{1}{\sqrt{10}}\right)} = \pm 10$



7. $6x^2 + 9y^2 = 54 \Rightarrow \frac{x^2}{9} + \frac{y^2}{6} = 1 \Rightarrow c = \sqrt{a^2 - b^2} = \sqrt{9-6} = \sqrt{3} \Rightarrow e = \frac{c}{a} = \frac{\sqrt{3}}{3}$; $F\left(\pm\sqrt{3}, 0\right)$; directrices are $x = 0 \pm \frac{a}{e} = \pm \frac{3}{\left(\frac{\sqrt{3}}{3}\right)} = \pm 3\sqrt{3}$



8. $169x^2 + 25y^2 = 4225 \Rightarrow \frac{x^2}{25} + \frac{y^2}{169} = 1 \Rightarrow c = \sqrt{a^2 - b^2}$
$= \sqrt{169 - 25} = 12 \Rightarrow e = \frac{c}{a} = \frac{12}{13}$; $F(0, \pm 12)$;
directrices are $y = 0 \pm \frac{a}{e} = \pm \frac{13}{\left(\frac{12}{13}\right)} = \pm \frac{169}{12}$



9. Foci: $(0, \pm 3)$, $e = 0.5 \Rightarrow c = 3$ and $a = \frac{c}{e} = \frac{3}{0.5} = 6 \Rightarrow b^2 = 36 - 9 = 27 \Rightarrow \frac{x^2}{27} + \frac{y^2}{36} = 1$

10. Foci: $(\pm 8, 0)$, $e = 0.2 \Rightarrow c = 8$ and $a = \frac{c}{e} = \frac{8}{0.2} = 40 \Rightarrow b^2 = 1600 - 64 = 1536 \Rightarrow \frac{x^2}{1600} + \frac{y^2}{1536} = 1$

11. Vertices: $(0, \pm 70)$, $e = 0.1 \Rightarrow a = 70$ and $c = ae = 70(0.1) = 7 \Rightarrow b^2 = 4900 - 49 = 4851 \Rightarrow \frac{x^2}{4851} + \frac{y^2}{4900} = 1$

12. Vertices: $(\pm 10, 0)$, $e = 0.24 \Rightarrow a = 10$ and $c = ae = 10(0.24) = 2.4 \Rightarrow b^2 = 100 - 5.76 = 94.24 \Rightarrow \frac{x^2}{100} + \frac{y^2}{94.24} = 1$

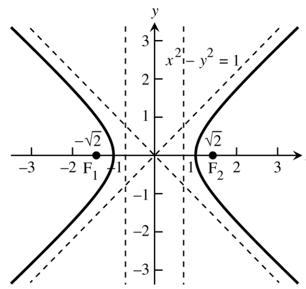
13. Focus: $\left(\sqrt{5}, 0\right)$, Directrix: $x = \frac{9}{\sqrt{5}} \Rightarrow c = ae = \sqrt{5}$ and $\frac{a}{e} = \frac{9}{\sqrt{5}} \Rightarrow \frac{ae}{e^2} = \frac{9}{\sqrt{5}} \Rightarrow \frac{\sqrt{5}}{e^2} = \frac{9}{\sqrt{5}} \Rightarrow e^2 = \frac{5}{9}$
$\Rightarrow e = \frac{\sqrt{5}}{3}$. Then $PF = \frac{\sqrt{5}}{3} PD \Rightarrow \sqrt{\left(x - \sqrt{5}\right)^2 + (y - 0)^2} = \frac{\sqrt{5}}{3}\left|x - \frac{9}{\sqrt{5}}\right| \Rightarrow \left(x - \sqrt{5}\right)^2 + y^2 = \frac{5}{9}\left(x - \frac{9}{\sqrt{5}}\right)^2$
$\Rightarrow x^2 - 2\sqrt{5}\,x + 5 + y^2 = \frac{5}{9}\left(x^2 - \frac{18}{\sqrt{5}}x + \frac{81}{5}\right) \Rightarrow \frac{4}{9}x^2 + y^2 = 4 \Rightarrow \frac{x^2}{9} + \frac{y^2}{4} = 1$

14. Focus: $(4, 0)$, Directrix: $x = \frac{16}{3} \Rightarrow c = ae = 4$ and $\frac{a}{e} = \frac{16}{3} \Rightarrow \frac{ae}{e^2} = \frac{16}{3} \Rightarrow \frac{4}{e^2} = \frac{16}{3} \Rightarrow e^2 = \frac{3}{4} \Rightarrow e = \frac{\sqrt{3}}{2}$. Then
$PF = \frac{\sqrt{3}}{2} PD \Rightarrow \sqrt{(x-4)^2 + (y-0)^2} = \frac{\sqrt{3}}{2}\left|x - \frac{16}{3}\right| \Rightarrow (x-4)^2 + y^2 = \frac{3}{4}\left(x - \frac{16}{3}\right)^2 \Rightarrow x^2 - 8x + 16 + y^2$
$= \frac{3}{4}\left(x^2 - \frac{32}{3}x + \frac{256}{9}\right) \Rightarrow \frac{1}{4}x^2 + y^2 = \frac{16}{3} \Rightarrow \frac{x^2}{\left(\frac{64}{3}\right)} + \frac{y^2}{\left(\frac{16}{3}\right)} = 1$

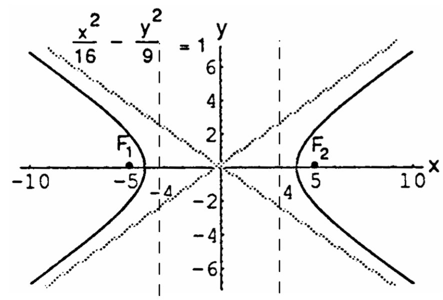
15. Focus: $(-4, 0)$, Directrix: $x = -16 \Rightarrow c = ae = 4$ and $\frac{a}{e} = 16 \Rightarrow \frac{ae}{e^2} = 16 \Rightarrow \frac{4}{e^2} = 16 \Rightarrow e^2 = \frac{1}{4} \Rightarrow e = \frac{1}{2}$. Then
$PF = \frac{1}{2} PD \Rightarrow \sqrt{(x+4)^2 + (y-0)^2} = \frac{1}{2}|x + 16| \Rightarrow (x+4)^2 + y^2 = \frac{1}{4}(x+16)^2 \Rightarrow x^2 + 8x + 16 + y^2$
$= \frac{1}{4}(x^2 + 32x + 256) \Rightarrow \frac{3}{4}x^2 + y^2 = 48 \Rightarrow \frac{x^2}{64} + \frac{y^2}{48} = 1$

16. Focus: $\left(-\sqrt{2}, 0\right)$, Directrix: $x = -2\sqrt{2} \Rightarrow c = ae = \sqrt{2}$ and $\frac{a}{e} = 2\sqrt{2} \Rightarrow \frac{ae}{e^2} = 2\sqrt{2} \Rightarrow \frac{\sqrt{2}}{e^2} = 2\sqrt{2} \Rightarrow e^2 = \frac{1}{2}$
$\Rightarrow e = \frac{1}{\sqrt{2}}$. Then $PF = \frac{1}{\sqrt{2}} PD \Rightarrow \sqrt{\left(x + \sqrt{2}\right)^2 + (y-0)^2} = \frac{1}{\sqrt{2}}\left|x + 2\sqrt{2}\right| \Rightarrow \left(x + \sqrt{2}\right)^2 + y^2$
$= \frac{1}{2}\left(x + 2\sqrt{2}\right)^2 \Rightarrow x^2 + 2\sqrt{2}\,x + 2 + y^2 = \frac{1}{2}\left(x^2 + 4\sqrt{2}\,x + 8\right) \Rightarrow \frac{1}{2}x^2 + y^2 = 2 \Rightarrow \frac{x^2}{4} + \frac{y^2}{2} = 1$

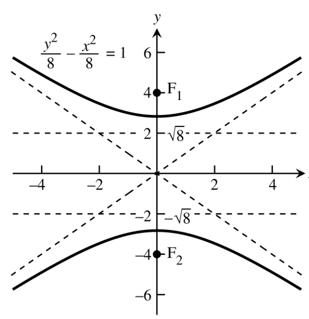
17. $x^2 - y^2 = 1 \Rightarrow c = \sqrt{a^2 + b^2} = \sqrt{1+1} = \sqrt{2} \Rightarrow e = \frac{c}{a}$
$= \frac{\sqrt{2}}{1} = \sqrt{2}$; asymptotes are $y = \pm x$; $F\left(\pm\sqrt{2}, 0\right)$;
directrices are $x = 0 \pm \frac{a}{e} = \pm \frac{1}{\sqrt{2}}$



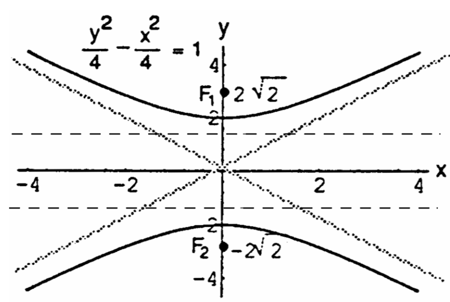
18. $9x^2 - 16y^2 = 144 \Rightarrow \frac{x^2}{16} - \frac{y^2}{9} = 1 \Rightarrow c = \sqrt{a^2 + b^2}$
$= \sqrt{16+9} = 5 \Rightarrow e = \frac{c}{a} = \frac{5}{4}$; asymptotes are
$y = \pm \frac{3}{4}x$; $F(\pm 5, 0)$; directrices are $x = 0 \pm \frac{a}{e}$
$= \pm \frac{16}{5}$



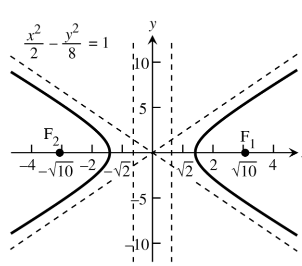
19. $y^2 - x^2 = 8 \Rightarrow \frac{y^2}{8} - \frac{x^2}{8} = 1 \Rightarrow c = \sqrt{a^2 + b^2}$
$= \sqrt{8+8} = 4 \Rightarrow e = \frac{c}{a} = \frac{4}{\sqrt{8}} = \sqrt{2}$; asymptotes are
$y = \pm x$; $F(0, \pm 4)$; directrices are $y = 0 \pm \frac{a}{e}$
$= \pm \frac{\sqrt{8}}{\sqrt{2}} = \pm 2$



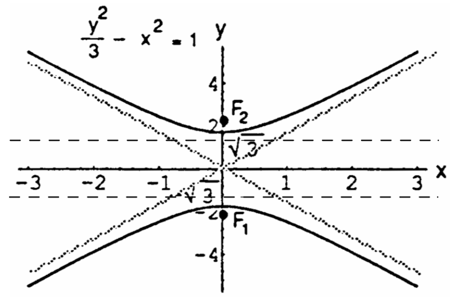
20. $y^2 - x^2 = 4 \Rightarrow \frac{y^2}{4} - \frac{x^2}{4} = 1 \Rightarrow c = \sqrt{a^2 + b^2}$
$= \sqrt{4+4} = 2\sqrt{2} \Rightarrow e = \frac{c}{a} = \frac{2\sqrt{2}}{2} = \sqrt{2}$; asymptotes
are $y = \pm x$; $F\left(0, \pm 2\sqrt{2}\right)$; directrices are $y = 0 \pm \frac{a}{e}$
$= \pm \frac{2}{\sqrt{2}} = \pm\sqrt{2}$



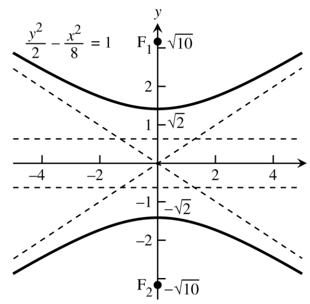
21. $8x^2 - 2y^2 = 16 \Rightarrow \frac{x^2}{2} - \frac{y^2}{8} = 1 \Rightarrow c = \sqrt{a^2 + b^2}$
$= \sqrt{2+8} = \sqrt{10} \Rightarrow e = \frac{c}{a} = \frac{\sqrt{10}}{\sqrt{2}} = \sqrt{5}$; asymptotes
are $y = \pm 2x$; $F\left(\pm\sqrt{10}, 0\right)$; directrices are $x = 0 \pm \frac{a}{e}$
$= \pm \frac{\sqrt{2}}{\sqrt{5}} = \pm \frac{2}{\sqrt{10}}$



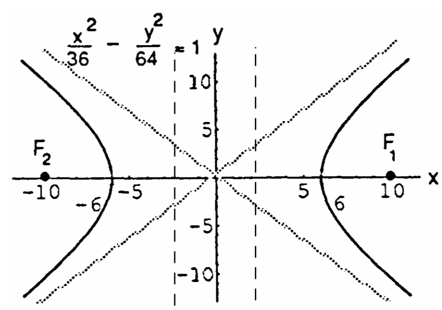
22. $y^2 - 3x^2 = 3 \Rightarrow \frac{y^2}{3} - x^2 = 1 \Rightarrow c = \sqrt{a^2 + b^2}$ $= \sqrt{3+1} = 2 \Rightarrow e = \frac{c}{a} = \frac{2}{\sqrt{3}}$; asymptotes are $y = \pm\sqrt{3}\,x$; $F(0, \pm 2)$; directrices are $y = 0 \pm \frac{a}{e}$ $= \pm \frac{\sqrt{3}}{\left(\frac{2}{\sqrt{3}}\right)} = \pm \frac{3}{2}$



23. $8y^2 - 2x^2 = 16 \Rightarrow \frac{y^2}{2} - \frac{x^2}{8} = 1 \Rightarrow c = \sqrt{a^2 + b^2}$ $= \sqrt{2+8} = \sqrt{10} \Rightarrow e = \frac{c}{a} = \frac{\sqrt{10}}{\sqrt{2}} = \sqrt{5}$; asymptotes are $y = \pm \frac{x}{2}$; $F\left(0, \pm\sqrt{10}\right)$; directrices are $y = 0 \pm \frac{a}{e}$ $= \pm \frac{\sqrt{2}}{\sqrt{5}} = \pm \frac{2}{\sqrt{10}}$



24. $64x^2 - 36y^2 = 2304 \Rightarrow \frac{x^2}{36} - \frac{y^2}{64} = 1 \Rightarrow c = \sqrt{a^2 + b^2}$ $= \sqrt{36+64} = 10 \Rightarrow e = \frac{c}{a} = \frac{10}{6} = \frac{5}{3}$; asymptotes are $y = \pm \frac{4}{3}x$; $F(\pm 10, 0)$; directrices are $x = 0 \pm \frac{a}{e}$ $= \pm \frac{6}{\left(\frac{5}{3}\right)} = \pm \frac{18}{5}$



25. Vertices $(0, \pm 1)$ and $e = 3 \Rightarrow a = 1$ and $e = \frac{c}{a} = 3 \Rightarrow c = 3a = 3 \Rightarrow b^2 = c^2 - a^2 = 9 - 1 = 8 \Rightarrow y^2 - \frac{x^2}{8} = 1$

26. Vertices $(\pm 2, 0)$ and $e = 2 \Rightarrow a = 2$ and $e = \frac{c}{a} = 2 \Rightarrow c = 2a = 4 \Rightarrow b^2 = c^2 - a^2 = 16 - 4 = 12 \Rightarrow \frac{x^2}{4} - \frac{y^2}{12} = 1$

27. Foci $(\pm 3, 0)$ and $e = 3 \Rightarrow c = 3$ and $e = \frac{c}{a} = 3 \Rightarrow c = 3a \Rightarrow a = 1 \Rightarrow b^2 = c^2 - a^2 = 9 - 1 = 8 \Rightarrow x^2 - \frac{y^2}{8} = 1$

28. Foci $(0, \pm 5)$ and $e = 1.25 \Rightarrow c = 5$ and $e = \frac{c}{a} = 1.25 = \frac{5}{4} \Rightarrow c = \frac{5}{4}a \Rightarrow 5 = \frac{5}{4}a \Rightarrow a = 4 \Rightarrow b^2 = c^2 - a^2$ $= 25 - 16 = 9 \Rightarrow \frac{y^2}{16} - \frac{x^2}{9} = 1$

29. $e = 1, x = 2 \Rightarrow k = 2 \Rightarrow r = \frac{2(1)}{1 + (1)\cos\theta} = \frac{2}{1+\cos\theta}$

30. $e = 1, y = 2 \Rightarrow k = 2 \Rightarrow r = \frac{2(1)}{1 + (1)\sin\theta} = \frac{2}{1+\sin\theta}$

31. $e = 5, y = -6 \Rightarrow k = 6 \Rightarrow r = \frac{6(5)}{1 - 5\sin\theta} = \frac{30}{1-5\sin\theta}$

32. $e = 2, x = 4 \Rightarrow k = 4 \Rightarrow r = \frac{4(2)}{1 + 2\cos\theta} = \frac{8}{1+2\cos\theta}$

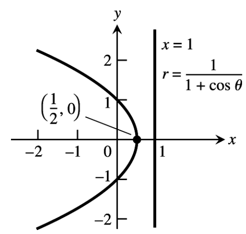
33. $e = \frac{1}{2}, x = 1 \Rightarrow k = 1 \Rightarrow r = \frac{\left(\frac{1}{2}\right)(1)}{1 + \left(\frac{1}{2}\right)\cos\theta} = \frac{1}{2+\cos\theta}$

34. $e = \frac{1}{4}, x = -2 \Rightarrow k = 2 \Rightarrow r = \frac{\left(\frac{1}{4}\right)(2)}{1 - \left(\frac{1}{4}\right)\cos\theta} = \frac{2}{4 - \cos\theta}$

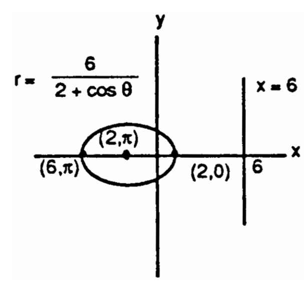
35. $e = \frac{1}{5}, y = -10 \Rightarrow k = 10 \Rightarrow r = \frac{\left(\frac{1}{5}\right)(10)}{1 - \left(\frac{1}{5}\right)\sin\theta} = \frac{10}{5 - \sin\theta}$

36. $e = \frac{1}{3}, y = 6 \Rightarrow k = 6 \Rightarrow r = \frac{\left(\frac{1}{3}\right)(6)}{1 + \left(\frac{1}{3}\right)\sin\theta} = \frac{6}{3 + \sin\theta}$

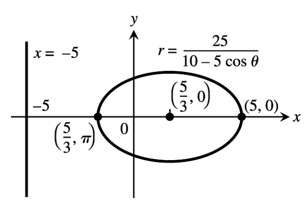
37. $r = \frac{1}{1 + \cos\theta} \Rightarrow e = 1, k = 1 \Rightarrow x = 1$



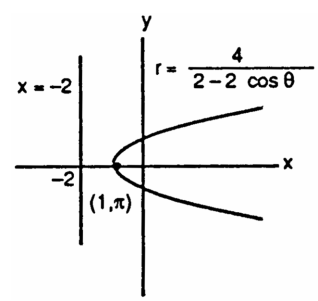
38. $r = \frac{6}{2 + \cos\theta} = \frac{3}{1 + \left(\frac{1}{2}\right)\cos\theta} \Rightarrow e = \frac{1}{2}, k = 6 \Rightarrow x = 6;$
$a(1 - e^2) = ke \Rightarrow a\left[1 - \left(\frac{1}{2}\right)^2\right] = 3 \Rightarrow \frac{3}{4}a = 3$
$\Rightarrow a = 4 \Rightarrow ea = 2$



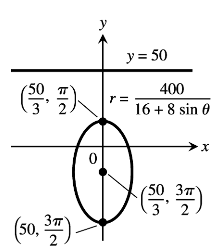
39. $r = \frac{25}{10 - 5\cos\theta} \Rightarrow r = \frac{\left(\frac{25}{10}\right)}{1 - \left(\frac{5}{10}\right)\cos\theta} = \frac{\left(\frac{5}{2}\right)}{1 - \left(\frac{1}{2}\right)\cos\theta}$
$\Rightarrow e = \frac{1}{2}, k = 5 \Rightarrow x = -5; a(1 - e^2) = ke$
$\Rightarrow a\left[1 - \left(\frac{1}{2}\right)^2\right] = \frac{5}{2} \Rightarrow \frac{3}{4}a = \frac{5}{2} \Rightarrow a = \frac{10}{3} \Rightarrow ea = \frac{5}{3}$



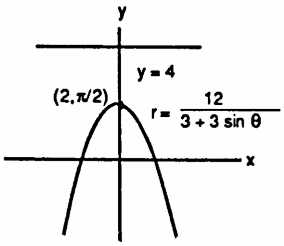
40. $r = \frac{4}{2 - 2\cos\theta} \Rightarrow r = \frac{2}{1 - \cos\theta} \Rightarrow e = 1, k = 2 \Rightarrow x = -2$



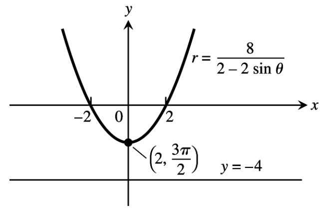
41. $r = \frac{400}{16 + 8\sin\theta} \Rightarrow r = \frac{\left(\frac{400}{16}\right)}{1 + \left(\frac{8}{16}\right)\sin\theta} \Rightarrow r = \frac{25}{1 + \left(\frac{1}{2}\right)\sin\theta}$
$e = \frac{1}{2}, k = 50 \Rightarrow y = 50; a(1 - e^2) = ke$
$\Rightarrow a\left[1 - \left(\frac{1}{2}\right)^2\right] = 25 \Rightarrow \frac{3}{4}a = 25 \Rightarrow a = \frac{100}{3}$
$\Rightarrow ea = \frac{50}{3}$



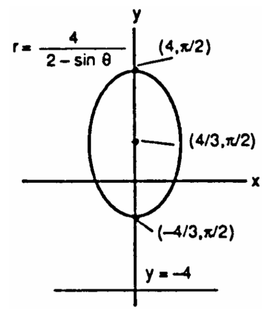
42. $r = \frac{12}{3+3\sin\theta} \Rightarrow r = \frac{4}{1+\sin\theta} \Rightarrow e = 1,$
$k = 4 \Rightarrow y = 4$



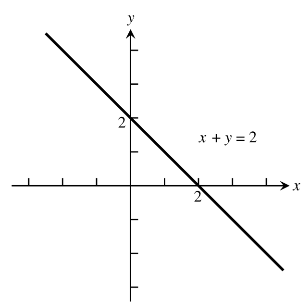
43. $r = \frac{8}{2-2\sin\theta} \Rightarrow r = \frac{4}{1-\sin\theta} \Rightarrow e = 1,$
$k = 4 \Rightarrow y = -4$



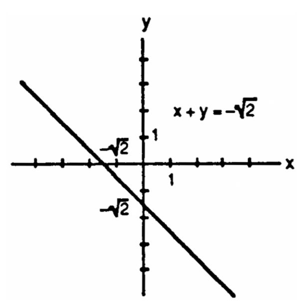
44. $r = \frac{4}{2-\sin\theta} \Rightarrow r = \frac{2}{1-\left(\frac{1}{2}\right)\sin\theta} \Rightarrow e = \frac{1}{2}, k = 4$
$\Rightarrow y = -4; a\left(1-e^2\right) = ke \Rightarrow a\left[1-\left(\frac{1}{2}\right)^2\right] = 2$
$\Rightarrow \frac{3}{4}a = 2 \Rightarrow a = \frac{8}{3} \Rightarrow ea = \frac{4}{3}$



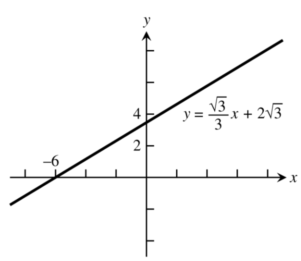
45. $r\cos\left(\theta-\frac{\pi}{4}\right) = \sqrt{2} \Rightarrow r\left(\cos\theta\cos\frac{\pi}{4}+\sin\theta\sin\frac{\pi}{4}\right)$
$= \sqrt{2} \Rightarrow \frac{1}{\sqrt{2}}r\cos\theta + \frac{1}{\sqrt{2}}r\sin\theta = \sqrt{2} \Rightarrow \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y$
$= \sqrt{2} \Rightarrow x + y = 2 \Rightarrow y = 2 - x$



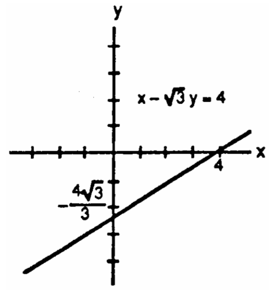
46. $r\cos\left(\theta+\frac{3\pi}{4}\right) = 1 \Rightarrow r\left(\cos\theta\cos\frac{3\pi}{4}-\sin\theta\sin\frac{3\pi}{4}\right) = 1$
$\Rightarrow -\frac{\sqrt{2}}{2}r\cos\theta - \frac{\sqrt{2}}{2}r\sin\theta = 1 \Rightarrow x + y = -\sqrt{2}$
$\Rightarrow y = -x - \sqrt{2}$



47. $r\cos\left(\theta-\frac{2\pi}{3}\right) = 3 \Rightarrow r\left(\cos\theta\cos\frac{2\pi}{3}+\sin\theta\sin\frac{2\pi}{3}\right) = 3$
$\Rightarrow -\frac{1}{2}r\cos\theta + \frac{\sqrt{3}}{2}r\sin\theta = 3 \Rightarrow -\frac{1}{2}x + \frac{\sqrt{3}}{2}y = 3$
$\Rightarrow -x + \sqrt{3}y = 6 \Rightarrow y = \frac{\sqrt{3}}{3}x + 2\sqrt{3}$



48. $r\cos\left(\theta+\frac{\pi}{3}\right)=2 \Rightarrow r\left(\cos\theta\cos\frac{\pi}{3}-\sin\theta\sin\frac{\pi}{3}\right)=2$
$\Rightarrow \frac{1}{2}r\cos\theta-\frac{\sqrt{3}}{2}r\sin\theta=2 \Rightarrow \frac{1}{2}x-\frac{\sqrt{3}}{2}y=2$
$\Rightarrow x-\sqrt{3}y=4 \Rightarrow y=\frac{\sqrt{3}}{3}x-\frac{4\sqrt{3}}{3}$



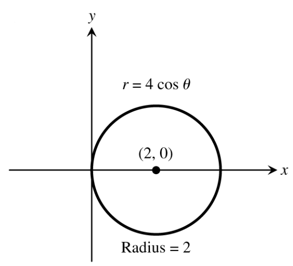
49. $\sqrt{2}x+\sqrt{2}y=6 \Rightarrow \sqrt{2}r\cos\theta+\sqrt{2}r\sin\theta=6 \Rightarrow r\left(\frac{\sqrt{2}}{2}\cos\theta+\frac{\sqrt{2}}{2}\sin\theta\right)=3 \Rightarrow r\left(\cos\frac{\pi}{4}\cos\theta+\sin\frac{\pi}{4}\sin\theta\right)$
$=3 \Rightarrow r\cos\left(\theta-\frac{\pi}{4}\right)=3$

50. $\sqrt{3}x-y=1 \Rightarrow \sqrt{3}r\cos\theta-r\sin\theta=1 \Rightarrow r\left(\frac{\sqrt{3}}{2}\cos\theta-\frac{1}{2}\sin\theta\right)=\frac{1}{2} \Rightarrow r\left(\cos\frac{\pi}{6}\cos\theta-\sin\frac{\pi}{6}\sin\theta\right)$
$=\frac{1}{2} \Rightarrow r\cos\left(\theta+\frac{\pi}{6}\right)=\frac{1}{2}$

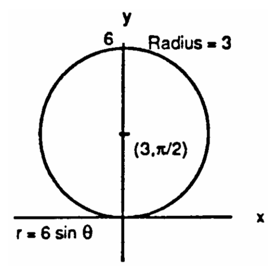
51. $y=-5 \Rightarrow r\sin\theta=-5 \Rightarrow -r\sin\theta=5 \Rightarrow r\sin(-\theta)=5 \Rightarrow r\cos\left(\frac{\pi}{2}-(-\theta)\right)=5 \Rightarrow r\cos\left(\theta+\frac{\pi}{2}\right)=5$

52. $x=-4 \Rightarrow r\cos\theta=-4 \Rightarrow -r\cos\theta=4 \Rightarrow r\cos(\theta-\pi)=4$

53.



54.



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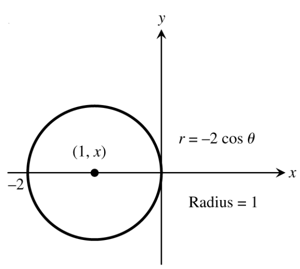

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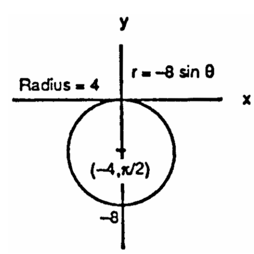
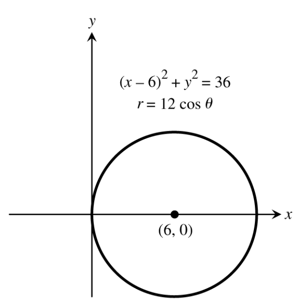
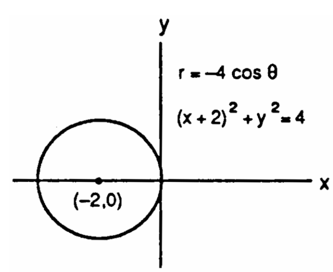


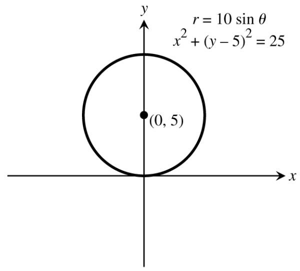
57. $(x-6)^2+y^2=36 \Rightarrow C=(6,0),\ a=6$
$\Rightarrow r=12\cos\theta$ is the polar equation



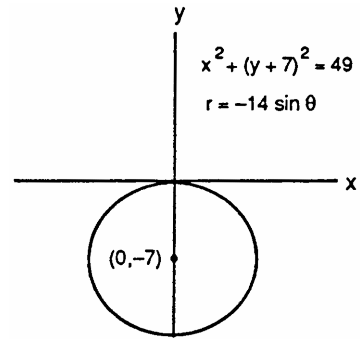
58. $(x+2)^2+y^2=4 \Rightarrow C=(-2,0),\ a=2$
$\Rightarrow r=-4\cos\theta$ is the polar equation



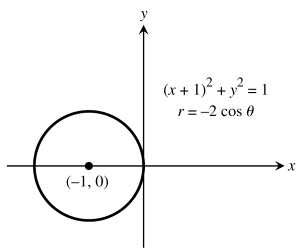
59. $x^2 + (y - 5)^2 = 25 \Rightarrow C = (0, 5), a = 5$
$\Rightarrow r = 10 \sin \theta$ is the polar equation



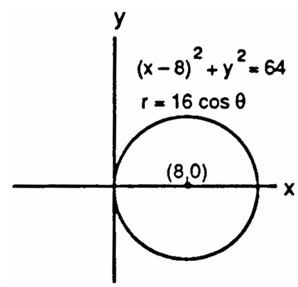
60. $x^2 + (y + 7)^2 = 49 \Rightarrow C = (0, -7), a = 7$
$\Rightarrow r = -14 \sin \theta$ is the polar equation



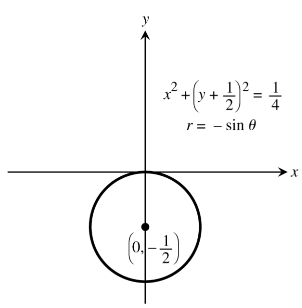
61. $x^2 + 2x + y^2 = 0 \Rightarrow (x + 1)^2 + y^2 = 1$
$\Rightarrow C = (-1, 0), a = 1 \Rightarrow r = -2 \cos \theta$ is the polar equation



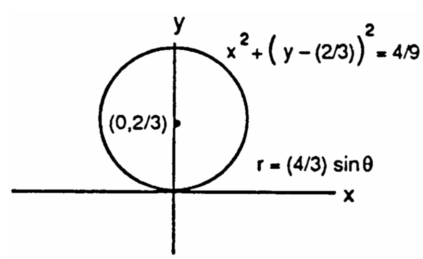
62. $x^2 - 16x + y^2 = 0 \Rightarrow (x - 8)^2 + y^2 = 64$
$\Rightarrow C = (8, 0), a = 8 \Rightarrow r = 16 \cos \theta$ is the polar equation



63. $x^2 + y^2 + y = 0 \Rightarrow x^2 + \left(y + \frac{1}{2}\right)^2 = \frac{1}{4}$
$\Rightarrow C = \left(0, -\frac{1}{2}\right), a = \frac{1}{2} \Rightarrow r = -\sin \theta$ is the polar equation



64. $x^2 + y^2 - \frac{4}{3} y = 0 \Rightarrow x^2 + \left(y - \frac{2}{3}\right)^2 = \frac{4}{9}$
$\Rightarrow C = \left(0, \frac{2}{3}\right), a = \frac{2}{3} \Rightarrow r = \frac{4}{3} \sin \theta$ is the polar equation



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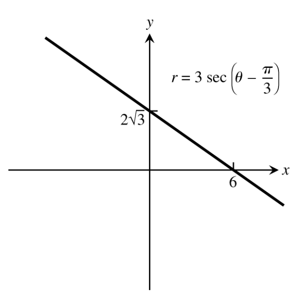

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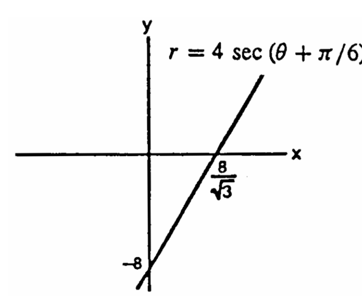

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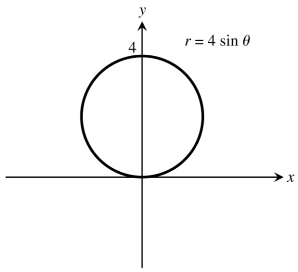

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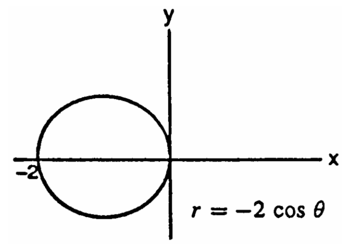

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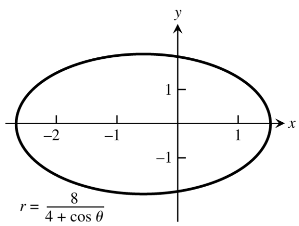

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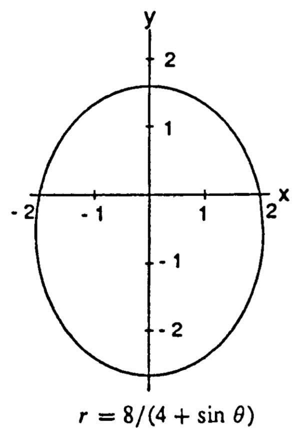

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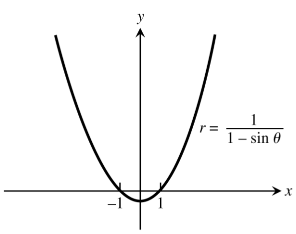

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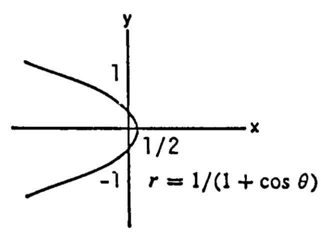

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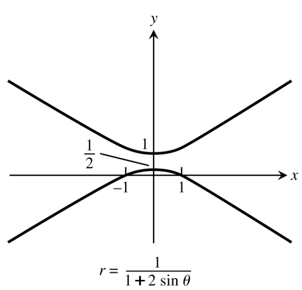

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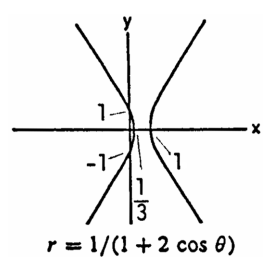


75. (a) Perihelion $= a - ae = a(1 - e)$, Aphelion $= ea + a = a(1 + e)$

(b)

Planet	Perihelion	Aphelion
Mercury	0.3075 AU	0.4667 AU
Venus	0.7184 AU	0.7282 AU
Earth	0.9833 AU	1.0167 AU
Mars	1.3817 AU	1.6663 AU
Jupiter	4.9512 AU	5.4548 AU
Saturn	9.0210 AU	10.0570 AU
Uranus	18.2977 AU	20.0623 AU
Neptune	29.8135 AU	30.3065 AU

76. Mercury: $r = \frac{(0.3871)(1 - 0.2056^2)}{1 + 0.2056 \cos\theta} = \frac{0.3707}{1 + 0.2056 \cos\theta}$

Venus: $r = \frac{(0.7233)(1 - 0.0068^2)}{1 + 0.0068 \cos\theta} = \frac{0.7233}{1 + 0.0068 \cos\theta}$

Earth: $r = \frac{1(1 - 0.0167^2)}{1 + 0.0167 \cos\theta} = \frac{0.9997}{1 + 0.0617 \cos\theta}$

Mars: $r = \frac{(1.524)(1 - 0.0934^2)}{1 + 0.0934 \cos\theta} = \frac{1.511}{1 + 0.0934 \cos\theta}$

Jupiter: $r = \frac{(5.203)(1 - 0.0484^2)}{1 + 0.0484 \cos\theta} = \frac{5.191}{1 + 0.0484 \cos\theta}$

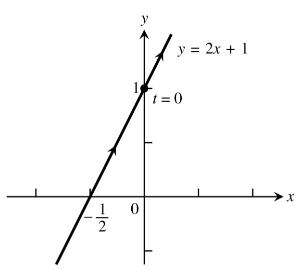
Saturn: $r = \frac{(9.539)(1 - 0.0543^2)}{1 + 0.0543 \cos\theta} = \frac{9.511}{1 + 0.0543 \cos\theta}$

Uranus: $r = \frac{(19.18)(1 - 0.0460^2)}{1 + 0.0460 \cos\theta} = \frac{19.14}{1 + 0.0460 \cos\theta}$

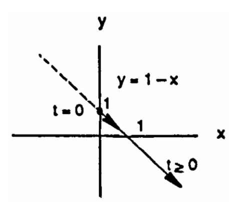
Neptune: $r = \frac{(30.06)(1 - 0.0082^2)}{1 + 0.0082 \cos\theta} = \frac{30.06}{1 + 0.0082 \cos\theta}$

CHAPTER 11 PRACTICE EXERCISES

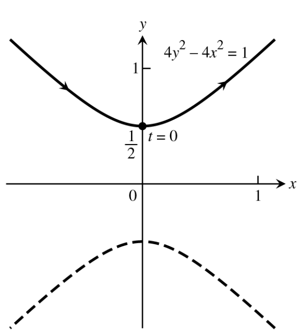
1. $x = \frac{t}{2}$ and $y = t + 1 \Rightarrow 2x = t \Rightarrow y = 2x + 1$



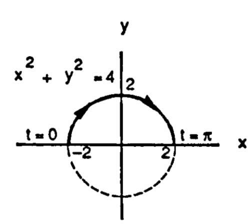
2. $x = \sqrt{t}$ and $y = 1 - \sqrt{t} \Rightarrow y = 1 - x$



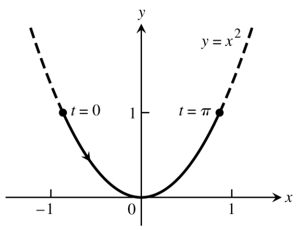
3. $x = \frac{1}{2} \tan t$ and $y = \frac{1}{2} \sec t \Rightarrow x^2 = \frac{1}{4} \tan^2 t$ and $y^2 = \frac{1}{4} \sec^2 t \Rightarrow 4x^2 = \tan^2 t$ and $4y^2 = \sec^2 t \Rightarrow 4x^2 + 1 = 4y^2 \Rightarrow 4y^2 - 4x^2 = 1$



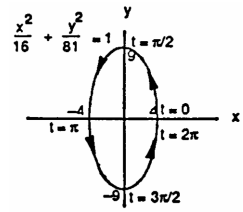
4. $x = -2 \cos t$ and $y = 2 \sin t \Rightarrow x^2 = 4 \cos^2 t$ and $y^2 = 4 \sin^2 t \Rightarrow x^2 + y^2 = 4$



5. $x = -\cos t$ and $y = \cos^2 t \;\Rightarrow\; y = (-x)^2 = x^2$



6. $x = 4\cos t$ and $y = 9\sin t \;\Rightarrow\; x^2 = 6\cos^2 t$ and $y^2 = 81\sin^2 t \;\Rightarrow\; \frac{x^2}{16} + \frac{y^2}{81} = 1$



7. $16x^2 + 9y^2 = 144 \;\Rightarrow\; \frac{x^2}{9} + \frac{y^2}{16} = 1 \;\Rightarrow\; a = 3$ and $b = 4 \;\Rightarrow\; x = 3\cos t$ and $y = 4\sin t,\ 0 \le t \le 2\pi$

8. $x^2 + y^2 = 4 \;\Rightarrow\; x = -2\cos t$ and $y = 2\sin t,\ 0 \le t \le 6\pi$

9. $x = \frac{1}{2}\tan t,\ y = \frac{1}{2}\sec t \;\Rightarrow\; \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{1}{2}\sec t\tan t}{\frac{1}{2}\sec^2 t} = \frac{\tan t}{\sec t} = \sin t \;\Rightarrow\; \left.\frac{dy}{dx}\right|_{t=\pi/3} = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2};\ t = \frac{\pi}{3}$
$\Rightarrow\; x = \frac{1}{2}\tan\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $y = \frac{1}{2}\sec\frac{\pi}{3} = 1 \;\Rightarrow\; y = \frac{\sqrt{3}}{2}x + \frac{1}{4};\ \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{\cos t}{\frac{1}{2}\sec^2 t} = 2\cos^3 t \;\Rightarrow\; \left.\frac{d^2y}{dx^2}\right|_{t=\pi/3}$
$= 2\cos^3\left(\frac{\pi}{3}\right) = \frac{1}{4}$

10. $x = 1 + \frac{1}{t^2},\ y = 1 - \frac{3}{t} \;\Rightarrow\; \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\left(\frac{3}{t^2}\right)}{\left(-\frac{2}{t^3}\right)} = -\frac{3}{2}t \;\Rightarrow\; \left.\frac{dy}{dx}\right|_{t=2} = -\frac{3}{2}(2) = -3;\ t = 2 \;\Rightarrow\; x = 1 + \frac{1}{2^2} = \frac{5}{4}$ and
$y = 1 - \frac{3}{2} = -\frac{1}{2} \;\Rightarrow\; y = -3x + \frac{13}{4};\ \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{\left(-\frac{3}{2}\right)}{\left(-\frac{2}{t^3}\right)} = \frac{3}{4}t^3 \;\Rightarrow\; \left.\frac{d^2y}{dx^2}\right|_{t=2} = \frac{3}{4}(2)^3 = 6$

11. (a) $x = 4t^2,\ y = t^3 - 1 \Rightarrow t = \pm\frac{\sqrt{x}}{2} \Rightarrow y = \left(\pm\frac{\sqrt{x}}{2}\right)^3 - 1 = \pm\frac{x^{3/2}}{8} - 1$
(b) $x = \cos t,\ y = \tan t \Rightarrow \sec t = \frac{1}{x} \Rightarrow \tan^2 t + 1 = \sec^2 t \Rightarrow y^2 = \frac{1}{x^2} - 1 = \frac{1-x^2}{x^2} \Rightarrow y = \pm\frac{\sqrt{1-x^2}}{x}$

12. (a) The line through $(1, -2)$ with slope 3 is $y = 3x - 5 \Rightarrow x = t,\ y = 3t - 5,\ -\infty < t < \infty$
(b) $(x-1)^2 + (y+2)^2 = 9 \Rightarrow x - 1 = 3\cos t,\ y + 2 = 3\sin t \Rightarrow x = 1 + 3\cos t,\ y = -2 + 3\sin t,\ 0 \le t \le 2\pi$
(c) $y = 4x^2 - x \Rightarrow x = t,\ y = 4t^2 - t,\ -\infty < t < \infty$
(d) $9x^2 + 4y^2 = 36 \Rightarrow \frac{x^2}{4} + \frac{y^2}{9} = 1 \Rightarrow x = 2\cos t,\ y = 3\sin t,\ 0 \le t \le 2\pi$

13. $y = x^{1/2} - \frac{x^{3/2}}{3} \;\Rightarrow\; \frac{dy}{dx} = \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{1/2} \;\Rightarrow\; \left(\frac{dy}{dx}\right)^2 = \frac{1}{4}\left(\frac{1}{x} - 2 + x\right) \;\Rightarrow\; L = \int_1^4 \sqrt{1 + \frac{1}{4}\left(\frac{1}{x} - 2 + x\right)}\,dx$
$\Rightarrow\; L = \int_1^4 \sqrt{\frac{1}{4}\left(\frac{1}{x} + 2 + x\right)}\,dx = \int_1^4 \sqrt{\frac{1}{4}\left(x^{-1/2} + x^{1/2}\right)^2}\,dx = \int_1^4 \frac{1}{2}\left(x^{-1/2} + x^{1/2}\right)dx = \frac{1}{2}\left[2x^{1/2} + \frac{2}{3}x^{3/2}\right]_1^4$
$= \frac{1}{2}\left[\left(4 + \frac{2}{3}\cdot 8\right) - \left(2 + \frac{2}{3}\right)\right] = \frac{1}{2}\left(2 + \frac{14}{3}\right) = \frac{10}{3}$

14. $x = y^{2/3} \;\Rightarrow\; \frac{dx}{dy} = \frac{2}{3}x^{-1/3} \;\Rightarrow\; \left(\frac{dx}{dy}\right)^2 = \frac{4x^{-2/3}}{9} \;\Rightarrow\; L = \int_1^8 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy = \int_1^8 \sqrt{1 + \frac{4}{9x^{2/3}}}\,dy$
$= \int_1^8 \frac{\sqrt{9x^{2/3}+4}}{3x^{1/3}}\,dx = \frac{1}{3}\int_1^8 \sqrt{9x^{2/3}+4}\,\left(x^{-1/3}\right)dx;\ \left[u = 9x^{2/3} + 4 \;\Rightarrow\; du = 6y^{-1/3}\,dy;\ x = 1 \;\Rightarrow\; u = 13,\right.$
$\left.x = 8 \;\Rightarrow\; u = 40\right] \;\rightarrow\; L = \frac{1}{18}\int_{13}^{40} u^{1/2}\,du = \frac{1}{18}\left[\frac{2}{3}u^{3/2}\right]_{13}^{40} = \frac{1}{27}\left[40^{3/2} - 13^{3/2}\right] \approx 7.634$

15. $y = \frac{5}{12}x^{6/5} - \frac{5}{8}x^{4/5} \;\Rightarrow\; \frac{dy}{dx} = \frac{1}{2}x^{1/5} - \frac{1}{2}x^{-1/5} \;\Rightarrow\; \left(\frac{dy}{dx}\right)^2 = \frac{1}{4}\left(x^{2/5} - 2 + x^{-2/5}\right)$
$\Rightarrow\; L = \int_1^{32} \sqrt{1 + \frac{1}{4}\left(x^{2/5} - 2 + x^{-2/5}\right)}\,dx \;\Rightarrow\; L = \int_1^{32} \sqrt{\frac{1}{4}\left(x^{2/5} + 2 + x^{-2/5}\right)}\,dx = \int_1^{32} \sqrt{\frac{1}{4}\left(x^{1/5} + x^{-1/5}\right)^2}\,dx$

$= \int_1^{32} \frac{1}{2}\left(x^{1/5} + x^{-1/5}\right) dx = \frac{1}{2}\left[\frac{5}{6}x^{6/5} + \frac{5}{4}x^{4/5}\right]_1^{32} = \frac{1}{2}\left[\left(\frac{5}{6}\cdot 2^6 + \frac{5}{4}\cdot 2^4\right) - \left(\frac{5}{6} + \frac{5}{4}\right)\right] = \frac{1}{2}\left(\frac{315}{6} + \frac{75}{4}\right)$
$= \frac{1}{48}(1260 + 450) = \frac{1710}{48} = \frac{285}{8}$

16. $x = \frac{1}{12}y^3 + \frac{1}{y} \Rightarrow \frac{dx}{dy} = \frac{1}{4}y^2 - \frac{1}{y^2} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{1}{16}y^4 - \frac{1}{2} + \frac{1}{y^4} \Rightarrow L = \int_1^2 \sqrt{1 + \left(\frac{1}{16}y^4 - \frac{1}{2} + \frac{1}{y^4}\right)}\, dy$
$= \int_1^2 \sqrt{\frac{1}{16}y^4 + \frac{1}{2} + \frac{1}{y^4}}\, dy = \int_1^2 \sqrt{\left(\frac{1}{4}y^2 + \frac{1}{y^2}\right)^2}\, dy = \int_1^2 \left(\frac{1}{4}y^2 + \frac{1}{y^2}\right) dy = \left[\frac{1}{12}y^3 - \frac{1}{y}\right]_1^2$
$= \left(\frac{8}{12} - \frac{1}{2}\right) - \left(\frac{1}{12} - 1\right) = \frac{7}{12} + \frac{1}{2} = \frac{13}{12}$

17. $\frac{dx}{dt} = -5\sin t + 5\sin 5t$ and $\frac{dy}{dt} = 5\cos t - 5\cos 5t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$
$= \sqrt{(-5\sin t + 5\sin 5t)^2 + (5\cos t - 5\cos 5t)^2}$
$= 5\sqrt{\sin^2 5t - 2\sin t\sin 5t + \sin^2 t + \cos^2 t - 2\cos t\cos 5t + \cos^2 5t} = 5\sqrt{2 - 2(\sin t\sin 5t + \cos t\cos 5\,t)}$
$= 5\sqrt{2(1-\cos 4t)} = 5\sqrt{4\left(\frac{1}{2}\right)(1-\cos 4t)} = 10\sqrt{\sin^2 2t} = 10|\sin 2t| = 10\sin 2t$ (since $0 \le t \le \frac{\pi}{2}$)
$\Rightarrow$ Length $= \int_0^{\pi/2} 10\sin 2t\, dt = [-5\cos 2t]_0^{\pi/2} = (-5)(-1) - (-5)(1) = 10$

18. $\frac{dx}{dt} = 3t^2 - 12t$ and $\frac{dy}{dt} = 3t^2 + 12t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(3t^2-12t)^2 + (3t^2+12t)^2} = \sqrt{288t^2 + 18t^4}$
$= 3\sqrt{2}\,|t|\sqrt{16+t^2} \Rightarrow$ Length $= \int_0^1 3\sqrt{2}\,|t|\sqrt{16+t^2}\, dt = 3\sqrt{2}\int_0^1 t\sqrt{16+t^2}\, dt$; $\big[u = 16 + t^2 \Rightarrow du = 2t\, dt$
$\Rightarrow \frac{1}{2}du = t\, dt; t = 0 \Rightarrow u = 16; t = 1 \Rightarrow u = 17\big]$; $\frac{3\sqrt{2}}{2}\int_{16}^{17}\sqrt{u}\, du = \frac{3\sqrt{2}}{2}\left[\frac{2}{3}u^{3/2}\right]_{16}^{17} = \frac{3\sqrt{2}}{2}\left(\frac{2}{3}(17)^{3/2} - \frac{2}{3}(16)^{3/2}\right)$
$= \frac{3\sqrt{2}}{2}\cdot\frac{2}{3}\left((17)^{3/2} - 64\right) = \sqrt{2}\left((17)^{3/2} - 64\right) \approx 8.617.$

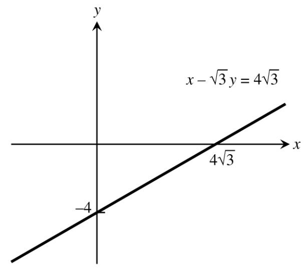
19. $\frac{dx}{d\theta} = -3\sin\theta$ and $\frac{dy}{d\theta} = 3\cos\theta \Rightarrow \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = \sqrt{(-3\sin\theta)^2 + (3\cos\theta)^2} = \sqrt{3(\sin^2\theta + \cos^2\theta)} = 3$
$\Rightarrow$ Length $= \int_0^{3\pi/2} 3\, d\theta = 3\int_0^{3\pi/2} d\theta = 3\left(\frac{3\pi}{2} - 0\right) = \frac{9\pi}{2}$

20. $x = t^2$ and $y = \frac{t^3}{3} - t, -\sqrt{3} \le t \le \sqrt{3} \Rightarrow \frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = t^2 - 1 \Rightarrow$ Length $= \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{(2t)^2 + (t^2-1)^2}\, dt$
$= \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{t^4 + 2t^2 + 1}\, dt = \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{t^4 + 2t^2 + 1}\, dt = \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{(t^2+1)^2}\, dt = \int_{-\sqrt{3}}^{\sqrt{3}} (t^2+1)\, dt = \left[\frac{t^3}{3} + t\right]_{-\sqrt{3}}^{\sqrt{3}}$
$= 4\sqrt{3}$

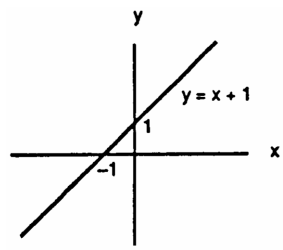
21. $x = \frac{t^2}{2}$ and $y = 2t, 0 \le t \le \sqrt{5} \Rightarrow \frac{dx}{dt} = t$ and $\frac{dy}{dt} = 2 \Rightarrow$ Surface Area $= \int_0^{\sqrt{5}} 2\pi(2t)\sqrt{t^2+4}\, dt = \int_4^9 2\pi u^{1/2}\, du$
$= 2\pi\left[\frac{2}{3}u^{3/2}\right]_4^9 = \frac{76\pi}{3}$, where $u = t^2 + 4 \Rightarrow du = 2t\, dt; t = 0 \Rightarrow u = 4, t = \sqrt{5} \Rightarrow u = 9$

22. $x = t^2 + \frac{1}{2t}$ and $y = 4\sqrt{t}, \frac{1}{\sqrt{2}} \le t \le 1 \Rightarrow \frac{dx}{dt} = 2t - \frac{1}{2t^2}$ and $\frac{dy}{dt} = \frac{2}{\sqrt{t}}$
$\Rightarrow$ Surface Area $= \int_{1/\sqrt{2}}^1 2\pi\left(t^2 + \frac{1}{2t}\right)\sqrt{\left(2t - \frac{1}{2t^2}\right)^2 + \left(\frac{2}{\sqrt{t}}\right)^2}\, dt = 2\pi\int_{1/\sqrt{2}}^1 \left(t^2 + \frac{1}{2t}\right)\sqrt{\left(2t + \frac{1}{2t^2}\right)^2}\, dt$
$= 2\pi\int_{1/\sqrt{2}}^1 \left(t^2 + \frac{1}{2t}\right)\left(2t + \frac{1}{2t^2}\right) dt = 2\pi\int_{1/\sqrt{2}}^1 \left(2t^3 + \frac{3}{2} + \frac{1}{4}t^{-3}\right) dt = 2\pi\left[\frac{1}{2}t^4 + \frac{3}{2}t - \frac{1}{8}t^{-2}\right]_{1/\sqrt{2}}^1$
$= 2\pi\left(2 - \frac{3\sqrt{2}}{4}\right)$

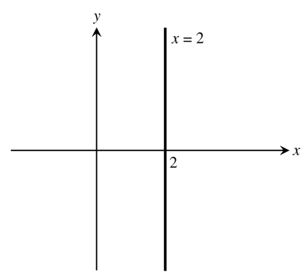
23. $r\cos\left(\theta+\frac{\pi}{3}\right)=2\sqrt{3} \Rightarrow r\left(\cos\theta\cos\frac{\pi}{3}-\sin\theta\sin\frac{\pi}{3}\right)$
$=2\sqrt{3} \Rightarrow \frac{1}{2}r\cos\theta-\frac{\sqrt{3}}{2}r\sin\theta=2\sqrt{3}$
$\Rightarrow r\cos\theta-\sqrt{3}r\sin\theta=4\sqrt{3} \Rightarrow x-\sqrt{3}y=4\sqrt{3}$
$\Rightarrow y=\frac{\sqrt{3}}{3}x-4$



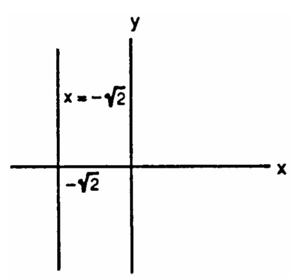
24. $r\cos\left(\theta-\frac{3\pi}{4}\right)=\frac{\sqrt{2}}{2} \Rightarrow r\left(\cos\theta\cos\frac{3\pi}{4}+\sin\theta\sin\frac{3\pi}{4}\right)$
$=\frac{\sqrt{2}}{2} \Rightarrow -\frac{\sqrt{2}}{2}r\cos\theta+\frac{\sqrt{2}}{2}r\sin\theta=\frac{\sqrt{2}}{2} \Rightarrow -x+y=1$
$\Rightarrow y=x+1$



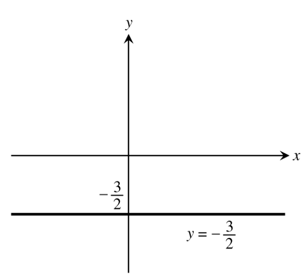
25. $r=2\sec\theta \Rightarrow r=\frac{2}{\cos\theta} \Rightarrow r\cos\theta=2 \Rightarrow x=2$



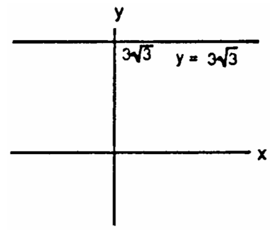
26. $r=-\sqrt{2}\sec\theta \Rightarrow r\cos\theta=-\sqrt{2} \Rightarrow x=-\sqrt{2}$



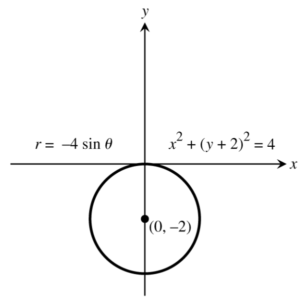
27. $r=-\frac{3}{2}\csc\theta \Rightarrow r\sin\theta=-\frac{3}{2} \Rightarrow y=-\frac{3}{2}$



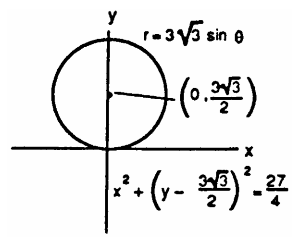
28. $r = 3\sqrt{3}\csc\theta \Rightarrow r\sin\theta = 3\sqrt{3} \Rightarrow y = 3\sqrt{3}$



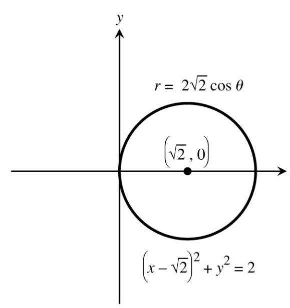
29. $r = -4\sin\theta \Rightarrow r^2 = -4r\sin\theta \Rightarrow x^2 + y^2 + 4y = 0$
$\Rightarrow x^2 + (y+2)^2 = 4$; circle with center $(0, -2)$ and radius 2.



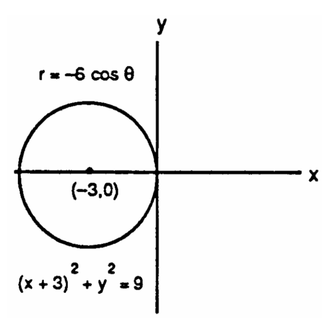
30. $r = 3\sqrt{3}\sin\theta \Rightarrow r^2 = 3\sqrt{3}\,r\sin\theta$
$\Rightarrow x^2 + y^2 - 3\sqrt{3}\,y = 0 \Rightarrow x^2 + \left(y - \frac{3\sqrt{3}}{2}\right)^2 = \frac{27}{4}$;
circle with center $\left(0, \frac{3\sqrt{3}}{2}\right)$ and radius $\frac{3\sqrt{3}}{2}$



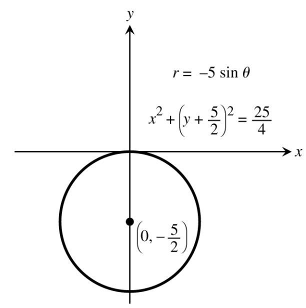
31. $r = 2\sqrt{2}\cos\theta \Rightarrow r^2 = 2\sqrt{2}\,r\cos\theta$
$\Rightarrow x^2 + y^2 - 2\sqrt{2}\,x = 0 \Rightarrow \left(x - \sqrt{2}\right)^2 + y^2 = 2$;
circle with center $\left(\sqrt{2}, 0\right)$ and radius $\sqrt{2}$



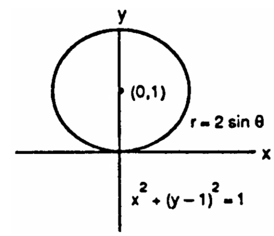
32. $r = -6\cos\theta \Rightarrow r^2 = -6r\cos\theta \Rightarrow x^2 + y^2 + 6x = 0$
$\Rightarrow (x+3)^2 + y^2 = 9$; circle with center $(-3, 0)$ and radius 3



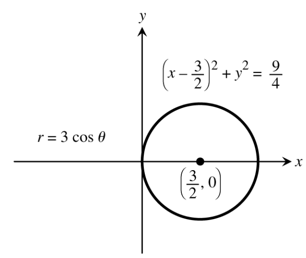
33. $x^2 + y^2 + 5y = 0 \Rightarrow x^2 + \left(y + \frac{5}{2}\right)^2 = \frac{25}{4} \Rightarrow C = \left(0, -\frac{5}{2}\right)$ and $a = \frac{5}{2}$; $r^2 + 5r \sin\theta = 0 \Rightarrow r = -5 \sin\theta$



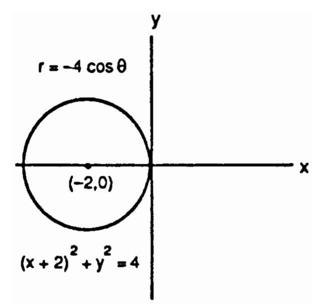
34. $x^2 + y^2 - 2y = 0 \Rightarrow x^2 + (y-1)^2 = 1 \Rightarrow C = (0,1)$ and $a = 1$; $r^2 - 2r\sin\theta = 0 \Rightarrow r = 2\sin\theta$



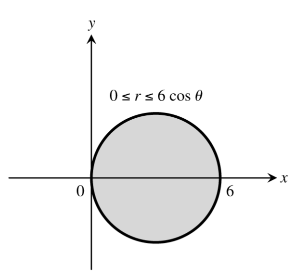
35. $x^2 + y^2 - 3x = 0 \Rightarrow \left(x - \frac{3}{2}\right)^2 + y^2 = \frac{9}{4} \Rightarrow C = \left(\frac{3}{2}, 0\right)$ and $a = \frac{3}{2}$; $r^2 - 3r\cos\theta = 0 \Rightarrow r = 3\cos\theta$



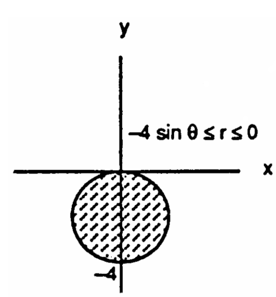
36. $x^2 + y^2 + 4x = 0 \Rightarrow (x+2)^2 + y^2 = 4 \Rightarrow C = (-2,0)$ and $a = 2$; $r^2 + 4r\cos\theta = 0 \Rightarrow r = -4\cos\theta$



37.



38.



39. d 40. e 41. l 42. f

43. k 44. h 45. i 46. j

47. $A = 2\int_0^{\pi} \frac{1}{2} r^2 \, d\theta = \int_0^{\pi} (2 - \cos\theta)^2 \, d\theta = \int_0^{\pi} (4 - 4\cos\theta + \cos^2\theta) \, d\theta = \int_0^{\pi} \left(4 - 4\cos\theta + \frac{1+\cos 2\theta}{2}\right) d\theta$
$= \int_0^{\pi} \left(\frac{9}{2} - 4\cos\theta + \frac{\cos 2\theta}{2}\right) d\theta = \left[\frac{9}{2}\theta - 4\sin\theta + \frac{\sin 2\theta}{4}\right]_0^{\pi} = \frac{9}{2}\pi$

48. $A = \int_0^{\pi/3} \frac{1}{2}(\sin^2 3\theta) \, d\theta = \int_0^{\pi/3} \left(\frac{1-\cos 6\theta}{2}\right) d\theta = \frac{1}{4}\left[\theta - \frac{1}{6}\sin 6\theta\right]_0^{\pi/3} = \frac{\pi}{12}$

49. $r = 1 + \cos 2\theta$ and $r = 1 \Rightarrow 1 = 1 + \cos 2\theta \Rightarrow 0 = \cos 2\theta \Rightarrow 2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$; therefore
$A = 4\int_0^{\pi/4} \frac{1}{2}\left[(1+\cos 2\theta)^2 - 1^2\right] d\theta = 2\int_0^{\pi/4} (1 + 2\cos 2\theta + \cos^2 2\theta - 1) \, d\theta$
$= 2\int_0^{\pi/4} \left(2\cos 2\theta + \frac{1}{2} + \frac{\cos 4\theta}{2}\right) d\theta = 2\left[\sin 2\theta + \frac{1}{2}\theta + \frac{\sin 4\theta}{8}\right]_0^{\pi/4} = 2\left(1 + \frac{\pi}{8} + 0\right) = 2 + \frac{\pi}{4}$

50. The circle lies interior to the cardioid. Thus,
$A = 2\int_{-\pi/2}^{\pi/2} \frac{1}{2}\left[2(1+\sin\theta)\right]^2 d\theta - \pi$ (the integral is the area of the cardioid minus the area of the circle)
$= \int_{-\pi/2}^{\pi/2} 4(1 + 2\sin\theta + \sin^2\theta) \, d\theta - \pi = \int_{-\pi/2}^{\pi/2} (6 + 8\sin\theta - 2\cos 2\theta) \, d\theta - \pi = \left[6\theta - 8\cos\theta - \sin 2\theta\right]_{-\pi/2}^{\pi/2} - \pi$
$= [3\pi - (-3\pi)] - \pi = 5\pi$

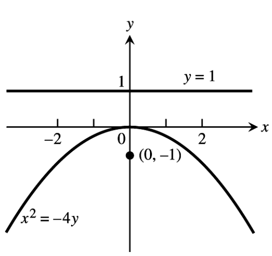
51. $r = -1 + \cos\theta \Rightarrow \frac{dr}{d\theta} = -\sin\theta$; Length $= \int_0^{2\pi} \sqrt{(-1+\cos\theta)^2 + (-\sin\theta)^2} \, d\theta = \int_0^{2\pi} \sqrt{2 - 2\cos\theta} \, d\theta$
$= \int_0^{2\pi} \sqrt{\frac{4(1-\cos\theta)}{2}} \, d\theta = \int_0^{2\pi} 2\sin\frac{\theta}{2} \, d\theta = \left[-4\cos\frac{\theta}{2}\right]_0^{2\pi} = (-4)(-1) - (-4)(1) = 8$

52. $r = 2\sin\theta + 2\cos\theta, \ 0 \le \theta \le \frac{\pi}{2} \Rightarrow \frac{dr}{d\theta} = 2\cos\theta - 2\sin\theta$; $r^2 + \left(\frac{dr}{d\theta}\right)^2 = (2\sin\theta + 2\cos\theta)^2 + (2\cos\theta - 2\sin\theta)^2$
$= 8(\sin^2\theta + \cos^2\theta) = 8 \Rightarrow L = \int_0^{\pi/2} \sqrt{8} \, d\theta = \left[2\sqrt{2}\,\theta\right]_0^{\pi/2} = 2\sqrt{2}\left(\frac{\pi}{2}\right) = \pi\sqrt{2}$

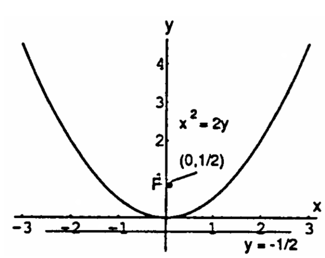
53. $r = 8\sin^3\left(\frac{\theta}{3}\right), \ 0 \le \theta \le \frac{\pi}{4} \Rightarrow \frac{dr}{d\theta} = 8\sin^2\left(\frac{\theta}{3}\right)\cos\left(\frac{\theta}{3}\right)$; $r^2 + \left(\frac{dr}{d\theta}\right)^2 = \left[8\sin^3\left(\frac{\theta}{3}\right)\right]^2 + \left[8\sin^2\left(\frac{\theta}{3}\right)\cos\left(\frac{\theta}{3}\right)\right]^2$
$= 64\sin^4\left(\frac{\theta}{3}\right) \Rightarrow L = \int_0^{\pi/4} \sqrt{64\sin^4\left(\frac{\theta}{3}\right)} \, d\theta = \int_0^{\pi/4} 8\sin^2\left(\frac{\theta}{3}\right) d\theta = \int_0^{\pi/4} 8\left[\frac{1-\cos\left(\frac{2\theta}{3}\right)}{2}\right] d\theta$
$= \int_0^{\pi/4} \left[4 - 4\cos\left(\frac{2\theta}{3}\right)\right] d\theta = \left[4\theta - 6\sin\left(\frac{2\theta}{3}\right)\right]_0^{\pi/4} = 4\left(\frac{\pi}{4}\right) - 6\sin\left(\frac{\pi}{6}\right) - 0 = \pi - 3$

54. $r = \sqrt{1 + \cos 2\theta} \Rightarrow \frac{dr}{d\theta} = \frac{1}{2}(1+\cos 2\theta)^{-1/2}(-2\sin 2\theta) = \frac{-\sin 2\theta}{\sqrt{1+\cos 2\theta}} \Rightarrow \left(\frac{dr}{d\theta}\right)^2 = \frac{\sin^2 2\theta}{1+\cos 2\theta}$
$\Rightarrow r^2 + \left(\frac{dr}{d\theta}\right)^2 = 1 + \cos 2\theta + \frac{\sin^2 2\theta}{1+\cos 2\theta} = \frac{(1+\cos 2\theta)^2 + \sin^2 2\theta}{1+\cos 2\theta} = \frac{1 + 2\cos 2\theta + \cos^2 2\theta + \sin^2 2\theta}{1+\cos 2\theta}$
$= \frac{2 + 2\cos 2\theta}{1+\cos 2\theta} = 2 \Rightarrow L = \int_{-\pi/2}^{\pi/2} \sqrt{2} \, d\theta = \sqrt{2}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = \sqrt{2}\,\pi$

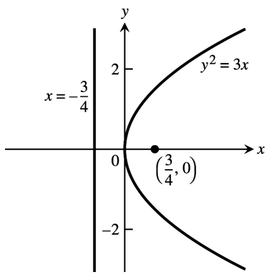
55. $x^2 = -4y \Rightarrow y = -\frac{x^2}{4} \Rightarrow 4p = 4 \Rightarrow p = 1$;
therefore Focus is $(0, -1)$, Directrix is $y = 1$



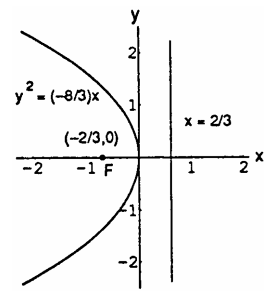
56. $x^2 = 2y \Rightarrow \frac{x^2}{2} = y \Rightarrow 4p = 2 \Rightarrow p = \frac{1}{2}$;
therefore Focus is $\left(0, \frac{1}{2}\right)$; Directrix is $y = -\frac{1}{2}$



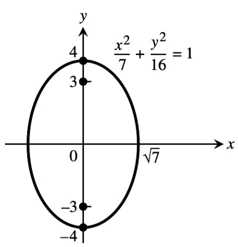
57. $y^2 = 3x \Rightarrow x = \frac{y^2}{3} \Rightarrow 4p = 3 \Rightarrow p = \frac{3}{4}$; therefore Focus is $\left(\frac{3}{4}, 0\right)$, Directrix is $x = -\frac{3}{4}$



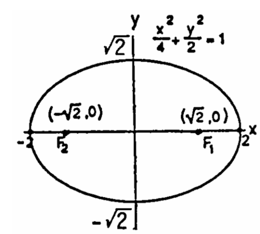
58. $y^2 = -\frac{8}{3}x \Rightarrow x = -\frac{y^2}{\left(\frac{8}{3}\right)} \Rightarrow 4p = \frac{8}{3} \Rightarrow p = \frac{2}{3}$; therefore Focus is $\left(-\frac{2}{3}, 0\right)$, Directrix is $x = \frac{2}{3}$



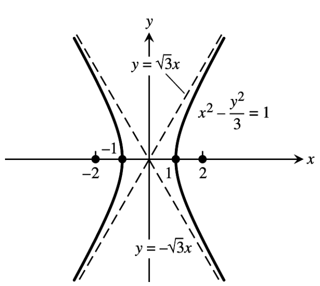
59. $16x^2 + 7y^2 = 112 \Rightarrow \frac{x^2}{7} + \frac{y^2}{16} = 1$
$\Rightarrow c^2 = 16 - 7 = 9 \Rightarrow c = 3; e = \frac{c}{a} = \frac{3}{4}$



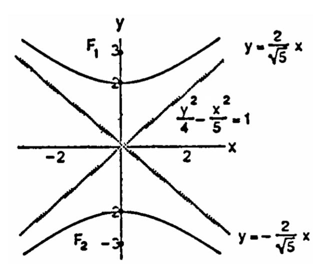
60. $x^2 + 2y^2 = 4 \Rightarrow \frac{x^2}{4} + \frac{y^2}{2} = 1 \Rightarrow c^2 = 4 - 2 = 2$
$\Rightarrow c = \sqrt{2}; e = \frac{c}{a} = \frac{\sqrt{2}}{2}$



61. $3x^2 - y^2 = 3 \Rightarrow x^2 - \frac{y^2}{3} = 1 \Rightarrow c^2 = 1 + 3 = 4$
$\Rightarrow c = 2; e = \frac{c}{a} = \frac{2}{1} = 2$; the asymptotes are
$y = \pm\sqrt{3}\,x$



62. $5y^2 - 4x^2 = 20 \Rightarrow \frac{y^2}{4} - \frac{x^2}{5} = 1 \Rightarrow c^2 = 4 + 5 = 9$
$\Rightarrow c = 3, e = \frac{c}{a} = \frac{3}{2}$; the asymptotes are $y = \pm\frac{2}{\sqrt{5}}x$



63. $x^2 = -12y \Rightarrow -\frac{x^2}{12} = y \Rightarrow 4p = 12 \Rightarrow p = 3 \Rightarrow$ focus is $(0, -3)$, directrix is $y = 3$, vertex is $(0, 0)$; therefore new vertex is $(2, 3)$, new focus is $(2, 0)$, new directrix is $y = 6$, and the new equation is $(x - 2)^2 = -12(y - 3)$

64. $y^2 = 10x \Rightarrow \frac{y^2}{10} = x \Rightarrow 4p = 10 \Rightarrow p = \frac{5}{2} \Rightarrow$ focus is $\left(\frac{5}{2}, 0\right)$, directrix is $x = -\frac{5}{2}$, vertex is $(0, 0)$; therefore new vertex is $\left(-\frac{1}{2}, -1\right)$, new focus is $(2, -1)$, new directrix is $x = -3$, and the new equation is $(y + 1)^2 = 10\left(x + \frac{1}{2}\right)$

65. $\frac{x^2}{9} + \frac{y^2}{25} = 1 \Rightarrow a = 5$ and $b = 3 \Rightarrow c = \sqrt{25 - 9} = 4 \Rightarrow$ foci are $(0, \pm 4)$, vertices are $(0, \pm 5)$, center is $(0, 0)$; therefore the new center is $(-3, -5)$, new foci are $(-3, -1)$ and $(-3, -9)$, new vertices are $(-3, -10)$ and $(-3, 0)$, and the new equation is $\frac{(x+3)^2}{9} + \frac{(y+5)^2}{25} = 1$

66. $\frac{x^2}{169} + \frac{y^2}{144} = 1 \Rightarrow a = 13$ and $b = 12 \Rightarrow c = \sqrt{169 - 144} = 5 \Rightarrow$ foci are $(\pm 5, 0)$, vertices are $(\pm 13, 0)$, center is $(0, 0)$; therefore the new center is $(5, 12)$, new foci are $(10, 12)$ and $(0, 12)$, new vertices are $(18, 12)$ and $(-8, 12)$, and the new equation is $\frac{(x-5)^2}{169} + \frac{(y-12)^2}{144} = 1$

67. $\frac{y^2}{8} - \frac{x^2}{2} = 1 \Rightarrow a = 2\sqrt{2}$ and $b = \sqrt{2} \Rightarrow c = \sqrt{8+2} = \sqrt{10} \Rightarrow$ foci are $\left(0, \pm\sqrt{10}\right)$, vertices are $\left(0, \pm 2\sqrt{2}\right)$, center is $(0, 0)$, and the asymptotes are $y = \pm 2x$; therefore the new center is $\left(2, 2\sqrt{2}\right)$, new foci are $\left(2, 2\sqrt{2} \pm \sqrt{10}\right)$, new vertices are $\left(2, 4\sqrt{2}\right)$ and $(2, 0)$, the new asymptotes are $y = 2x - 4 + 2\sqrt{2}$ and $y = -2x + 4 + 2\sqrt{2}$; the new equation is $\frac{\left(y - 2\sqrt{2}\right)^2}{8} - \frac{(x-2)^2}{2} = 1$

68. $\frac{x^2}{36} - \frac{y^2}{64} = 1 \Rightarrow a = 6$ and $b = 8 \Rightarrow c = \sqrt{36 + 64} = 10 \Rightarrow$ foci are $(\pm 10, 0)$, vertices are $(\pm 6, 0)$, the center is $(0, 0)$ and the asymptotes are $\frac{y}{8} = \pm\frac{x}{6}$ or $y = \pm\frac{4}{3}x$; therefore the new center is $(-10, -3)$, the new foci are $(-20, -3)$ and $(0, -3)$, the new vertices are $(-16, -3)$ and $(-4, -3)$, the new asymptotes are $y = \frac{4}{3}x + \frac{31}{3}$ and $y = -\frac{4}{3}x - \frac{49}{3}$; the new equation is $\frac{(x+10)^2}{36} - \frac{(y+3)^2}{64} = 1$

69. $x^2 - 4x - 4y^2 = 0 \Rightarrow x^2 - 4x + 4 - 4y^2 = 4 \Rightarrow (x-2)^2 - 4y^2 = 4 \Rightarrow \frac{(x-2)^2}{4} - y^2 = 1$, a hyperbola; $a = 2$ and $b = 1 \Rightarrow c = \sqrt{1+4} = \sqrt{5}$; the center is $(2, 0)$, the vertices are $(0, 0)$ and $(4, 0)$; the foci are $\left(2 \pm \sqrt{5}, 0\right)$ and the asymptotes are $y = \pm\frac{x-2}{2}$

70. $4x^2 - y^2 + 4y = 8 \Rightarrow 4x^2 - y^2 + 4y - 4 = 4 \Rightarrow 4x^2 - (y-2)^2 = 4 \Rightarrow x^2 - \frac{(y-2)^2}{4} = 1$, a hyperbola; $a = 1$ and $b = 2 \Rightarrow c = \sqrt{1+4} = \sqrt{5}$; the center is $(0, 2)$, the vertices are $(1, 2)$ and $(-1, 2)$, the foci are $\left(\pm\sqrt{5}, 2\right)$ and the asymptotes are $y = \pm 2x + 2$

71. $y^2 - 2y + 16x = -49 \Rightarrow y^2 - 2y + 1 = -16x - 48 \Rightarrow (y-1)^2 = -16(x+3)$, a parabola; the vertex is $(-3, 1)$; $4p = 16 \Rightarrow p = 4 \Rightarrow$ the focus is $(-7, 1)$ and the directrix is $x = 1$

72. $x^2 - 2x + 8y = -17 \Rightarrow x^2 - 2x + 1 = -8y - 16 \Rightarrow (x-1)^2 = -8(y+2)$, a parabola; the vertex is $(1, -2)$; $4p = 8 \Rightarrow p = 2 \Rightarrow$ the focus is $(1, -4)$ and the directrix is $y = 0$

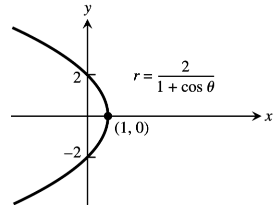
73. $9x^2 + 16y^2 + 54x - 64y = -1 \Rightarrow 9(x^2 + 6x) + 16(y^2 - 4y) = -1 \Rightarrow 9(x^2 + 6x + 9) + 16(y^2 - 4y + 4) = 144$ $\Rightarrow 9(x+3)^2 + 16(y-2)^2 = 144 \Rightarrow \frac{(x+3)^2}{16} + \frac{(y-2)^2}{9} = 1$, an ellipse; the center is $(-3, 2)$; $a = 4$ and $b = 3$ $\Rightarrow c = \sqrt{16 - 9} = \sqrt{7}$; the foci are $\left(-3 \pm \sqrt{7}, 2\right)$; the vertices are $(1, 2)$ and $(-7, 2)$

74. $25x^2 + 9y^2 - 100x + 54y = 44 \Rightarrow 25(x^2 - 4x) + 9(y^2 + 6y) = 44 \Rightarrow 25(x^2 - 4x + 4) + 9(y^2 + 6y + 9) = 225$ $\Rightarrow \frac{(x-2)^2}{9} + \frac{(y+3)^2}{25} = 1$, an ellipse; the center is $(2, -3)$; $a = 5$ and $b = 3 \Rightarrow c = \sqrt{25 - 9} = 4$; the foci are $(2, 1)$ and $(2, -7)$; the vertices are $(2, 2)$ and $(2, -8)$

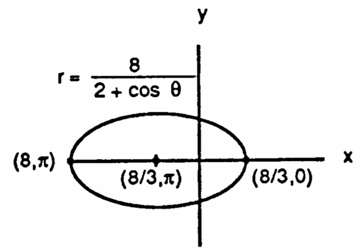
75. $x^2 + y^2 - 2x - 2y = 0 \Rightarrow x^2 - 2x + 1 + y^2 - 2y + 1 = 2 \Rightarrow (x-1)^2 + (y-1)^2 = 2$, a circle with center $(1, 1)$ and radius $= \sqrt{2}$

76. $x^2 + y^2 + 4x + 2y = 1 \Rightarrow x^2 + 4x + 4 + y^2 + 2y + 1 = 6 \Rightarrow (x+2)^2 + (y+1)^2 = 6$, a circle with center $(-2, -1)$ and radius $= \sqrt{6}$

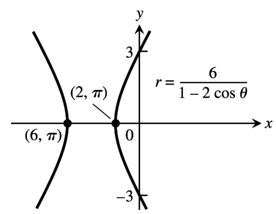
77. $r = \frac{2}{1+\cos\theta} \Rightarrow e = 1 \Rightarrow$ parabola with vertex at $(1, 0)$



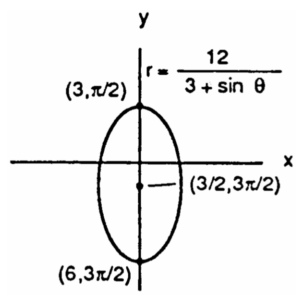
78. $r = \frac{8}{2+\cos\theta} \Rightarrow r = \frac{4}{1+\left(\frac{1}{2}\right)\cos\theta} \Rightarrow e = \frac{1}{2} \Rightarrow$ ellipse; $ke = 4 \Rightarrow \frac{1}{2}k = 4 \Rightarrow k = 8$; $k = \frac{a}{e} - ea \Rightarrow 8 = \frac{a}{\left(\frac{1}{2}\right)} - \frac{1}{2}a$ $\Rightarrow a = \frac{16}{3} \Rightarrow ea = \left(\frac{1}{2}\right)\left(\frac{16}{3}\right) = \frac{8}{3}$; therefore the center is $\left(\frac{8}{3}, \pi\right)$; vertices are $(8, \pi)$ and $\left(\frac{8}{3}, 0\right)$



79. $r = \frac{6}{1-2\cos\theta} \Rightarrow e = 2 \Rightarrow$ hyperbola; $ke = 6 \Rightarrow 2k = 6$ $\Rightarrow k = 3 \Rightarrow$ vertices are $(2, \pi)$ and $(6, \pi)$



80. $r = \frac{12}{3+\sin\theta} \Rightarrow r = \frac{4}{1+\left(\frac{1}{3}\right)\sin\theta} \Rightarrow e = \frac{1}{3}$; $ke = 4$ $\Rightarrow \frac{1}{3}k = 4 \Rightarrow k = 12$; $a\left(1 - e^2\right) = 4 \Rightarrow a\left[1 - \left(\frac{1}{3}\right)^2\right]$ $= 4 \Rightarrow a = \frac{9}{2} \Rightarrow ea = \left(\frac{1}{3}\right)\left(\frac{9}{2}\right) = \frac{3}{2}$; therefore the center is $\left(\frac{3}{2}, \frac{3\pi}{2}\right)$; vertices are $\left(3, \frac{\pi}{2}\right)$ and $\left(6, \frac{3\pi}{2}\right)$



81. $e = 2$ and $r\cos\theta = 2 \Rightarrow x = 2$ is directrix $\Rightarrow k = 2$; the conic is a hyperbola; $r = \frac{ke}{1+e\cos\theta} \Rightarrow r = \frac{(2)(2)}{1+2\cos\theta}$ $\Rightarrow r = \frac{4}{1+2\cos\theta}$

82. $e = 1$ and $r\cos\theta = -4 \Rightarrow x = -4$ is directrix $\Rightarrow k = 4$; the conic is a parabola; $r = \frac{ke}{1-e\cos\theta} \Rightarrow r = \frac{(4)(1)}{1-\cos\theta}$ $\Rightarrow r = \frac{4}{1-\cos\theta}$

83. $e = \frac{1}{2}$ and $r\sin\theta = 2 \Rightarrow y = 2$ is directrix $\Rightarrow k = 2$; the conic is an ellipse; $r = \frac{ke}{1+e\sin\theta} \Rightarrow r = \frac{(2)\left(\frac{1}{2}\right)}{1+\left(\frac{1}{2}\right)\sin\theta}$ $\Rightarrow r = \frac{2}{2+\sin\theta}$

84. $e = \frac{1}{3}$ and $r\sin\theta = -6 \Rightarrow y = -6$ is directrix $\Rightarrow k = 6$; the conic is an ellipse; $r = \frac{ke}{1-e\sin\theta} \Rightarrow r = \frac{(6)\left(\frac{1}{3}\right)}{1-\left(\frac{1}{3}\right)\sin\theta}$ $\Rightarrow r = \frac{6}{3-\sin\theta}$

85. (a) Around the x-axis: $9x^2 + 4y^2 = 36 \Rightarrow y^2 = 9 - \frac{9}{4}x^2 \Rightarrow y = \pm\sqrt{9 - \frac{9}{4}x^2}$ and we use the positive root:

$$V = 2\int_0^2 \pi\left(\sqrt{9 - \frac{9}{4}x^2}\right)^2 dx = 2\int_0^2 \pi\left(9 - \frac{9}{4}x^2\right) dx = 2\pi\left[9x - \frac{3}{4}x^3\right]_0^2 = 24\pi$$

(b) Around the y-axis: $9x^2 + 4y^2 = 36 \Rightarrow x^2 = 4 - \frac{4}{9}y^2 \Rightarrow x = \pm\sqrt{4 - \frac{4}{9}y^2}$ and we use the positive root:

$$V = 2\int_0^3 \pi\left(\sqrt{4 - \frac{4}{9}y^2}\right)^2 dy = 2\int_0^3 \pi\left(4 - \frac{4}{9}y^2\right) dy = 2\pi\left[4y - \frac{4}{27}y^3\right]_0^3 = 16\pi$$

86. $9x^2 - 4y^2 = 36,\ x = 4 \Rightarrow y^2 = \frac{9x^2 - 36}{4} \Rightarrow y = \frac{3}{2}\sqrt{x^2 - 4}$; $V = \int_2^4 \pi\left(\frac{3}{2}\sqrt{x^2 - 4}\right)^2 dx = \frac{9\pi}{4}\int_2^4 (x^2 - 4)\,dx$
$= \frac{9\pi}{4}\left[\frac{x^3}{3} - 4x\right]_2^4 = \frac{9\pi}{4}\left[\left(\frac{64}{3} - 16\right) - \left(\frac{8}{3} - 8\right)\right] = \frac{9\pi}{4}\left(\frac{56}{3} - \frac{24}{3}\right) = \frac{3\pi}{4}(32) = 24\pi$

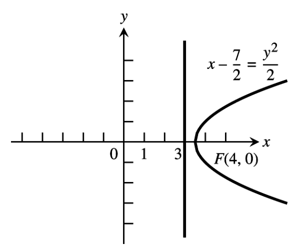
87. (a) $r = \frac{k}{1 + e\cos\theta} \Rightarrow r + er\cos\theta = k \Rightarrow \sqrt{x^2 + y^2} + ex = k \Rightarrow \sqrt{x^2 + y^2} = k - ex \Rightarrow x^2 + y^2$
$= k^2 - 2kex + e^2x^2 \Rightarrow x^2 - e^2x^2 + y^2 + 2kex - k^2 = 0 \Rightarrow (1 - e^2)x^2 + y^2 + 2kex - k^2 = 0$

(b) $e = 0 \Rightarrow x^2 + y^2 - k^2 = 0 \Rightarrow x^2 + y^2 = k^2 \Rightarrow$ circle;
$0 < e < 1 \Rightarrow e^2 < 1 \Rightarrow e^2 - 1 < 0 \Rightarrow B^2 - 4AC = 0^2 - 4(1 - e^2)(1) = 4(e^2 - 1) < 0 \Rightarrow$ ellipse;
$e = 1 \Rightarrow B^2 - 4AC = 0^2 - 4(0)(1) = 0 \Rightarrow$ parabola;
$e > 1 \Rightarrow e^2 > 1 \Rightarrow B^2 - 4AC = 0^2 - 4(1 - e^2)(1) = 4e^2 - 4 > 0 \Rightarrow$ hyperbola

88. Let (r_1, θ_1) be a point on the graph where $r_1 = a\theta_1$. Let (r_2, θ_2) be on the graph where $r_2 = a\theta_2$ and $\theta_2 = \theta_1 + 2\pi$. Then r_1 and r_2 lie on the same ray on consecutive turns of the spiral and the distance between the two points is $r_2 - r_1 = a\theta_2 - a\theta_1 = a(\theta_2 - \theta_1) = 2\pi a$, which is constant.

CHAPTER 11 ADDITIONAL AND ADVANCED EXERCISES

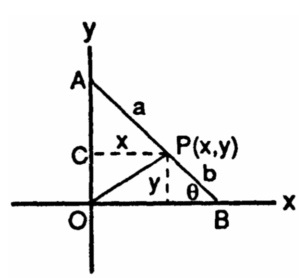
1. Directrix $x = 3$ and focus $(4, 0) \Rightarrow$ vertex is $\left(\frac{7}{2}, 0\right)$
$\Rightarrow p = \frac{1}{2} \Rightarrow$ the equation is $x - \frac{7}{2} = \frac{y^2}{2}$



2. $x^2 - 6x - 12y + 9 = 0 \Rightarrow x^2 - 6x + 9 = 12y \Rightarrow \frac{(x-3)^2}{12} = y \Rightarrow$ vertex is $(3, 0)$ and $p = 3 \Rightarrow$ focus is $(3, 3)$ and the directrix is $y = -3$

3. $x^2 = 4y \Rightarrow$ vertex is $(0, 0)$ and $p = 1 \Rightarrow$ focus is $(0, 1)$; thus the distance from $P(x, y)$ to the vertex is $\sqrt{x^2 + y^2}$ and the distance from P to the focus is $\sqrt{x^2 + (y - 1)^2} \Rightarrow \sqrt{x^2 + y^2} = 2\sqrt{x^2 + (y - 1)^2}$
$\Rightarrow x^2 + y^2 = 4\left[x^2 + (y - 1)^2\right] \Rightarrow x^2 + y^2 = 4x^2 + 4y^2 - 8y + 4 \Rightarrow 3x^2 + 3y^2 - 8y + 4 = 0$, which is a circle

4. Let the segment $a + b$ intersect the y-axis in point A and intersect the x-axis in point B so that $PB = b$ and $PA = a$ (see figure). Draw the horizontal line through P and let it intersect the y-axis in point C. Let $\angle PBO = \theta$
$\Rightarrow \angle APC = \theta$. Then $\sin\theta = \frac{y}{b}$ and $\cos\theta = \frac{x}{a}$
$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2\theta + \sin^2\theta = 1.$



5. Vertices are $(0, \pm 2) \Rightarrow a = 2$; $e = \frac{c}{a} \Rightarrow 0.5 = \frac{c}{2} \Rightarrow c = 1 \Rightarrow$ foci are $(0, \pm 1)$

6. Let the center of the ellipse be $(x, 0)$; directrix $x = 2$, focus $(4, 0)$, and $e = \frac{2}{3} \Rightarrow \frac{a}{e} - c = 2 \Rightarrow \frac{a}{e} = 2 + c$ $\Rightarrow a = \frac{2}{3}(2 + c)$. Also $c = ae = \frac{2}{3}a \Rightarrow a = \frac{2}{3}\left(2 + \frac{2}{3}a\right) \Rightarrow a = \frac{4}{3} + \frac{4}{9}a \Rightarrow \frac{5}{9}a = \frac{4}{3} \Rightarrow a = \frac{12}{5}$; $x - 2 = \frac{a}{e}$ $\Rightarrow x - 2 = \left(\frac{12}{5}\right)\left(\frac{3}{2}\right) = \frac{18}{5} \Rightarrow x = \frac{28}{5} \Rightarrow$ the center is $\left(\frac{28}{5}, 0\right)$; $x - 4 = c \Rightarrow c = \frac{28}{5} - 4 = \frac{8}{5}$ so that $c^2 = a^2 - b^2$ $= \left(\frac{12}{5}\right)^2 - \left(\frac{8}{5}\right)^2 = \frac{80}{25}$; therefore the equation is $\frac{\left(x - \frac{28}{5}\right)^2}{\left(\frac{144}{25}\right)} + \frac{y^2}{\left(\frac{80}{25}\right)} = 1$ or $\frac{25\left(x - \frac{28}{5}\right)^2}{144} + \frac{5y^2}{16} = 1$

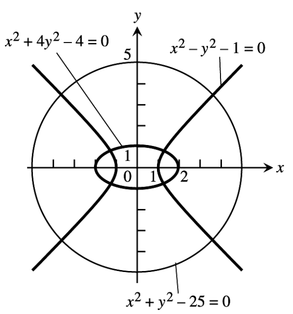
7. Let the center of the hyperbola be $(0, y)$.
   (a) Directrix $y = -1$, focus $(0, -7)$ and $e = 2 \Rightarrow c - \frac{a}{e} = 6 \Rightarrow \frac{a}{e} = c - 6 \Rightarrow a = 2c - 12$. Also $c = ae = 2a$ $\Rightarrow a = 2(2a) - 12 \Rightarrow a = 4 \Rightarrow c = 8$; $y - (-1) = \frac{a}{e} = \frac{4}{2} = 2 \Rightarrow y = 1 \Rightarrow$ the center is $(0, 1)$; $c^2 = a^2 + b^2$ $\Rightarrow b^2 = c^2 - a^2 = 64 - 16 = 48$; therefore the equation is $\frac{(y-1)^2}{16} - \frac{x^2}{48} = 1$
   (b) $e = 5 \Rightarrow c - \frac{a}{e} = 6 \Rightarrow \frac{a}{e} = c - 6 \Rightarrow a = 5c - 30$. Also, $c = ae = 5a \Rightarrow a = 5(5a) - 30 \Rightarrow 24a = 30 \Rightarrow a = \frac{5}{4}$ $\Rightarrow c = \frac{25}{4}$; $y - (-1) = \frac{a}{e} = \frac{\left(\frac{5}{4}\right)}{5} = \frac{1}{4} \Rightarrow y = -\frac{3}{4} \Rightarrow$ the center is $\left(0, -\frac{3}{4}\right)$; $c^2 = a^2 + b^2 \Rightarrow b^2 = c^2 - a^2$ $= \frac{625}{16} - \frac{25}{16} = \frac{75}{2}$; therefore the equation is $\frac{\left(y + \frac{3}{4}\right)^2}{\left(\frac{25}{16}\right)} - \frac{x^2}{\left(\frac{75}{2}\right)} = 1$ or $\frac{16\left(y + \frac{3}{4}\right)^2}{25} - \frac{2x^2}{75} = 1$

8. The center is $(0, 0)$ and $c = 2 \Rightarrow 4 = a^2 + b^2 \Rightarrow b^2 = 4 - a^2$. The equation is $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \Rightarrow \frac{49}{a^2} - \frac{144}{b^2} = 1$ $\Rightarrow \frac{49}{a^2} - \frac{144}{(4-a^2)} = 1 \Rightarrow 49\left(4 - a^2\right) - 144a^2 = a^2\left(4 - a^2\right) \Rightarrow 196 - 49a^2 - 144a^2 = 4a^2 - a^4 \Rightarrow a^4 - 197a^2 + 196$ $= 0 \Rightarrow \left(a^2 - 196\right)\left(a^2 - 1\right) = 0 \Rightarrow a = 14$ or $a = 1$; $a = 14 \Rightarrow b^2 = 4 - (14)^2 < 0$ which is impossible; $a = 1$ $\Rightarrow b^2 = 4 - 1 = 3$; therefore the equation is $y^2 - \frac{x^2}{3} = 1$

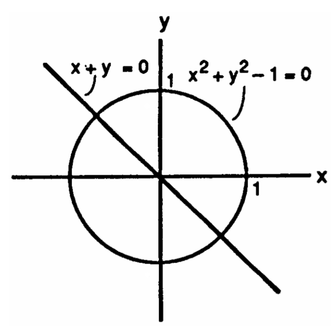
9. $b^2x^2 + a^2y^2 = a^2b^2 \Rightarrow \frac{dy}{dx} = -\frac{b^2x}{a^2y}$; at (x_1, y_1) the tangent line is $y - y_1 = \left(-\frac{b^2x_1}{a^2y_1}\right)(x - x_1)$ $\Rightarrow a^2yy_1 + b^2xx_1 = b^2x_1^2 + a^2y_1^2 = a^2b^2 \Rightarrow b^2xx_1 + a^2yy_1 - a^2b^2 = 0$

10. $b^2x^2 - a^2y^2 = a^2b^2 \Rightarrow \frac{dy}{dx} = \frac{b^2x}{a^2y}$; at (x_1, y_1) the tangent line is $y - y_1 = \left(\frac{b^2x_1}{a^2y_1}\right)(x - x_1)$ $\Rightarrow b^2xx_1 - a^2yy_1 = b^2x_1^2 - a^2y_1^2 = a^2b^2 \Rightarrow b^2xx_1 - a^2yy_1 - a^2b^2 = 0$

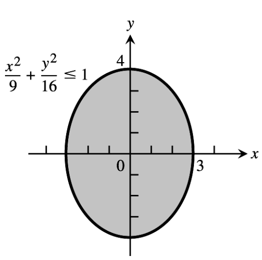
11.



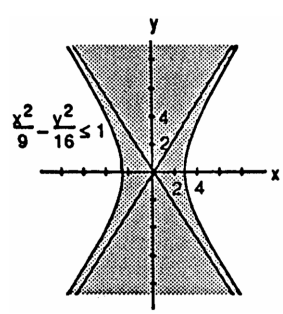
12.



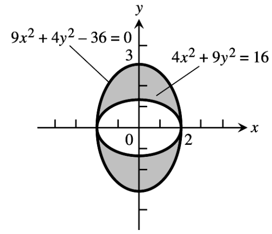
13.



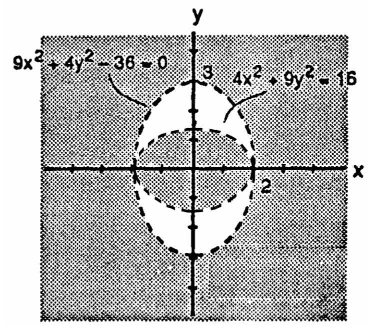
14.



15. $(9x^2 + 4y^2 - 36)(4x^2 + 9y^2 - 16) \le 0$
$\Rightarrow\ 9x^2 + 4y^2 - 36 \le 0$ and $4x^2 + 9y^2 - 16 \ge 0$
or $9x^2 + 4y^2 - 36 \ge 0$ and $4x^2 + 9y^2 - 16 \le 0$

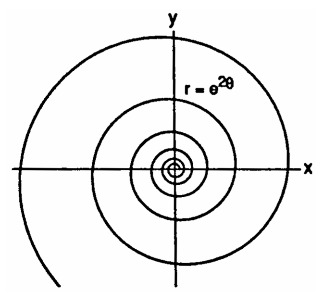


16. $(9x^2 + 4y^2 - 36)(4x^2 + 9y^2 - 16) > 0$, which is the complement of the set in Exercise 15



17. (a) $x = e^{2t}\cos t$ and $y = e^{2t}\sin t \ \Rightarrow\ x^2 + y^2 = e^{4t}\cos^2 t + e^{4t}\sin^2 t = e^{4t}$. Also $\frac{y}{x} = \frac{e^{2t}\sin t}{e^{2t}\cos t} = \tan t$
$\Rightarrow\ t = \tan^{-1}\left(\frac{y}{x}\right) \ \Rightarrow\ x^2 + y^2 = e^{4\tan^{-1}(y/x)}$ is the Cartesian equation. Since $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$, the polar equation is $r^2 = e^{4\theta}$ or $r = e^{2\theta}$ for $r > 0$

(b) $ds^2 = r^2\, d\theta^2 + dr^2;\ r = e^{2\theta} \ \Rightarrow\ dr = 2e^{2\theta}\, d\theta$
$\Rightarrow\ ds^2 = r^2\, d\theta^2 + \left(2e^{2\theta}\, d\theta\right)^2 = \left(e^{2\theta}\right)^2 d\theta^2 + 4e^{4\theta}\, d\theta^2$
$= 5e^{4\theta}\, d\theta^2 \ \Rightarrow\ ds = \sqrt{5}\, e^{2\theta}\, d\theta \Rightarrow L = \int_0^{2\pi} \sqrt{5}\, e^{2\theta}\, d\theta$
$= \left[\frac{\sqrt{5}\, e^{2\theta}}{2}\right]_0^{2\pi} = \frac{\sqrt{5}}{2}\left(e^{4\pi} - 1\right)$



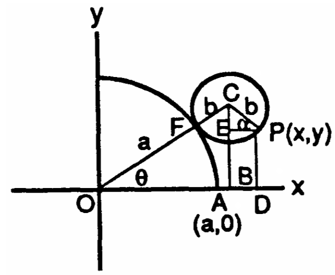
18. $r = 2\sin^3\left(\frac{\theta}{3}\right) \ \Rightarrow\ dr = 2\sin^2\left(\frac{\theta}{3}\right)\cos\left(\frac{\theta}{3}\right) d\theta \ \Rightarrow\ ds^2 = r^2\, d\theta^2 + dr^2 = \left[2\sin^3\left(\frac{\theta}{3}\right)\right]^2 d\theta^2 + \left[2\sin^2\left(\frac{\theta}{3}\right)\cos\left(\frac{\theta}{3}\right) d\theta\right]^2$
$= 4\sin^6\left(\frac{\theta}{3}\right) d\theta^2 + 4\sin^4\left(\frac{\theta}{3}\right)\cos^2\left(\frac{\theta}{3}\right) d\theta^2 = \left[4\sin^4\left(\frac{\theta}{3}\right)\right]\left[\sin^2\left(\frac{\theta}{3}\right) + \cos^2\left(\frac{\theta}{3}\right)\right] d\theta^2 = 4\sin^4\left(\frac{\theta}{3}\right) d\theta^2$
$\Rightarrow\ ds = 2\sin^2\left(\frac{\theta}{3}\right) d\theta$. Then $L = \int_0^{3\pi} 2\sin^2\left(\frac{\theta}{3}\right) d\theta = \int_0^{3\pi}\left[1 - \cos\left(\frac{2\theta}{3}\right)\right] d\theta = \left[\theta - \frac{3}{2}\sin\left(\frac{2\theta}{3}\right)\right]_0^{3\pi} = 3\pi$

19. $e = 2$ and $r\cos\theta = 2 \ \Rightarrow\ x = 2$ is the directrix $\Rightarrow\ k = 2$; the conic is a hyperbola with $r = \frac{ke}{1 + e\cos\theta}$
$\Rightarrow\ r = \frac{(2)(2)}{1 + 2\cos\theta} = \frac{4}{1 + 2\cos\theta}$

20. $e = 1$ and $r\cos\theta = -4 \ \Rightarrow\ x = -4$ is the directrix $\Rightarrow\ k = 4$; the conic is a parabola with $r = \frac{ke}{1 - e\cos\theta}$
$\Rightarrow\ r = \frac{(4)(1)}{1 - \cos\theta} = \frac{4}{1 - \cos\theta}$

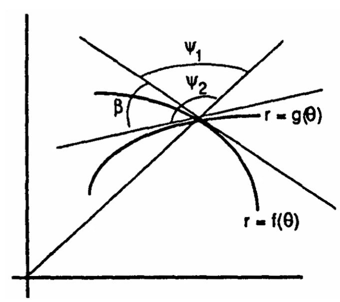
21. $e = \frac{1}{2}$ and $r\sin\theta = 2 \ \Rightarrow\ y = 2$ is the directrix $\Rightarrow\ k = 2$; the conic is an ellipse with $r = \frac{ke}{1 + e\sin\theta}$
$\Rightarrow\ r = \frac{2\left(\frac{1}{2}\right)}{1 + \left(\frac{1}{2}\right)\sin\theta} = \frac{2}{2 + \sin\theta}$

22. $e = \frac{1}{3}$ and $r\sin\theta = -6 \ \Rightarrow\ y = -6$ is the directrix $\Rightarrow\ k = 6$; the conic is an ellipse with $r = \frac{ke}{1 - e\sin\theta}$
$\Rightarrow\ r = \frac{6\left(\frac{1}{3}\right)}{1 - \left(\frac{1}{3}\right)\sin\theta} = \frac{6}{3 - \sin\theta}$



23. Arc PF = Arc AF since each is the distance rolled;
$\angle PCF = \frac{\text{Arc PF}}{b} \Rightarrow \text{Arc PF} = b(\angle PCF);\ \theta = \frac{\text{Arc AF}}{a}$
$\Rightarrow \text{Arc AF} = a\theta \Rightarrow a\theta = b(\angle PCF) \Rightarrow \angle PCF = \left(\frac{a}{b}\right)\theta;$
$\angle OCB = \frac{\pi}{2} - \theta$ and $\angle OCB = \angle PCF - \angle PCE$
$= \angle PCF - \left(\frac{\pi}{2} - \alpha\right) = \left(\frac{a}{b}\right)\theta - \left(\frac{\pi}{2} - \alpha\right) \Rightarrow \frac{\pi}{2} - \theta$
$= \left(\frac{a}{b}\right)\theta - \left(\frac{\pi}{2} - \alpha\right) \Rightarrow \frac{\pi}{2} - \theta = \left(\frac{a}{b}\right)\theta - \frac{\pi}{2} + \alpha$
$\Rightarrow \alpha = \pi - \theta - \left(\frac{a}{b}\right)\theta \Rightarrow \alpha = \pi - \left(\frac{a+b}{b}\right)\theta.$
Now $x = OB + BD = OB + EP = (a+b)\cos\theta + b\cos\alpha = (a+b)\cos\theta + b\cos\left(\pi - \left(\frac{a+b}{b}\right)\theta\right)$
$= (a+b)\cos\theta + b\cos\pi\cos\left(\left(\frac{a+b}{b}\right)\theta\right) + b\sin\pi\sin\left(\left(\frac{a+b}{b}\right)\theta\right) = (a+b)\cos\theta - b\cos\left(\left(\frac{a+b}{b}\right)\theta\right)$ and
$y = PD = CB - CE = (a+b)\sin\theta - b\sin\alpha = (a+b)\sin\theta - b\sin\left(\left(\frac{a+b}{b}\right)\theta\right)$
$= (a+b)\sin\theta - b\sin\pi\cos\left(\left(\frac{a+b}{b}\right)\theta\right) + b\cos\pi\sin\left(\left(\frac{a+b}{b}\right)\theta\right) = (a+b)\sin\theta - b\sin\left(\left(\frac{a+b}{b}\right)\theta\right);$
therefore $x = (a+b)\cos\theta - b\cos\left(\left(\frac{a+b}{b}\right)\theta\right)$ and $y = (a+b)\sin\theta - b\sin\left(\left(\frac{a+b}{b}\right)\theta\right)$

24. $x = a(t - \sin t) \Rightarrow \frac{dx}{dt} = a(1 - \cos t)$ and let $\delta = 1 \Rightarrow dm = dA = y\,dx = y\left(\frac{dx}{dt}\right)dt$
$= a(1-\cos t)\,a(1-\cos t)\,dt = a^2(1-\cos t)^2\,dt$; then $A = \int_0^{2\pi} a^2(1-\cos t)^2\,dt$
$= a^2\int_0^{2\pi}(1 - 2\cos t + \cos^2 t)\,dt = a^2\int_0^{2\pi}\left(1 - 2\cos t + \frac{1}{2} + \frac{1}{2}\cos 2t\right)dt = a^2\left[\frac{3}{2}t - 2\sin t + \frac{\sin 2t}{4}\right]_0^{2\pi}$
$= 3\pi a^2$; $\tilde{x} = x = a(t - \sin t)$ and $\tilde{y} = \frac{1}{2}y = \frac{1}{2}a(1-\cos t) \Rightarrow M_x = \int \tilde{y}\,dm = \int \tilde{y}\,\delta\,dA$
$= \int_0^{2\pi}\frac{1}{2}a(1-\cos t)\,a^2(1-\cos t)^2\,dt = \frac{1}{2}a^3\int_0^{2\pi}(1-\cos t)^3\,dt = \frac{a^3}{2}\int_0^{2\pi}(1 - 3\cos t + 3\cos^2 t - \cos^3 t)\,dt$
$= \frac{a^3}{2}\int_0^{2\pi}\left[1 - 3\cos t + \frac{3}{2} + \frac{3\cos 2t}{2} - (1 - \sin^2 t)(\cos t)\right]dt = \frac{a^3}{2}\left[\frac{5}{2}t - 3\sin t + \frac{3\sin 2t}{4} - \sin t + \frac{\sin^3 t}{3}\right]_0^{2\pi}$
$= \frac{5\pi a^3}{2}$. Therefore $\overline{y} = \frac{M_x}{M} = \frac{\left(\frac{5\pi a^3}{2}\right)}{3\pi a^2} = \frac{5}{6}a$. Also, $M_y = \int \tilde{x}\,dm = \int \tilde{x}\,\delta\,dA$
$= \int_0^{2\pi} a(t - \sin t)\,a^2(1-\cos t)^2\,dt = a^3\int_0^{2\pi}(t - 2t\cos t + t\cos^2 t - \sin t + 2\sin t\cos t - \sin t\cos^2 t)\,dt$
$= a^3\left[\frac{t^2}{2} - 2\cos t - 2t\sin t + \frac{1}{4}t^2 + \frac{1}{8}\cos 2t + \frac{t}{4}\sin 2t + \cos t + \sin^2 t + \frac{\cos^3 t}{3}\right]_0^{2\pi} = 3\pi^2 a^3$. Thus
$\overline{x} = \frac{M_y}{M} = \frac{3\pi^2 a^3}{3\pi a^2} = \pi a \Rightarrow \left(\pi a, \frac{5}{6}a\right)$ is the center of mass.

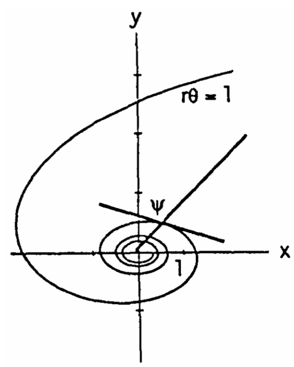


25. $\beta = \psi_2 - \psi_1 \Rightarrow \tan\beta = \tan(\psi_2 - \psi_1) = \frac{\tan\psi_2 - \tan\psi_1}{1 + \tan\psi_2\tan\psi_1}$;
the curves will be orthogonal when $\tan\beta$ is undefined, or
when $\tan\psi_2 = \frac{-1}{\tan\psi_1} \Rightarrow \frac{r}{g'(\theta)} = \frac{-1}{\left[\frac{r}{f'(\theta)}\right]}$
$\Rightarrow r^2 = -f'(\theta)\,g'(\theta)$

26. $r = \sin^4\left(\frac{\theta}{4}\right) \Rightarrow \frac{dr}{d\theta} = \sin^3\left(\frac{\theta}{4}\right)\cos\left(\frac{\theta}{4}\right) \Rightarrow \tan\psi = \frac{\sin^4\left(\frac{\theta}{4}\right)}{\sin^3\left(\frac{\theta}{4}\right)\cos\left(\frac{\theta}{4}\right)} = \tan\left(\frac{\theta}{4}\right)$

27. $r = 2a\sin 3\theta \Rightarrow \frac{dr}{d\theta} = 6a\cos 3\theta \Rightarrow \tan\psi = \frac{r}{\left(\frac{dr}{d\theta}\right)} = \frac{2a\sin 3\theta}{6a\cos 3\theta} = \frac{1}{3}\tan 3\theta$; when $\theta = \frac{\pi}{6}$, $\tan\psi = \frac{1}{3}\tan\frac{\pi}{2} \Rightarrow \psi = \frac{\pi}{2}$

28. (a)



(b) $r\theta = 1 \Rightarrow r = \theta^{-1} \Rightarrow \frac{dr}{d\theta} = -\theta^{-2} \Rightarrow \tan \psi|_{\theta=1} = \frac{\theta^{-1}}{-\theta^{-2}} = -\theta \Rightarrow \lim_{\theta \to \infty} \tan \psi = -\infty$ $\Rightarrow \psi \to \frac{\pi}{2}$ from the right as the spiral winds in around the origin.

29. $\tan \psi_1 = \frac{\sqrt{3}\cos\theta}{-\sqrt{3}\sin\theta} = -\cot\theta$ is $-\frac{1}{\sqrt{3}}$ at $\theta = \frac{\pi}{3}$; $\tan \psi_2 = \frac{\sin\theta}{\cos\theta} = \tan\theta$ is $\sqrt{3}$ at $\theta = \frac{\pi}{3}$; since the product of these slopes is -1, the tangents are perpendicular

30. $\tan \psi = \frac{r}{\left(\frac{dr}{d\theta}\right)} = \frac{a(1-\cos\theta)}{a\sin\theta}$ is 1 at $\theta = \frac{\pi}{2} \Rightarrow \psi = \frac{\pi}{4}$

NOTES:

CHAPTER 12 VECTORS AND THE GEOMETRY OF SPACE

12.1 THREE-DIMENSIONAL COORDINATE SYSTEMS

1. The line through the point $(2, 3, 0)$ parallel to the z-axis

2. The line through the point $(-1, 0, 0)$ parallel to the y-axis

3. The x-axis

4. The line through the point $(1, 0, 0)$ parallel to the z-axis

5. The circle $x^2 + y^2 = 4$ in the xy-plane

6. The circle $x^2 + y^2 = 4$ in the plane $z = -2$

7. The circle $x^2 + z^2 = 4$ in the xz-plane

8. The circle $y^2 + z^2 = 1$ in the yz-plane

9. The circle $y^2 + z^2 = 1$ in the yz-plane

10. The circle $x^2 + z^2 = 9$ in the plane $y = -4$

11. The circle $x^2 + y^2 = 16$ in the xy-plane

12. The circle $x^2 + z^2 = 3$ in the xz-plane

13. The ellipse formed by the intersection of the cylinder $x^2 + y^2 = 4$ and the plane $z = y$.

14. The circle formed by the intersection of the sphere $x^2 + y^2 + z^2 = 4$ and the plane $y = x$.

15. The parabola $y = x^2$ in the the xy-plane.

16. The parabola $z = y^2$ in the the plane $x = 1$.

17. (a) The first quadrant of the xy-plane (b) The fourth quadrant of the xy-plane

18. (a) The slab bounded by the planes $x = 0$ and $x = 1$
    (b) The square column bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$
    (c) The unit cube in the first octant having one vertex at the origin

19. (a) The solid ball of radius 1 centered at the origin
    (b) The exterior of the sphere of radius 1 centered at the origin

20. (a) The circumference and interior of the circle $x^2 + y^2 = 1$ in the xy-plane
    (b) The circumference and interior of the circle $x^2 + y^2 = 1$ in the plane $z = 3$

(c) A solid cylindrical column of radius 1 whose axis is the z-axis

21. (a) The solid enclosed between the sphere of radius 1 and radius 2 centered at the origin
(b) The solid upper hemisphere of radius 1 centered at the origin

22. (a) The line $y = x$ in the xy-plane
(b) The plane $y = x$ consisting of all points of the form (x, x, z)

23. (a) The region on or inside the parabola $y = x^2$ in the xy-plane and all points above this region.
(b) The region on or to the left of the parabola $x = y^2$ in the xy-plane and all points above it that are 2 units or less away from the xy-plane.

24. (a) All the points the lie on the plane $z = 1 - y$.
(b) All points that lie on the curve $z = y^3$ in the plane $x = -2$.

25. (a) $x = 3$ (b) $y = -1$ (c) $z = -2$

26. (a) $x = 3$ (b) $y = -1$ (c) $z = 2$

27. (a) $z = 1$ (b) $x = 3$ (c) $y = -1$

28. (a) $x^2 + y^2 = 4, z = 0$ (b) $y^2 + z^2 = 4, x = 0$ (c) $x^2 + z^2 = 4, y = 0$

29. (a) $x^2 + (y-2)^2 = 4, z = 0$ (b) $(y-2)^2 + z^2 = 4, x = 0$ (c) $x^2 + z^2 = 4, y = 2$

30. (a) $(x+3)^2 + (y-4)^2 = 1, z = 1$ (b) $(y-4)^2 + (z-1)^2 = 1, x = -3$
(c) $(x+3)^2 + (z-1)^2 = 1, y = 4$

31. (a) $y = 3, z = -1$ (b) $x = 1, z = -1$ (c) $x = 1, y = 3$

32. $\sqrt{x^2 + y^2 + z^2} = \sqrt{x^2 + (y-2)^2 + z^2} \Rightarrow x^2 + y^2 + z^2 = x^2 + (y-2)^2 + z^2 \Rightarrow y^2 = y^2 - 4y + 4 \Rightarrow y = 1$

33. $x^2 + y^2 + z^2 = 25, z = 3 \Rightarrow x^2 + y^2 = 16$ in the plane $z = 3$

34. $x^2 + y^2 + (z-1)^2 = 4$ and $x^2 + y^2 + (z+1)^2 = 4 \Rightarrow x^2 + y^2 + (z-1)^2 = x^2 + y^2 + (z+1)^2 \Rightarrow z = 0, x^2 + y^2 = 3$

35. $0 \le z \le 1$

36. $0 \le x \le 2, 0 \le y \le 2, 0 \le z \le 2$

37. $z \le 0$

38. $z = \sqrt{1 - x^2 - y^2}$

39. (a) $(x-1)^2 + (y-1)^2 + (z-1)^2 < 1$ (b) $(x-1)^2 + (y-1)^2 + (z-1)^2 > 1$

40. $1 \le x^2 + y^2 + z^2 \le 4$

41. $|P_1P_2| = \sqrt{(3-1)^2 + (3-1)^2 + (0-1)^2} = \sqrt{9} = 3$

42. $|P_1P_2| = \sqrt{(2+1)^2 + (5-1)^2 + (0-5)^2} = \sqrt{50} = 5\sqrt{2}$

43. $|P_1P_2| = \sqrt{(4-1)^2 + (-2-4)^2 + (7-5)^2} = \sqrt{49} = 7$

44. $|P_1P_2| = \sqrt{(2-3)^2 + (3-4)^2 + (4-5)^2} = \sqrt{3}$

45. $|P_1P_2| = \sqrt{(2-0)^2 + (-2-0)^2 + (-2-0)^2} = \sqrt{3 \cdot 4} = 2\sqrt{3}$

46. $|P_1P_2| = \sqrt{(0-5)^2 + (0-3)^2 + (0+2)^2} = \sqrt{38}$

47. center $(-2, 0, 2)$, radius $2\sqrt{2}$

48. center $\left(1, -\frac{1}{2}, -3\right)$, radius 5

49. center $\left(\sqrt{2}, \sqrt{2}, -\sqrt{2}\right)$, radius $\sqrt{2}$

50. center $\left(0, -\frac{1}{3}, \frac{1}{3}\right)$, radius $\frac{4}{3}$

51. $(x-1)^2 + (y-2)^2 + (z-3)^2 = 14$

52. $x^2 + (y+1)^2 + (z-5)^2 = 4$

53. $(x+1)^2 + \left(y - \frac{1}{2}\right)^2 + \left(z + \frac{2}{3}\right)^2 = \frac{16}{81}$

54. $x^2 + (y+7)^2 + z^2 = 49$

55. $x^2 + y^2 + z^2 + 4x - 4z = 0 \Rightarrow (x^2 + 4x + 4) + y^2 + (z^2 - 4z + 4) = 4 + 4$
$\Rightarrow (x+2)^2 + (y-0)^2 + (z-2)^2 = \left(\sqrt{8}\right)^2 \Rightarrow$ the center is at $(-2, 0, 2)$ and the radius is $\sqrt{8}$

56. $x^2 + y^2 + z^2 - 6y + 8z = 0 \Rightarrow x^2 + (y^2 - 6y + 9) + (z^2 + 8z + 16) = 9 + 16 \Rightarrow (x-0)^2 + (y-3)^2 + (z+4)^2 = 5^2$
$\Rightarrow$ the center is at $(0, 3, -4)$ and the radius is 5

57. $2x^2 + 2y^2 + 2z^2 + x + y + z = 9 \Rightarrow x^2 + \frac{1}{2}x + y^2 + \frac{1}{2}y + z^2 + \frac{1}{2}z = \frac{9}{2}$
$\Rightarrow \left(x^2 + \frac{1}{2}x + \frac{1}{16}\right) + \left(y^2 + \frac{1}{2}y + \frac{1}{16}\right) + \left(z^2 + \frac{1}{2}z + \frac{1}{16}\right) = \frac{9}{2} + \frac{3}{16} \Rightarrow \left(x + \frac{1}{4}\right)^2 + \left(y + \frac{1}{4}\right)^2 + \left(z + \frac{1}{4}\right)^2 = \left(\frac{5\sqrt{3}}{4}\right)^2$
$\Rightarrow$ the center is at $\left(-\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}\right)$ and the radius is $\frac{5\sqrt{3}}{4}$

58. $3x^2 + 3y^2 + 3z^2 + 2y - 2z = 9 \Rightarrow x^2 + y^2 + \frac{2}{3}y + z^2 - \frac{2}{3}z = 3 \Rightarrow x^2 + \left(y^2 + \frac{2}{3}y + \frac{1}{9}\right) + \left(z^2 - \frac{2}{3}z + \frac{1}{9}\right) = 3 + \frac{2}{9}$
$\Rightarrow (x-0)^2 + \left(y + \frac{1}{3}\right)^2 + \left(z - \frac{1}{3}\right)^2 = \left(\frac{\sqrt{29}}{3}\right)^2 \Rightarrow$ the center is at $\left(0, -\frac{1}{3}, \frac{1}{3}\right)$ and the radius is $\frac{\sqrt{29}}{3}$

59. (a) the distance between (x, y, z) and $(x, 0, 0)$ is $\sqrt{y^2 + z^2}$
(b) the distance between (x, y, z) and $(0, y, 0)$ is $\sqrt{x^2 + z^2}$
(c) the distance between (x, y, z) and $(0, 0, z)$ is $\sqrt{x^2 + y^2}$

60. (a) the distance between (x, y, z) and $(x, y, 0)$ is z
(b) the distance between (x, y, z) and $(0, y, z)$ is x
(c) the distance between (x, y, z) and $(x, 0, z)$ is y

61. $|AB| = \sqrt{(1-(-1))^2 + (-1-2)^2 + (3-1)^2} = \sqrt{4+9+4} = \sqrt{17}$
$|BC| = \sqrt{(3-1)^2 + (4-(-1))^2 + (5-3)^2} = \sqrt{4+25+4} = \sqrt{33}$
$|CA| = \sqrt{(-1-3)^2 + (2-4)^2 + (1-5)^2} = \sqrt{16+4+16} = \sqrt{36} = 6$
Thus the perimeter of triangle ABC is $\sqrt{17} + \sqrt{33} + 6$.

62. $|PA| = \sqrt{(2-3)^2 + (-1-1)^2 + (3-2)^2} = \sqrt{1+4+1} = \sqrt{6}$
$|PB| = \sqrt{(4-3)^2 + (3-1)^2 + (1-2)^2} = \sqrt{1+4+1} = \sqrt{6}$
Thus P is equidistant from A and B.

63. $\sqrt{(x-x)^2 + (y-(-1))^2 + (z-z)^2} = \sqrt{(x-x)^2 + (y-3)^2 + (z-z)^2} \Rightarrow (y+1)^2 = (y-3)^2 \Rightarrow 2y+1 = -6y+9$
$\Rightarrow y = 1$

64. $\sqrt{(x-0)^2 + (y-0)^2 + (z-2)^2} = \sqrt{(x-x)^2 + (y-y)^2 + (z-0)^2} \Rightarrow x^2 + y^2 + (z-2)^2 = z^2$
$\Rightarrow x^2 + y^2 - 4z + 4 = 0 \Rightarrow z = \frac{x^2}{4} + \frac{y^2}{4} + 1$

65. (a) Since the entire sphere is below the xy-plane, the point on the sphere closest to the xy-plane is the point at the top of the sphere, which occurs when $x = 0$ and $y = 3 \Rightarrow 0^2 + (3-3)^2 + (z+5)^2 = 4 \Rightarrow z = -5 \pm 2 \Rightarrow z = -3$
$\Rightarrow (0, 3, -3)$.
(b) Both the center $(0, 3, -5)$ and the the point $(0, 7, -5)$ lie in the plane $z = -5$, so the point on the sphere closest to $(0, 7, -5)$ should also be in the same plane. In fact it should lie on the line segment between $(0, 3, -5)$ and $(0, 7, -5)$, thus the point occurs when $x = 0$ and $z = -5 \Rightarrow 0^2 + (y-3)^2 + (-5+5)^2 = 4 \Rightarrow y = 3 \pm 2 \Rightarrow y = 5$
$\Rightarrow (0, 5, -5)$.

66. $\sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2} = \sqrt{(x-0)^2 + (y-4)^2 + (z-0)^2} = \sqrt{(x-3)^2 + (y-0)^2 + (z-0)^2}$
$= \sqrt{(x-2)^2 + (y-2)^2 + (z+3)^2}$
$\Rightarrow x^2 + y^2 + z^2 = x^2 + y^2 - 8y + 16 + z^2 = x^2 - 6x + 9 + y^2 + z^2 = x^2 - 4x + y^2 - 4y + z^2 + 6z + 17$
Solve: $x^2 + y^2 + z^2 = x^2 + y^2 - 8y + 16 + z^2 \Rightarrow 0 = -8y + 16 \Rightarrow y = 2$
Solve: $x^2 + y^2 + z^2 = x^2 - 6x + 9 + y^2 + z^2 \Rightarrow 0 = -6x + 9 \Rightarrow x = \frac{3}{2}$
Solve: $x^2 + y^2 + z^2 = x^2 - 4x + y^2 - 4y + z^2 + 6z + 17 \Rightarrow 0 = -4x - 4y + 6z + 17 \Rightarrow 0 = -4\left(\frac{3}{2}\right) - 4(2) + 6z + 17$
$\Rightarrow z = -\frac{1}{2} \Rightarrow \left(\frac{3}{2}, 2, -\frac{1}{2}\right)$

12.2 VECTORS

1. (a) $\langle 3(3), 3(-2) \rangle = \langle 9, -6 \rangle$
(b) $\sqrt{9^2 + (-6)^2} = \sqrt{117} = 3\sqrt{13}$

2. (a) $\langle -2(-2), -2(5) \rangle = \langle 4, -10 \rangle$
(b) $\sqrt{4^2 + (-10)^2} = \sqrt{116} = 2\sqrt{29}$

3. (a) $\langle 3 + (-2), -2 + 5 \rangle = \langle 1, 3 \rangle$
(b) $\sqrt{1^2 + 3^2} = \sqrt{10}$

4. (a) $\langle 3 - (-2), -2 - 5 \rangle = \langle 5, -7 \rangle$
(b) $\sqrt{5^2 + (-7)^2} = \sqrt{74}$

5. (a) $2\mathbf{u} = \langle 2(3), 2(-2) \rangle = \langle 6, -4 \rangle$
$3\mathbf{v} = \langle 3(-2), 3(5) \rangle = \langle -6, 15 \rangle$
$2\mathbf{u} - 3\mathbf{v} = \langle 6 - (-6), -4 - 15 \rangle = \langle 12, -19 \rangle$
(b) $\sqrt{12^2 + (-19)^2} = \sqrt{505}$

6. (a) $-2\mathbf{u} = \langle -2(3), -2(-2) \rangle = \langle -6, 4 \rangle$
$5\mathbf{v} = \langle 5(-2), 5(5) \rangle = \langle -10, 25 \rangle$
$-2\mathbf{u} + 5\mathbf{v} = \langle -6 + (-10), 4 + 25 \rangle = \langle -16, 29 \rangle$
(b) $\sqrt{(-16)^2 + 29^2} = \sqrt{1097}$

7. (a) $\frac{3}{5}\mathbf{u} = \left\langle \frac{3}{5}(3), \frac{3}{5}(-2) \right\rangle = \left\langle \frac{9}{5}, -\frac{6}{5} \right\rangle$

$\frac{4}{5}\mathbf{v} = \left\langle \frac{4}{5}(-2), \frac{4}{5}(5) \right\rangle = \left\langle -\frac{8}{5}, 4 \right\rangle$

$\frac{3}{5}\mathbf{u} + \frac{4}{5}\mathbf{v} = \left\langle \frac{9}{5} + \left(-\frac{8}{5}\right), -\frac{6}{5} + 4 \right\rangle = \left\langle \frac{1}{5}, \frac{14}{5} \right\rangle$

(b) $\sqrt{\left(\frac{1}{5}\right)^2 + \left(\frac{14}{5}\right)^2} = \frac{\sqrt{197}}{5}$

8. (a) $-\frac{5}{13}\mathbf{u} = \left\langle -\frac{5}{13}(3), -\frac{5}{13}(-2) \right\rangle = \left\langle -\frac{15}{13}, \frac{10}{13} \right\rangle$

$\frac{12}{13}\mathbf{v} = \left\langle \frac{12}{13}(-2), \frac{12}{13}(5) \right\rangle = \left\langle -\frac{24}{13}, \frac{60}{13} \right\rangle$

$-\frac{5}{13}\mathbf{u} + \frac{12}{13}\mathbf{v} = \left\langle -\frac{15}{13} + \left(-\frac{24}{13}\right), \frac{10}{13} + \frac{60}{13} \right\rangle = \left\langle -3, \frac{70}{13} \right\rangle$

(b) $\sqrt{(-3)^2 + \left(\frac{70}{13}\right)^2} = \frac{\sqrt{6421}}{13}$

9. $\langle 2-1, -1-3 \rangle = \langle 1, -4 \rangle$

10. $\left\langle \frac{2+(-4)}{2} - 0, \frac{-1+3}{2} - 0 \right\rangle = \langle -1, 1 \rangle$

11. $\langle 0-2, 0-3 \rangle = \langle -2, -3 \rangle$

12. $\overrightarrow{AB} = \langle 2-1, 0-(-1) \rangle = \langle 1, 1 \rangle$, $\overrightarrow{CD} = \langle -2-(-1), 2-3 \rangle = \langle -1, -1 \rangle$, $\overrightarrow{AB} + \overrightarrow{CD} = \langle 0, 0 \rangle$

13. $\left\langle \cos\frac{2\pi}{3}, \sin\frac{2\pi}{3} \right\rangle = \left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$

14. $\left\langle \cos\left(-\frac{3\pi}{4}\right), \sin\left(-\frac{3\pi}{4}\right) \right\rangle = \left\langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$

15. This is the unit vector which makes an angle of $120° + 90° = 210°$ with the positive x-axis;
$\langle \cos 210°, \sin 210° \rangle = \left\langle -\frac{\sqrt{3}}{2}, -\frac{1}{2} \right\rangle$

16. $\langle \cos 135°, \sin 135° \rangle = \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$

17. $\overrightarrow{P_1P_2} = (2-5)\mathbf{i} + (9-7)\mathbf{j} + (-2-(-1))\mathbf{k} = -3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

18. $\overrightarrow{P_1P_2} = (-3-1)\mathbf{i} + (0-2)\mathbf{j} + (5-0)\mathbf{k} = -4\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$

19. $\overrightarrow{AB} = (-10-(-7))\mathbf{i} + (8-(-8))\mathbf{j} + (1-1)\mathbf{k} = -3\mathbf{i} + 16\mathbf{j}$

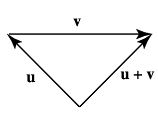
20. $\overrightarrow{AB} = (-1-1)\mathbf{i} + (4-0)\mathbf{j} + (5-3)\mathbf{k} = -2\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$

21. $5\mathbf{u} - \mathbf{v} = 5\langle 1, 1, -1 \rangle - \langle 2, 0, 3 \rangle = \langle 5, 5, -5 \rangle - \langle 2, 0, 3 \rangle = \langle 5-2, 5-0, -5-3 \rangle = \langle 3, 5, -8 \rangle = 3\mathbf{i} + 5\mathbf{j} - 8\mathbf{k}$

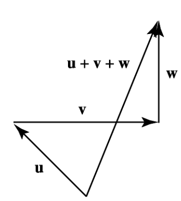
22. $-2\mathbf{u} + 3\mathbf{v} = -2\langle -1, 0, 2 \rangle + 3\langle 1, 1, 1 \rangle = \langle 2, 0, -4 \rangle + \langle 3, 3, 3 \rangle = \langle 5, 3, -1 \rangle = 5\mathbf{i} + 3\mathbf{j} - \mathbf{k}$

23. The vector **v** is horizontal and 1 in. long. The vectors **u** and **w** are $\frac{11}{16}$ in. long. **w** is vertical and **u** makes a 45° angle with the horizontal. All vectors must be drawn to scale.

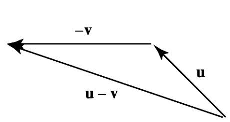
(a)



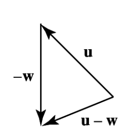
(b)



(c)

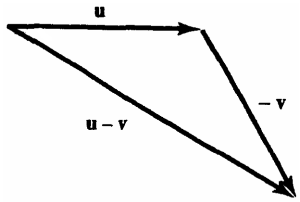


(d)

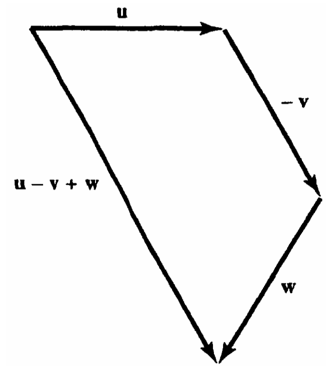


24. The angle between the vectors is 120° and vector **u** is horizontal. They are all 1 in. long. Draw to scale.

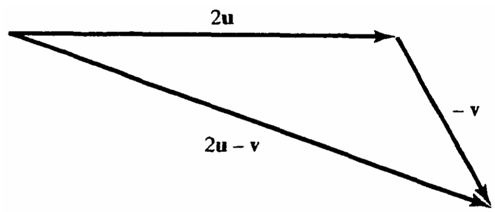
(a)



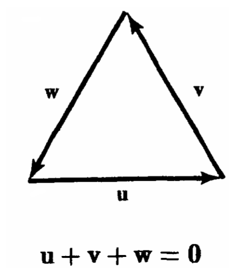
(b)



(c)



(d)



25. length $= |2\mathbf{i}+\mathbf{j}-2\mathbf{k}| = \sqrt{2^2+1^2+(-2)^2} = 3$, the direction is $\frac{2}{3}\,\mathbf{i}+\frac{1}{3}\,\mathbf{j}-\frac{2}{3}\,\mathbf{k} \;\Rightarrow\; 2\mathbf{i}+\mathbf{j}-2\mathbf{k} = 3\left(\frac{2}{3}\,\mathbf{i}+\frac{1}{3}\,\mathbf{j}-\frac{2}{3}\,\mathbf{k}\right)$

26. length $= |9\mathbf{i}-2\mathbf{j}+6\mathbf{k}| = \sqrt{81+4+36} = 11$, the direction is $\frac{9}{11}\,\mathbf{i}-\frac{2}{11}\,\mathbf{j}+\frac{6}{11}\,\mathbf{k} \;\Rightarrow\; 9\mathbf{i}-2\mathbf{j}+6\mathbf{k}$
$= 11\left(\frac{9}{11}\,\mathbf{i}-\frac{2}{11}\,\mathbf{j}+\frac{6}{11}\,\mathbf{k}\right)$

27. length $= |5\mathbf{k}| = \sqrt{25} = 5$, the direction is $\mathbf{k} \;\Rightarrow\; 5\mathbf{k} = 5(\mathbf{k})$

28. length $= \left|\frac{3}{5}\,\mathbf{i}+\frac{4}{5}\,\mathbf{k}\right| = \sqrt{\frac{9}{25}+\frac{16}{25}} = 1$, the direction is $\frac{3}{5}\,\mathbf{i}+\frac{4}{5}\,\mathbf{k} \;\Rightarrow\; \frac{3}{5}\,\mathbf{i}+\frac{4}{5}\,\mathbf{k} = 1\left(\frac{3}{5}\,\mathbf{i}+\frac{4}{5}\,\mathbf{k}\right)$

29. length $= \left|\frac{1}{\sqrt{6}}\,\mathbf{i}-\frac{1}{\sqrt{6}}\,\mathbf{j}-\frac{1}{\sqrt{6}}\,\mathbf{k}\right| = \sqrt{3\left(\frac{1}{\sqrt{6}}\right)^2} = \sqrt{\frac{1}{2}}$, the direction is $\frac{1}{\sqrt{3}}\,\mathbf{i}-\frac{1}{\sqrt{3}}\,\mathbf{j}-\frac{1}{\sqrt{3}}\,\mathbf{k}$
$\Rightarrow\; \frac{1}{\sqrt{6}}\,\mathbf{i}-\frac{1}{\sqrt{6}}\,\mathbf{j}-\frac{1}{\sqrt{6}}\,\mathbf{k} = \sqrt{\frac{1}{2}}\left(\frac{1}{\sqrt{3}}\,\mathbf{i}-\frac{1}{\sqrt{3}}\,\mathbf{j}-\frac{1}{\sqrt{3}}\,\mathbf{k}\right)$

30. length $= \left|\frac{1}{\sqrt{3}}\,\mathbf{i}+\frac{1}{\sqrt{3}}\,\mathbf{j}+\frac{1}{\sqrt{3}}\,\mathbf{k}\right| = \sqrt{3\left(\frac{1}{\sqrt{3}}\right)^2} = 1$, the direction is $\frac{1}{\sqrt{3}}\,\mathbf{i}+\frac{1}{\sqrt{3}}\,\mathbf{j}+\frac{1}{\sqrt{3}}\,\mathbf{k}$
$\Rightarrow\; \frac{1}{\sqrt{3}}\,\mathbf{i}+\frac{1}{\sqrt{3}}\,\mathbf{j}+\frac{1}{\sqrt{3}}\,\mathbf{k} = 1\left(\frac{1}{\sqrt{3}}\,\mathbf{i}+\frac{1}{\sqrt{3}}\,\mathbf{j}+\frac{1}{\sqrt{3}}\,\mathbf{k}\right)$

31. (a) $2\mathbf{i}$ (b) $-\sqrt{3}\mathbf{k}$ (c) $\frac{3}{10}\,\mathbf{j}+\frac{2}{5}\,\mathbf{k}$ (d) $6\mathbf{i}-2\mathbf{j}+3\mathbf{k}$

32. (a) $-7\mathbf{j}$ (b) $-\frac{3\sqrt{2}}{5}\,\mathbf{i}-\frac{4\sqrt{2}}{5}\,\mathbf{k}$ (c) $\frac{1}{4}\,\mathbf{i}-\frac{1}{3}\,\mathbf{j}-\mathbf{k}$ (d) $\frac{a}{\sqrt{2}}\,\mathbf{i}+\frac{a}{\sqrt{3}}\,\mathbf{j}-\frac{a}{\sqrt{6}}\,\mathbf{k}$

33. $|\mathbf{v}| = \sqrt{12^2+5^2} = \sqrt{169} = 13$; $\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{13}\,\mathbf{v} = \frac{1}{13}\,(12\mathbf{i}-5\mathbf{k}) \;\Rightarrow\;$ the desired vector is $\frac{7}{13}\,(12\mathbf{i}-5\mathbf{k})$

34. $|\mathbf{v}| = \sqrt{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}} = \frac{\sqrt{3}}{2}$; $\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}$ $\Rightarrow$ the desired vector is $-3\left(\frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}\right)$ $= -\sqrt{3}\mathbf{i} + \sqrt{3}\mathbf{j} + \sqrt{3}\mathbf{k}$

35. (a) $3\mathbf{i}+4\mathbf{j}-5\mathbf{k} = 5\sqrt{2}\left(\frac{3}{5\sqrt{2}}\mathbf{i} + \frac{4}{5\sqrt{2}}\mathbf{j} - \frac{1}{\sqrt{2}}\mathbf{k}\right)$ $\Rightarrow$ the direction is $\frac{3}{5\sqrt{2}}\mathbf{i} + \frac{4}{5\sqrt{2}}\mathbf{j} - \frac{1}{\sqrt{2}}\mathbf{k}$
    (b) the midpoint is $\left(\frac{1}{2}, 3, \frac{5}{2}\right)$

36. (a) $3\mathbf{i}-6\mathbf{j}+2\mathbf{k} = 7\left(\frac{3}{7}\mathbf{i} - \frac{6}{7}\mathbf{j} + \frac{2}{7}\mathbf{k}\right)$ $\Rightarrow$ the direction is $\frac{3}{7}\mathbf{i} - \frac{6}{7}\mathbf{j} + \frac{2}{7}\mathbf{k}$
    (b) the midpoint is $\left(\frac{5}{2}, 1, 6\right)$

37. (a) $-\mathbf{i}-\mathbf{j}-\mathbf{k} = \sqrt{3}\left(-\frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}\right)$ $\Rightarrow$ the direction is $-\frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}$
    (b) the midpoint is $\left(\frac{5}{2}, \frac{7}{2}, \frac{9}{2}\right)$

38. (a) $2\mathbf{i}-2\mathbf{j}-2\mathbf{k} = 2\sqrt{3}\left(\frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}\right)$ $\Rightarrow$ the direction is $\frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}$
    (b) the midpoint is $(1, -1, -1)$

39. $\overrightarrow{AB} = (5-a)\mathbf{i} + (1-b)\mathbf{j} + (3-c)\mathbf{k} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ $\Rightarrow$ $5-a=1$, $1-b=4$, and $3-c=-2$ $\Rightarrow$ $a=4$, $b=-3$, and $c=5$ $\Rightarrow$ A is the point $(4, -3, 5)$

40. $\overrightarrow{AB} = (a+2)\mathbf{i} + (b+3)\mathbf{j} + (c-6)\mathbf{k} = -7\mathbf{i} + 3\mathbf{j} + 8\mathbf{k}$ $\Rightarrow$ $a+2=-7$, $b+3=3$, and $c-6=8$ $\Rightarrow$ $a=-9$, $b=0$, and $c=14$ $\Rightarrow$ B is the point $(-9, 0, 14)$

41. $2\mathbf{i}+\mathbf{j} = a(\mathbf{i}+\mathbf{j}) + b(\mathbf{i}-\mathbf{j}) = (a+b)\mathbf{i} + (a-b)\mathbf{j}$ $\Rightarrow$ $a+b=2$ and $a-b=1$ $\Rightarrow$ $2a=3$ $\Rightarrow$ $a=\frac{3}{2}$ and $b=a-1=\frac{1}{2}$

42. $\mathbf{i}-2\mathbf{j} = a(2\mathbf{i}+3\mathbf{j}) + b(\mathbf{i}+\mathbf{j}) = (2a+b)\mathbf{i} + (3a+b)\mathbf{j}$ $\Rightarrow$ $2a+b=1$ and $3a+b=-2$ $\Rightarrow$ $a=-3$ and $b=1-2a=7$ $\Rightarrow$ $\mathbf{u}_1 = a(2\mathbf{i}+3\mathbf{j}) = -6\mathbf{i}-9\mathbf{j}$ and $\mathbf{u}_2 = b(\mathbf{i}+\mathbf{j}) = 7\mathbf{i}+7\mathbf{j}$

43. 25° west of north is $90° + 25° = 115°$ north of east. $800\langle\cos 115°, \sin 115°\rangle \approx \langle -338.095, 725.046\rangle$

44. Let $\mathbf{u} = \langle x, y\rangle$ be represent the velocity of the plane alone, $\mathbf{v} = \langle 70\cos 60°, 70\sin 60°\rangle = \langle 35, 35\sqrt{3}\rangle$, and let the resultant $\mathbf{u}+\mathbf{v} = \langle 500, 0\rangle$. Then $\langle x, y\rangle + \langle 35, 35\sqrt{3}\rangle = \langle 500, 0\rangle \Rightarrow \langle x+35, y+35\sqrt{3}\rangle = \langle 500, 0\rangle$ $\Rightarrow x+35 = 500$ and $y + 35\sqrt{3} = 0 \Rightarrow x = 465$ and $y = -35\sqrt{3} \Rightarrow \mathbf{u} = \langle 465, -35\sqrt{3}\rangle$ $\Rightarrow |\mathbf{u}| = \sqrt{465^2 + \left(-35\sqrt{3}\right)^2} \approx 468.9$ mph, and $\tan\theta = \frac{-35\sqrt{3}}{465} \Rightarrow \theta \approx -7.4° \Rightarrow 7.4°$ south of east.

45. $\mathbf{F}_1 = \left\langle -|\mathbf{F}_1|\cos 30°, |\mathbf{F}_1|\sin 30°\right\rangle = \left\langle -\frac{\sqrt{3}}{2}|\mathbf{F}_1|, \frac{1}{2}|\mathbf{F}_1|\right\rangle$, $\mathbf{F}_2 = \left\langle |\mathbf{F}_2|\cos 45°, |\mathbf{F}_2|\sin 45°\right\rangle = \left\langle \frac{1}{\sqrt{2}}|\mathbf{F}_2|, \frac{1}{\sqrt{2}}|\mathbf{F}_2|\right\rangle$, and $\mathbf{w} = \langle 0, -100\rangle$. Since $\mathbf{F}_1 + \mathbf{F}_2 = \langle 0, 100\rangle \Rightarrow \left\langle -\frac{\sqrt{3}}{2}|\mathbf{F}_1| + \frac{1}{\sqrt{2}}|\mathbf{F}_2|, \frac{1}{2}|\mathbf{F}_1| + \frac{1}{\sqrt{2}}|\mathbf{F}_2|\right\rangle = \langle 0, 100\rangle$ $\Rightarrow -\frac{\sqrt{3}}{2}|\mathbf{F}_1| + \frac{1}{\sqrt{2}}|\mathbf{F}_2| = 0$ and $\frac{1}{2}|\mathbf{F}_1| + \frac{1}{\sqrt{2}}|\mathbf{F}_2| = 100$. Solving the first equation for $|\mathbf{F}_2|$ results in: $|\mathbf{F}_2| = \frac{\sqrt{6}}{2}|\mathbf{F}_1|$. Substituting this result into the second equation gives us: $\frac{1}{2}|\mathbf{F}_1| + \frac{1}{\sqrt{2}}\left(\frac{\sqrt{6}}{2}|\mathbf{F}_1|\right) = 100 \Rightarrow |\mathbf{F}_1| = \frac{200}{1+\sqrt{3}} \approx 73.205$ N $\Rightarrow |\mathbf{F}_2| = \frac{100\sqrt{6}}{1+\sqrt{3}} \approx 89.658$ N $\Rightarrow \mathbf{F}_1 \approx \langle -63.397, 36.603\rangle$ and $\mathbf{F}_2 = \approx \langle 63.397, 63.397\rangle$

46. $\mathbf{F}_1 = \langle -35\cos\alpha, 35\sin\alpha\rangle$, $\mathbf{F}_2 = \left\langle |\mathbf{F}_2|\cos 60°, |\mathbf{F}_2|\sin 60°\right\rangle = \left\langle \frac{1}{2}|\mathbf{F}_2|, \frac{\sqrt{3}}{2}|\mathbf{F}_2|\right\rangle$, and $\mathbf{w} = \langle 0, -50\rangle$. Since $\mathbf{F}_1 + \mathbf{F}_2 = \langle 0, 50\rangle \Rightarrow \langle -35\cos\alpha + \frac{1}{2}|\mathbf{F}_2|, 35\sin\alpha + \frac{\sqrt{3}}{2}|\mathbf{F}_2|\rangle = \langle 0, 50\rangle \Rightarrow -35\cos\alpha + \frac{1}{2}|\mathbf{F}_2| = 0$ and $35\sin\alpha + \frac{\sqrt{3}}{2}|\mathbf{F}_2| = 50$. Solving the first equation for $|\mathbf{F}_2|$ results in: $|\mathbf{F}_2| = 70\cos\alpha$. Substituting this result into the second equation gives us: $35\sin\alpha + 35\sqrt{3}\cos\alpha = 50 \Rightarrow \sqrt{3}\cos\alpha = \frac{10}{7} - \sin\alpha \Rightarrow 3\cos^2\alpha = \frac{100}{49} - \frac{20}{7}\sin\alpha + \sin^2\alpha$ $\Rightarrow 3(1-\sin^2\alpha) = \frac{100}{49} - \frac{20}{7}\sin\alpha + \sin^2\alpha \Rightarrow 196\sin^2\alpha - 140\sin\alpha - 47 = 0 \Rightarrow \sin\alpha = \frac{5\pm 6\sqrt{2}}{14}$. Since $\alpha > 0 \Rightarrow$ $\sin\alpha > 0 \Rightarrow \sin\alpha = \frac{5+6\sqrt{2}}{14} \Rightarrow \alpha \approx 74.42°$, and $|\mathbf{F}_2| = 70\cos\alpha \approx 18.81$ N.

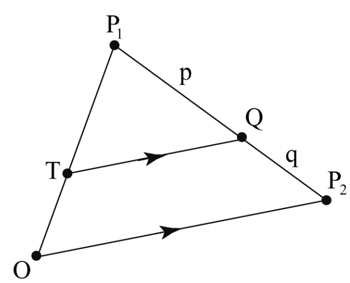
47. $\mathbf{F}_1 = \left\langle -|\mathbf{F}_1|\cos 40°, |\mathbf{F}_1|\sin 40°\right\rangle$, $\mathbf{F}_2 = \langle 100\cos 35°, 100\sin 35°\rangle$, and $\mathbf{w} = \langle 0, -w\rangle$. Since $\mathbf{F}_1 + \mathbf{F}_2 = \langle 0, w\rangle$ $\Rightarrow \left\langle -|\mathbf{F}_1|\cos 40° + 100\cos 35°, |\mathbf{F}_1|\sin 40° + 100\sin 35°\right\rangle = \langle 0, w\rangle \Rightarrow -|\mathbf{F}_1|\cos 40° + 100\cos 35° = 0$ and $|\mathbf{F}_1|\sin 40° + 100\sin 35° = w$. Solving the first equation for $|\mathbf{F}_1|$ results in: $|\mathbf{F}_1| = \frac{100\cos 35°}{\cos 40°} \approx 106.933$ N. Substituting this result into the second equation gives us: $w \approx 126.093$ N.

48. $\mathbf{F}_1 = \langle -|\mathbf{F}_1|\cos\alpha, |\mathbf{F}_1|\sin\alpha\rangle = \langle -75\cos\alpha, 75\sin\alpha\rangle$, $\mathbf{F}_2 = \langle |\mathbf{F}_2|\cos\beta, |\mathbf{F}_2|\sin\beta\rangle = \langle 75\cos\alpha, 75\sin\alpha\rangle$, and $\mathbf{w} = \langle 0, -25\rangle$. Since $\mathbf{F}_1 + \mathbf{F}_2 = \langle 0, 25\rangle \Rightarrow \langle -75\cos\alpha + 75\cos\alpha, 75\sin\alpha + 75\sin\alpha\rangle = \langle 0, 25\rangle \Rightarrow 150\sin\alpha = 25$ $\Rightarrow \alpha \approx 9.59°$.

49. (a) The tree is located at the tip of the vector $\overrightarrow{OP} = (5\cos 60°)\mathbf{i} + (5\sin 60°)\mathbf{j} = \frac{5}{2}\mathbf{i} + \frac{5\sqrt{3}}{2}\mathbf{j} \Rightarrow P = \left(\frac{5}{2}, \frac{5\sqrt{3}}{2}\right)$

(b) The telephone pole is located at the point Q, which is the tip of the vector $\overrightarrow{OP} + \overrightarrow{PQ}$ $= \left(\frac{5}{2}\mathbf{i} + \frac{5\sqrt{3}}{2}\mathbf{j}\right) + (10\cos 315°)\mathbf{i} + (10\sin 315°)\mathbf{j} = \left(\frac{5}{2} + \frac{10\sqrt{2}}{2}\right)\mathbf{i} + \left(\frac{5\sqrt{3}}{2} - \frac{10\sqrt{2}}{2}\right)\mathbf{j}$ $\Rightarrow Q = \left(\frac{5+10\sqrt{2}}{2}, \frac{5\sqrt{3}-10\sqrt{2}}{2}\right)$

50. Let $t = \frac{q}{p+q}$ and $s = \frac{p}{p+q}$. Choose T on $\overline{OP_1}$ so that $\overline{TQ}$ is parallel to $\overline{OP_2}$, so that $\triangle TP_1Q$ is similar to $\triangle OP_1P_2$. Then $\frac{|OT|}{|OP_1|} = t \Rightarrow \overrightarrow{OT} = t\,\overrightarrow{OP_1}$ so that $T = (t x_1, t y_1, t z_1)$.
Also, $\frac{|TQ|}{|OP_2|} = s \Rightarrow \overrightarrow{TQ} = s\,\overrightarrow{OP_2} = s\langle x_2, y_2, z_2\rangle$.
Letting $Q = (x, y, z)$, we have that
$\overrightarrow{TQ} = \langle x - t x_1, y - t y_1, z - t z_1\rangle = s\langle x_2, y_2, z_2\rangle$
Thus $x = t x_1 + s x_2$, $y = t y_1 + s y_2$, $z = t z_1 + s z_2$.
(Note that if Q is the midpoint, then $\frac{p}{q} = 1$ and $t = s = \frac{1}{2}$ so that $x = \frac{1}{2}x_1 + \frac{1}{2}x_2 = \frac{x_1+x_2}{2}$, $y = \frac{y_1+y_2}{2}$, $z = \frac{z_1+z_2}{2}$ so that this result agress with the midpoint formula.)



51. (a) the midpoint of AB is $M\left(\frac{5}{2}, \frac{5}{2}, 0\right)$ and $\overrightarrow{CM} = \left(\frac{5}{2} - 1\right)\mathbf{i} + \left(\frac{5}{2} - 1\right)\mathbf{j} + (0-3)\mathbf{k} = \frac{3}{2}\mathbf{i} + \frac{3}{2}\mathbf{j} - 3\mathbf{k}$

(b) the desired vector is $\left(\frac{2}{3}\right)\overrightarrow{CM} = \frac{2}{3}\left(\frac{3}{2}\mathbf{i} + \frac{3}{2}\mathbf{j} - 3\mathbf{k}\right) = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$

(c) the vector whose sum is the vector from the origin to C and the result of part (b) will terminate at the center of mass $\Rightarrow$ the terminal point of $(\mathbf{i} + \mathbf{j} + 3\mathbf{k}) + (\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ is the point $(2, 2, 1)$, which is the location of the center of mass

52. The midpoint of AB is $M\left(\frac{3}{2}, 0, \frac{5}{2}\right)$ and $\left(\frac{2}{3}\right)\overrightarrow{CM} = \frac{2}{3}\left[\left(\frac{3}{2} + 1\right)\mathbf{i} + (0-2)\mathbf{j} + \left(\frac{5}{2} + 1\right)\mathbf{k}\right] = \frac{2}{3}\left(\frac{5}{2}\mathbf{i} - 2\mathbf{j} + \frac{7}{2}\mathbf{k}\right)$ $= \frac{5}{3}\mathbf{i} - \frac{4}{3}\mathbf{j} + \frac{7}{3}\mathbf{k}$. The vector from the origin to the point of intersection of the medians is $\left(\frac{5}{3}\mathbf{i} - \frac{4}{3}\mathbf{j} + \frac{7}{3}\mathbf{k}\right) + \overrightarrow{OC}$ $= \left(\frac{5}{3}\mathbf{i} - \frac{4}{3}\mathbf{j} + \frac{7}{3}\mathbf{k}\right) + (-\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{4}{3}\mathbf{k}$.

53. Without loss of generality we identify the vertices of the quadrilateral such that $A(0,0,0)$, $B(x_b,0,0)$, $C(x_c,y_c,0)$ and $D(x_d,y_d,z_d)$ $\Rightarrow$ the midpoint of AB is $M_{AB}\left(\frac{x_b}{2},0,0\right)$, the midpoint of BC is $M_{BC}\left(\frac{x_b+x_c}{2},\frac{y_c}{2},0\right)$, the midpoint of CD is $M_{CD}\left(\frac{x_c+x_d}{2},\frac{y_c+y_d}{2},\frac{z_d}{2}\right)$ and the midpoint of AD is $M_{AD}\left(\frac{x_d}{2},\frac{y_d}{2},\frac{z_d}{2}\right)$ $\Rightarrow$ the midpoint of $M_{AB}M_{CD}$ is $\left(\frac{\frac{x_b}{2}+\frac{x_c+x_d}{2}}{2},\frac{y_c+y_d}{4},\frac{z_d}{4}\right)$ which is the same as the midpoint of $M_{AD}M_{BC} = \left(\frac{\frac{x_b+x_c}{2}+\frac{x_d}{2}}{2},\frac{y_c+y_d}{4},\frac{z_d}{4}\right)$.

54. Let $V_1, V_2, V_3, \ldots, V_n$ be the vertices of a regular n-sided polygon and $\mathbf{v}_i$ denote the vector from the center to V_i for $i = 1, 2, 3, \ldots, n$. If $\mathbf{S} = \sum_{i=1}^{n} \mathbf{v}_i$ and the polygon is rotated through an angle of $\frac{i(2\pi)}{n}$ where $i = 1, 2, 3, \ldots, n$, then $\mathbf{S}$ would remain the same. Since the vector $\mathbf{S}$ does not change with these rotations we conclude that $\mathbf{S} = \mathbf{0}$.

55. Without loss of generality we can coordinatize the vertices of the triangle such that $A(0,0)$, $B(b,0)$ and $C(x_c,y_c)$ $\Rightarrow$ a is located at $\left(\frac{b+x_c}{2},\frac{y_c}{2}\right)$, b is at $\left(\frac{x_c}{2},\frac{y_c}{2}\right)$ and c is at $\left(\frac{b}{2},0\right)$. Therefore, $\overrightarrow{Aa} = \left(\frac{b}{2}+\frac{x_c}{2}\right)\mathbf{i} + \left(\frac{y_c}{2}\right)\mathbf{j}$, $\overrightarrow{Bb} = \left(\frac{x_c}{2}-b\right)\mathbf{i} + \left(\frac{y_c}{2}\right)\mathbf{j}$, and $\overrightarrow{Cc} = \left(\frac{b}{2}-x_c\right)\mathbf{i} + (-y_c)\mathbf{j}$ $\Rightarrow$ $\overrightarrow{Aa} + \overrightarrow{Bb} + \overrightarrow{Cc} = \mathbf{0}$.

56. Let $\mathbf{u}$ be any unit vector in the plane. If $\mathbf{u}$ is positioned so that its initial point is at the origin and terminal point is at (x, y), then $\mathbf{u}$ makes an angle θ with $\mathbf{i}$, measured in the counter-clockwise direction. Since $|\mathbf{u}| = 1$, we have that $x = \cos\theta$ and $y = \sin\theta$. Thus $\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$. Since $\mathbf{u}$ was assumed to be any unit vector in the plane, this holds for <u>every</u> unit vector in the plane.

12.3 THE DOT PRODUCT

NOTE: In Exercises 1-8 below we calculate $\text{proj}_\mathbf{v}\,\mathbf{u}$ as the vector $\left(\frac{|\mathbf{u}|\cos\theta}{|\mathbf{v}|}\right)\mathbf{v}$, so the scalar multiplier of $\mathbf{v}$ is the number in column 5 divided by the number in column 2.

	$\mathbf{v}\cdot\mathbf{u}$	$\lvert\mathbf{v}\rvert$	$\lvert\mathbf{u}\rvert$	$\cos\theta$	$\lvert\mathbf{u}\rvert\cos\theta$	$\text{proj}_\mathbf{v}\,\mathbf{u}$
1.	-25	5	5	-1	-5	$-2\mathbf{i}+4\mathbf{j}-\sqrt{5}\mathbf{k}$
2.	3	1	13	$\frac{3}{13}$	3	$3\left(\frac{3}{5}\mathbf{i}+\frac{4}{5}\mathbf{k}\right)$
3.	25	15	5	$\frac{1}{3}$	$\frac{5}{3}$	$\frac{1}{9}(10\mathbf{i}+11\mathbf{j}-2\mathbf{k})$
4.	13	15	3	$\frac{13}{45}$	$\frac{13}{15}$	$\frac{13}{225}(2\mathbf{i}+10\mathbf{j}-11\mathbf{k})$
5.	2	$\sqrt{34}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}\sqrt{34}}$	$\frac{2}{\sqrt{34}}$	$\frac{1}{17}(5\mathbf{j}-3\mathbf{k})$
6.	$\sqrt{3}-\sqrt{2}$	$\sqrt{2}$	3	$\frac{\sqrt{3}-\sqrt{2}}{3\sqrt{2}}$	$\frac{\sqrt{3}-\sqrt{2}}{\sqrt{2}}$	$\frac{\sqrt{3}-\sqrt{2}}{2}(-\mathbf{i}+\mathbf{j})$
7.	$10+\sqrt{17}$	$\sqrt{26}$	$\sqrt{21}$	$\frac{10+\sqrt{17}}{\sqrt{546}}$	$\frac{10+\sqrt{17}}{\sqrt{26}}$	$\frac{10+\sqrt{17}}{26}(5\mathbf{i}+\mathbf{j})$
8.	$\frac{1}{6}$	$\frac{\sqrt{30}}{6}$	$\frac{\sqrt{30}}{6}$	$\frac{1}{5}$	$\frac{1}{\sqrt{30}}$	$\frac{1}{5}\left\langle\frac{1}{\sqrt{2}},\frac{1}{\sqrt{3}}\right\rangle$

9. $\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}\right) = \cos^{-1}\left(\frac{(2)(1)+(1)(2)+(0)(-1)}{\sqrt{2^2+1^2+0^2}\,\sqrt{1^2+2^2+(-1)^2}}\right) = \cos^{-1}\left(\frac{4}{\sqrt{5}\,\sqrt{6}}\right) = \cos^{-1}\left(\frac{4}{\sqrt{30}}\right) \approx 0.75$ rad

10. $\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}\right) = \cos^{-1}\left(\frac{(2)(3)+(-2)(0)+(1)(4)}{\sqrt{2^2+(-2)^2+1^2}\,\sqrt{3^2+0^2+4^2}}\right) = \cos^{-1}\left(\frac{10}{\sqrt{9}\,\sqrt{25}}\right) = \cos^{-1}\left(\frac{2}{3}\right) \approx 0.84$ rad

11. $\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}\right) = \cos^{-1}\left(\frac{\left(\sqrt{3}\right)\left(\sqrt{3}\right)+(-7)(1)+(0)(-2)}{\sqrt{\left(\sqrt{3}\right)^2+(-7)^2+0^2}\,\sqrt{\left(\sqrt{3}\right)^2+(1)^2+(-2)^2}}\right) = \cos^{-1}\left(\frac{3-7}{\sqrt{52}\,\sqrt{8}}\right)$

$= \cos^{-1}\left(\frac{-1}{\sqrt{26}}\right) \approx 1.77$ rad

12. $\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}\right) = \cos^{-1}\left(\frac{(1)(-1)+\left(\sqrt{2}\right)(1)+\left(-\sqrt{2}\right)(1)}{\sqrt{(1)^2+\left(\sqrt{2}\right)^2+\left(-\sqrt{2}\right)^2}\,\sqrt{(-1)^2+(1)^2+(1)^2}}\right) = \cos^{-1}\left(\frac{-1}{\sqrt{5}\,\sqrt{3}}\right)$

$= \cos^{-1}\left(\frac{-1}{\sqrt{15}}\right) \approx 1.83$ rad

13. $\overrightarrow{AB} = \langle 3, 1\rangle$, $\overrightarrow{BC} = \langle -1, -3\rangle$, and $\overrightarrow{AC} = \langle 2, -2\rangle$. $\overrightarrow{BA} = \langle -3, -1\rangle$, $\overrightarrow{CB} = \langle 1, 3\rangle$, $\overrightarrow{CA} = \langle -2, 2\rangle$.

$\left|\overrightarrow{AB}\right| = \left|\overrightarrow{BA}\right| = \sqrt{10}$, $\left|\overrightarrow{BC}\right| = \left|\overrightarrow{CB}\right| = \sqrt{10}$, $\left|\overrightarrow{AC}\right| = \left|\overrightarrow{CA}\right| = 2\sqrt{2}$,

Angle at A $= \cos^{-1}\left(\frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|}\right) = \cos^{-1}\left(\frac{3(2)+1(-2)}{\left(\sqrt{10}\right)\left(2\sqrt{2}\right)}\right) = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right) \approx 63.435°$

Angle at B $= \cos^{-1}\left(\frac{\overrightarrow{BC}\cdot\overrightarrow{BA}}{\left|\overrightarrow{BC}\right|\left|\overrightarrow{BA}\right|}\right) = \cos^{-1}\left(\frac{(-1)(-3)+(-3)(-1)}{\left(\sqrt{10}\right)\left(\sqrt{10}\right)}\right) = \cos^{-1}\left(\frac{3}{5}\right) \approx 53.130°$, and

Angle at C $= \cos^{-1}\left(\frac{\overrightarrow{CB}\cdot\overrightarrow{CA}}{\left|\overrightarrow{CB}\right|\left|\overrightarrow{CA}\right|}\right) = \cos^{-1}\left(\frac{1(-2)+3(2)}{\left(\sqrt{10}\right)\left(2\sqrt{2}\right)}\right) = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right) \approx 63.435°$

14. $\overrightarrow{AC} = \langle 2, 4\rangle$ and $\overrightarrow{BD} = \langle 4, -2\rangle$. $\overrightarrow{AC}\cdot\overrightarrow{BD} = 2(4)+4(-2) = 0$, so the angle measures are all 90°.

15. (a) $\cos\alpha = \frac{\mathbf{i}\cdot\mathbf{v}}{|\mathbf{i}|\,|\mathbf{v}|} = \frac{a}{|\mathbf{v}|}$, $\cos\beta = \frac{\mathbf{j}\cdot\mathbf{v}}{|\mathbf{j}|\,|\mathbf{v}|} = \frac{b}{|\mathbf{v}|}$, $\cos\gamma = \frac{\mathbf{k}\cdot\mathbf{v}}{|\mathbf{k}|\,|\mathbf{v}|} = \frac{c}{|\mathbf{v}|}$ and

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \left(\frac{a}{|\mathbf{v}|}\right)^2 + \left(\frac{b}{|\mathbf{v}|}\right)^2 + \left(\frac{c}{|\mathbf{v}|}\right)^2 = \frac{a^2+b^2+c^2}{|\mathbf{v}|\,|\mathbf{v}|} = \frac{|\mathbf{v}|\,|\mathbf{v}|}{|\mathbf{v}|\,|\mathbf{v}|} = 1$

(b) $|\mathbf{v}| = 1 \Rightarrow \cos\alpha = \frac{a}{|\mathbf{v}|} = a$, $\cos\beta = \frac{b}{|\mathbf{v}|} = b$ and $\cos\gamma = \frac{c}{|\mathbf{v}|} = c$ are the direction cosines of $\mathbf{v}$

16. $\mathbf{u} = 10\mathbf{i} + 2\mathbf{k}$ is parallel to the pipe in the north direction and $\mathbf{v} = 10\mathbf{j} + \mathbf{k}$ is parallel to the pipe in the east direction. The angle between the two pipes is $\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}\right) = \cos^{-1}\left(\frac{2}{\sqrt{104}\,\sqrt{101}}\right) \approx 1.55$ rad $\approx 88.88°$.

17. The sum of two vectors of equal length is *always* orthogonal to their difference, as we can see from the equation $(\mathbf{v}_1+\mathbf{v}_2)\cdot(\mathbf{v}_1-\mathbf{v}_2) = \mathbf{v}_1\cdot\mathbf{v}_1 + \mathbf{v}_2\cdot\mathbf{v}_1 - \mathbf{v}_1\cdot\mathbf{v}_2 - \mathbf{v}_2\cdot\mathbf{v}_2 = |\mathbf{v}_1|^2 - |\mathbf{v}_2|^2 = 0$

18. $\overrightarrow{CA}\cdot\overrightarrow{CB} = (-\mathbf{v}+(-\mathbf{u}))\cdot(-\mathbf{v}+\mathbf{u}) = \mathbf{v}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{v} - \mathbf{u}\cdot\mathbf{u} = |\mathbf{v}|^2 - |\mathbf{u}|^2 = 0$ because $|\mathbf{u}| = |\mathbf{v}|$ since both equal the radius of the circle. Therefore, $\overrightarrow{CA}$ and $\overrightarrow{CB}$ are orthogonal.

19. Let $\mathbf{u}$ and $\mathbf{v}$ be the sides of a rhombus $\Rightarrow$ the diagonals are $\mathbf{d}_1 = \mathbf{u}+\mathbf{v}$ and $\mathbf{d}_2 = -\mathbf{u}+\mathbf{v}$ $\Rightarrow \mathbf{d}_1\cdot\mathbf{d}_2 = (\mathbf{u}+\mathbf{v})\cdot(-\mathbf{u}+\mathbf{v}) = -\mathbf{u}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} = |\mathbf{v}|^2 - |\mathbf{u}|^2 = 0$ because $|\mathbf{u}| = |\mathbf{v}|$, since a rhombus has equal sides.

20. Suppose the diagonals of a rectangle are perpendicular, and let $\mathbf{u}$ and $\mathbf{v}$ be the sides of a rectangle $\Rightarrow$ the diagonals are $\mathbf{d}_1 = \mathbf{u} + \mathbf{v}$ and $\mathbf{d}_2 = -\mathbf{u} + \mathbf{v}$. Since the diagonals are perpendicular we have $\mathbf{d}_1 \cdot \mathbf{d}_2 = 0$
$\Leftrightarrow (\mathbf{u}+\mathbf{v})\cdot(-\mathbf{u}+\mathbf{v}) = -\mathbf{u}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} = 0 \Leftrightarrow |\mathbf{v}|^2 - |\mathbf{u}|^2 = 0 \Leftrightarrow (|\mathbf{v}| + |\mathbf{u}|)(|\mathbf{v}| - |\mathbf{u}|) = 0$
$\Leftrightarrow (|\mathbf{v}| + |\mathbf{u}|) = 0$ which is not possible, or $(|\mathbf{v}| - |\mathbf{u}|) = 0$ which is equivalent to $|\mathbf{v}| = |\mathbf{u}| \Rightarrow$ the rectangle is a square.

21. Clearly the diagonals of a rectangle are equal in length. What is not as obvious is the statement that equal diagonals happen only in a rectangle. We show this is true by letting the adjacent sides of a parallelogram be the vectors $(v_1\mathbf{i} + v_2\mathbf{j})$ and $(u_1\mathbf{i} + u_2\mathbf{j})$. The equal diagonals of the parallelogram are $\mathbf{d}_1 = (v_1\mathbf{i} + v_2\mathbf{j}) + (u_1\mathbf{i} + u_2\mathbf{j})$ and $\mathbf{d}_2 = (v_1\mathbf{i} + v_2\mathbf{j}) - (u_1\mathbf{i} + u_2\mathbf{j})$. Hence $|\mathbf{d}_1| = |\mathbf{d}_2| = |(v_1\mathbf{i} + v_2\mathbf{j}) + (u_1\mathbf{i} + u_2\mathbf{j})| = |(v_1\mathbf{i} + v_2\mathbf{j}) - (u_1\mathbf{i} + u_2\mathbf{j})|$
$\Rightarrow |(v_1+u_1)\mathbf{i} + (v_2+u_2)\mathbf{j}| = |(v_1-u_1)\mathbf{i} + (v_2-u_2)\mathbf{j}| \Rightarrow \sqrt{(v_1+u_1)^2 + (v_2+u_2)^2} = \sqrt{(v_1-u_1)^2 + (v_2-u_2)^2}$
$\Rightarrow v_1^2 + 2v_1u_1 + u_1^2 + v_2^2 + 2v_2u_2 + u_2^2 = v_1^2 - 2v_1u_1 + u_1^2 + v_2^2 - 2v_2u_2 + u_2^2 \Rightarrow 2(v_1u_1 + v_2u_2)$
$= -2(v_1u_1 + v_2u_2) \Rightarrow v_1u_1 + v_2u_2 = 0 \Rightarrow (v_1\mathbf{i} + v_2\mathbf{j})\cdot(u_1\mathbf{i} + u_2\mathbf{j}) = 0 \Rightarrow$ the vectors $(v_1\mathbf{i} + v_2\mathbf{j})$ and $(u_1\mathbf{i} + u_2\mathbf{j})$ are perpendicular and the parallelogram must be a rectangle.

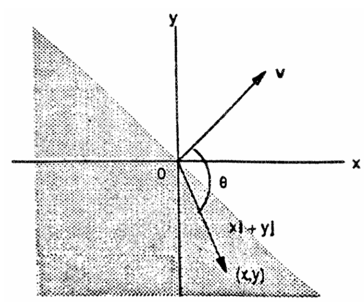
22. If $|\mathbf{u}| = |\mathbf{v}|$ and $\mathbf{u}+\mathbf{v}$ is the indicated diagonal, then $(\mathbf{u}+\mathbf{v})\cdot\mathbf{u} = \mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{u} = |\mathbf{u}|^2 + \mathbf{v}\cdot\mathbf{u} = \mathbf{u}\cdot\mathbf{v} + |\mathbf{v}|^2$
$= \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v} = (\mathbf{u}+\mathbf{v})\cdot\mathbf{v} \Rightarrow$ the angle $\cos^{-1}\left(\frac{(\mathbf{u}+\mathbf{v})\cdot\mathbf{u}}{|\mathbf{u}+\mathbf{v}|\,|\mathbf{u}|}\right)$ between the diagonal and $\mathbf{u}$ and the angle $\cos^{-1}\left(\frac{(\mathbf{u}+\mathbf{v})\cdot\mathbf{v}}{|\mathbf{u}+\mathbf{v}|\,|\mathbf{v}|}\right)$ between the diagonal and $\mathbf{v}$ are equal because the inverse cosine function is one-to-one. Therefore, the diagonal bisects the angle between $\mathbf{u}$ and $\mathbf{v}$.

23. horizontal component: $1200\cos(8°) \approx 1188$ ft/s; vertical component: $1200\sin(8°) \approx 167$ ft/s

24. $|\mathbf{w}|\cos(33° - 15°) = 2.5$ lb, so $|\mathbf{w}| = \frac{2.5 \text{ lb}}{\cos 18°}$. Then $\mathbf{w} = \frac{2.5 \text{ lb}}{\cos 18°}\langle \cos 33°, \sin 33° \rangle \approx \langle 2.205, 1.432 \rangle$

25. (a) Since $|\cos\theta| \le 1$, we have $|\mathbf{u}\cdot\mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|\,|\cos\theta| \le |\mathbf{u}|\,|\mathbf{v}|\,(1) = |\mathbf{u}|\,|\mathbf{v}|$.
(b) We have equality precisely when $|\cos\theta| = 1$ or when one or both of $\mathbf{u}$ and $\mathbf{v}$ is $\mathbf{0}$. In the case of nonzero vectors, we have equality when $\theta = 0$ or π, i.e., when the vectors are parallel.

26. $(x\mathbf{i} + y\mathbf{j})\cdot\mathbf{v} = |x\mathbf{i} + y\mathbf{j}|\,|\mathbf{v}|\cos\theta \le 0$ when $\frac{\pi}{2} \le \theta \le \pi$. This means (x, y) has to be a point whose position vector makes an angle with $\mathbf{v}$ that is a right angle or bigger.



27. $\mathbf{v}\cdot\mathbf{u}_1 = (a\mathbf{u}_1 + b\mathbf{u}_2)\cdot\mathbf{u}_1 = a\mathbf{u}_1\cdot\mathbf{u}_1 + b\mathbf{u}_2\cdot\mathbf{u}_1 = a|\mathbf{u}_1|^2 + b(\mathbf{u}_2\cdot\mathbf{u}_1) = a(1)^2 + b(0) = a$

28. No, $\mathbf{v}_1$ need not equal $\mathbf{v}_2$. For example, $\mathbf{i}+\mathbf{j} \ne \mathbf{i} + 2\mathbf{j}$ but $\mathbf{i}\cdot(\mathbf{i}+\mathbf{j}) = \mathbf{i}\cdot\mathbf{i} + \mathbf{i}\cdot\mathbf{j} = 1 + 0 = 1$ and $\mathbf{i}\cdot(\mathbf{i}+2\mathbf{j}) = \mathbf{i}\cdot\mathbf{i} + 2\mathbf{i}\cdot\mathbf{j} = 1 + 2\cdot 0 = 1$.

29. $\text{proj}_{\mathbf{v}}\,\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v} \Rightarrow \left(\mathbf{u} - \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v}\right)\cdot\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v}\right) = \mathbf{u}\cdot\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v}\right) - \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v}\right)\cdot\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v}\right) = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)(\mathbf{u}\cdot\mathbf{v}) - \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)^2(\mathbf{v}\cdot\mathbf{v})$
$= \frac{(\mathbf{u}\cdot\mathbf{v})^2}{|\mathbf{v}|^2} - \frac{(\mathbf{u}\cdot\mathbf{v})^2}{|\mathbf{v}|^4}|\mathbf{v}|^2 = 0$

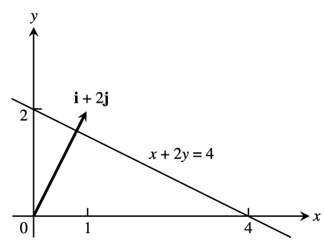
30. $\mathbf{F} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} - \mathbf{j} \Rightarrow \text{proj}_{\mathbf{v}}\,\mathbf{F} = \frac{\mathbf{F}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v} = \frac{5}{\left(\sqrt{10}\right)^2}(3\mathbf{i} - \mathbf{j}) = \frac{3}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$, is the vector parallel to $\mathbf{v}$.

$\mathbf{F} - \text{proj}_{\mathbf{v}}\,\mathbf{F} = (2\mathbf{i} + \mathbf{j} - 3\mathbf{k}) - \left(\frac{3}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}\right) = \frac{1}{2}\mathbf{i} + \frac{3}{2}\mathbf{j} - 3\mathbf{k}$ is the vector orthogonal to $\mathbf{v}$.

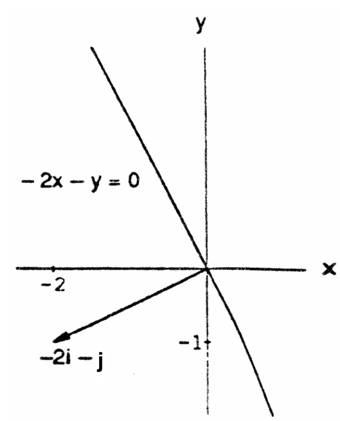
31. $P(x_1, y_1) = P\left(x_1, \frac{c}{b} - \frac{a}{b}x_1\right)$ and $Q(x_2, y_2) = Q\left(x_2, \frac{c}{b} - \frac{a}{b}x_2\right)$ are any two points P and Q on the line with $b \neq 0$ $\Rightarrow \overrightarrow{PQ} = (x_2 - x_1)\mathbf{i} + \frac{a}{b}(x_1 - x_2)\mathbf{j} \Rightarrow \overrightarrow{PQ} \cdot \mathbf{v} = \left[(x_2 - x_1)\mathbf{i} + \frac{a}{b}(x_1 - x_2)\mathbf{j}\right] \cdot (a\mathbf{i} + b\mathbf{j}) = a(x_2 - x_1) + b\left(\frac{a}{b}\right)(x_1 - x_2)$ $= 0 \Rightarrow \mathbf{v}$ is perpendicular to $\overrightarrow{PQ}$ for $b \neq 0$. If $b = 0$, then $\mathbf{v} = a\mathbf{i}$ is perpendicular to the vertical line $ax = c$. Alternatively, the slope of $\mathbf{v}$ is $\frac{b}{a}$ and the slope of the line $ax + by = c$ is $-\frac{a}{b}$, so the slopes are negative reciprocals $\Rightarrow$ the vector $\mathbf{v}$ and the line are perpendicular.

32. The slope of $\mathbf{v}$ is $\frac{b}{a}$ and the slope of $bx - ay = c$ is $\frac{b}{a}$, provided that $a \neq 0$. If $a = 0$, then $\mathbf{v} = b\mathbf{j}$ is parallel to the vertical line $bx = c$. In either case, the vector $\mathbf{v}$ is parallel to the line $bx - ay = c$.

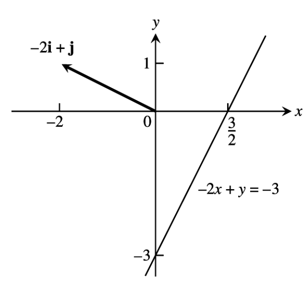
33. $\mathbf{v} = \mathbf{i} + 2\mathbf{j}$ is perpendicular to the line $x + 2y = c$;
$P(2, 1)$ on the line $\Rightarrow 2 + 2 = c \Rightarrow x + 2y = 4$



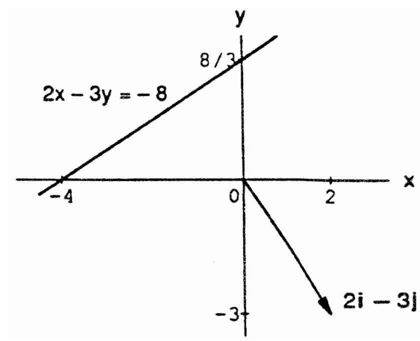
34. $\mathbf{v} = -2\mathbf{i} - \mathbf{j}$ is perpendicular to the line $-2x - y = c$;
$P(-1, 2)$ on the line $\Rightarrow (-2)(-1) - 2 = c$
$\Rightarrow -2x - y = 0$



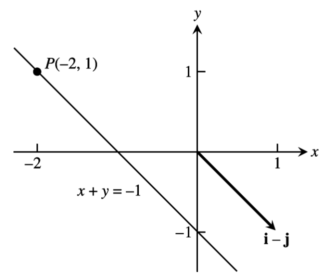
35. $\mathbf{v} = -2\mathbf{i} + \mathbf{j}$ is perpendicular to the line $-2x + y = c$;
$P(-2, -7)$ on the line $\Rightarrow (-2)(-2) - 7 = c$
$\Rightarrow -2x + y = -3$



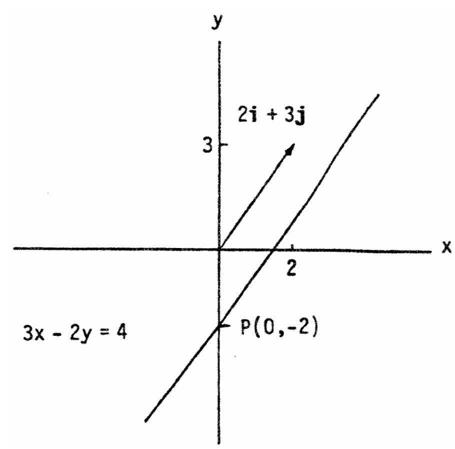
36. $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}$ is perpendicular to the line $2x - 3y = c$;
$P(11, 10)$ on the line $\Rightarrow (2)(11) - (3)(10) = c$
$\Rightarrow 2x - 3y = -8$



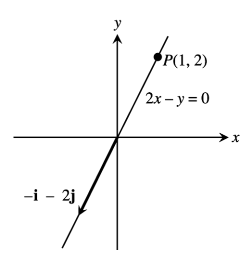
37. $\mathbf{v} = \mathbf{i} - \mathbf{j}$ is parallel to the line $-x - y = c$;
$P(-2, 1)$ on the line $\Rightarrow -(-2) - 1 = c \Rightarrow -x - y = 1$
or $x + y = -1$.



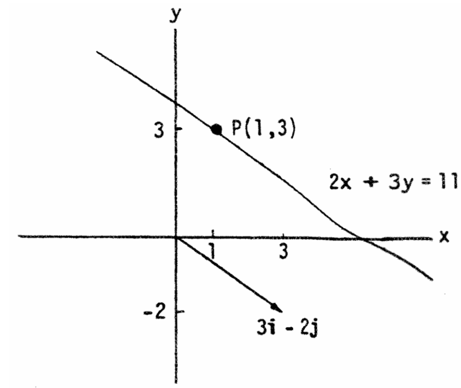
38. $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}$ is parallel to the line $3x - 2y = c$;
$P(0, -2)$ on the line $\Rightarrow 0 - 2(-2) = c \Rightarrow 3x - 2y = 4$



39. $\mathbf{v} = -\mathbf{i} - 2\mathbf{j}$ is parallel to the line $-2x + y = c$;
$P(1, 2)$ on the line $\Rightarrow -2(1) + 2 = c \Rightarrow -2x - y = 0$
or $2x - y = 0$.



40. $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j}$ is parallel to the line $-2x - 3y = c$; $P(1, 3)$ on the line $\Rightarrow$ $(-2)(1) - (3)(3) = c$ $\Rightarrow$ $-2x - 3y = -11$ or $2x + 3y = 11$



41. $P(0, 0)$, $Q(1, 1)$ and $\mathbf{F} = 5\mathbf{j}$ $\Rightarrow$ $\overrightarrow{PQ} = \mathbf{i} + \mathbf{j}$ and $\mathbf{W} = \mathbf{F} \cdot \overrightarrow{PQ} = (5\mathbf{j}) \cdot (\mathbf{i} + \mathbf{j}) = 5\ \text{N} \cdot \text{m} = 5\ \text{J}$

42. $\mathbf{W} = |\mathbf{F}|\,(\text{distance}) \cos\theta = (602{,}148\ \text{N})(605\ \text{km})(\cos 0) = 364{,}299{,}540\ \text{N} \cdot \text{km} = (364{,}299{,}540)(1000)\ \text{N} \cdot \text{m}$ $= 3.6429954 \times 10^{11}\ \text{J}$

43. $\mathbf{W} = |\mathbf{F}| \left|\overrightarrow{PQ}\right| \cos\theta = (200)(20)(\cos 30°) = 2000\sqrt{3} = 3464.10\ \text{N} \cdot \text{m} = 3464.10\ \text{J}$

44. $\mathbf{W} = |\mathbf{F}| \left|\overrightarrow{PQ}\right| \cos\theta = (1000)(5280)(\cos 60°) = 2{,}640{,}000\ \text{ft} \cdot \text{lb}$

In Exercises 45-50 we use the fact that $\mathbf{n} = a\mathbf{i} + b\mathbf{j}$ is normal to the line $ax + by = c$.

45. $\mathbf{n}_1 = 3\mathbf{i} + \mathbf{j}$ and $\mathbf{n}_2 = 2\mathbf{i} - \mathbf{j}$ $\Rightarrow$ $\theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{6-1}{\sqrt{10}\,\sqrt{5}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

46. $\mathbf{n}_1 = -\sqrt{3}\mathbf{i} + \mathbf{j}$ and $\mathbf{n}_2 = \sqrt{3}\mathbf{i} + \mathbf{j}$ $\Rightarrow$ $\theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{-3+1}{\sqrt{4}\,\sqrt{4}}\right) = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$

47. $\mathbf{n}_1 = \sqrt{3}\mathbf{i} - \mathbf{j}$ and $\mathbf{n}_2 = \mathbf{i} - \sqrt{3}\mathbf{j}$ $\Rightarrow$ $\theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{\sqrt{3}+\sqrt{3}}{\sqrt{4}\,\sqrt{4}}\right) = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$

48. $\mathbf{n}_1 = \mathbf{i} + \sqrt{3}\mathbf{j}$ and $\mathbf{n}_2 = \left(1 - \sqrt{3}\right)\mathbf{i} + \left(1 + \sqrt{3}\right)\mathbf{j}$ $\Rightarrow$ $\theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right)$
$= \cos^{-1}\left(\frac{1-\sqrt{3}+\sqrt{3}+3}{\sqrt{1+3}\,\sqrt{1-2\sqrt{3}+3+1+2\sqrt{3}+3}}\right) = \cos^{-1}\left(\frac{4}{2\sqrt{8}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

49. $\mathbf{n}_1 = 3\mathbf{i} - 4\mathbf{j}$ and $\mathbf{n}_2 = \mathbf{i} - \mathbf{j}$ $\Rightarrow$ $\theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{3+4}{\sqrt{25}\,\sqrt{2}}\right) = \cos^{-1}\left(\frac{7}{5\sqrt{2}}\right) \approx 0.14\ \text{rad}$

50. $\mathbf{n}_1 = 12\mathbf{i} + 5\mathbf{j}$ and $\mathbf{n}_2 = 2\mathbf{i} - 2\mathbf{j}$ $\Rightarrow$ $\theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{24-10}{\sqrt{169}\,\sqrt{8}}\right) = \cos^{-1}\left(\frac{14}{26\sqrt{2}}\right) \approx 1.18\ \text{rad}$

12.4 THE CROSS PRODUCT

1. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & -1 \\ 1 & 0 & -1 \end{vmatrix} = 3\left(\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}\right)$ $\Rightarrow$ length $= 3$ and the direction is $\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$;
$\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = -3\left(\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}\right)$ $\Rightarrow$ length $= 3$ and the direction is $-\frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$

2. $\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 0 \\ -1 & 1 & 0 \end{vmatrix} = 5(\mathbf{k}) \Rightarrow$ length $= 5$ and the direction is $\mathbf{k}$

$\mathbf{v}\times\mathbf{u} = -(\mathbf{u}\times\mathbf{v}) = -5(\mathbf{k}) \Rightarrow$ length $= 5$ and the direction is $-\mathbf{k}$

3. $\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & 4 \\ -1 & 1 & -2 \end{vmatrix} = \mathbf{0} \Rightarrow$ length $= 0$ and has no direction

$\mathbf{v}\times\mathbf{u} = -(\mathbf{u}\times\mathbf{v}) = \mathbf{0} \Rightarrow$ length $= 0$ and has no direction

4. $\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -1 \\ 0 & 0 & 0 \end{vmatrix} = \mathbf{0} \Rightarrow$ length $= 0$ and has no direction

$\mathbf{v}\times\mathbf{u} = -(\mathbf{u}\times\mathbf{v}) = \mathbf{0} \Rightarrow$ length $= 0$ and has no direction

5. $\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 0 \\ 0 & -3 & 0 \end{vmatrix} = -6(\mathbf{k}) \Rightarrow$ length $= 6$ and the direction is $-\mathbf{k}$

$\mathbf{v}\times\mathbf{u} = -(\mathbf{u}\times\mathbf{v}) = 6(\mathbf{k}) \Rightarrow$ length $= 6$ and the direction is $\mathbf{k}$

6. $\mathbf{u}\times\mathbf{v} = (\mathbf{i}\times\mathbf{j})\times(\mathbf{j}\times\mathbf{k}) = \mathbf{k}\times\mathbf{i} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = \mathbf{j} \Rightarrow$ length $= 1$ and the direction is $\mathbf{j}$

$\mathbf{v}\times\mathbf{u} = -(\mathbf{u}\times\mathbf{v}) = -\mathbf{j} \Rightarrow$ length $= 1$ and the direction is $-\mathbf{j}$

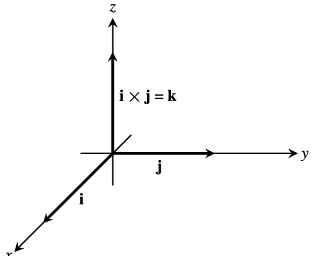
7. $\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -8 & -2 & -4 \\ 2 & 2 & 1 \end{vmatrix} = 6\mathbf{i} - 12\mathbf{k} \Rightarrow$ length $= 6\sqrt{5}$ and the direction is $\frac{1}{\sqrt{5}}\mathbf{i} - \frac{2}{\sqrt{5}}\mathbf{k}$

$\mathbf{v}\times\mathbf{u} = -(\mathbf{u}\times\mathbf{v}) = -(6\mathbf{i} - 12\mathbf{k}) \Rightarrow$ length $= 6\sqrt{5}$ and the direction is $-\frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{k}$

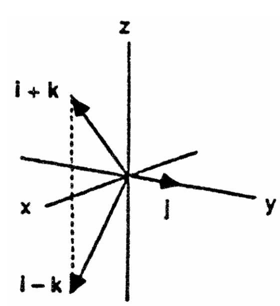
8. $\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{3}{2} & -\frac{1}{2} & 1 \\ 1 & 1 & 2 \end{vmatrix} = -2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k} \Rightarrow$ length $= 2\sqrt{3}$ and the direction is $-\frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$

$\mathbf{v}\times\mathbf{u} = -(\mathbf{u}\times\mathbf{v}) = -(-2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) \Rightarrow$ length $= 2\sqrt{3}$ and the direction is $\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}$

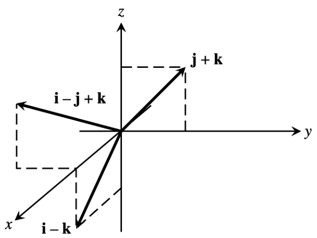
9. $\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \mathbf{k}$



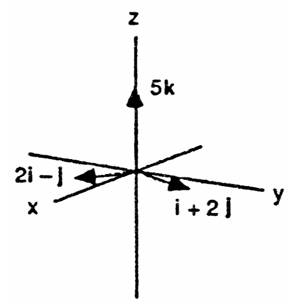
10. $\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{vmatrix} = \mathbf{i} + \mathbf{k}$



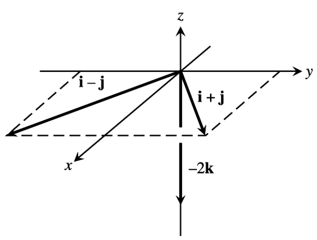
11. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{vmatrix} = \mathbf{i} - \mathbf{j} + \mathbf{k}$



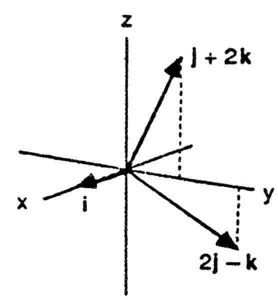
12. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 0 \\ 1 & 2 & 0 \end{vmatrix} = 5\mathbf{k}$



13. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix} = -2\mathbf{k}$



14. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{vmatrix} = 2\mathbf{j} - \mathbf{k}$



15. (a) $\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -3 \\ -1 & 3 & -1 \end{vmatrix} = 8\mathbf{i} + 4\mathbf{j} + 4\mathbf{k} \Rightarrow \text{Area} = \frac{1}{2}\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| = \frac{1}{2}\sqrt{64+16+16} = 2\sqrt{6}$

(b) $\mathbf{u} = \frac{\overrightarrow{PQ} \times \overrightarrow{PR}}{\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right|} = \frac{1}{\sqrt{6}}(2\mathbf{i} + \mathbf{j} + \mathbf{k})$

16. (a) $\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2 \\ 2 & -2 & 0 \end{vmatrix} = 4\mathbf{i} + 4\mathbf{j} - 2\mathbf{k} \Rightarrow \text{Area} = \frac{1}{2}\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| = \frac{1}{2}\sqrt{16+16+4} = 3$

(b) $\mathbf{u} = \frac{\overrightarrow{PQ} \times \overrightarrow{PR}}{\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right|} = \frac{1}{3}(2\mathbf{i} + 2\mathbf{j} - \mathbf{k})$

17. (a) $\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -\mathbf{i} + \mathbf{j} \Rightarrow \text{Area} = \frac{1}{2}\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| = \frac{1}{2}\sqrt{1+1} = \frac{\sqrt{2}}{2}$

(b) $\mathbf{u} = \frac{\overrightarrow{PQ} \times \overrightarrow{PR}}{\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right|} = \frac{1}{\sqrt{2}}(-\mathbf{i} + \mathbf{j}) = -\frac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{j})$

18. (a) $\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & -1 \\ 1 & 0 & -2 \end{vmatrix} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k} \Rightarrow \text{Area} = \frac{1}{2}\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| = \frac{1}{2}\sqrt{4+9+1} = \frac{\sqrt{14}}{2}$

(b) $\mathbf{u} = \frac{\overrightarrow{PQ} \times \overrightarrow{PR}}{\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right|} = \frac{1}{\sqrt{14}}(2\mathbf{i} + 3\mathbf{j} + \mathbf{k})$

19. If $\mathbf{u} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{v} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, and $\mathbf{w} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$, then $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$,

$(\mathbf{v}\times\mathbf{w})\cdot\mathbf{u} = \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$ and $(\mathbf{w}\times\mathbf{u})\cdot\mathbf{v} = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$ which all have the same absolute value, since the interchanging of two rows in a determinant does not change its absolute value $\Rightarrow$ the volume is

$|(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}| = \text{abs}\begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 8$

20. $|(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}| = \text{abs}\begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -2 \\ -1 & 2 & -1 \end{vmatrix} = 4$ (for details about verification, see Exercise 19)

21. $|(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}| = \text{abs}\begin{vmatrix} 2 & 1 & 0 \\ 2 & -1 & 1 \\ 1 & 0 & 2 \end{vmatrix} = |-7| = 7$ (for details about verification, see Exercise 19)

22. $|(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}| = \text{abs}\begin{vmatrix} 1 & 1 & -2 \\ -1 & 0 & -1 \\ 2 & 4 & -2 \end{vmatrix} = 8$ (for details about verification, see Exercise 19)

23. (a) $\mathbf{u}\cdot\mathbf{v} = -6$, $\mathbf{u}\cdot\mathbf{w} = -81$, $\mathbf{v}\cdot\mathbf{w} = 18 \Rightarrow$ none are perpendicular

(b) $\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & -1 & 1 \\ 0 & 1 & -5 \end{vmatrix} \neq \mathbf{0}$, $\mathbf{u}\times\mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & -1 & 1 \\ -15 & 3 & -3 \end{vmatrix} = \mathbf{0}$, $\mathbf{v}\times\mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & -5 \\ -15 & 3 & -3 \end{vmatrix} \neq \mathbf{0}$

$\Rightarrow$ **u** and **w** are parallel

24. (a) $\mathbf{u}\cdot\mathbf{v} = 0$, $\mathbf{u}\times\mathbf{w} = 0$, $\mathbf{u}\cdot\mathbf{r} = -3\pi$, $\mathbf{v}\cdot\mathbf{w} = 0$, $\mathbf{v}\cdot\mathbf{r} = 0$, $\mathbf{w}\cdot\mathbf{r} = 0 \Rightarrow \mathbf{u}\perp\mathbf{v}$, $\mathbf{u}\perp\mathbf{w}$, $\mathbf{v}\perp\mathbf{w}$, $\mathbf{v}\perp\mathbf{r}$ and $\mathbf{w}\perp\mathbf{r}$

(b) $\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ -1 & 1 & 1 \end{vmatrix} \neq \mathbf{0}$, $\mathbf{u}\times\mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ 1 & 0 & 1 \end{vmatrix} \neq \mathbf{0}$, $\mathbf{u}\times\mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ -\frac{\pi}{2} & -\pi & \frac{\pi}{2} \end{vmatrix} = \mathbf{0}$

$\mathbf{v}\times\mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} \neq \mathbf{0}$, $\mathbf{v}\times\mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 1 \\ -\frac{\pi}{2} & -\pi & \frac{\pi}{2} \end{vmatrix} \neq \mathbf{0}$, $\mathbf{w}\times\mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ -\frac{\pi}{2} & -\pi & \frac{\pi}{2} \end{vmatrix} \neq \mathbf{0}$

$\Rightarrow$ **u** and **r** are parallel

25. $\left|\overrightarrow{PQ}\times\mathbf{F}\right| = \left|\overrightarrow{PQ}\right| |\mathbf{F}| \sin(60°) = \frac{2}{3}\cdot 30\cdot\frac{\sqrt{3}}{2}$ ft · lb $= 10\sqrt{3}$ ft · lb

26. $\left|\overrightarrow{PQ}\times\mathbf{F}\right| = \left|\overrightarrow{PQ}\right| |\mathbf{F}| \sin(135°) = \frac{2}{3}\cdot 30\cdot\frac{\sqrt{2}}{2}$ ft · lb $= 10\sqrt{2}$ ft · lb

27. (a) true, $|\mathbf{u}| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{\mathbf{u}\cdot\mathbf{u}}$

(b) not always true, $\mathbf{u}\cdot\mathbf{u} = |\mathbf{u}|^2$

(c) true, $\mathbf{u}\times\mathbf{0} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ 0 & 0 & 0 \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$ and $\mathbf{0}\times\mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 0 \\ u_1 & u_2 & u_3 \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$

(d) true, $\mathbf{u}\times(-\mathbf{u}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ -u_1 & -u_2 & -u_3 \end{vmatrix} = (-u_2u_3 + u_2u_3)\mathbf{i} - (-u_1u_3 + u_1u_3)\mathbf{j} + (-u_1u_2 + u_1u_2)\mathbf{k} = \mathbf{0}$

(e) not always true, $\mathbf{i}\times\mathbf{j} = \mathbf{k} \neq -\mathbf{k} = \mathbf{j}\times\mathbf{i}$ for example

(f) true, distributive property of the cross product

(g) true, $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{v} = \mathbf{u}\cdot(\mathbf{v}\times\mathbf{v}) = \mathbf{u}\cdot\mathbf{0} = 0$

(h) true, the volume of a parallelpiped with **u**, **v**, and **w** along the three edges is the same whether the plane containing **u** and **v** or the plane containing **v** and **w** is used as the base plane, and the dot product is commutative.

28. (a) true, $\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3 = v_1u_1 + v_2u_2 + v_3u_3 = \mathbf{v}\cdot\mathbf{u}$

(b) true, $\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = -\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{vmatrix} = -(\mathbf{v}\times\mathbf{u})$

(c) true, $(-\mathbf{u})\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -u_1 & -u_2 & -u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = -\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = -(\mathbf{u}\times\mathbf{v})$

(d) true, $(c\mathbf{u})\cdot\mathbf{v} = (cu_1)v_1 + (cu_2)v_2 + (cu_3)v_3 = u_1(cv_1) + u_2(cv_2) + u_3(cv_3) = \mathbf{u}\cdot(c\mathbf{v}) = c(u_1v_1 + u_2v_2 + u_3v_3)$
$= c(\mathbf{u}\cdot\mathbf{v})$

(e) true, $c(\mathbf{u}\times\mathbf{v}) = c\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ cu_1 & cu_2 & cu_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (c\mathbf{u})\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ cv_1 & cv_2 & cv_3 \end{vmatrix} = \mathbf{u}\times(c\mathbf{v})$

(f) true, $\mathbf{u}\cdot\mathbf{u} = u_1^2 + u_2^2 + u_3^2 = \left(\sqrt{u_1^2 + u_2^2 + u_3^2}\right)^2 = |\mathbf{u}|^2$

(g) true, $(\mathbf{u}\times\mathbf{u})\cdot\mathbf{u} = \mathbf{0}\cdot\mathbf{u} = 0$

(h) true, $\mathbf{u}\times\mathbf{v} \perp \mathbf{u}$ and $\mathbf{u}\times\mathbf{v} \perp \mathbf{v} \Rightarrow (\mathbf{u}\times\mathbf{v})\cdot\mathbf{u} = \mathbf{v}\cdot(\mathbf{u}\times\mathbf{v}) = 0$

29. (a) $\text{proj}_{\mathbf{v}}\,\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}||\mathbf{v}|}\right)\mathbf{v}$ (b) $(\mathbf{u}\times\mathbf{v})$ (c) $((\mathbf{u}\times\mathbf{v})\times\mathbf{w})$ (d) $|(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}|$

(e) $(\mathbf{u}\times\mathbf{v})\times(\mathbf{u}\times\mathbf{w})$ (f) $|\mathbf{u}|\,\frac{\mathbf{v}}{|\mathbf{v}|}$

30. $(\mathbf{i}\times\mathbf{j})\times\mathbf{j} = \mathbf{k}\times\mathbf{j} = -\mathbf{i}$; $\mathbf{i}\times(\mathbf{j}\times\mathbf{j}) = \mathbf{i}\times\mathbf{0} = \mathbf{0}$. The cross product is not associative.

31. (a) yes, $\mathbf{u}\times\mathbf{v}$ and **w** are both vectors (b) no, **u** is a vector but $\mathbf{v}\cdot\mathbf{w}$ is a scalar

(c) yes, **u** and $\mathbf{u}\times\mathbf{w}$ are both vectors (d) no, **u** is a vector but $\mathbf{v}\cdot\mathbf{w}$ is a scalar

32. $(\mathbf{u}\times\mathbf{v})\times\mathbf{w}$ is perpendicular to $\mathbf{u}\times\mathbf{v}$, and $\mathbf{u}\times\mathbf{v}$ is perpendicular to both **u** and **v** $\Rightarrow$ $(\mathbf{u}\times\mathbf{v})\times\mathbf{w}$ is parallel to a vector in the plane of **u** and **v** which means it lies in the plane determined by **u** and **v**. The situation is degenerate if **u** and **v** are parallel so $\mathbf{u}\times\mathbf{v} = \mathbf{0}$ and the vectors do not determine a plane. Similar reasoning shows that $\mathbf{u}\times(\mathbf{v}\times\mathbf{w})$ lies in the plane of **v** and **w** provided **v** and **w** are nonparallel.

33. No, **v** need not equal **w**. For example, $\mathbf{i}+\mathbf{j} \neq -\mathbf{i}+\mathbf{j}$, but $\mathbf{i}\times(\mathbf{i}+\mathbf{j}) = \mathbf{i}\times\mathbf{i} + \mathbf{i}\times\mathbf{j} = \mathbf{0}+\mathbf{k} = \mathbf{k}$ and $\mathbf{i}\times(-\mathbf{i}+\mathbf{j}) = \mathbf{i}\times(-\mathbf{i}) + \mathbf{i}\times\mathbf{j} = \mathbf{0}+\mathbf{k} = \mathbf{k}$.

34. Yes. If $\mathbf{u}\times\mathbf{v} = \mathbf{u}\times\mathbf{w}$ and $\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{w}$, then $\mathbf{u}\times(\mathbf{v}-\mathbf{w}) = \mathbf{0}$ and $\mathbf{u}\cdot(\mathbf{v}-\mathbf{w}) = 0$. Suppose now that $\mathbf{v}\neq\mathbf{w}$. Then $\mathbf{u}\times(\mathbf{v}-\mathbf{w}) = \mathbf{0}$ implies that $\mathbf{v}-\mathbf{w} = k\mathbf{u}$ for some real number $k \neq 0$. This in turn implies that $\mathbf{u}\cdot(\mathbf{v}-\mathbf{w}) = \mathbf{u}\cdot(k\mathbf{u}) = k\,|\mathbf{u}|^2 = 0$, which implies that $\mathbf{u} = \mathbf{0}$. Since $\mathbf{u}\neq\mathbf{0}$, it cannot be true that $\mathbf{v}\neq\mathbf{w}$, so $\mathbf{v} = \mathbf{w}$.

35. $\overrightarrow{AB} = -\mathbf{i}+\mathbf{j}$ and $\overrightarrow{AD} = -\mathbf{i}-\mathbf{j} \Rightarrow \overrightarrow{AB}\times\overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 0 \\ -1 & -1 & 0 \end{vmatrix} = 2\mathbf{k} \Rightarrow \text{area} = \left|\overrightarrow{AB}\times\overrightarrow{AD}\right| = 2$

36. $\overrightarrow{AB} = 7\mathbf{i} + 3\mathbf{j}$ and $\overrightarrow{AD} = 2\mathbf{i} + 5\mathbf{j} \Rightarrow \overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 7 & 3 & 0 \\ 2 & 5 & 0 \end{vmatrix} = 29\mathbf{k} \Rightarrow \text{area} = \left|\overrightarrow{AB} \times \overrightarrow{AD}\right| = 29$

37. $\overrightarrow{AB} = 3\mathbf{i} - 2\mathbf{j}$ and $\overrightarrow{AD} = 5\mathbf{i} + \mathbf{j} \Rightarrow \overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 0 \\ 5 & 1 & 0 \end{vmatrix} = 13\mathbf{k} \Rightarrow \text{area} = \left|\overrightarrow{AB} \times \overrightarrow{AD}\right| = 13$

38. $\overrightarrow{AB} = 7\mathbf{i} - 4\mathbf{j}$ and $\overrightarrow{AD} = 2\mathbf{i} + 5\mathbf{j} \Rightarrow \overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 7 & -4 & 0 \\ 2 & 5 & 0 \end{vmatrix} = 43\mathbf{k} \Rightarrow \text{area} = \left|\overrightarrow{AB} \times \overrightarrow{AD}\right| = 43$

39. $\overrightarrow{AB} = 3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$ and $\overrightarrow{DC} = 3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k} \Rightarrow \overrightarrow{AB}$ is parallel to $\overrightarrow{DC}$; $\overrightarrow{BC} = 2\mathbf{i} - \mathbf{j}$ and $\overrightarrow{AD} = 2\mathbf{i} - \mathbf{j} \Rightarrow \overrightarrow{BC}$ is parallel to $\overrightarrow{AD}$. $\overrightarrow{AB} \times \overrightarrow{BC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 4 \\ 2 & -1 & 0 \end{vmatrix} = 4\mathbf{i} + 8\mathbf{j} - 7\mathbf{k} \Rightarrow \text{area} = |\overrightarrow{AB} \times \overrightarrow{BC}| = \sqrt{129}$

40. $\overrightarrow{AC} = \mathbf{i} + 4\mathbf{j}$ and $\overrightarrow{DB} = \mathbf{i} + 4\mathbf{j} \Rightarrow \overrightarrow{AC}$ is parallel to $\overrightarrow{DB}$; $\overrightarrow{AD} = -\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$ and $\overrightarrow{CB} = -\mathbf{i} + 3\mathbf{j} + 3\mathbf{k} \Rightarrow \overrightarrow{AD}$ is parallel to $\overrightarrow{CB}$. $\overrightarrow{AC} \times \overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & 0 \\ -1 & 3 & 3 \end{vmatrix} = 12\mathbf{i} - 3\mathbf{j} + 7\mathbf{k} \Rightarrow \text{area} = |\overrightarrow{AC} \times \overrightarrow{AD}| = \sqrt{202}$

41. $\overrightarrow{AB} = -2\mathbf{i} + 3\mathbf{j}$ and $\overrightarrow{AC} = 3\mathbf{i} + \mathbf{j} \Rightarrow \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 3 & 0 \\ 3 & 1 & 0 \end{vmatrix} = -11\mathbf{k} \Rightarrow \text{area} = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{11}{2}$

42. $\overrightarrow{AB} = 4\mathbf{i} + 4\mathbf{j}$ and $\overrightarrow{AC} = 3\mathbf{i} + 2\mathbf{j} \Rightarrow \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 4 & 0 \\ 3 & 2 & 0 \end{vmatrix} = -4\mathbf{k} \Rightarrow \text{area} = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = 2$

43. $\overrightarrow{AB} = 6\mathbf{i} - 5\mathbf{j}$ and $\overrightarrow{AC} = 11\mathbf{i} - 5\mathbf{j} \Rightarrow \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & -5 & 0 \\ 11 & -5 & 0 \end{vmatrix} = 25\mathbf{k} \Rightarrow \text{area} = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{25}{2}$

44. $\overrightarrow{AB} = 16\mathbf{i} - 5\mathbf{j}$ and $\overrightarrow{AC} = 4\mathbf{i} + 4\mathbf{j} \Rightarrow \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 16 & -5 & 0 \\ 4 & 4 & 0 \end{vmatrix} = 84\mathbf{k} \Rightarrow \text{area} = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = 42$

45. $\overrightarrow{AB} = -\mathbf{i} + 2\mathbf{j}$ and $\overrightarrow{AC} = -\mathbf{i} - \mathbf{k} \Rightarrow \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 0 \\ -1 & 0 & -1 \end{vmatrix} = -2\mathbf{i} - \mathbf{j} + 2\mathbf{k} \Rightarrow \text{area} = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{3}{2}$

46. $\overrightarrow{AB} = -\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\overrightarrow{AC} = 3\mathbf{i} + 3\mathbf{k} \Rightarrow \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & -1 \\ 3 & 0 & 3 \end{vmatrix} = 3\mathbf{i} - 3\mathbf{k} \Rightarrow \text{area} = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{3\sqrt{2}}{2}$

47. $\overrightarrow{AB} = -\mathbf{i} + 2\mathbf{j}$ and $\overrightarrow{AC} = \mathbf{j} - 2\mathbf{k} \Rightarrow \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 0 \\ 0 & 1 & -2 \end{vmatrix} = -4\mathbf{i} - 2\mathbf{j} - \mathbf{k} \Rightarrow \text{area} = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{\sqrt{21}}{2}$

48. $\overrightarrow{AB} = \mathbf{i} + 2\mathbf{j}$, $\overrightarrow{AC} = -3\mathbf{j} + 2\mathbf{k}$ and $\overrightarrow{AD} = 3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k} \Rightarrow \left(\overrightarrow{AB} \times \overrightarrow{AC}\right) \cdot \overrightarrow{AD} = \begin{vmatrix} 1 & 2 & 0 \\ 0 & -3 & 2 \\ 3 & -4 & 5 \end{vmatrix} = 5$

$\Rightarrow$ volume $= \left|\left(\overrightarrow{AB} \times \overrightarrow{AC}\right) \cdot \overrightarrow{AD}\right| = 5$

49. If $\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{B} = b_1\mathbf{i} + b_2\mathbf{j}$, then $\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & 0 \\ b_1 & b_2 & 0 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$ and the triangle's area is $\frac{1}{2}|\mathbf{A} \times \mathbf{B}| = \pm\frac{1}{2}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$. The applicable sign is $(+)$ if the acute angle from $\mathbf{A}$ to $\mathbf{B}$ runs counterclockwise in the xy-plane, and $(-)$ if it runs clockwise, because the area must be a nonnegative number.

50. If $\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j}$, $\mathbf{B} = b_1\mathbf{i} + b_2\mathbf{j}$, and $\mathbf{C} = c_1\mathbf{i} + c_2\mathbf{j}$, then the area of the triangle is $\frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$. Now,

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 - a_1 & b_2 - a_2 & 0 \\ c_1 - a_1 & c_2 - a_2 & 0 \end{vmatrix} = \begin{vmatrix} b_1 - a_1 & b_2 - a_2 \\ c_1 - a_1 & c_2 - a_2 \end{vmatrix} \mathbf{k} \Rightarrow \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$$
$$= \frac{1}{2}|(b_1 - a_1)(c_2 - a_2) - (c_1 - a_1)(b_2 - a_2)| = \frac{1}{2}|a_1(b_2 - c_2) + a_2(c_1 - b_1) + (b_1c_2 - c_1b_2)|$$
$$= \pm\frac{1}{2}\begin{vmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{vmatrix}.$$ The applicable sign ensures the area formula gives a nonnegative number.

12.5 LINES AND PLANES IN SPACE

1. The direction $\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $P(3, -4, -1) \Rightarrow x = 3 + t, y = -4 + t, z = -1 + t$

2. The direction $\overrightarrow{PQ} = -2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ and $P(1, 2, -1) \Rightarrow x = 1 - 2t, y = 2 - 2t, z = -1 + 2t$

3. The direction $\overrightarrow{PQ} = 5\mathbf{i} + 5\mathbf{j} - 5\mathbf{k}$ and $P(-2, 0, 3) \Rightarrow x = -2 + 5t, y = 5t, z = 3 - 5t$

4. The direction $\overrightarrow{PQ} = -\mathbf{j} - \mathbf{k}$ and $P(1, 2, 0) \Rightarrow x = 1, y = 2 - t, z = -t$

5. The direction $2\mathbf{j} + \mathbf{k}$ and $P(0, 0, 0) \Rightarrow x = 0, y = 2t, z = t$

6. The direction $2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $P(3, -2, 1) \Rightarrow x = 3 + 2t, y = -2 - t, z = 1 + 3t$

7. The direction $\mathbf{k}$ and $P(1, 1, 1) \Rightarrow x = 1, y = 1, z = 1 + t$

8. The direction $3\mathbf{i} + 7\mathbf{j} - 5\mathbf{k}$ and $P(2, 4, 5) \Rightarrow x = 2 + 3t, y = 4 + 7t, z = 5 - 5t$

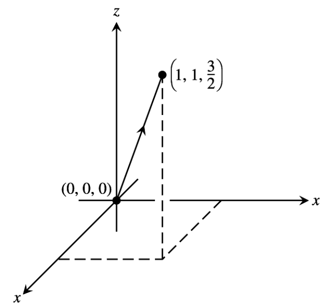
9. The direction $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $P(0, -7, 0) \Rightarrow x = t, y = -7 + 2t, z = 2t$

10. The direction is $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 3 & 4 & 5 \end{vmatrix} = -2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ and $P(2, 3, 0) \Rightarrow x = 2 - 2t, y = 3 + 4t, z = -2t$

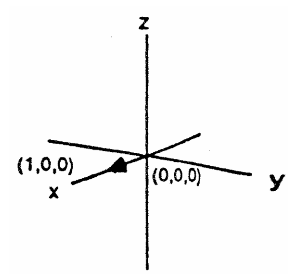
11. The direction $\mathbf{i}$ and $P(0, 0, 0) \Rightarrow x = t, y = 0, z = 0$

12. The direction $\mathbf{k}$ and $P(0,0,0) \Rightarrow x = 0, y = 0, z = t$

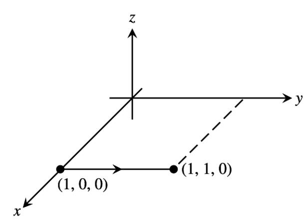
13. The direction $\overrightarrow{PQ} = \mathbf{i} + \mathbf{j} + \frac{3}{2}\mathbf{k}$ and $P(0,0,0) \Rightarrow x = t,$ $y = t, z = \frac{3}{2}t$, where $0 \le t \le 1$



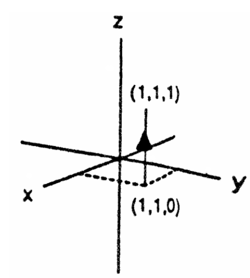
14. The direction $\overrightarrow{PQ} = \mathbf{i}$ and $P(0,0,0) \Rightarrow x = t, y = 0, z = 0,$ where $0 \le t \le 1$



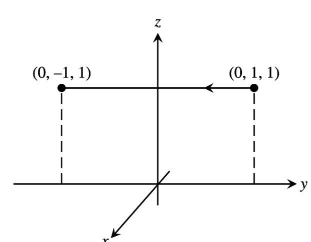
15. The direction $\overrightarrow{PQ} = \mathbf{j}$ and $P(1,1,0) \Rightarrow x = 1, y = 1 + t,$ $z = 0$, where $-1 \le t \le 0$



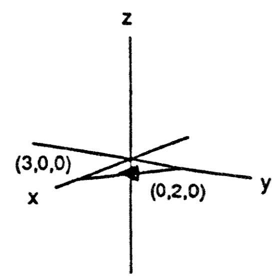
16. The direction $\overrightarrow{PQ} = \mathbf{k}$ and $P(1,1,0) \Rightarrow x = 1, y = 1, z = t,$ where $0 \le t \le 1$



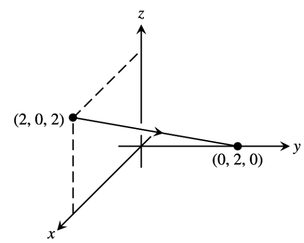
17. The direction $\overrightarrow{PQ} = -2\mathbf{j}$ and $P(0,1,1) \Rightarrow x = 0,$ $y = 1 - 2t, z = 1$, where $0 \le t \le 1$



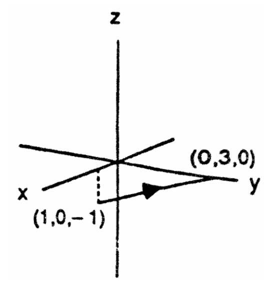
18. The direction $\overrightarrow{PQ} = 3\mathbf{i} - 2\mathbf{j}$ and $P(0,2,0) \Rightarrow x = 3t$, $y = 2 - 2t$, $z = 0$, where $0 \le t \le 1$



19. The direction $\overrightarrow{PQ} = -2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ and $P(2,0,2)$ $\Rightarrow x = 2 - 2t$, $y = 2t$, $z = 2 - 2t$, where $0 \le t \le 1$



20. The direction $\overrightarrow{PQ} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and $P(1,0,-1)$ $\Rightarrow x = 1 - t$, $y = 3t$, $z = -1 + t$, where $0 \le t \le 1$



21. $3(x-0) + (-2)(y-2) + (-1)(z+1) = 0 \Rightarrow 3x - 2y - z = -3$

22. $3(x-1) + (1)(y+1) + (1)(z-3) = 0 \Rightarrow 3x + y + z = 5$

23. $\overrightarrow{PQ} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$, $\overrightarrow{PS} = -\mathbf{i} - 3\mathbf{j} + 2\mathbf{k} \Rightarrow \overrightarrow{PQ} \times \overrightarrow{PS} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 3 \\ -1 & -3 & 2 \end{vmatrix} = 7\mathbf{i} - 5\mathbf{j} - 4\mathbf{k}$ is normal to the plane

$\Rightarrow 7(x-2) + (-5)(y-0) + (-4)(z-2) = 0 \Rightarrow 7x - 5y - 4z = 6$

24. $\overrightarrow{PQ} = -\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, $\overrightarrow{PS} = -3\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} \Rightarrow \overrightarrow{PQ} \times \overrightarrow{PS} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 2 \\ -3 & 2 & 3 \end{vmatrix} = -\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ is normal to the plane

$\Rightarrow (-1)(x-1) + (-3)(y-5) + (1)(z-7) = 0 \Rightarrow x + 3y - z = 9$

25. $\mathbf{n} = \mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$, $P(2,4,5) \Rightarrow (1)(x-2) + (3)(y-4) + (4)(z-5) = 0 \Rightarrow x + 3y + 4z = 34$

26. $\mathbf{n} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $P(1,-2,1) \Rightarrow (1)(x-1) + (-2)(y+2) + (1)(z-1) = 0 \Rightarrow x - 2y + z = 6$

27. $\begin{cases} x = 2t + 1 = s + 2 \\ y = 3t + 2 = 2s + 4 \end{cases} \Rightarrow \begin{cases} 2t - s = 1 \\ 3t - 2s = 2 \end{cases} \Rightarrow \begin{cases} 4t - 2s = 2 \\ 3t - 2s = 2 \end{cases} \Rightarrow t = 0$ and $s = -1$; then $z = 4t + 3 = -4s - 1$

$\Rightarrow 4(0) + 3 = (-4)(-1) - 1$ is satisfied $\Rightarrow$ the lines intersect when $t = 0$ and $s = -1 \Rightarrow$ the point of intersection is $x = 1$, $y = 2$, and $z = 3$ or $P(1,2,3)$. A vector normal to the plane determined by these lines is

$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 4 \\ 1 & 2 & -4 \end{vmatrix} = -20\mathbf{i} + 12\mathbf{j} + \mathbf{k}$, where $\mathbf{n}_1$ and $\mathbf{n}_2$ are directions of the lines $\Rightarrow$ the plane containing the lines is represented by$(-20)(x-1) + (12)(y-2) + (1)(z-3) = 0 \Rightarrow -20x + 12y + z = 7$.

28. $\begin{cases} x = t = 2s + 2 \\ y = -t + 2 = s + 3 \end{cases} \Rightarrow \begin{cases} t - 2s = 2 \\ -t - s = 1 \end{cases} \Rightarrow s = -1$ and $t = 0$; then $z = t + 1 = 5s + 6 \Rightarrow 0 + 1 = 5(-1) + 6$ is satisfied $\Rightarrow$ the lines do intersect when $s = -1$ and $t = 0 \Rightarrow$ the point of intersection is $x = 0$, $y = 2$ and $z = 1$ or $P(0, 2, 1)$. A vector normal to the plane determined by these lines is $\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 2 & 1 & 5 \end{vmatrix} = -6\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$, where $\mathbf{n}_1$ and $\mathbf{n}_2$ are directions of the lines $\Rightarrow$ the plane containing the lines is represented by $(-6)(x-0) + (-3)(y-2) + (3)(z-1) = 0 \Rightarrow 6x + 3y - 3z = 3$.

29. The cross product of $\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $-4\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ has the same direction as the normal to the plane $\Rightarrow \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -1 \\ -4 & 2 & -2 \end{vmatrix} = 6\mathbf{j} + 6\mathbf{k}$. Select a point on either line, such as $P(-1, 2, 1)$. Since the lines are given to intersect, the desired plane is $0(x+1) + 6(y-2) + 6(z-1) = 0 \Rightarrow 6y + 6z = 18 \Rightarrow y + z = 3$.

30. The cross product of $\mathbf{i} - 3\mathbf{j} - \mathbf{k}$ and $\mathbf{i} + \mathbf{j} + \mathbf{k}$ has the same direction as the normal to the plane $\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -3 & -1 \\ 1 & 1 & 1 \end{vmatrix} = -2\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$. Select a point on either line, such as $P(0, 3, -2)$. Since the lines are given to intersect, the desired plane is $(-2)(x-0) + (-2)(y-3) + (4)(z+2) = 0 \Rightarrow -2x - 2y + 4z = -14 \Rightarrow x + y - 2z = 7$.

31. $\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 3\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$ is a vector in the direction of the line of intersection of the planes $\Rightarrow 3(x-2) + (-3)(y-1) + 3(z+1) = 0 \Rightarrow 3x - 3y + 3z = 0 \Rightarrow x - y + z = 0$ is the desired plane containing $P_0(2, 1, -1)$

32. A vector normal to the desired plane is $\overrightarrow{P_1P_2} \times \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -2 \\ 4 & -1 & 2 \end{vmatrix} = -2\mathbf{i} - 12\mathbf{j} - 2\mathbf{k}$; choosing $P_1(1, 2, 3)$ as a point on the plane $\Rightarrow (-2)(x-1) + (-12)(y-2) + (-2)(z-3) = 0 \Rightarrow -2x - 12y - 2z = -32 \Rightarrow x + 6y + z = 16$ is the desired plane

33. $S(0, 0, 12)$, $P(0, 0, 0)$ and $\mathbf{v} = 4\mathbf{i} - 2\mathbf{j} + 2\mathbf{k} \Rightarrow \overrightarrow{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 12 \\ 4 & -2 & 2 \end{vmatrix} = 24\mathbf{i} + 48\mathbf{j} = 24(\mathbf{i} + 2\mathbf{j})$ $\Rightarrow d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{24\sqrt{1+4}}{\sqrt{16+4+4}} = \frac{24\sqrt{5}}{\sqrt{24}} = \sqrt{5 \cdot 24} = 2\sqrt{30}$ is the distance from S to the line

34. $S(0, 0, 0)$, $P(5, 5, -3)$ and $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k} \Rightarrow \overrightarrow{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -5 & -5 & 3 \\ 3 & 4 & -5 \end{vmatrix} = 13\mathbf{i} - 16\mathbf{j} - 5\mathbf{k}$ $\Rightarrow d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{169+256+25}}{\sqrt{9+16+25}} = \frac{\sqrt{450}}{\sqrt{50}} = \sqrt{9} = 3$ is the distance from S to the line

35. S(2, 1, 3), P(2, 1, 3) and $\mathbf{v} = 2\mathbf{i} + 6\mathbf{j} \Rightarrow \overrightarrow{PS} \times \mathbf{v} = \mathbf{0} \Rightarrow d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{0}{\sqrt{40}} = 0$ is the distance from S to the line (i.e., the point S lies on the line)

36. S(2, 1, −1), P(0, 1, 0) and $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \Rightarrow \overrightarrow{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 2 & 2 & 2 \end{vmatrix} = 2\mathbf{i} - 6\mathbf{j} + 4\mathbf{k}$

$\Rightarrow d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{4+36+16}}{\sqrt{4+4+4}} = \frac{\sqrt{56}}{\sqrt{12}} = \sqrt{\frac{14}{3}}$ is the distance from S to the line

37. S(3, −1, 4), P(4, 3, −5) and $\mathbf{v} = -\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} \Rightarrow \overrightarrow{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -4 & 9 \\ -1 & 2 & 3 \end{vmatrix} = -30\mathbf{i} - 6\mathbf{j} - 6\mathbf{k}$

$\Rightarrow d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{900+36+36}}{\sqrt{1+4+9}} = \frac{\sqrt{972}}{\sqrt{14}} = \frac{\sqrt{486}}{\sqrt{7}} = \frac{\sqrt{81 \cdot 6}}{\sqrt{7}} = \frac{9\sqrt{42}}{7}$ is the distance from S to the line

38. S(−1, 4, 3), P(10, −3, 0) and $\mathbf{v} = 4\mathbf{i} + 4\mathbf{k} \Rightarrow \overrightarrow{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -11 & 7 & 3 \\ 4 & 0 & 4 \end{vmatrix} = 28\mathbf{i} + 56\mathbf{j} - 28\mathbf{k} = 28(\mathbf{i} + 2\mathbf{j} - \mathbf{k})$

$\Rightarrow d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{28\sqrt{1+4+1}}{4\sqrt{1+1}} = 7\sqrt{3}$ is the distance from S to the line

39. S(2, −3, 4), x + 2y + 2z = 13 and P(13, 0, 0) is on the plane $\Rightarrow \overrightarrow{PS} = -11\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$ and $\mathbf{n} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$
$\Rightarrow d = \left|\overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|}\right| = \left|\frac{-11-6+8}{\sqrt{1+4+4}}\right| = \left|\frac{-9}{\sqrt{9}}\right| = 3$

40. S(0, 0, 0), 3x + 2y + 6z = 6 and P(2, 0, 0) is on the plane $\Rightarrow \overrightarrow{PS} = -2\mathbf{i}$ and $\mathbf{n} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$
$\Rightarrow d = \left|\overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|}\right| = \left|\frac{-6}{\sqrt{9+4+36}}\right| = \frac{6}{\sqrt{49}} = \frac{6}{7}$

41. S(0, 1, 1), 4y + 3z = −12 and P(0, −3, 0) is on the plane $\Rightarrow \overrightarrow{PS} = 4\mathbf{j} + \mathbf{k}$ and $\mathbf{n} = 4\mathbf{j} + 3\mathbf{k}$
$\Rightarrow d = \left|\overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|}\right| = \left|\frac{16+3}{\sqrt{16+9}}\right| = \frac{19}{5}$

42. S(2, 2, 3), 2x + y + 2z = 4 and P(2, 0, 0) is on the plane $\Rightarrow \overrightarrow{PS} = 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{n} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$
$\Rightarrow d = \left|\overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|}\right| = \left|\frac{2+6}{\sqrt{4+1+4}}\right| = \frac{8}{3}$

43. S(0, −1, 0), 2x + y + 2z = 4 and P(2, 0, 0) is on the plane $\Rightarrow \overrightarrow{PS} = -2\mathbf{i} - \mathbf{j}$ and $\mathbf{n} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$
$\Rightarrow d = \left|\overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|}\right| = \left|\frac{-4-1+0}{\sqrt{4+1+4}}\right| = \frac{5}{3}$

44. S(1, 0, −1), −4x + y + z = 4 and P(−1, 0, 0) is on the plane $\Rightarrow \overrightarrow{PS} = 2\mathbf{i} - \mathbf{k}$ and $\mathbf{n} = -4\mathbf{i} + \mathbf{j} + \mathbf{k}$
$\Rightarrow d = \left|\overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|}\right| = \left|\frac{-8-1}{\sqrt{16+1+1}}\right| = \frac{9}{\sqrt{18}} = \frac{3\sqrt{2}}{2}$

45. The point P(1, 0, 0) is on the first plane and S(10, 0, 0) is a point on the second plane $\Rightarrow \overrightarrow{PS} = 9\mathbf{i}$, and $\mathbf{n} = \mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$ is normal to the first plane $\Rightarrow$ the distance from S to the first plane is $d = \left|\overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|}\right| = \left|\frac{9}{\sqrt{1+4+36}}\right| = \frac{9}{\sqrt{41}}$, which is also the distance between the planes.

46. The line is parallel to the plane since $\mathbf{v} \cdot \mathbf{n} = \left(\mathbf{i} + \mathbf{j} - \frac{1}{2}\mathbf{k}\right) \cdot (\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}) = 1 + 2 - 3 = 0$. Also the point $S(1, 0, 0)$ when $t = -1$ lies on the line, and the point $P(10, 0, 0)$ lies on the plane $\Rightarrow \overrightarrow{PS} = -9\mathbf{i}$. The distance from S to the plane is $d = \left|\overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|}\right| = \left|\frac{-9}{\sqrt{1+4+36}}\right| = \frac{9}{\sqrt{41}}$, which is also the distance from the line to the plane.

47. $\mathbf{n}_1 = \mathbf{i} + \mathbf{j}$ and $\mathbf{n}_2 = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{2+1}{\sqrt{2}\sqrt{9}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

48. $\mathbf{n}_1 = 5\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{n}_2 = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{5-2-3}{\sqrt{27}\sqrt{14}}\right) = \cos^{-1}(0) = \frac{\pi}{2}$

49. $\mathbf{n}_1 = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\mathbf{n}_2 = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{4-4-2}{\sqrt{12}\sqrt{9}}\right) = \cos^{-1}\left(\frac{-1}{3\sqrt{3}}\right) \approx 1.76$ rad

50. $\mathbf{n}_1 = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{n}_2 = \mathbf{k} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{1}{\sqrt{3}\sqrt{1}}\right) \approx 0.96$ rad

51. $\mathbf{n}_1 = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{n}_2 = \mathbf{i} + 2\mathbf{j} + \mathbf{k} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{2+4-1}{\sqrt{9}\sqrt{6}}\right) = \cos^{-1}\left(\frac{5}{3\sqrt{6}}\right) \approx 0.82$ rad

52. $\mathbf{n}_1 = 4\mathbf{j} + 3\mathbf{k}$ and $\mathbf{n}_2 = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{8+18}{\sqrt{25}\sqrt{49}}\right) = \cos^{-1}\left(\frac{26}{35}\right) \approx 0.73$ rad

53. $2x - y + 3z = 6 \Rightarrow 2(1-t) - (3t) + 3(1+t) = 6 \Rightarrow -2t + 5 = 6 \Rightarrow t = -\frac{1}{2} \Rightarrow x = \frac{3}{2}, y = -\frac{3}{2}$ and $z = \frac{1}{2}$
$\Rightarrow \left(\frac{3}{2}, -\frac{3}{2}, \frac{1}{2}\right)$ is the point

54. $6x + 3y - 4z = -12 \Rightarrow 6(2) + 3(3+2t) - 4(-2-2t) = -12 \Rightarrow 14t + 29 = -12 \Rightarrow t = -\frac{41}{14} \Rightarrow x = 2, y = 3 - \frac{41}{7}$, and $z = -2 + \frac{41}{7} \Rightarrow \left(2, -\frac{20}{7}, \frac{27}{7}\right)$ is the point

55. $x + y + z = 2 \Rightarrow (1+2t) + (1+5t) + (3t) = 2 \Rightarrow 10t + 2 = 2 \Rightarrow t = 0 \Rightarrow x = 1, y = 1$ and $z = 0$
$\Rightarrow (1, 1, 0)$ is the point

56. $2x - 3z = 7 \Rightarrow 2(-1+3t) - 3(5t) = 7 \Rightarrow -9t - 2 = 7 \Rightarrow t = -1 \Rightarrow x = -1 - 3, y = -2$ and $z = -5$
$\Rightarrow (-4, -2, -5)$ is the point

57. $\mathbf{n}_1 = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{n}_2 = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -\mathbf{i} + \mathbf{j}$, the direction of the desired line; $(1, 1, -1)$ is on both planes $\Rightarrow$ the desired line is $x = 1 - t, y = 1 + t, z = -1$

58. $\mathbf{n}_1 = 3\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}$ and $\mathbf{n}_2 = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k} \Rightarrow \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix} = 14\mathbf{i} + 2\mathbf{j} + 15\mathbf{k}$, the direction of the desired line; $(1, 0, 0)$ is on both planes $\Rightarrow$ the desired line is $x = 1 + 14t, y = 2t, z = 15t$

59. $\mathbf{n}_1 = \mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ and $\mathbf{n}_2 = \mathbf{i} + \mathbf{j} - 2\mathbf{k} \Rightarrow \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 4 \\ 1 & 1 & -2 \end{vmatrix} = 6\mathbf{j} + 3\mathbf{k}$, the direction of the desired line; $(4, 3, 1)$ is on both planes $\Rightarrow$ the desired line is $x = 4, y = 3 + 6t, z = 1 + 3t$

60. $\mathbf{n}_1 = 5\mathbf{i} - 2\mathbf{j}$ and $\mathbf{n}_2 = 4\mathbf{j} - 5\mathbf{k} \Rightarrow \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & -2 & 0 \\ 0 & 4 & -5 \end{vmatrix} = 10\mathbf{i} + 25\mathbf{j} + 20\mathbf{k}$, the direction of the desired line; $(1, -3, 1)$ is on both planes $\Rightarrow$ the desired line is $x = 1 + 10t$, $y = -3 + 25t$, $z = 1 + 20t$

61. L1 & L2: $x = 3 + 2t = 1 + 4s$ and $y = -1 + 4t = 1 + 2s \Rightarrow \begin{cases} 2t - 4s = -2 \\ 4t - 2s = 2 \end{cases} \Rightarrow \begin{cases} 2t - 4s = -2 \\ 2t - s = 1 \end{cases}$

$\Rightarrow -3s = -3 \Rightarrow s = 1$ and $t = 1 \Rightarrow$ on L1, $z = 1$ and on L2, $z = 1 \Rightarrow$ L1 and L2 intersect at $(5, 3, 1)$.

L2 & L3: The direction of L2 is $\frac{1}{6}(4\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}) = \frac{1}{3}(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$ which is the same as the direction $\frac{1}{3}(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$ of L3; hence L2 and L3 are parallel.

L1 & L3: $x = 3 + 2t = 3 + 2r$ and $y = -1 + 4t = 2 + r \Rightarrow \begin{cases} 2t - 2r = 0 \\ 4t - r = 3 \end{cases} \Rightarrow \begin{cases} t - r = 0 \\ 4t - r = 3 \end{cases} \Rightarrow 3t = 3$

$\Rightarrow t = 1$ and $r = 1 \Rightarrow$ on L1, $z = 2$ while on L3, $z = 0 \Rightarrow$ L1 and L2 do not intersect. The direction of L1 is $\frac{1}{\sqrt{21}}(2\mathbf{i} + 4\mathbf{j} - \mathbf{k})$ while the direction of L3 is $\frac{1}{3}(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$ and neither is a multiple of the other; hence L1 and L3 are skew.

62. L1 & L2: $x = 1 + 2t = 2 - s$ and $y = -1 - t = 3s \Rightarrow \begin{cases} 2t + s = 1 \\ -t - 3s = 1 \end{cases} \Rightarrow -5s = 3 \Rightarrow s = -\frac{3}{5}$ and $t = \frac{4}{5} \Rightarrow$ on L1, $z = \frac{12}{5}$ while on L2, $z = 1 - \frac{3}{5} = \frac{2}{5} \Rightarrow$ L1 and L2 do not intersect. The direction of L1 is $\frac{1}{\sqrt{14}}(2\mathbf{i} - \mathbf{j} + 3\mathbf{k})$ while the direction of L2 is $\frac{1}{\sqrt{11}}(-\mathbf{i} + 3\mathbf{j} + \mathbf{k})$ and neither is a multiple of the other; hence, L1 and L2 are skew.

L2 & L3: $x = 2 - s = 5 + 2r$ and $y = 3s = 1 - r \Rightarrow \begin{cases} -s - 2r = 3 \\ 3s + r = 1 \end{cases} \Rightarrow 5s = 5 \Rightarrow s = 1$ and $r = -2 \Rightarrow$ on L2, $z = 2$ and on L3, $z = 2 \Rightarrow$ L2 and L3 intersect at $(1, 3, 2)$.

L1 & L3: L1 and L3 have the same direction $\frac{1}{\sqrt{14}}(2\mathbf{i} - \mathbf{j} + 3\mathbf{k})$; hence L1 and L3 are parallel.

63. $x = 2 + 2t$, $y = -4 - t$, $z = 7 + 3t$; $x = -2 - t$, $y = -2 + \frac{1}{2}t$, $z = 1 - \frac{3}{2}t$

64. $1(x - 4) - 2(y - 1) + 1(z - 5) = 0 \Rightarrow x - 4 - 2y + 2 + z - 5 = 0 \Rightarrow x - 2y + z = 7$;
$-\sqrt{2}(x - 3) + 2\sqrt{2}(y + 2) - \sqrt{2}(z - 0) = 0 \Rightarrow -\sqrt{2}x + 2\sqrt{2}y - \sqrt{2}z = -7\sqrt{2}$

65. $x = 0 \Rightarrow t = -\frac{1}{2}, y = -\frac{1}{2}, z = -\frac{3}{2} \Rightarrow \left(0, -\frac{1}{2}, -\frac{3}{2}\right)$; $y = 0 \Rightarrow t = -1, x = -1, z = -3 \Rightarrow (-1, 0, -3)$; $z = 0 \Rightarrow t = 0, x = 1, y = -1 \Rightarrow (1, -1, 0)$

66. The line contains $(0, 0, 3)$ and $\left(\sqrt{3}, 1, 3\right)$ because the projection of the line onto the xy-plane contains the origin and intersects the positive x-axis at a 30° angle. The direction of the line is $\sqrt{3}\mathbf{i} + \mathbf{j} + 0\mathbf{k} \Rightarrow$ the line in question is $x = \sqrt{3}t$, $y = t$, $z = 3$.

67. With substitution of the line into the plane we have $2(1 - 2t) + (2 + 5t) - (-3t) = 8 \Rightarrow 2 - 4t + 2 + 5t + 3t = 8 \Rightarrow 4t + 4 = 8 \Rightarrow t = 1 \Rightarrow$ the point $(-1, 7, -3)$ is contained in both the line and plane, so they are not parallel.

68. The planes are parallel when either vector $A_1\mathbf{i} + B_1\mathbf{j} + C_1\mathbf{k}$ or $A_2\mathbf{i} + B_2\mathbf{j} + C_2\mathbf{k}$ is a multiple of the other or when $(A_1\mathbf{i} + B_1\mathbf{j} + C_1\mathbf{k}) \times (A_2\mathbf{i} + B_2\mathbf{j} + C_2\mathbf{k}) = \mathbf{0}$. The planes are perpendicular when their normals are perpendicular, or$(A_1\mathbf{i} + B_1\mathbf{j} + C_1\mathbf{k}) \cdot (A_2\mathbf{i} + B_2\mathbf{j} + C_2\mathbf{k}) = 0$.

69. There are many possible answers. One is found as follows: eliminate t to get $t = x - 1 = 2 - y = \frac{z-3}{2}$ $\Rightarrow$ $x - 1 = 2 - y$ and $2 - y = \frac{z-3}{2}$ $\Rightarrow$ $x + y = 3$ and $2y + z = 7$ are two such planes.

70. Since the plane passes through the origin, its general equation is of the form $Ax + By + Cz = 0$. Since it meets the plane M at a right angle, their normal vectors are perpendicular $\Rightarrow$ $2A + 3B + C = 0$. One choice satisfying this equation is $A = 1$, $B = -1$ and $C = 1$ $\Rightarrow$ $x - y + z = 0$. Any plane $Ax + By + Cz = 0$ with $2A + 3B + C = 0$ will pass through the origin and be perpendicular to M.

71. The points $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ are the x, y, and z intercepts of the plane. Since a, b, and c are all nonzero, the plane must intersect all three coordinate axes and cannot pass through the origin. Thus, $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ describes all planes <u>except</u> those through the origin or parallel to a coordinate axis.

72. Yes. If $\mathbf{v}_1$ and $\mathbf{v}_2$ are nonzero vectors parallel to the lines, then $\mathbf{v}_1 \times \mathbf{v}_2 \neq \mathbf{0}$ is perpendicular to the lines.

73. (a) $\overrightarrow{EP} = c\overrightarrow{EP_1}$ $\Rightarrow$ $-x_0\mathbf{i} + y\mathbf{j} + z\mathbf{k} = c\left[(x_1 - x_0)\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}\right]$ $\Rightarrow$ $-x_0 = c(x_1 - x_0)$, $y = cy_1$ and $z = cz_1$, where c is a positive real number

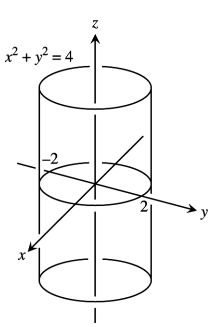
(b) At $x_1 = 0$ $\Rightarrow$ $c = 1$ $\Rightarrow$ $y = y_1$ and $z = z_1$; at $x_1 = x_0$ $\Rightarrow$ $x_0 = 0, y = 0, z = 0$; $\lim\limits_{x_0 \to \infty} c = \lim\limits_{x_0 \to \infty} \frac{-x_0}{x_1 - x_0}$ $= \lim\limits_{x_0 \to \infty} \frac{-1}{-1} = 1$ $\Rightarrow$ $c \to 1$ so that $y \to y_1$ and $z \to z_1$

74. The plane which contains the triangular plane is $x + y + z = 2$. The line containing the endpoints of the line segment is $x = 1 - t$, $y = 2t$, $z = 2t$. The plane and the line intersect at $\left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$. The visible section of the line segment is $\sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2} = 1$ unit in length. The length of the line segment is $\sqrt{1^2 + 2^2 + 2^2} = 3$ $\Rightarrow$ $\frac{2}{3}$ of the line segment is hidden from view.

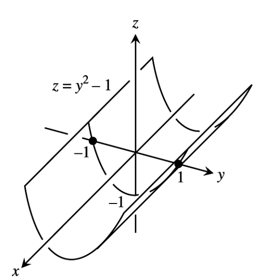
12.6 CYLINDERS AND QUADRIC SURFACES

1. d, ellipsoid
2. i, hyperboloid
3. a, cylinder
4. g, cone
5. l, hyperbolic paraboloid
6. e, paraboloid
7. b, cylinder
8. j, hyperboloid
9. k, hyperbolic paraboloid
10. f, paraboloid
11. h, cone
12. c, ellipsoid

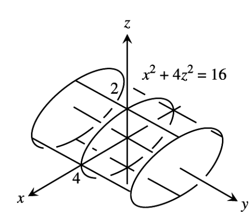
13. $x^2 + y^2 = 4$



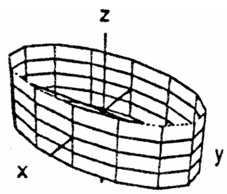
14. $z = y^2 - 1$



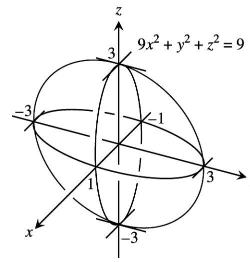
15. $x^2 + 4z^2 = 16$



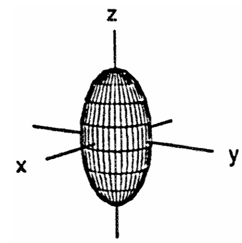
16. $4x^2 + y^2 = 36$



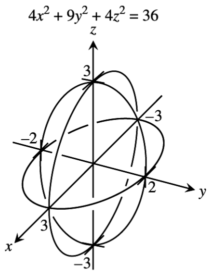
17. $9x^2 + y^2 + z^2 = 9$



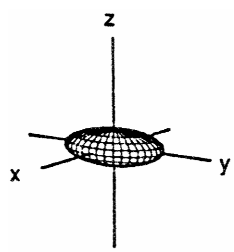
18. $4x^2 + 4y^2 + z^2 = 16$



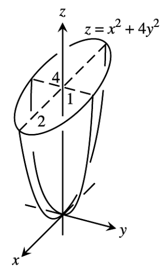
19. $4x^2 + 9y^2 + 4z^2 = 36$



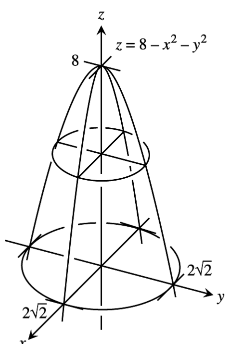
20. $9x^2 + 4y^2 + 36z^2 = 36$



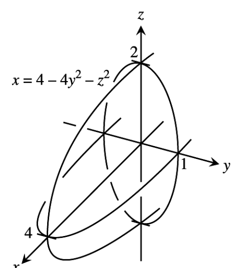
21. $x^2 + 4y^2 = z$



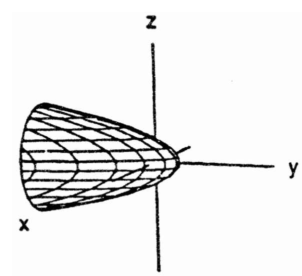
22. $z = 8 - x^2 - y^2$



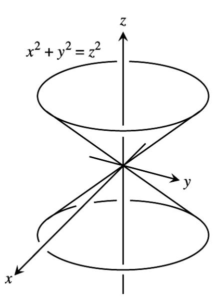
23. $x = 4 - 4y^2 - z^2$



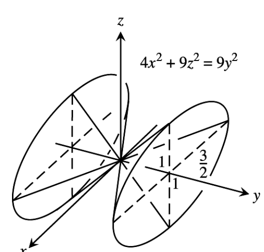
24. $y = 1 - x^2 - z^2$



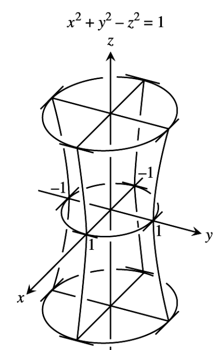
25. $x^2 + y^2 = z^2$



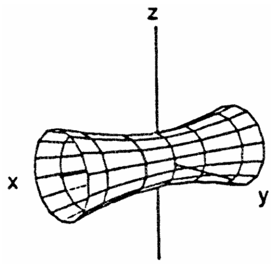
26. $4x^2 + 9z^2 = 9y^2$



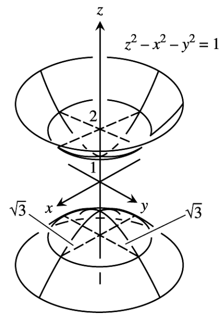
27. $x^2 + y^2 - z^2 = 1$



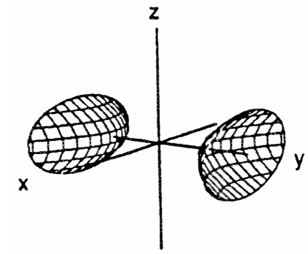
28. $y^2 + z^2 - x^2 = 1$



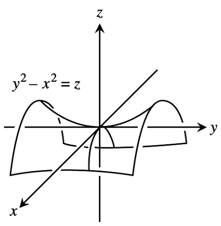
29. $z^2 - x^2 - y^2 = 1$



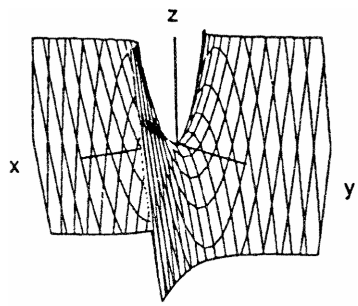
30. $\frac{y^2}{4} - \frac{x^2}{4} - z^2 = 1$



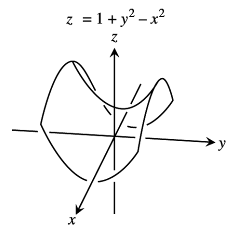
31. $y^2 - x^2 = z$



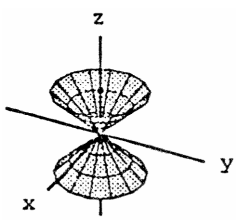
32. $x^2 - y^2 = z$



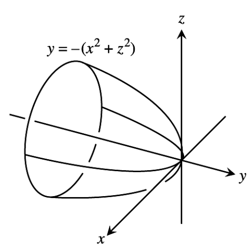
33. $z = 1 + y^2 - x^2$



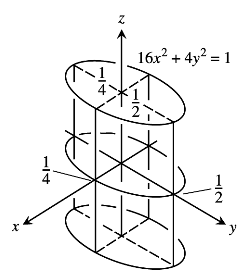
34. $4x^2 + 4y^2 = z^2$



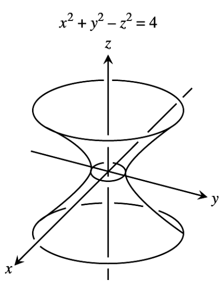
35. $y = -(x^2 + z^2)$



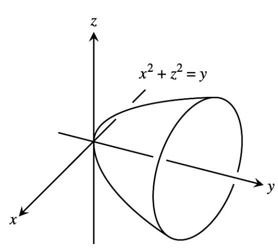
36. $16x^2 + 4y^2 = 1$



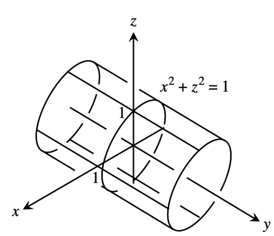
37. $x^2 + y^2 - z^2 = 4$



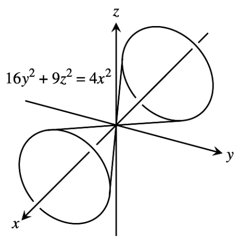
38. $x^2 + z^2 = y$



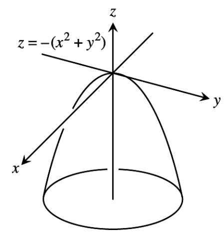
39. $x^2 + z^2 = 1$



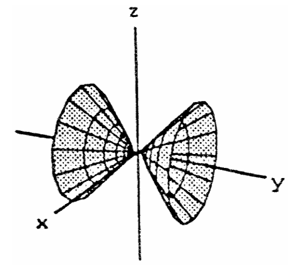
40. $16y^2 + 9z^2 = 4x^2$



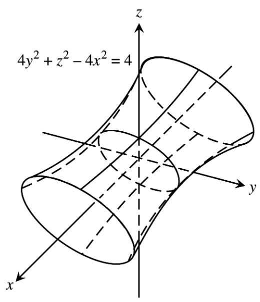
41. $z = -(x^2 + y^2)$



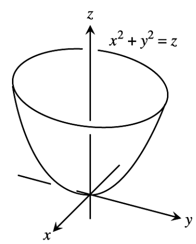
42. $y^2 - x^2 - z^2 = 1$



43. $4y^2 + z^2 - 4x^2 = 4$



44. $x^2 + y^2 = z$



45. (a) If $x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1$ and $z = c$, then $x^2 + \frac{y^2}{4} = \frac{9-c^2}{9} \Rightarrow \frac{x^2}{\left(\frac{9-c^2}{9}\right)} + \frac{y^2}{\left[\frac{4(9-c^2)}{9}\right]} = 1 \Rightarrow A = ab\pi$

$= \pi\left(\frac{\sqrt{9-c^2}}{3}\right)\left(\frac{2\sqrt{9-c^2}}{3}\right) = \frac{2\pi(9-c^2)}{9}$

(b) From part (a), each slice has the area $\frac{2\pi(9-z^2)}{9}$, where $-3 \le z \le 3$. Thus $V = 2\int_0^3 \frac{2\pi}{9}(9-z^2)\,dz$

$= \frac{4\pi}{9}\int_0^3 (9-z^2)\,dz = \frac{4\pi}{9}\left[9z - \frac{z^3}{3}\right]_0^3 = \frac{4\pi}{9}(27-9) = 8\pi$

(c) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \Rightarrow \frac{x^2}{\left[\frac{a^2(c^2-z^2)}{c^2}\right]} + \frac{y^2}{\left[\frac{b^2(c^2-z^2)}{c^2}\right]} = 1 \Rightarrow A = \pi\left(\frac{a\sqrt{c^2-z^2}}{c}\right)\left(\frac{b\sqrt{c^2-z^2}}{c}\right)$

$\Rightarrow V = 2\int_0^c \frac{\pi ab}{c^2}(c^2-z^2)\,dz = \frac{2\pi ab}{c^2}\left[c^2 z - \frac{z^3}{3}\right]_0^c = \frac{2\pi ab}{c^2}\left(\frac{2}{3}c^3\right) = \frac{4\pi abc}{3}$. Note that if $r = a = b = c$, then $V = \frac{4\pi r^3}{3}$, which is the volume of a sphere.

46. The ellipsoid has the form $\frac{x^2}{R^2} + \frac{y^2}{R^2} + \frac{z^2}{c^2} = 1$. To determine c^2 we note that the point $(0, r, h)$ lies on the surface of the barrel. Thus, $\frac{r^2}{R^2} + \frac{h^2}{c^2} = 1 \Rightarrow c^2 = \frac{h^2R^2}{R^2-r^2}$. We calculate the volume by the disk method: $V = \pi\int_{-h}^{h} y^2\,dz$. Now, $\frac{y^2}{R^2} + \frac{z^2}{c^2} = 1 \Rightarrow y^2 = R^2\left(1 - \frac{z^2}{c^2}\right) = R^2\left[1 - \frac{z^2(R^2-r^2)}{h^2R^2}\right] = R^2 - \left(\frac{R^2-r^2}{h^2}\right)z^2$

$\Rightarrow V = \pi\int_{-h}^{h}\left[R^2 - \left(\frac{R^2-r^2}{h^2}\right)z^2\right]dz = \pi\left[R^2 z - \frac{1}{3}\left(\frac{R^2-r^2}{h^2}\right)z^3\right]_{-h}^{h} = 2\pi\left[R^2h - \frac{1}{3}(R^2-r^2)h\right] = 2\pi\left(\frac{2R^2h}{3} + \frac{r^2h}{3}\right)$

$= \frac{4}{3}\pi R^2 h + \frac{2}{3}\pi r^2 h$, the volume of the barrel. If $r = R$, then $V = 2\pi R^2 h$ which is the volume of a cylinder of radius R and height 2h. If $r = 0$ and $h = R$, then $V = \frac{4}{3}\pi R^3$ which is the volume of a sphere.

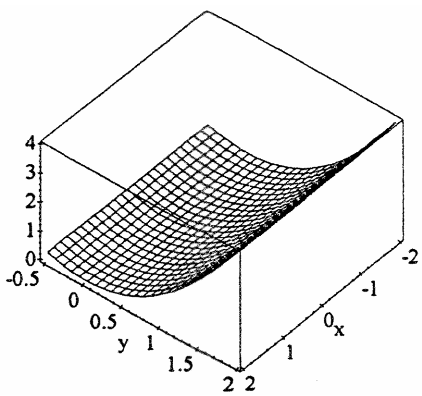
47. We calculate the volume by the slicing method, taking slices parallel to the xy-plane. For fixed z, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$ gives the ellipse $\frac{x^2}{\left(\frac{za^2}{c}\right)} + \frac{y^2}{\left(\frac{zb^2}{c}\right)} = 1$. The area of this ellipse is $\pi\left(a\sqrt{\frac{z}{c}}\right)\left(b\sqrt{\frac{z}{c}}\right) = \frac{\pi abz}{c}$ (see Exercise 45a). Hence the volume is given by $V = \int_0^h \frac{\pi abz}{c}\,dz = \left[\frac{\pi abz^2}{2c}\right]_0^h = \frac{\pi abh^2}{c}$. Now the area of the elliptic base when $z = h$ is $A = \frac{\pi abh}{c}$, as determined previously. Thus, $V = \frac{\pi abh^2}{c} = \frac{1}{2}\left(\frac{\pi abh}{c}\right)h = \frac{1}{2}$(base)(altitude), as claimed.

48. (a) For each fixed value of z, the hyperboloid $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ results in a cross-sectional ellipse $\frac{x^2}{\left[\frac{a^2(c^2+z^2)}{c^2}\right]} + \frac{y^2}{\left[\frac{b^2(c^2+z^2)}{c^2}\right]} = 1$. The area of the cross-sectional ellipse (see Exercise 45a) is $A(z) = \pi\left(\frac{a}{c}\sqrt{c^2+z^2}\right)\left(\frac{b}{c}\sqrt{c^2+z^2}\right) = \frac{\pi ab}{c^2}\left(c^2+z^2\right)$. The volume of the solid by the method of slices is $V = \int_0^h A(z)\,dz = \int_0^h \frac{\pi ab}{c^2}\left(c^2+z^2\right)dz = \frac{\pi ab}{c^2}\left[c^2z + \frac{1}{3}z^3\right]_0^h = \frac{\pi ab}{c^2}\left(c^2h + \frac{1}{3}h^3\right) = \frac{\pi abh}{3c^2}\left(3c^2+h^2\right)$

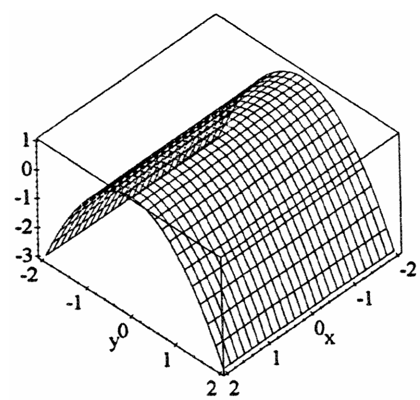
(b) $A_0 = A(0) = \pi ab$ and $A_h = A(h) = \frac{\pi ab}{c^2}\left(c^2+h^2\right)$, from part (a) $\Rightarrow$ $V = \frac{\pi abh}{3c^2}\left(3c^2+h^2\right)$ $= \frac{\pi abh}{3}\left(2+1+\frac{h^2}{c^2}\right) = \frac{\pi abh}{3}\left(2+\frac{c^2+h^2}{c^2}\right) = \frac{h}{3}\left[2\pi ab + \frac{\pi ab}{c^2}\left(c^2+h^2\right)\right] = \frac{h}{3}\left(2A_0 + A_h\right)$

(c) $A_m = A\left(\frac{h}{2}\right) = \frac{\pi ab}{c^2}\left(c^2+\frac{h^2}{4}\right) = \frac{\pi ab}{4c^2}\left(4c^2+h^2\right)$ $\Rightarrow$ $\frac{h}{6}\left(A_0 + 4A_m + A_h\right)$ $= \frac{h}{6}\left[\pi ab + \frac{\pi ab}{c^2}\left(4c^2+h^2\right) + \frac{\pi ab}{c^2}\left(c^2+h^2\right)\right] = \frac{\pi abh}{6c^2}\left(c^2+4c^2+h^2+c^2+h^2\right) = \frac{\pi abh}{6c^2}\left(6c^2+2h^2\right)$ $= \frac{\pi abh}{3c^2}\left(3c^2+h^2\right) = V$ from part (a)

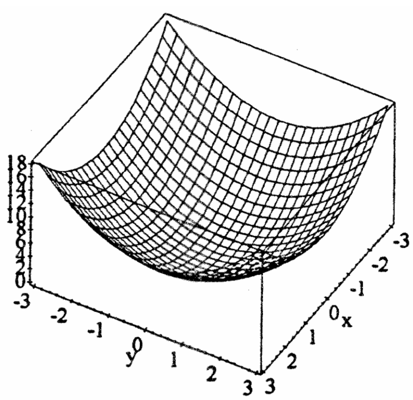
49. $z = y^2$



50. $z = 1 - y^2$

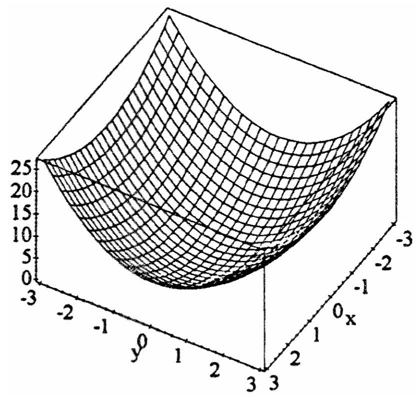


51. $z = x^2 + y^2$

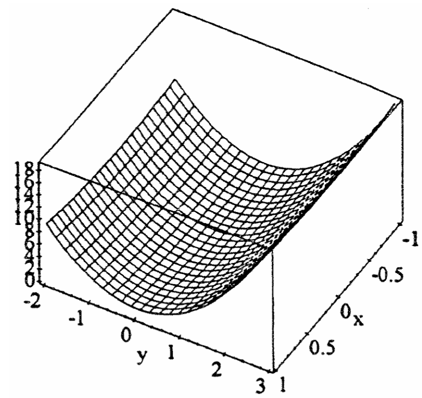


52. $z = x^2 + 2y^2$

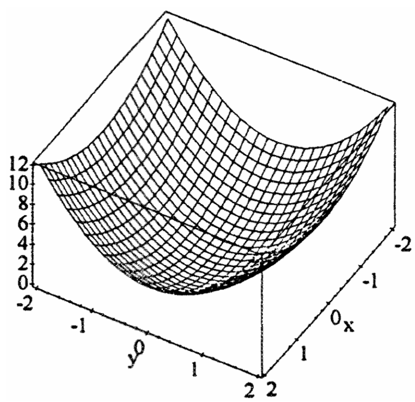
(a)



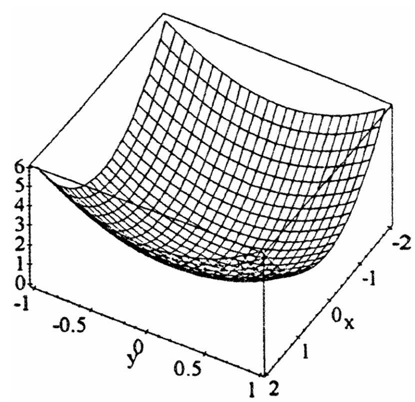
(b)



(c)



(d)



53-58. Example CAS commands:

Maple:

```
with( plots );
eq := x^2/9 + y^2/36 = 1 - z^2/25;
implicitplot3d( eq, x=-3..3, y=-6..6, z=-5..5, scaling=constrained,
                shading=zhue, axes=boxed, title="#89 (Section 11.6)" );
```

Mathematica: (functions and domains may vary):

In the following chapter, you will consider contours or level curves for surfaces in three dimensions. For the purposes of plotting the functions of two variables expressed implicitly in this section, we will call upon the function **ContourPlot3D**. To insert the stated function, write all terms on the same side of the equal sign and the default contour equating that expression to zero will be plotted.

This built-in function requires the loading of a special graphics package.

```
<<Graphics`ContourPlot3D`
Clear[x, y, z]
ContourPlot3D[x^2/9 − y^2/16 − z^2/2 − 1, {x, −9, 9}, {y, −12, 12}, {z, −5, 5},
       Axes → True, AxesLabel → {x, y, z}, Boxed → False,
       PlotLabel → "Elliptic Hyperboloid of Two Sheets"]
```

Your identification of the plot may or may not be able to be done without considering the graph.

CHAPTER 12 PRACTICE EXERCISES

1. (a) $3\langle -3, 4\rangle - 4\langle 2, -5\rangle = \langle -9 - 8, 12 + 20\rangle = \langle -17, 32\rangle$

   (b) $\sqrt{17^2 + 32^2} = \sqrt{1313}$

2. (a) $\langle -3+2, 4-5\rangle = \langle -1, -1\rangle$
   (b) $\sqrt{(-1)^2+(-1)^2} = \sqrt{2}$

3. (a) $\langle -2(-3), -2(4)\rangle = \langle 6, -8\rangle$
   (b) $\sqrt{6^2+(-8)^2} = 10$

4. (a) $\langle 5(2), 5(-5)\rangle = \langle 10, -25\rangle$
   (b) $\sqrt{10^2+(-25)^2} = \sqrt{725} = 5\sqrt{29}$

5. $\frac{\pi}{6}$ radians below the negative x-axis: $\left\langle -\frac{\sqrt{3}}{2}, -\frac{1}{2}\right\rangle$ [assuming counterclockwise].

6. $\left\langle \frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle$

7. $2\left(\frac{1}{\sqrt{4^2+1^2}}\right)(4\mathbf{i}-\mathbf{j}) = \left(\frac{8}{\sqrt{17}}\mathbf{i} - \frac{2}{\sqrt{17}}\mathbf{j}\right)$

8. $-5\left(\frac{1}{\sqrt{\left(\frac{3}{5}\right)^2+\left(\frac{4}{5}\right)^2}}\right)\left(\frac{3}{5}\mathbf{i}+\frac{4}{5}\mathbf{j}\right) = -3\mathbf{i}-4\mathbf{j}$

9. length $= \left|\sqrt{2}\mathbf{i}+\sqrt{2}\mathbf{j}\right| = \sqrt{2+2} = 2$, $\sqrt{2}\mathbf{i}+\sqrt{2}\mathbf{j} = 2\left(\frac{1}{\sqrt{2}}\mathbf{i}+\frac{1}{\sqrt{2}}\mathbf{j}\right) \Rightarrow$ the direction is $\frac{1}{\sqrt{2}}\mathbf{i}+\frac{1}{\sqrt{2}}\mathbf{j}$

10. length $= |-\mathbf{i}-\mathbf{j}| = \sqrt{1+1} = \sqrt{2}$, $-\mathbf{i}-\mathbf{j} = \sqrt{2}\left(-\frac{1}{\sqrt{2}}\mathbf{i}-\frac{1}{\sqrt{2}}\mathbf{j}\right) \Rightarrow$ the direction is $-\frac{1}{\sqrt{2}}\mathbf{i}-\frac{1}{\sqrt{2}}\mathbf{j}$

11. $t = \frac{\pi}{2} \Rightarrow \mathbf{v} = (-2\sin\frac{\pi}{2})\mathbf{i} + \left(2\cos\frac{\pi}{2}\right)\mathbf{j} = -2\mathbf{i}$; length $= |-2\mathbf{i}| = \sqrt{4+0} = 2$; $-2\mathbf{i} = 2(-\mathbf{i}) \Rightarrow$ the direction is $-\mathbf{i}$

12. $t = \ln 2 \Rightarrow \mathbf{v} = \left(e^{\ln 2}\cos(\ln 2) - e^{\ln 2}\sin(\ln 2)\right)\mathbf{i} + \left(e^{\ln 2}\sin(\ln 2) + e^{\ln 2}\cos(\ln 2)\right)\mathbf{j}$
$= (2\cos(\ln 2) - 2\sin(\ln 2))\,\mathbf{i} + (2\sin(\ln 2) + 2\cos(\ln 2))\,\mathbf{j} = 2[\,(\cos(\ln 2) - \sin(\ln 2))\,\mathbf{i} + (\sin(\ln 2) + \cos(\ln 2))\,\mathbf{j}\,]$
length $= |2[\,(\cos(\ln 2) - \sin(\ln 2))\,\mathbf{i} + (\sin(\ln 2) + \cos(\ln 2))\,\mathbf{j}\,]| = 2\sqrt{(\cos(\ln 2) - \sin(\ln 2))^2 + (\cos(\ln 2) + \sin(\ln 2))^2}$
$= 2\sqrt{2\cos^2(\ln 2) + 2\sin^2(\ln 2)} = 2\sqrt{2}$;
$2[\,(\cos(\ln 2) - \sin(\ln 2))\,\mathbf{i} + (\sin(\ln 2) + \cos(\ln 2))\,\mathbf{j}\,] = 2\sqrt{2}\left(\frac{(\cos(\ln 2) - \sin(\ln 2))\,\mathbf{i} + (\sin(\ln 2) + \cos(\ln 2))\,\mathbf{j}}{\sqrt{2}}\right)$
$\Rightarrow$ direction $= \frac{(\cos(\ln 2) - \sin(\ln 2))}{\sqrt{2}}\mathbf{i} + \frac{(\sin(\ln 2) + \cos(\ln 2))}{\sqrt{2}}\mathbf{j}$

13. length $= |2\mathbf{i}-3\mathbf{j}+6\mathbf{k}| = \sqrt{4+9+36} = 7$, $2\mathbf{i}-3\mathbf{j}+6\mathbf{k} = 7\left(\frac{2}{7}\mathbf{i}-\frac{3}{7}\mathbf{j}+\frac{6}{7}\mathbf{k}\right) \Rightarrow$ the direction is $\frac{2}{7}\mathbf{i}-\frac{3}{7}\mathbf{j}+\frac{6}{7}\mathbf{k}$

14. length $= |\mathbf{i}+2\mathbf{j}-\mathbf{k}| = \sqrt{1+4+1} = \sqrt{6}$, $\mathbf{i}+2\mathbf{j}-\mathbf{k} = \sqrt{6}\left(\frac{1}{\sqrt{6}}\mathbf{i}+\frac{2}{\sqrt{6}}\mathbf{j}-\frac{1}{\sqrt{6}}\mathbf{k}\right) \Rightarrow$ the direction is
$\frac{1}{\sqrt{6}}\mathbf{i}+\frac{2}{\sqrt{6}}\mathbf{j}-\frac{1}{\sqrt{6}}\mathbf{k}$

15. $2\frac{\mathbf{v}}{|\mathbf{v}|} = 2\cdot\frac{4\mathbf{i}-\mathbf{j}+4\mathbf{k}}{\sqrt{4^2+(-1)^2+4^2}} = 2\cdot\frac{4\mathbf{i}-\mathbf{j}+4\mathbf{k}}{\sqrt{33}} = \frac{8}{\sqrt{33}}\mathbf{i}-\frac{2}{\sqrt{33}}\mathbf{j}+\frac{8}{\sqrt{33}}\mathbf{k}$

16. $-5\frac{\mathbf{v}}{|\mathbf{v}|} = -5\cdot\frac{\left(\frac{3}{5}\right)\mathbf{i}+\left(\frac{4}{5}\right)\mathbf{k}}{\sqrt{\left(\frac{3}{5}\right)^2+\left(\frac{4}{5}\right)^2}} = -5\cdot\frac{\left(\frac{3}{5}\right)\mathbf{i}+\left(\frac{4}{5}\right)\mathbf{k}}{\sqrt{\frac{9}{25}+\frac{16}{25}}} = -3\mathbf{i}-4\mathbf{k}$

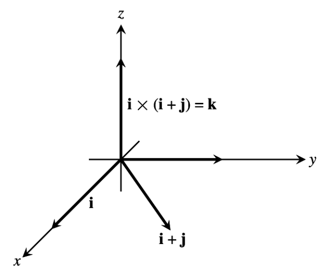
17. $|\mathbf{v}| = \sqrt{1+1} = \sqrt{2}$, $|\mathbf{u}| = \sqrt{4+1+4} = 3$, $\mathbf{v}\cdot\mathbf{u} = 3$, $\mathbf{u}\cdot\mathbf{v} = 3$, $\mathbf{v}\times\mathbf{u} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 1 & 0\\ 2 & 1 & -2\end{vmatrix} = -2\mathbf{i}+2\mathbf{j}-\mathbf{k}$,
$\mathbf{u}\times\mathbf{v} = -(\mathbf{v}\times\mathbf{u}) = 2\mathbf{i}-2\mathbf{j}+\mathbf{k}$, $|\mathbf{v}\times\mathbf{u}| = \sqrt{4+4+1} = 3$, $\theta = \cos^{-1}\left(\frac{\mathbf{v}\cdot\mathbf{u}}{|\mathbf{v}|\,|\mathbf{u}|}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$,
$|\mathbf{u}|\cos\theta = \frac{3}{\sqrt{2}}$, $\text{proj}_{\mathbf{v}}\,\mathbf{u} = \left(\frac{\mathbf{v}\cdot\mathbf{u}}{|\mathbf{v}||\mathbf{v}|}\right)\mathbf{v} = \frac{3}{2}(\mathbf{i}+\mathbf{j})$

18. $|\mathbf{v}| = \sqrt{1^2+1^2+2^2} = \sqrt{6}$, $|\mathbf{u}| = \sqrt{(-1)^2+(-1)^2} = \sqrt{2}$, $\mathbf{v}\cdot\mathbf{u} = (1)(-1)+(1)(0)+(2)(-1) = -3$,
$\mathbf{u}\cdot\mathbf{v} = -3$, $\mathbf{v}\times\mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 2 \\ -1 & 0 & -1 \end{vmatrix} = -\mathbf{i}-\mathbf{j}+\mathbf{k}$, $\mathbf{u}\times\mathbf{v} = -(\mathbf{v}\times\mathbf{u}) = \mathbf{i}+\mathbf{j}-\mathbf{k}$,
$|\mathbf{v}\times\mathbf{u}| = \sqrt{(-1)^2+(-1)^2+1^2} = \sqrt{3}$, $\theta = \cos^{-1}\left(\frac{\mathbf{v}\cdot\mathbf{u}}{|\mathbf{v}|\,|\mathbf{u}|}\right) = \cos^{-1}\left(\frac{-3}{\sqrt{6}\sqrt{2}}\right) = \cos^{-1}\left(\frac{-3}{\sqrt{12}}\right)$
$= \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$, $|\mathbf{u}|\cos\theta = \sqrt{2}\cdot\left(\frac{-\sqrt{3}}{2}\right) = -\frac{\sqrt{6}}{2}$, $\text{proj}_{\mathbf{v}}\,\mathbf{u} = \left(\frac{\mathbf{v}\cdot\mathbf{u}}{|\mathbf{v}||\mathbf{v}|}\right)\mathbf{v} = \frac{-3}{6}(\mathbf{i}+\mathbf{j}+2\mathbf{k}) = -\frac{1}{2}(\mathbf{i}+\mathbf{j}+\mathbf{k})$

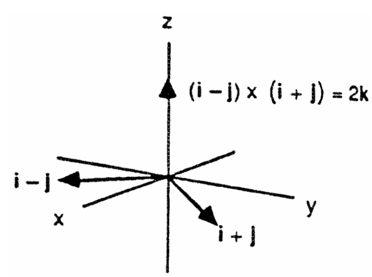
19. $\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{v}\cdot\mathbf{u}}{|\mathbf{v}||\mathbf{v}|}\right)\mathbf{v} = \frac{4}{3}(2\mathbf{i}+\mathbf{j}-\mathbf{k})$ where $\mathbf{v}\cdot\mathbf{u} = 8$ and $\mathbf{v}\cdot\mathbf{v} = 6$

20. $\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{v}\cdot\mathbf{u}}{|\mathbf{v}||\mathbf{v}|}\right)\mathbf{v} = -\frac{1}{3}(\mathbf{i}-2\mathbf{j})$ where $\mathbf{v}\cdot\mathbf{u} = -1$ and $\mathbf{v}\cdot\mathbf{v} = 3$

21. $\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{vmatrix} = \mathbf{k}$



22. $\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{vmatrix} = 2\mathbf{k}$



23. Let $\mathbf{v} = v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k}$ and $\mathbf{w} = w_1\mathbf{i}+w_2\mathbf{j}+w_3\mathbf{k}$. Then $|\mathbf{v}-2\mathbf{w}|^2 = |(v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k}) - 2(w_1\mathbf{i}+w_2\mathbf{j}+w_3\mathbf{k})|^2$
$= |(v_1-2w_1)\mathbf{i}+(v_2-2w_2)\mathbf{j}+(v_3-2w_3)\mathbf{k}|^2 = \left(\sqrt{(v_1-2w_1)^2+(v_2-2w_2)^2+(v_3-2w_3)^2}\right)^2$
$= (v_1^2+v_2^2+v_3^2) - 4(v_1w_1+v_2w_2+v_3w_3) + 4(w_1^2+w_2^2+w_3^2) = |\mathbf{v}|^2 - 4\mathbf{v}\cdot\mathbf{w} + 4|\mathbf{w}|^2$
$= |\mathbf{v}|^2 - 4|\mathbf{v}|\,|\mathbf{w}|\cos\theta + 4|\mathbf{w}|^2 = 4 - 4(2)(3)\left(\cos\frac{\pi}{3}\right) + 36 = 40 - 24\left(\frac{1}{2}\right) = 40 - 12 = 28 \Rightarrow |\mathbf{v}-2\mathbf{w}| = \sqrt{28}$
$= 2\sqrt{7}$

24. $\mathbf{u}$ and $\mathbf{v}$ are parallel when $\mathbf{u}\times\mathbf{v} = \mathbf{0} \Rightarrow \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 4 & -5 \\ -4 & -8 & a \end{vmatrix} = \mathbf{0} \Rightarrow (4a-40)\mathbf{i}+(20-2a)\mathbf{j}+(0)\mathbf{k} = \mathbf{0}$
$\Rightarrow 4a-40 = 0$ and $20-2a = 0 \Rightarrow a = 10$

25. (a) area $= |\mathbf{u}\times\mathbf{v}| = \text{abs}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -1 \\ 2 & 1 & 1 \end{vmatrix} = |2\mathbf{i}-3\mathbf{j}-\mathbf{k}| = \sqrt{4+9+1} = \sqrt{14}$

(b) volume $= (\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = \begin{vmatrix} 1 & 1 & -1 \\ 2 & 1 & 1 \\ -1 & -2 & 3 \end{vmatrix} = 1(3+2) - 1(6-(-1)) - 1(-4+1) = 1$

26. (a) area $= |\mathbf{u} \times \mathbf{v}| = \text{abs}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{vmatrix} = |\mathbf{k}| = 1$

(b) volume $= (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 1(1-0) - 1(0-0) + 0 = 1$

27. The desired vector is $\mathbf{n} \times \mathbf{v}$ or $\mathbf{v} \times \mathbf{n}$ since $\mathbf{n} \times \mathbf{v}$ is perpendicular to both $\mathbf{n}$ and $\mathbf{v}$ and, therefore, also parallel to the plane.

28. If $a = 0$ and $b \neq 0$, then the line $by = c$ and $\mathbf{i}$ are parallel. If $a \neq 0$ and $b = 0$, then the line $ax = c$ and $\mathbf{j}$ are parallel. If a and b are both $\neq 0$, then $ax + by = c$ contains the points $\left(\frac{c}{a}, 0\right)$ and $\left(0, \frac{c}{b}\right) \Rightarrow$ the vector $ab\left(\frac{c}{a}\,\mathbf{i} - \frac{c}{b}\,\mathbf{j}\right) = c(b\mathbf{i} - a\mathbf{j})$ and the line are parallel. Therefore, the vector $b\mathbf{i} - a\mathbf{j}$ is parallel to the line $ax + by = c$ in every case.

29. The line L passes through the point $P(0, 0, -1)$ parallel to $\mathbf{v} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$. With $\overrightarrow{PS} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\overrightarrow{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 1 \\ -1 & 1 & 1 \end{vmatrix} = (2-1)\mathbf{i} - (2+1)\mathbf{j} + (2+2)\mathbf{k} = \mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$, we find the distance $d = \frac{\left|\overrightarrow{PS} \times \mathbf{v}\right|}{|\mathbf{v}|} = \frac{\sqrt{1+9+16}}{\sqrt{1+1+1}} = \frac{\sqrt{26}}{\sqrt{3}} = \frac{\sqrt{78}}{3}$.

30. The line L passes through the point $P(2, 2, 0)$ parallel to $\mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$. With $\overrightarrow{PS} = -2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\overrightarrow{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix} = (2-1)\mathbf{i} - (-2-1)\mathbf{j} + (-2-2)\mathbf{k} = \mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$, we find the distance $d = \frac{\left|\overrightarrow{PS} \times \mathbf{v}\right|}{|\mathbf{v}|} = \frac{\sqrt{1+9+16}}{\sqrt{1+1+1}} = \frac{\sqrt{26}}{\sqrt{3}} = \frac{\sqrt{78}}{3}$.

31. Parametric equations for the line are $x = 1 - 3t$, $y = 2$, $z = 3 + 7t$.

32. The line is parallel to $\overrightarrow{PQ} = 0\mathbf{i} + \mathbf{j} - \mathbf{k}$ and contains the point $P(1, 2, 0) \Rightarrow$ parametric equations are $x = 1$, $y = 2 + t$, $z = -t$ for $0 \le t \le 1$.

33. The point $P(4, 0, 0)$ lies on the plane $x - y = 4$, and $\overrightarrow{PS} = (6-4)\mathbf{i} + 0\mathbf{j} + (-6+0)\mathbf{k} = 2\mathbf{i} - 6\mathbf{k}$ with $\mathbf{n} = \mathbf{i} - \mathbf{j}$ $\Rightarrow d = \frac{\left|\mathbf{n} \cdot \overrightarrow{PS}\right|}{|\mathbf{n}|} = \left|\frac{2+0+0}{\sqrt{1+1+0}}\right| = \frac{2}{\sqrt{2}} = \sqrt{2}$.

34. The point $P(0, 0, 2)$ lies on the plane $2x + 3y + z = 2$, and $\overrightarrow{PS} = (3-0)\mathbf{i} + (0-0)\mathbf{j} + (10+2)\mathbf{k} = 3\mathbf{i} + 8\mathbf{k}$ with $\mathbf{n} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k} \Rightarrow d = \frac{\left|\mathbf{n} \cdot \overrightarrow{PS}\right|}{|\mathbf{n}|} = \left|\frac{6+0+8}{\sqrt{4+9+1}}\right| = \frac{14}{\sqrt{14}} = \sqrt{14}$.

35. $P(3, -2, 1)$ and $\mathbf{n} = 2\mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow (2)(x-3) + (1)(y-(-2)) + (1)(z-1) = 0 \Rightarrow 2x + y + z = 5$

36. $P(-1, 6, 0)$ and $\mathbf{n} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k} \Rightarrow (1)(x-(-1)) + (-2)(y-6) + (3)(z-0) = 0 \Rightarrow x - 2y + 3z = -13$

37. P(1, −1, 2), Q(2, 1, 3) and R(−1, 2, −1) $\Rightarrow$ $\overrightarrow{PQ} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$, $\overrightarrow{PR} = -2\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$ and $\overrightarrow{PQ} \times \overrightarrow{PR}$
$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ -2 & 3 & -3 \end{vmatrix} = -9\mathbf{i} + \mathbf{j} + 7\mathbf{k}$ is normal to the plane $\Rightarrow$ $(-9)(x-1) + (1)(y+1) + (7)(z-2) = 0$
$\Rightarrow$ $-9x + y + 7z = 4$

38. P(1, 0, 0), Q(0, 1, 0) and R(0, 0, 1) $\Rightarrow$ $\overrightarrow{PQ} = -\mathbf{i} + \mathbf{j}$, $\overrightarrow{PR} = -\mathbf{i} + \mathbf{k}$ and $\overrightarrow{PQ} \times \overrightarrow{PR}$
$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ is normal to the plane $\Rightarrow$ $(1)(x-1) + (1)(y-0) + (1)(z-0) = 0$
$\Rightarrow$ $x + y + z = 1$

39. $\left(0, -\frac{1}{2}, -\frac{3}{2}\right)$, since $t = -\frac{1}{2}$, $y = -\frac{1}{2}$ and $z = -\frac{3}{2}$ when $x = 0$; $(-1, 0, -3)$, since $t = -1$, $x = -1$ and $z = -3$ when $y = 0$; $(1, -1, 0)$, since $t = 0$, $x = 1$ and $y = -1$ when $z = 0$

40. $x = 2t$, $y = -t$, $z = -t$ represents a line containing the origin and perpendicular to the plane $2x - y - z = 4$; this line intersects the plane $3x - 5y + 2z = 6$ when t is the solution of $3(2t) - 5(-t) + 2(-t) = 6$
$\Rightarrow$ $t = \frac{2}{3}$ $\Rightarrow$ $\left(\frac{4}{3}, -\frac{2}{3}, -\frac{2}{3}\right)$ is the point of intersection

41. $\mathbf{n}_1 = \mathbf{i}$ and $\mathbf{n}_2 = \mathbf{i} + \mathbf{j} + \sqrt{2}\mathbf{k}$ $\Rightarrow$ the desired angle is $\cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

42. $\mathbf{n}_1 = \mathbf{i} + \mathbf{j}$ and $\mathbf{n}_2 = \mathbf{j} + \mathbf{k}$ $\Rightarrow$ the desired angle is $\cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

43. The direction of the line is $\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 1 & -1 & 2 \end{vmatrix} = 5\mathbf{i} - \mathbf{j} - 3\mathbf{k}$. Since the point $(-5, 3, 0)$ is on both planes, the desired line is $x = -5 + 5t$, $y = 3 - t$, $z = -3t$.

44. The direction of the intersection is $\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -2 \\ 5 & -2 & -1 \end{vmatrix} = -6\mathbf{i} - 9\mathbf{j} - 12\mathbf{k} = -3(2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k})$ and is the same as the direction of the given line.

45. (a) The corresponding normals are $\mathbf{n}_1 = 3\mathbf{i} + 6\mathbf{k}$ and $\mathbf{n}_2 = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and since $\mathbf{n}_1 \cdot \mathbf{n}_2$ $= (3)(2) + (0)(2) + (6)(-1) = 6 + 0 - 6 = 0$, we have that the planes are orthogonal

(b) The line of intersection is parallel to $\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 0 & 6 \\ 2 & 2 & -1 \end{vmatrix} = -12\mathbf{i} + 15\mathbf{j} + 6\mathbf{k}$. Now to find a point in the intersection, solve $\begin{cases} 3x + 6z = 1 \\ 2x + 2y - z = 3 \end{cases}$ $\Rightarrow$ $\begin{cases} 3x + 6z = 1 \\ 12x + 12y - 6z = 18 \end{cases}$ $\Rightarrow$ $15x + 12y = 19$ $\Rightarrow$ $x = 0$ and $y = \frac{19}{12}$
$\Rightarrow$ $\left(0, \frac{19}{12}, \frac{1}{6}\right)$ is a point on the line we seek. Therefore, the line is $x = -12t$, $y = \frac{19}{12} + 15t$ and $z = \frac{1}{6} + 6t$.

46. A vector in the direction of the plane's normal is $\mathbf{n} = \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 1 \\ 1 & -1 & 2 \end{vmatrix} = 7\mathbf{i} - 3\mathbf{j} - 5\mathbf{k}$ and P(1, 2, 3) on the plane $\Rightarrow$ $7(x-1) - 3(y-2) - 5(z-3) = 0$ $\Rightarrow$ $7x - 3y - 5z = -14$.

47. Yes; $\mathbf{v}\cdot\mathbf{n} = (2\mathbf{i} - 4\mathbf{j} + \mathbf{k})\cdot(2\mathbf{i} + \mathbf{j} + 0\mathbf{k}) = 2\cdot 2 - 4\cdot 1 + 1\cdot 0 = 0 \Rightarrow$ the vector is orthogonal to the plane's normal $\Rightarrow$ $\mathbf{v}$ is parallel to the plane

48. $\mathbf{n}\cdot\overrightarrow{PP_0} > 0$ represents the half-space of points lying on one side of the plane in the direction which the normal $\mathbf{n}$ points

49. A normal to the plane is $\mathbf{n} = \overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 2 & -1 & 0 \end{vmatrix} = -\mathbf{i} - 2\mathbf{j} - 2\mathbf{k} \Rightarrow$ the distance is $d = \left|\frac{\overrightarrow{AP}\cdot\mathbf{n}}{\mathbf{n}}\right|$

$= \left|\frac{(\mathbf{i}+4\mathbf{j})\cdot(-\mathbf{i}-2\mathbf{j}-2\mathbf{k})}{\sqrt{1+4+4}}\right| = \left|\frac{-1-8+0}{3}\right| = 3$

50. $P(0,0,0)$ lies on the plane $2x + 3y + 5z = 0$, and $\overrightarrow{PS} = 2\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ with $\mathbf{n} = 2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k} \Rightarrow$
$d = \left|\frac{\mathbf{n}\cdot\overrightarrow{PS}}{|\mathbf{n}|}\right| = \left|\frac{4+6+15}{\sqrt{4+9+25}}\right| = \frac{25}{\sqrt{38}}$

51. $\mathbf{n} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$ is normal to the plane $\Rightarrow$ $\mathbf{n}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = 0\mathbf{i} - 3\mathbf{j} + 3\mathbf{k} = -3\mathbf{j} + 3\mathbf{k}$ is orthogonal

to $\mathbf{v}$ and parallel to the plane

52. The vector $\mathbf{B}\times\mathbf{C}$ is normal to the plane of $\mathbf{B}$ and $\mathbf{C} \Rightarrow \mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ is orthogonal to $\mathbf{A}$ and parallel to the plane of $\mathbf{B}$ and $\mathbf{C}$:

$\mathbf{B}\times\mathbf{C} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 1 & 1 & -2 \end{vmatrix} = -5\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ and $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ -5 & 3 & -1 \end{vmatrix} = -2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$

$\Rightarrow |\mathbf{A}\times(\mathbf{B}\times\mathbf{C})| = \sqrt{4+9+1} = \sqrt{14}$ and $\mathbf{u} = \frac{1}{\sqrt{14}}(-2\mathbf{i} - 3\mathbf{j} + \mathbf{k})$ is the desired unit vector.

53. A vector parallel to the line of intersection is $\mathbf{v} = \mathbf{n}_1\times\mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 1 & -1 & 2 \end{vmatrix} = 5\mathbf{i} - \mathbf{j} - 3\mathbf{k}$

$\Rightarrow |\mathbf{v}| = \sqrt{25+1+9} = \sqrt{35} \Rightarrow 2\left(\frac{\mathbf{v}}{|\mathbf{v}|}\right) = \frac{2}{\sqrt{35}}(5\mathbf{i} - \mathbf{j} - 3\mathbf{k})$ is the desired vector.

54. The line containing $(0,0,0)$ normal to the plane is represented by $x = 2t$, $y = -t$, and $z = -t$. This line intersects the plane $3x - 5y + 2z = 6$ when $3(2t) - 5(-t) + 2(-t) = 6 \Rightarrow t = \frac{2}{3} \Rightarrow$ the point is $\left(\frac{4}{3}, -\frac{2}{3}, -\frac{2}{3}\right)$.

55. The line is represented by $x = 3 + 2t$, $y = 2 - t$, and $z = 1 + 2t$. It meets the plane $2x - y + 2z = -2$ when $2(3+2t) - (2-t) + 2(1+2t) = -2 \Rightarrow t = -\frac{8}{9} \Rightarrow$ the point is $\left(\frac{11}{9}, \frac{26}{9}, -\frac{7}{9}\right)$.

56. The direction of the intersection is $\mathbf{v} = \mathbf{n}_1\times\mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 1 & 1 & 2 \end{vmatrix} = 3\mathbf{i} - 5\mathbf{j} + \mathbf{k} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{v}\cdot\mathbf{i}}{|\mathbf{v}|\,|\mathbf{i}|}\right)$

$= \cos^{-1}\left(\frac{3}{\sqrt{35}}\right) \approx 59.5°$

57. The intersection occurs when $(3+2t) + 3(2t) - t = -4 \Rightarrow t = -1 \Rightarrow$ the point is $(1, -2, -1)$. The required line must be perpendicular to both the given line and to the normal, and hence is parallel to $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 1 \\ 1 & 3 & -1 \end{vmatrix}$

$= -5\mathbf{i} + 3\mathbf{j} + 4\mathbf{k} \Rightarrow$ the line is represented by $x = 1 - 5t$, $y = -2 + 3t$, and $z = -1 + 4t$.

58. If $P(a, b, c)$ is a point on the line of intersection, then P lies in both planes $\Rightarrow$ $a - 2b + c + 3 = 0$ and $2a - b - c + 1 = 0 \Rightarrow (a - 2b + c + 3) + k(2a - b - c + 1) = 0$ for all k.

59. The vector $\overrightarrow{AB} \times \overrightarrow{CD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 4 \\ \frac{26}{5} & 0 & -\frac{26}{5} \end{vmatrix} = \frac{26}{5}(2\mathbf{i} + 7\mathbf{j} + 2\mathbf{k})$ is normal to the plane and $A(-2, 0, -3)$ lies on the plane $\Rightarrow 2(x + 2) + 7(y - 0) + 2(z - (-3)) = 0 \Rightarrow 2x + 7y + 2z + 10 = 0$ is an equation of the plane.

60. Yes; the line's direction vector is $2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$ which is parallel to the line and also parallel to the normal $-4\mathbf{i} - 6\mathbf{j} + 10\mathbf{k}$ to the plane $\Rightarrow$ the line is orthogonal to the plane.

61. The vector $\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ -3 & 0 & 1 \end{vmatrix} = -\mathbf{i} - 11\mathbf{j} - 3\mathbf{k}$ is normal to the plane.

(a) No, the plane is not orthogonal to $\overrightarrow{PQ} \times \overrightarrow{PR}$.

(b) No, these equations represent a line, not a plane.

(c) No, the plane $(x + 2) + 11(y - 1) - 3z = 0$ has normal $\mathbf{i} + 11\mathbf{j} - 3\mathbf{k}$ which is not parallel to $\overrightarrow{PQ} \times \overrightarrow{PR}$.

(d) No, this vector equation is equivalent to the equations $3y + 3z = 3$, $3x - 2z = -6$, and $3x + 2y = -4$ $\Rightarrow x = -\frac{4}{3} - \frac{2}{3}t$, $y = t$, $z = 1 - t$, which represents a line, not a plane.

(e) Yes, this is a plane containing the point $R(-2, 1, 0)$ with normal $\overrightarrow{PQ} \times \overrightarrow{PR}$.

62. (a) The line through A and B is $x = 1 + t$, $y = -t$, $z = -1 + 5t$; the line through C and D must be parallel and is L_1: $x = 1 + t$, $y = 2 - t$, $z = 3 + 5t$. The line through B and C is $x = 1$, $y = 2 + 2s$, $z = 3 + 4s$; the line through A and D must be parallel and is L_2: $x = 2$, $y = -1 + 2s$, $z = 4 + 4s$. The lines L_1 and L_2 intersect at $D(2, 1, 8)$ where $t = 1$ and $s = 1$.

(b) $\cos\theta = \frac{(2\mathbf{j} + 4\mathbf{k}) \cdot (\mathbf{i} - \mathbf{j} + 5\mathbf{k})}{\sqrt{20}\sqrt{27}} = \frac{3}{\sqrt{15}}$

(c) $\left(\frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{\overrightarrow{BC} \cdot \overrightarrow{BC}}\right)\overrightarrow{BC} = \frac{18}{20}\overrightarrow{BC} = \frac{9}{5}(\mathbf{j} + 2\mathbf{k})$ where $\overrightarrow{BA} = \mathbf{i} - \mathbf{j} + 5\mathbf{k}$ and $\overrightarrow{BC} = 2\mathbf{j} + 4\mathbf{k}$

(d) area $= |(2\mathbf{j} + 4\mathbf{k}) \times (\mathbf{i} - \mathbf{j} + 5\mathbf{k})| = |14\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}| = 6\sqrt{6}$

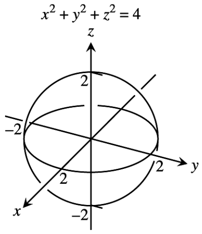
(e) From part (d), $\mathbf{n} = 14\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ is normal to the plane $\Rightarrow 14(x - 1) + 4(y - 0) - 2(z + 1) = 0$ $\Rightarrow 7x + 2y - z = 8$.

(f) From part (d), $\mathbf{n} = 14\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ $\Rightarrow$ the area of the projection on the yz-plane is $|\mathbf{n} \cdot \mathbf{i}| = 14$; the area of the projection on the xy-plane is $|\mathbf{n} \cdot \mathbf{j}| = 4$; and the area of the projection on the xy-plane is $|\mathbf{n} \cdot \mathbf{k}| = 2$.

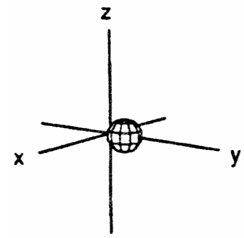
63. $\overrightarrow{AB} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\overrightarrow{CD} = \mathbf{i} + 4\mathbf{j} - \mathbf{k}$, and $\overrightarrow{AC} = 2\mathbf{i} + \mathbf{j} \Rightarrow \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 1 \\ 1 & 4 & -1 \end{vmatrix} = -5\mathbf{i} - \mathbf{j} - 9\mathbf{k} \Rightarrow$ the distance is $d = \left|\frac{(2\mathbf{i} + \mathbf{j}) \cdot (-5\mathbf{i} - \mathbf{j} - 9\mathbf{k})}{\sqrt{25 + 1 + 81}}\right| = \frac{11}{\sqrt{107}}$

64. $\overrightarrow{AB} = -2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$, $\overrightarrow{CD} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$, and $\overrightarrow{AC} = -3\mathbf{i} + 3\mathbf{j} \Rightarrow \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 4 & -1 \\ 1 & -1 & 2 \end{vmatrix} = 7\mathbf{i} + 3\mathbf{j} - 2\mathbf{k} \Rightarrow$ the distance is $d = \left|\frac{(-3\mathbf{i} + 3\mathbf{j}) \cdot (7\mathbf{i} + 3\mathbf{j} - 2\mathbf{k})}{\sqrt{49 + 9 + 4}}\right| = \frac{12}{\sqrt{62}}$

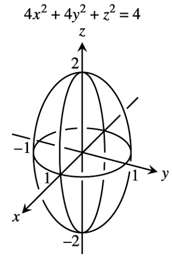
65. $x^2 + y^2 + z^2 = 4$



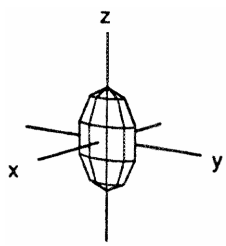
66. $x^2 + (y - 1)^2 + z^2 = 1$



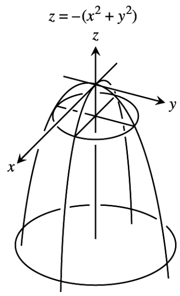
67. $4x^2 + 4y^2 + z^2 = 4$



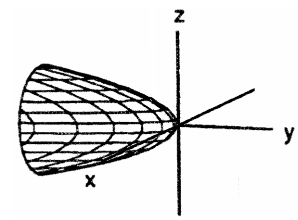
68. $36x^2 + 9y^2 + 4z^2 = 36$



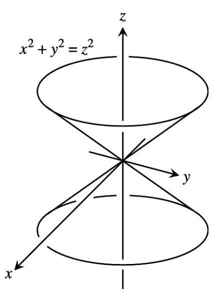
69. $z = -(x^2 + y^2)$



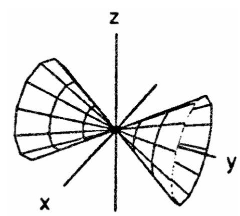
70. $y = -(x^2 + z^2)$



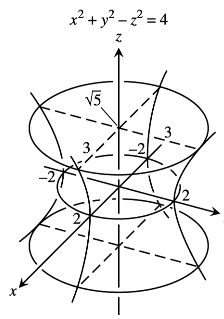
71. $x^2 + y^2 = z^2$



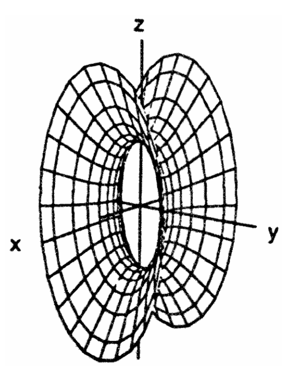
72. $x^2 + z^2 = y^2$



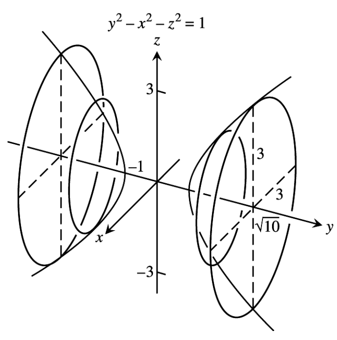
73. $x^2 + y^2 - z^2 = 4$



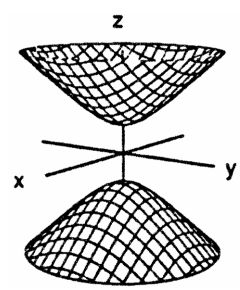
74. $4y^2 + z^2 - 4x^2 = 4$



75. $y^2 - x^2 - z^2 = 1$



76. $z^2 - x^2 - y^2 = 1$



CHAPTER 12 ADDITIONAL AND ADVANCED EXERCISES

1. Information from ship A indicates the submarine is now on the line L_1: $x = 4 + 2t$, $y = 3t$, $z = -\frac{1}{3}t$; information from ship B indicates the submarine is now on the line L_2: $x = 18s$, $y = 5 - 6s$, $z = -s$. The current position of the sub is $\left(6, 3, -\frac{1}{3}\right)$ and occurs when the lines intersect at $t = 1$ and $s = \frac{1}{3}$. The straight line path of the submarine contains both points $P\left(2, -1, -\frac{1}{3}\right)$ and $Q\left(6, 3, -\frac{1}{3}\right)$; the line representing this path is L: $x = 2 + 4t$, $y = -1 + 4t$, $z = -\frac{1}{3}$. The submarine traveled the distance between P and Q in 4 minutes $\Rightarrow$ a speed of $\frac{|\overrightarrow{PQ}|}{4} = \frac{\sqrt{32}}{4} = \sqrt{2}$ thousand ft/min. In 20 minutes the submarine will move $20\sqrt{2}$ thousand ft from Q along the line L $\Rightarrow 20\sqrt{2} = \sqrt{(2 + 4t - 6)^2 + (-1 + 4t - 3)^2 + 0^2} \Rightarrow 800 = 16(t-1)^2 + 16(t-1)^2 = 32(t-1)^2 \Rightarrow (t-1)^2 = \frac{800}{32} = 25 \Rightarrow t = 6 \Rightarrow$ the submarine will be located at $\left(26, 23, -\frac{1}{3}\right)$ in 20 minutes.

2. H_2 stops its flight when $6 + 110t = 446 \Rightarrow t = 4$ hours. After 6 hours, H_1 is at $P(246, 57, 9)$ while H_2 is at $(446, 13, 0)$. The distance between P and Q is $\sqrt{(246 - 446)^2 + (57 - 13)^2 + (9 - 0)^2} \approx 204.98$ miles. At 150 mph, it would take about 1.37 hours for H_1 to reach H_2.

3. Torque $= \left|\overrightarrow{PQ} \times \mathbf{F}\right| \Rightarrow 15$ ft-lb $= \left|\overrightarrow{PQ}\right| |\mathbf{F}| \sin\frac{\pi}{2} = \frac{3}{4}$ ft $\cdot |\mathbf{F}| \Rightarrow |\mathbf{F}| = 20$ lb

4. Let $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ be the vector from O to A and $\mathbf{b} = \mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ be the vector from O to B. The vector $\mathbf{v}$ orthogonal to $\mathbf{a}$ and $\mathbf{b} \Rightarrow \mathbf{v}$ is parallel to $\mathbf{b} \times \mathbf{a}$ (since the rotation is clockwise). Now $\mathbf{b} \times \mathbf{a} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$; $\text{proj}_{\mathbf{a}}\mathbf{b} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\right)\mathbf{a} = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ $\Rightarrow (2, 2, 2)$ is the center of the circular path $(1, 3, 2)$ takes $\Rightarrow$ radius $= \sqrt{1^2 + (-1)^2 + 0^2} = \sqrt{2} \Rightarrow$ arc length per second covered by the point is $\frac{3}{2}\sqrt{2}$ units/sec $= |\mathbf{v}|$ (velocity is constant). A unit vector in the direction of $\mathbf{v}$ is $\frac{\mathbf{b}\times\mathbf{a}}{|\mathbf{b}\times\mathbf{a}|} = \frac{1}{\sqrt{6}}\mathbf{i} + \frac{1}{\sqrt{6}}\mathbf{j} - \frac{2}{\sqrt{6}}\mathbf{k} \Rightarrow \mathbf{v} = |\mathbf{v}|\left(\frac{\mathbf{b}\times\mathbf{a}}{|\mathbf{b}\times\mathbf{a}|}\right) = \frac{3}{2}\sqrt{2}\left(\frac{1}{\sqrt{6}}\mathbf{i} + \frac{1}{\sqrt{6}}\mathbf{j} - \frac{2}{\sqrt{6}}\mathbf{k}\right) = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j} - \sqrt{3}\mathbf{k}$

5. (a) By the Law of Cosines we have $\cos\alpha = \frac{3^2 + 5^2 - 4^2}{2(3)(5)} = \frac{3}{5}$ and $\cos\beta = \frac{4^2 + 5^2 - 3^2}{2(4)(5)} = \frac{4}{5} \Rightarrow \sin\alpha = \frac{4}{5}$ and $\sin\beta = \frac{3}{5}$ $\Rightarrow \mathbf{F}_1 = \left\langle -|\mathbf{F}_1|\cos\alpha, |\mathbf{F}_1|\sin\alpha \right\rangle = \left\langle -\frac{3}{5}|\mathbf{F}_1|, \frac{4}{5}|\mathbf{F}_1| \right\rangle$, $\mathbf{F}_2 = \left\langle |\mathbf{F}_2|\cos\beta, |\mathbf{F}_2|\sin\beta \right\rangle = \left\langle \frac{4}{5}|\mathbf{F}_2|, \frac{3}{5}|\mathbf{F}_2| \right\rangle$, and $\mathbf{w} = \langle 0, -100 \rangle$. Since $\mathbf{F}_1 + \mathbf{F}_2 = \langle 0, 100 \rangle \Rightarrow \left\langle -\frac{3}{5}|\mathbf{F}_1| + \frac{4}{5}|\mathbf{F}_2|, \frac{4}{5}|\mathbf{F}_1| + \frac{3}{5}|\mathbf{F}_2| \right\rangle = \langle 0, 100 \rangle \Rightarrow -\frac{3}{5}|\mathbf{F}_1| + \frac{4}{5}|\mathbf{F}_2| = 0$ and $\frac{4}{5}|\mathbf{F}_1| + \frac{3}{5}|\mathbf{F}_2| = 100$. Solving the first equation for $|\mathbf{F}_2|$ results in: $|\mathbf{F}_2| = \frac{3}{4}|\mathbf{F}_1|$. Substituting this result into the second equation gives us: $\frac{4}{5}|\mathbf{F}_1| + \frac{9}{20}|\mathbf{F}_1| = 100 \Rightarrow |\mathbf{F}_1| = 80$ lb. $\Rightarrow |\mathbf{F}_2| = 60$ lb. $\Rightarrow \mathbf{F}_1 = \langle -48, 64 \rangle$ and $\mathbf{F}_2 = \langle 48, 36 \rangle$, and $\alpha = \tan^{-1}\left(\frac{4}{3}\right)$ and $\beta = \tan^{-1}\left(\frac{3}{4}\right)$

   (b) By the Law of Cosines we have $\cos\alpha = \frac{5^2 + 13^2 - 12^2}{2(5)(13)} = \frac{5}{13}$ and $\cos\beta = \frac{12^2 + 13^2 - 5^2}{2(12)(13)} = \frac{12}{13} \Rightarrow \sin\alpha = \frac{12}{13}$ and $\sin\beta = \frac{5}{13}$ $\Rightarrow \mathbf{F}_1 = \left\langle -|\mathbf{F}_1|\cos\alpha, |\mathbf{F}_1|\sin\alpha \right\rangle = \left\langle -\frac{5}{13}|\mathbf{F}_1|, \frac{12}{13}|\mathbf{F}_1| \right\rangle$, $\mathbf{F}_2 = \left\langle |\mathbf{F}_2|\cos\beta, |\mathbf{F}_2|\sin\beta \right\rangle = \left\langle \frac{12}{13}|\mathbf{F}_2|, \frac{5}{13}|\mathbf{F}_2| \right\rangle$, and $\mathbf{w} = \langle 0, -200 \rangle$. Since $\mathbf{F}_1 + \mathbf{F}_2 = \langle 0, 200 \rangle \Rightarrow \left\langle -\frac{5}{13}|\mathbf{F}_1| + \frac{12}{13}|\mathbf{F}_2|, \frac{12}{13}|\mathbf{F}_1| + \frac{5}{13}|\mathbf{F}_2| \right\rangle = \langle 0, 200 \rangle$ $\Rightarrow -\frac{5}{13}|\mathbf{F}_1| + \frac{12}{13}|\mathbf{F}_2| = 0$ and $\frac{12}{13}|\mathbf{F}_1| + \frac{5}{13}|\mathbf{F}_2| = 200$. Solving the first equation for $|\mathbf{F}_2|$ results in: $|\mathbf{F}_2| = \frac{5}{12}|\mathbf{F}_1|$. Substituting this result into the second equation gives us: $\frac{12}{13}|\mathbf{F}_1| + \frac{25}{156}|\mathbf{F}_1| = 200 \Rightarrow |\mathbf{F}_1| = \frac{2400}{13} \approx 184.615$ lb. $\Rightarrow |\mathbf{F}_2| = \frac{1000}{13} \approx 76.923$ lb. $\Rightarrow \mathbf{F}_1 = \left\langle -\frac{12000}{1169}, \frac{28800}{1169} \right\rangle \approx \langle -71.006, 170.414 \rangle$ and $\mathbf{F}_2 = \left\langle \frac{12000}{1169}, \frac{5000}{1169} \right\rangle$ $\approx \langle 71.006, 29.586 \rangle$.

6. (a) $\mathbf{T}_1 = \left\langle -|\mathbf{T}_1|\cos\alpha, |\mathbf{T}_1|\sin\alpha \right\rangle$, $\mathbf{T}_2 = \left\langle |\mathbf{T}_2|\cos\beta, |\mathbf{T}_2|\sin\beta \right\rangle$, and $\mathbf{w} = \langle 0, -w \rangle$. Since $\mathbf{T}_1 + \mathbf{T}_2 = \langle 0, w \rangle \Rightarrow$ $\left\langle -|\mathbf{T}_1|\cos\alpha + |\mathbf{T}_2|\cos\beta, |\mathbf{T}_1|\sin\alpha + |\mathbf{T}_2|\sin\beta \right\rangle = \langle 0, w \rangle \Rightarrow -|\mathbf{T}_1|\cos\alpha + |\mathbf{T}_2|\cos\beta = 0$ and $|\mathbf{T}_1|\sin\alpha + |\mathbf{T}_2|\sin\beta = w$. Solving the first equation for $|\mathbf{T}_2|$ results in: $|\mathbf{T}_2| = \frac{\cos\alpha}{\cos\beta}|\mathbf{T}_1|$. Substituting this result into

the second equation gives us: $|\mathbf{T}_1|\sin\alpha + \frac{\cos\alpha\sin\beta}{\cos\beta}|\mathbf{T}_1| = w \Rightarrow |\mathbf{T}_1| = \frac{w\cos\beta}{\sin\alpha\cos\beta+\cos\alpha\sin\beta} = \frac{w\cos\beta}{\sin(\alpha+\beta)}$ and $|\mathbf{T}_2| = \frac{w\cos\alpha}{\sin(\alpha+\beta)}$

(b) $\frac{d}{d\alpha}(|\mathbf{T}_1|) = \frac{d}{d\alpha}\left(\frac{w\cos\beta}{\sin(\alpha+\beta)}\right) = \frac{-w\cos\beta\cos(\alpha+\beta)}{\sin^2(\alpha+\beta)}$; $\frac{d}{d\alpha}(|\mathbf{T}_1|) = 0 \Rightarrow -w\cos\beta\cos(\alpha+\beta) = 0 \Rightarrow \cos(\alpha+\beta) = 0$ $\Rightarrow \alpha+\beta = \frac{\pi}{2} \Rightarrow \alpha = \frac{\pi}{2} - \beta$; $\frac{d^2}{d\alpha^2}(|\mathbf{T}_1|) = \frac{d}{d\alpha}\left(\frac{-w\cos\beta\cos(\alpha+\beta)}{\sin^2(\alpha+\beta)}\right) = \frac{w\cos\beta\left(\cos^2(\alpha+\beta)+1\right)}{\sin^3(\alpha+\beta)}$;

$\left.\frac{d^2}{d\alpha^2}(|\mathbf{T}_1|)\right|_{\alpha=\frac{\pi}{2}-\beta} = w\cos\beta > 0 \Rightarrow$ local minimum when $\alpha = \frac{\pi}{2} - \beta$

(c) $\frac{d}{d\beta}(|\mathbf{T}_2|) = \frac{d}{d\beta}\left(\frac{w\cos\alpha}{\sin(\alpha+\beta)}\right) = \frac{-w\cos\alpha\cos(\alpha+\beta)}{\sin^2(\alpha+\beta)}$; $\frac{d}{d\beta}(|\mathbf{T}_2|) = 0 \Rightarrow -w\cos\alpha\cos(\alpha+\beta) = 0 \Rightarrow \cos(\alpha+\beta) = 0$ $\Rightarrow \alpha+\beta = \frac{\pi}{2} \Rightarrow \beta = \frac{\pi}{2} - \alpha$; $\frac{d^2}{d\beta^2}(|\mathbf{T}_2|) = \frac{d}{d\beta}\left(\frac{-w\cos\alpha\cos(\alpha+\beta)}{\sin^2(\alpha+\beta)}\right) = \frac{w\cos\alpha\left(\cos^2(\alpha+\beta)+1\right)}{\sin^3(\alpha+\beta)}$;

$\left.\frac{d^2}{d\alpha^2}(|\mathbf{T}_2|)\right|_{\beta=\frac{\pi}{2}-\alpha} = w\cos\alpha > 0 \Rightarrow$ local minimum when $\beta = \frac{\pi}{2} - \alpha$

7. (a) If $P(x, y, z)$ is a point in the plane determined by the three points $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$ and $P_3(x_3, y_3, z_3)$, then the vectors $\overrightarrow{PP_1}$, $\overrightarrow{PP_2}$ and $\overrightarrow{PP_3}$ all lie in the plane. Thus $\overrightarrow{PP_1} \cdot (\overrightarrow{PP_2} \times \overrightarrow{PP_3}) = 0$ $\Rightarrow \begin{vmatrix} x_1 - x & y_1 - y & z_1 - z \\ x_2 - x & y_2 - y & z_2 - z \\ x_3 - x & y_3 - y & z_3 - z \end{vmatrix} = 0$ by the determinant formula for the triple scalar product in Section 12.4.

(b) Subtract row 1 from rows 2, 3, and 4 and evaluate the resulting determinant (which has the same value as the given determinant) by cofactor expansion about column 4. This expansion is exactly the determinant in part (a) so we have all points $P(x, y, z)$ in the plane determined by $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, and $P_3(x_3, y_3, z_3)$.

8. Let L_1: $x = a_1 s + b_1$, $y = a_2 s + b_2$, $z = a_3 s + b_3$ and L_2: $x = c_1 t + d_1$, $y = c_2 t + d_2$, $z = c_3 t + d_3$. If $L_1 \parallel L_2$, then for some k, $a_i = kc_i$, $i = 1, 2, 3$ and the determinant $\begin{vmatrix} a_1 & c_1 & b_1 - d_1 \\ a_2 & c_2 & b_2 - d_2 \\ a_3 & c_3 & b_3 - d_3 \end{vmatrix} = \begin{vmatrix} kc_1 & c_1 & b_1 - d_1 \\ kc_2 & c_2 & b_2 - d_2 \\ kc_3 & c_3 & b_3 - d_3 \end{vmatrix} = 0$, since the first column is a multiple of the second column. The lines L_1 and L_2 intersect if and only if the system $\begin{cases} a_1 s - c_1 t + (b_1 - d_1) = 0 \\ a_2 s - c_2 t + (b_2 - d_2) = 0 \\ a_3 s - c_3 t + (b_3 - d_3) = 0 \end{cases}$ has a nontrivial solution $\Leftrightarrow$ the determinant of the coefficients is zero.

9. (a) Place the tetrahedron so that A is at (0, 0, 0), the point P is on the y-axis, and $\triangle ABC$ lies in the xy-plane. Since $\triangle ABC$ is an equilateral triangle, all the angles in the triangle are 60° and since AP bisects BC $\Rightarrow \triangle ABP$ is a 30°- 60°- 90° trinagle. Thus the coordinates of P are $\left(0, \sqrt{3}, 0\right)$, the coordinates of B are $\left(1, \sqrt{3}, 0\right)$, and the coordinates of C are $\left(-1, \sqrt{3}, 0\right)$. Let the coordinates of D be given by (a, b, c). Since all of the faces are equilateral trinagles $\Rightarrow$ all the angles in each of the triangles are 60° $\Rightarrow \cos(\angle DAB) = \cos(60°) = \frac{\overrightarrow{AD}\cdot\overrightarrow{AB}}{|\overrightarrow{AD}||\overrightarrow{AB}|} = \frac{a+b\sqrt{3}}{(2)(2)} = \frac{1}{2}$ $\Rightarrow a + b\sqrt{3} = 2$ and $\cos(\angle DAC) = \cos(60°) = \frac{\overrightarrow{AD}\cdot\overrightarrow{AC}}{|\overrightarrow{AD}||\overrightarrow{AC}|} = \frac{-a+b\sqrt{3}}{(2)(2)} = \frac{1}{2} \Rightarrow -a + b\sqrt{3} = 2$. Add the two equations to obtain: $2b\sqrt{3} = 4 \Rightarrow b = \frac{2}{\sqrt{3}}$. Substituting this value for b in the first equation gives us: $a + \left(\frac{2}{\sqrt{3}}\right)\sqrt{3} = 2$ $\Rightarrow a = 0$. Since $|\overrightarrow{AD}| = \sqrt{a^2 + b^2 + c^2} = 2 \Rightarrow 0^2 + \left(\frac{2}{\sqrt{3}}\right)^2 + c^2 = 4 \Rightarrow c = \frac{2\sqrt{2}}{\sqrt{3}}$. Thus the coordinates of D are $\left(0, \frac{2}{\sqrt{3}}, \frac{2\sqrt{2}}{\sqrt{3}}\right)$. $\cos\theta = \cos(\angle DAP) = \frac{\overrightarrow{AD}\cdot\overrightarrow{AP}}{|\overrightarrow{AD}||\overrightarrow{AP}|} = \frac{2}{2\sqrt{3}} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \Rightarrow 57.74°$.

(b) Since $\triangle ABC$ lies in the xy-plane $\Rightarrow$ the normal to the face given by $\triangle ABC$ is $\mathbf{n}_1 = \mathbf{k}$. The face given by $\triangle BCD$ is an adjacent face. The vectors $\overrightarrow{DB} = \mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} - \frac{2\sqrt{2}}{\sqrt{3}}\mathbf{k}$ and $\overrightarrow{DC} = -\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} - \frac{2\sqrt{2}}{\sqrt{3}}\mathbf{k}$ both lie in the plane containing

$\triangle$BCD. The normal to this plane is given by $\mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & \frac{1}{\sqrt{3}} & -\frac{2\sqrt{2}}{\sqrt{3}} \\ -1 & \frac{1}{\sqrt{3}} & -\frac{2\sqrt{2}}{\sqrt{3}} \end{vmatrix} = \frac{4\sqrt{2}}{\sqrt{3}}\mathbf{j} + \frac{2}{\sqrt{3}}\mathbf{k}$. The angle θ between two adjacent faces is given by $\cos\theta = \cos(\angle \mathrm{DAP}) = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = \frac{2/\sqrt{3}}{(1)\left(6/\sqrt{3}\right)} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{3}\right) \approx 70.53°$.

10. Extend $\overrightarrow{CD}$ to $\overrightarrow{CG}$ so that $\overrightarrow{CD} = \overrightarrow{DG}$. Then $\overrightarrow{CG} = t\,\overrightarrow{CF} = \overrightarrow{CB} + \overrightarrow{BG}$ and $t\,\overrightarrow{CF} = 3\,\overrightarrow{CE} + \overrightarrow{CA}$, since ACBG is a parallelogram. If $t\,\overrightarrow{CF} - 3\,\overrightarrow{CE} - \overrightarrow{CA} = \mathbf{0}$, then $t - 3 - 1 = 0 \Rightarrow t = 4$, since F, E, and A are collinear. Therefore, $\overrightarrow{CG} = 4\,\overrightarrow{CF} \Rightarrow \overrightarrow{CD} = 2\,\overrightarrow{CF} \Rightarrow$ F is the midpoint of $\overline{CD}$.

11. If Q(x, y) is a point on the line ax + by = c, then $\overrightarrow{P_1Q} = (x - x_1)\mathbf{i} + (y - y_1)\mathbf{j}$, and $\mathbf{n} = a\mathbf{i} + b\mathbf{j}$ is normal to the line. The distance is $\left|\mathrm{proj}_{\mathbf{n}}\, \overrightarrow{P_1Q}\right| = \left|\frac{[(x-x_1)\mathbf{i} + (y-y_1)\mathbf{j}]\cdot(a\mathbf{i}+b\mathbf{j})}{\sqrt{a^2+b^2}}\right| = \frac{|a(x-x_1)+b(y-y_1)|}{\sqrt{a^2+b^2}}$ $= \frac{|ax_1+by_1-c|}{\sqrt{a^2+b^2}}$, since c = ax + by.

12. (a) Let Q(x, y, z) be any point on Ax + By + Cz − D = 0. Let $\overrightarrow{QP_1} = (x - x_1)\mathbf{i} + (y - y_1)\mathbf{j} + (z - z_1)\mathbf{k}$, and $\mathbf{n} = \frac{A\mathbf{i}+B\mathbf{j}+C\mathbf{k}}{\sqrt{A^2+B^2+C^2}}$. The distance is $\left|\mathrm{proj}_{\mathbf{n}}\, \overrightarrow{QP_1}\right| = \left|((x-x_1)\mathbf{i} + (y-y_1)\mathbf{j} + (z-z_1)\mathbf{k}) \cdot \left(\frac{A\mathbf{i}+B\mathbf{j}+C\mathbf{k}}{\sqrt{A^2+B^2+C^2}}\right)\right|$ $= \frac{|Ax_1+By_1+Cz_1-(Ax+By+Cz)|}{\sqrt{A^2+B^2+C^2}} = \frac{|Ax_1+By_1+Cz_1-D|}{\sqrt{A^2+B^2+C^2}}$.

(b) Since both tangent planes are parallel, one-half of the distance between them is equal to the radius of the sphere, i.e., $r = \frac{1}{2}\frac{|3-9|}{\sqrt{1+1+1}} = \sqrt{3}$ (see also Exercise 12a). Clearly, the points (1, 2, 3) and (−1, −2, −3) are on the line containing the sphere's center. Hence, the line containing the center is x = 1 + 2t, y = 2 + 4t, z = 3 + 6t. The distance from the plane x + y + z − 3 = 0 to the center is $\sqrt{3}$ $\Rightarrow \frac{|(1+2t)+(2+4t)+(3+6t)-3|}{\sqrt{1+1+1}} = \sqrt{3}$ from part (a) $\Rightarrow t = 0 \Rightarrow$ the center is at (1, 2, 3). Therefore an equation of the sphere is $(x-1)^2 + (y-2)^2 + (z-3)^2 = 3$.

13. (a) If (x_1, y_1, z_1) is on the plane $Ax + By + Cz = D_1$, then the distance d between the planes is $d = \frac{|Ax_1+By_1+Cz_1-D_2|}{\sqrt{A^2+B^2+C^2}} = \frac{|D_1-D_2|}{|A\mathbf{i}+B\mathbf{j}+C\mathbf{k}|}$, since $Ax_1 + By_1 + Cz_1 = D_1$, by Exercise 12(a).

(b) $d = \frac{|12-6|}{\sqrt{4+9+1}} = \frac{6}{\sqrt{14}}$

(c) $\frac{|2(3)+(-1)(2)+2(-1)+4|}{\sqrt{14}} = \frac{|2(3)+(-1)(2)+2(-1)-D|}{\sqrt{14}} \Rightarrow D = 8$ or $-4 \Rightarrow$ the desired plane is 2x − y + 2x = 8

(d) Choose the point (2, 0, 1) on the plane. Then $\frac{|3-D|}{\sqrt{6}} = 5 \Rightarrow D = 3 \pm 5\sqrt{6} \Rightarrow$ the desired planes are $x - 2y + z = 3 + 5\sqrt{6}$ and $x - 2y + z = 3 - 5\sqrt{6}$.

14. Let $\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{BC}$ and D(x, y, z) be any point in the plane determined by A, B and C. Then the point D lies in this plane if and only if $\overrightarrow{AD} \cdot \mathbf{n} = 0 \Leftrightarrow \overrightarrow{AD} \cdot (\overrightarrow{AB} \times \overrightarrow{BC}) = 0$.

15. $\mathbf{n} = \mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$ is normal to the plane x + 2y + 6z = 6; $\mathbf{v} \times \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & 2 & 6 \end{vmatrix} = 4\mathbf{i} - 5\mathbf{j} + \mathbf{k}$ is parallel to the plane and perpendicular to the plane of $\mathbf{v}$ and $\mathbf{n} \Rightarrow \mathbf{w} = \mathbf{n} \times (\mathbf{v} \times \mathbf{n}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 6 \\ 4 & -5 & 1 \end{vmatrix} = 32\mathbf{i} + 23\mathbf{j} - 13\mathbf{k}$ is a

vector parallel to the plane $x + 2y + 6z = 6$ in the direction of the projection vector $\text{proj}_P\, \mathbf{v}$. Therefore,
$\text{proj}_P\, \mathbf{v} = \text{proj}_{\mathbf{w}}\, \mathbf{v} = \left(\mathbf{v} \cdot \frac{\mathbf{w}}{|\mathbf{w}|}\right) \frac{\mathbf{w}}{|\mathbf{w}|} = \left(\frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{w}|^2}\right) \mathbf{w} = \left(\frac{32+23-13}{32^2+23^2+13^2}\right) \mathbf{w} = \frac{42}{1722}\, \mathbf{w} = \frac{1}{41}\, \mathbf{w} = \frac{32}{41}\, \mathbf{i} + \frac{23}{41}\, \mathbf{j} - \frac{13}{41}\, \mathbf{k}$

16. $\text{proj}_{\mathbf{z}}\, \mathbf{w} = -\text{proj}_{\mathbf{z}}\, \mathbf{v}$ and $\mathbf{w} - \text{proj}_{\mathbf{z}}\, \mathbf{w} = \mathbf{v} - \text{proj}_{\mathbf{z}}\, \mathbf{v} \Rightarrow \mathbf{w} = (\mathbf{w} - \text{proj}_{\mathbf{z}}\, \mathbf{w}) + \text{proj}_{\mathbf{z}}\, \mathbf{w} = (\mathbf{v} - \text{proj}_{\mathbf{z}}\, \mathbf{v}) + \text{proj}_{\mathbf{z}}\, \mathbf{w}$
$= \mathbf{v} - 2\,\text{proj}_{\mathbf{z}}\, \mathbf{v} = \mathbf{v} - 2\left(\frac{\mathbf{v}\cdot\mathbf{z}}{|\mathbf{z}|^2}\right) \mathbf{z}$

17. (a) $\mathbf{u} \times \mathbf{v} = 2\mathbf{i} \times 2\mathbf{j} = 4\mathbf{k} \Rightarrow (\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \mathbf{0}$; $(\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u} = 0\mathbf{v} - 0\mathbf{u} = \mathbf{0}$; $\mathbf{v} \times \mathbf{w} = 4\mathbf{i} \Rightarrow \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{0}$;
$(\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} = 0\mathbf{v} - 0\mathbf{w} = \mathbf{0}$

(b) $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 2 & 1 & -2 \end{vmatrix} = \mathbf{i} + 4\mathbf{j} + 3\mathbf{k} \Rightarrow (\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & 3 \\ -1 & 2 & -1 \end{vmatrix} = -10\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$;
$(\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u} = -4(2\mathbf{i} + \mathbf{j} - 2\mathbf{k}) - 2(\mathbf{i} - \mathbf{j} + \mathbf{k}) = -10\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$;
$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -2 \\ -1 & 2 & -1 \end{vmatrix} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k} \Rightarrow \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 3 & 4 & 5 \end{vmatrix} = -9\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}$;
$(\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} = -4(2\mathbf{i} + \mathbf{j} - 2\mathbf{k}) - (-1)(-\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = -9\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}$

(c) $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 0 \\ 2 & -1 & 1 \end{vmatrix} = \mathbf{i} - 2\mathbf{j} - 4\mathbf{k} \Rightarrow (\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & -4 \\ 1 & 0 & 2 \end{vmatrix} = -4\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$;
$(\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u} = 2(2\mathbf{i} - \mathbf{j} + \mathbf{k}) - 4(2\mathbf{i} + \mathbf{j}) = -4\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$;
$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ 1 & 0 & 2 \end{vmatrix} = -2\mathbf{i} - 3\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 0 \\ -2 & -3 & 1 \end{vmatrix} = \mathbf{i} - 2\mathbf{j} - 4\mathbf{k}$;
$(\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} = 2(2\mathbf{i} - \mathbf{j} + \mathbf{k}) - 3(\mathbf{i} + 2\mathbf{k}) = \mathbf{i} - 2\mathbf{j} - 4\mathbf{k}$

(d) $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -2 \\ -1 & 0 & -1 \end{vmatrix} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k} \Rightarrow (\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 3 & 1 \\ 2 & 4 & -2 \end{vmatrix} = -10\mathbf{i} - 10\mathbf{k}$;
$(\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u} = 10(-\mathbf{i} - \mathbf{k}) - 0(\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = -10\mathbf{i} - 10\mathbf{k}$;
$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & -1 \\ 2 & 4 & -2 \end{vmatrix} = 4\mathbf{i} - 4\mathbf{j} - 4\mathbf{k} \Rightarrow \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -2 \\ 4 & -4 & -4 \end{vmatrix} = -12\mathbf{i} - 4\mathbf{j} - 8\mathbf{k}$;
$(\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} = 10(-\mathbf{i} - \mathbf{k}) - 1(2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) = -12\mathbf{i} - 4\mathbf{j} - 8\mathbf{k}$

18. (a) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) + \mathbf{v} \times (\mathbf{w} \times \mathbf{u}) + \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} + (\mathbf{v} \cdot \mathbf{u})\mathbf{w} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u} + (\mathbf{w} \cdot \mathbf{v})\mathbf{u} - (\mathbf{w} \cdot \mathbf{u})\mathbf{v} = \mathbf{0}$

(b) $[\mathbf{u} \cdot (\mathbf{v} \times \mathbf{i})]\mathbf{i} + [(\mathbf{u} \cdot (\mathbf{v} \times \mathbf{j})]\mathbf{j} + [(\mathbf{u} \cdot (\mathbf{v} \times \mathbf{k})]\mathbf{k} = [(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{i}]\mathbf{i} + [(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{j}]\mathbf{j} + [(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{k}]\mathbf{k} = \mathbf{u} \times \mathbf{v}$

(c) $(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{w} \times \mathbf{r}) = \mathbf{u} \cdot [\mathbf{v} \times (\mathbf{w} \times \mathbf{r})] = \mathbf{u} \cdot [(\mathbf{v} \cdot \mathbf{r})\mathbf{w} - (\mathbf{v} \cdot \mathbf{w})\mathbf{r}] = (\mathbf{u} \cdot \mathbf{w})(\mathbf{v} \cdot \mathbf{r}) - (\mathbf{u} \cdot \mathbf{r})(\mathbf{v} \cdot \mathbf{w}) = \begin{vmatrix} \mathbf{u} \cdot \mathbf{w} & \mathbf{v} \cdot \mathbf{w} \\ \mathbf{u} \cdot \mathbf{r} & \mathbf{v} \cdot \mathbf{r} \end{vmatrix}$

19. The formula is always true; $\mathbf{u} \times [\mathbf{u} \times (\mathbf{u} \times \mathbf{v})] \cdot \mathbf{w} = \mathbf{u} \times [(\mathbf{u} \cdot \mathbf{v})\mathbf{u} - (\mathbf{u} \cdot \mathbf{u})\mathbf{v}] \cdot \mathbf{w}$
$= [(\mathbf{u} \cdot \mathbf{v})\mathbf{u} \times \mathbf{u} - (\mathbf{u} \cdot \mathbf{u})\mathbf{u} \times \mathbf{v}] \cdot \mathbf{w} = -|\mathbf{u}|^2 \mathbf{u} \times \mathbf{v} \cdot \mathbf{w} = -|\mathbf{u}|^2 \mathbf{u} \cdot \mathbf{v} \times \mathbf{w}$

20. If $\mathbf{u} = (\cos B)\mathbf{i} + (\sin B)\mathbf{j}$ and $\mathbf{v} = (\cos A)\mathbf{i} + (\sin A)\mathbf{j}$, where $A > B$, then $\mathbf{u} \times \mathbf{v} = [|\mathbf{u}|\,|\mathbf{v}| \sin(A - B)]\,\mathbf{k}$
$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos B & \sin B & 0 \\ \cos A & \sin A & 0 \end{vmatrix} = (\cos B \sin A - \sin B \cos A)\mathbf{k} \Rightarrow \sin(A - B) = \cos B \sin A - \sin B \cos A$, since
$|\mathbf{u}| = 1$ and $|\mathbf{v}| = 1$.

21. If $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$ and $\mathbf{v} = c\mathbf{i} + d\mathbf{j}$, then $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\theta \Rightarrow ac + bd = \sqrt{a^2+b^2}\,\sqrt{c^2+d^2}\cos\theta$
$\Rightarrow (ac+bd)^2 = (a^2+b^2)(c^2+d^2)\cos^2\theta \Rightarrow (ac+bd)^2 \le (a^2+b^2)(c^2+d^2)$, since $\cos^2\theta \le 1$.

22. If $\mathbf{u} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, then $\mathbf{u} \cdot \mathbf{u} = a^2 + b^2 + c^2 \ge 0$ and $\mathbf{u} \cdot \mathbf{u} = 0$ iff $a = b = c = 0$.

23. $|\mathbf{u}+\mathbf{v}|^2 = (\mathbf{u}+\mathbf{v}) \cdot (\mathbf{u}+\mathbf{v}) = \mathbf{u}\cdot\mathbf{u} + 2\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v} \le |\mathbf{u}|^2 + 2\,|\mathbf{u}|\,|\mathbf{v}| + |\mathbf{v}|^2 = (|\mathbf{u}| + |\mathbf{v}|)^2 \Rightarrow |\mathbf{u}+\mathbf{v}| \le |\mathbf{u}| + |\mathbf{v}|$

24. Let α denote the angle between $\mathbf{w}$ and $\mathbf{u}$, and β the angle between $\mathbf{w}$ and $\mathbf{v}$. Let $a = |\mathbf{u}|$ and $b = |\mathbf{v}|$. Then
$\cos\alpha = \frac{\mathbf{w}\cdot\mathbf{u}}{|\mathbf{w}|\,|\mathbf{u}|} = \frac{(a\mathbf{v}+b\mathbf{u})\cdot\mathbf{u}}{|\mathbf{w}|\,|\mathbf{u}|} = \frac{(a\mathbf{v}\cdot\mathbf{u}+b\mathbf{u}\cdot\mathbf{u})}{|\mathbf{w}|\,|\mathbf{u}|} = \frac{(a\mathbf{v}\cdot\mathbf{u}+b\mathbf{u}\cdot\mathbf{u})}{|\mathbf{w}|\,|\mathbf{u}|} = \frac{(a\mathbf{v}\cdot\mathbf{u}+ba^2)}{|\mathbf{w}|\,a} = \frac{\mathbf{v}\cdot\mathbf{u}+ba}{|\mathbf{w}|}$, and likewise, $\cos\beta = \frac{\mathbf{u}\cdot\mathbf{v}+ba}{|\mathbf{w}|}$.
Since the angle between $\mathbf{u}$ and $\mathbf{v}$ is always $\le \frac{\pi}{2}$ and $\cos\alpha = \cos\beta$, we have that $\alpha = \beta \Rightarrow \mathbf{w}$ bisects the angle between $\mathbf{u}$ and $\mathbf{v}$.

25. $(|\mathbf{u}|\mathbf{v} + |\mathbf{v}|\mathbf{u}) \cdot (|\mathbf{v}|\mathbf{u} - |\mathbf{u}|\mathbf{v}) = |\mathbf{u}|\mathbf{v} \cdot |\mathbf{v}|\mathbf{u} + |\mathbf{v}|\mathbf{u} \cdot |\mathbf{v}|\mathbf{u} - |\mathbf{u}|\mathbf{v} \cdot |\mathbf{u}|\mathbf{v} - |\mathbf{v}|\mathbf{u} \cdot |\mathbf{u}|\mathbf{v}$
$= |\mathbf{v}|\mathbf{u} \cdot |\mathbf{u}|\mathbf{v} + |\mathbf{v}|^2\mathbf{u}\cdot\mathbf{u} - |\mathbf{u}|^2\mathbf{v}\cdot\mathbf{v} - |\mathbf{v}|\mathbf{u} \cdot |\mathbf{u}|\mathbf{v} = |\mathbf{v}|^2|\mathbf{u}|^2 - |\mathbf{u}|^2|\mathbf{v}|^2 = 0$

CHAPTER 13 VECTOR-VALUED FUNCTIONS AND MOTION IN SPACE

13.1 CURVES IN SPACE AND THEIR TANGENTS

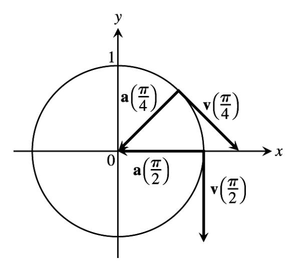
1. $x = t + 1$ and $y = t^2 - 1 \Rightarrow y = (x-1)^2 - 1 = x^2 - 2x$; $\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} \Rightarrow \mathbf{a} = \frac{d\mathbf{v}}{dt} = 2\mathbf{j} \Rightarrow \mathbf{v} = \mathbf{i} + 2\mathbf{j}$ and $\mathbf{a} = 2\mathbf{j}$ at $t = 1$

2. $x = \frac{t}{t+1}$ and $y = \frac{1}{t} \Rightarrow x = \frac{\frac{1}{y}}{\frac{1}{y}+1} = \frac{1}{1+y} \Rightarrow y = \frac{1}{x} - 1$; $\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{1}{(t+1)^2}\mathbf{i} - \frac{1}{t^2}\mathbf{j} \Rightarrow \mathbf{a} = \frac{d\mathbf{v}}{dt} = -\frac{2}{(t+1)^3}\mathbf{i} + \frac{2}{t^3}\mathbf{j}$
$\Rightarrow \mathbf{v} = 4\mathbf{i} - 4\mathbf{j}$ and $\mathbf{a} = -16\mathbf{i} - 16\mathbf{j}$ at $t = -\frac{1}{2}$

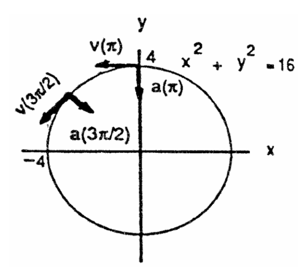
3. $x = e^t$ and $y = \frac{2}{9}e^{2t} \Rightarrow y = \frac{2}{9}x^2$; $\mathbf{v} = \frac{d\mathbf{r}}{dt} = e^t\mathbf{i} + \frac{4}{9}e^{2t}\mathbf{j} \Rightarrow \mathbf{a} = e^t\mathbf{i} + \frac{8}{9}e^{2t}\mathbf{j} \Rightarrow \mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{a} = 3\mathbf{i} + 8\mathbf{j}$ at $t = \ln 3$

4. $x = \cos 2t$ and $y = 3\sin 2t \Rightarrow x^2 + \frac{1}{9}y^2 = 1$; $\mathbf{v} = \frac{d\mathbf{r}}{dt} = (-2\sin 2t)\mathbf{i} + (6\cos 2t)\mathbf{j} \Rightarrow \mathbf{a} = \frac{d\mathbf{v}}{dt}$
$= (-4\cos 2t)\mathbf{i} + (-12\sin 2t)\mathbf{j} \Rightarrow \mathbf{v} = 6\mathbf{j}$ and $\mathbf{a} = -4\mathbf{i}$ at $t = 0$

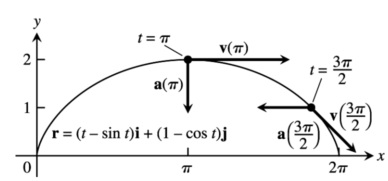
5. $\mathbf{v} = \frac{d\mathbf{r}}{dt} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j}$ and $\mathbf{a} = \frac{d\mathbf{v}}{dt} = -(\sin t)\mathbf{i} - (\cos t)\mathbf{j}$
$\Rightarrow$ for $t = \frac{\pi}{4}$, $\mathbf{v}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$ and
$\mathbf{a}\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$; for $t = \frac{\pi}{2}$, $\mathbf{v}\left(\frac{\pi}{2}\right) = -\mathbf{j}$ and
$\mathbf{a}\left(\frac{\pi}{2}\right) = -\mathbf{i}$



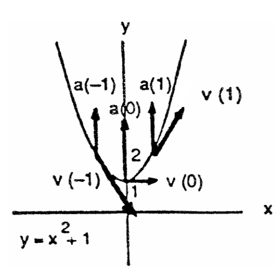
6. $\mathbf{v} = \frac{d\mathbf{r}}{dt} = \left(-2\sin\frac{t}{2}\right)\mathbf{i} + \left(2\cos\frac{t}{2}\right)\mathbf{j}$ and $\mathbf{a} = \frac{d\mathbf{v}}{dt}$
$= \left(-\cos\frac{t}{2}\right)\mathbf{i} + \left(-\sin\frac{t}{2}\right)\mathbf{j} \Rightarrow$ for $t = \pi$, $\mathbf{v}(\pi) = -2\mathbf{i}$ and
$\mathbf{a}(\pi) = -\mathbf{j}$; for $t = \frac{3\pi}{2}$, $\mathbf{v}\left(\frac{3\pi}{2}\right) = -\sqrt{2}\,\mathbf{i} - \sqrt{2}\,\mathbf{j}$ and
$\mathbf{a}\left(\frac{3\pi}{2}\right) = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$



7. $\mathbf{v} = \frac{d\mathbf{r}}{dt} = (1 - \cos t)\mathbf{i} + (\sin t)\mathbf{j}$ and $\mathbf{a} = \frac{d\mathbf{v}}{dt}$
$= (\sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow$ for $t = \pi$, $\mathbf{v}(\pi) = 2\mathbf{i}$ and $\mathbf{a}(\pi) = -\mathbf{j}$;
for $t = \frac{3\pi}{2}$, $\mathbf{v}\left(\frac{3\pi}{2}\right) = \mathbf{i} - \mathbf{j}$ and $\mathbf{a}\left(\frac{3\pi}{2}\right) = -\mathbf{i}$



8. $\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j}$ and $\mathbf{a} = \frac{d\mathbf{v}}{dt} = 2\mathbf{j} \Rightarrow$ for $t = -1$,
$\mathbf{v}(-1) = \mathbf{i} - 2\mathbf{j}$ and $\mathbf{a}(-1) = 2\mathbf{j}$; for $t = 0$, $\mathbf{v}(0) = \mathbf{i}$ and
$\mathbf{a}(0) = 2\mathbf{j}$; for $t = 1$, $\mathbf{v}(1) = \mathbf{i} + 2\mathbf{j}$ and $\mathbf{a}(1) = 2\mathbf{j}$



9. $\mathbf{r} = (t+1)\mathbf{i} + (t^2-1)\,\mathbf{j} + 2t\mathbf{k} \Rightarrow \mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 2\mathbf{k} \Rightarrow \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = 2\mathbf{j}$; Speed: $|\mathbf{v}(1)| = \sqrt{1^2 + (2(1))^2 + 2^2} = 3$; Direction: $\frac{\mathbf{v}(1)}{|\mathbf{v}(1)|} = \frac{\mathbf{i}+2(1)\mathbf{j}+2\mathbf{k}}{3} = \frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k} \Rightarrow \mathbf{v}(1) = 3\left(\frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}\right)$

10. $\mathbf{r} = (1+t)\mathbf{i} + \frac{t^2}{\sqrt{2}}\mathbf{j} + \frac{t^3}{3}\mathbf{k} \Rightarrow \mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i} + \frac{2t}{\sqrt{2}}\mathbf{j} + t^2\mathbf{k} \Rightarrow \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = \frac{2}{\sqrt{2}}\mathbf{j} + 2t\mathbf{k}$; Speed: $|\mathbf{v}(1)| = \sqrt{1^2 + \left(\frac{2(1)}{\sqrt{2}}\right)^2 + (1^2)^2} = 2$; Direction: $\frac{\mathbf{v}(1)}{|\mathbf{v}(1)|} = \frac{\mathbf{i} + \frac{2(1)}{\sqrt{2}}\mathbf{j} + (1^2)\mathbf{k}}{2} = \frac{1}{2}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{2}\mathbf{k} \Rightarrow \mathbf{v}(1) = 2\left(\frac{1}{2}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{2}\mathbf{k}\right)$

11. $\mathbf{r} = (2\cos t)\mathbf{i} + (3\sin t)\mathbf{j} + 4t\mathbf{k} \Rightarrow \mathbf{v} = \frac{d\mathbf{r}}{dt} = (-2\sin t)\mathbf{i} + (3\cos t)\mathbf{j} + 4\mathbf{k} \Rightarrow \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = (-2\cos t)\mathbf{i} - (3\sin t)\mathbf{j}$; Speed: $\left|\mathbf{v}\left(\frac{\pi}{2}\right)\right| = \sqrt{\left(-2\sin\frac{\pi}{2}\right)^2 + \left(3\cos\frac{\pi}{2}\right)^2 + 4^2} = 2\sqrt{5}$; Direction: $\frac{\mathbf{v}\left(\frac{\pi}{2}\right)}{\left|\mathbf{v}\left(\frac{\pi}{2}\right)\right|} = \left(-\frac{2}{2\sqrt{5}}\sin\frac{\pi}{2}\right)\mathbf{i} + \left(\frac{3}{2\sqrt{5}}\cos\frac{\pi}{2}\right)\mathbf{j} + \frac{4}{2\sqrt{5}}\mathbf{k} = -\frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{k} \Rightarrow \mathbf{v}\left(\frac{\pi}{2}\right) = 2\sqrt{5}\left(-\frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{k}\right)$

12. $\mathbf{r} = (\sec t)\mathbf{i} + (\tan t)\mathbf{j} + \frac{4}{3}t\mathbf{k} \Rightarrow \mathbf{v} = \frac{d\mathbf{r}}{dt} = (\sec t\tan t)\mathbf{i} + (\sec^2 t)\,\mathbf{j} + \frac{4}{3}\mathbf{k} \Rightarrow \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = (\sec t\tan^2 t + \sec^3 t)\,\mathbf{i} + (2\sec^2 t\tan t)\,\mathbf{j}$; Speed: $\left|\mathbf{v}\left(\frac{\pi}{6}\right)\right| = \sqrt{\left(\sec\frac{\pi}{6}\tan\frac{\pi}{6}\right)^2 + \left(\sec^2\frac{\pi}{6}\right)^2 + \left(\frac{4}{3}\right)^2} = 2$; Direction: $\frac{\mathbf{v}\left(\frac{\pi}{6}\right)}{\left|\mathbf{v}\left(\frac{\pi}{6}\right)\right|} = \frac{\left(\sec\frac{\pi}{6}\tan\frac{\pi}{6}\right)\mathbf{i} + \left(\sec^2\frac{\pi}{6}\right)\mathbf{j} + \frac{4}{3}\mathbf{k}}{2} = \frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k} \Rightarrow \mathbf{v}\left(\frac{\pi}{6}\right) = 2\left(\frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}\right)$

13. $\mathbf{r} = (2\ln(t+1))\mathbf{i} + t^2\mathbf{j} + \frac{t^2}{2}\mathbf{k} \Rightarrow \mathbf{v} = \frac{d\mathbf{r}}{dt} = \left(\frac{2}{t+1}\right)\mathbf{i} + 2t\mathbf{j} + t\mathbf{k} \Rightarrow \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = \left[\frac{-2}{(t+1)^2}\right]\mathbf{i} + 2\mathbf{j} + \mathbf{k}$; Speed: $|\mathbf{v}(1)| = \sqrt{\left(\frac{2}{1+1}\right)^2 + (2(1))^2 + 1^2} = \sqrt{6}$; Direction: $\frac{\mathbf{v}(1)}{|\mathbf{v}(1)|} = \frac{\left(\frac{2}{1+1}\right)\mathbf{i} + 2(1)\mathbf{j} + (1)\mathbf{k}}{\sqrt{6}} = \frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} + \frac{1}{\sqrt{6}}\mathbf{k} \Rightarrow \mathbf{v}(1) = \sqrt{6}\left(\frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} + \frac{1}{\sqrt{6}}\mathbf{k}\right)$

14. $\mathbf{r} = (e^{-t})\,\mathbf{i} + (2\cos 3t)\mathbf{j} + (2\sin 3t)\mathbf{k} \Rightarrow \mathbf{v} = \frac{d\mathbf{r}}{dt} = (-e^{-t})\,\mathbf{i} - (6\sin 3t)\mathbf{j} + (6\cos 3t)\mathbf{k} \Rightarrow \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = (e^{-t})\,\mathbf{i} - (18\cos 3t)\mathbf{j} - (18\sin 3t)\mathbf{k}$; Speed: $|\mathbf{v}(0)| = \sqrt{(-e^0)^2 + [-6\sin 3(0)]^2 + [6\cos 3(0)]^2} = \sqrt{37}$; Direction: $\frac{\mathbf{v}(0)}{|\mathbf{v}(0)|} = \frac{(-e^0)\,\mathbf{i} - 6\sin 3(0)\mathbf{j} + 6\cos 3(0)\mathbf{k}}{\sqrt{37}} = -\frac{1}{\sqrt{37}}\mathbf{i} + \frac{6}{\sqrt{37}}\mathbf{k} \Rightarrow \mathbf{v}(0) = \sqrt{37}\left(-\frac{1}{\sqrt{37}}\mathbf{i} + \frac{6}{\sqrt{37}}\mathbf{k}\right)$

15. $\mathbf{v} = 3\mathbf{i} + \sqrt{3}\,\mathbf{j} + 2t\mathbf{k}$ and $\mathbf{a} = 2\mathbf{k} \Rightarrow \mathbf{v}(0) = 3\mathbf{i} + \sqrt{3}\,\mathbf{j}$ and $\mathbf{a}(0) = 2\mathbf{k} \Rightarrow |\mathbf{v}(0)| = \sqrt{3^2 + \left(\sqrt{3}\right)^2 + 0^2} = \sqrt{12}$ and $|\mathbf{a}(0)| = \sqrt{2^2} = 2$; $\mathbf{v}(0)\cdot\mathbf{a}(0) = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$

16. $\mathbf{v} = \frac{\sqrt{2}}{2}\mathbf{i} + \left(\frac{\sqrt{2}}{2} - 32t\right)\mathbf{j}$ and $\mathbf{a} = -32\mathbf{j} \Rightarrow \mathbf{v}(0) = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$ and $\mathbf{a}(0) = -32\mathbf{j} \Rightarrow |\mathbf{v}(0)| = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2} = 1$ and $|\mathbf{a}(0)| = \sqrt{(-32)^2} = 32$; $\mathbf{v}(0)\cdot\mathbf{a}(0) = \left(\frac{\sqrt{2}}{2}\right)(-32) = -16\sqrt{2} \Rightarrow \cos\theta = \frac{-16\sqrt{2}}{1(32)} = -\frac{\sqrt{2}}{2} \Rightarrow \theta = \frac{3\pi}{4}$

17. $\mathbf{v} = \left(\frac{2t}{t^2+1}\right)\mathbf{i} + \left(\frac{1}{t^2+1}\right)\mathbf{j} + t(t^2+1)^{-1/2}\mathbf{k}$ and $\mathbf{a} = \left[\frac{-2t^2+2}{(t^2+1)^2}\right]\mathbf{i} - \left[\frac{2t}{(t^2+1)^2}\right]\mathbf{j} + \left[\frac{1}{(t^2+1)^{3/2}}\right]\mathbf{k} \Rightarrow \mathbf{v}(0) = \mathbf{j}$ and $\mathbf{a}(0) = 2\mathbf{i} + \mathbf{k} \Rightarrow |\mathbf{v}(0)| = 1$ and $|\mathbf{a}(0)| = \sqrt{2^2 + 1^2} = \sqrt{5}$; $\mathbf{v}(0)\cdot\mathbf{a}(0) = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$

18. $\mathbf{v} = \frac{2}{3}(1+t)^{1/2}\,\mathbf{i} - \frac{2}{3}(1-t)^{1/2}\,\mathbf{j} + \frac{1}{3}\mathbf{k}$ and $\mathbf{a} = \frac{1}{3}(1+t)^{-1/2}\,\mathbf{i} + \frac{1}{3}(1-t)^{-1/2}\,\mathbf{j} \Rightarrow \mathbf{v}(0) = \frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}$ and $\mathbf{a}(0) = \frac{1}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} \Rightarrow |\mathbf{v}(0)| = \sqrt{\left(\frac{2}{3}\right)^2 + \left(-\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2} = 1$ and $|\mathbf{a}(0)| = \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2} = \frac{\sqrt{2}}{3}$; $\mathbf{v}(0)\cdot\mathbf{a}(0) = \frac{2}{9} - \frac{2}{9} = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$

19. $\mathbf{r}(t) = (\sin t)\mathbf{i} + (t^2 - \cos t)\,\mathbf{j} + e^t\mathbf{k} \Rightarrow \mathbf{v}(t) = (\cos t)\mathbf{i} + (2t + \sin t)\mathbf{j} + e^t\mathbf{k}$; $t_0 = 0 \Rightarrow \mathbf{v}(t_0) = \mathbf{i} + \mathbf{k}$ and $\mathbf{r}(t_0) = P_0 = (0, -1, 1) \Rightarrow x = 0 + t = t,\ y = -1$, and $z = 1 + t$ are parametric equations of the tangent line

20. $\mathbf{r}(t) = t^2\,\mathbf{i} + (2t - 1)\,\mathbf{j} + t^3\mathbf{k} \Rightarrow \mathbf{v}(t) = 2t\,\mathbf{i} + 2\,\mathbf{j} + 3t^2\mathbf{k}$; $t_0 = 2 \Rightarrow \mathbf{v}(2) = 4\,\mathbf{i} + 2\,\mathbf{j} + 12\mathbf{k}$ and $\mathbf{r}(t_0) = P_0 = (4, 3, 8) \Rightarrow x = 4 + 4t,\ y = 3 + 2t$, and $z = 8 + 12t$ are parametric equations of the tangent line

21. $\mathbf{r}(t) = (\ln t)\mathbf{i} + \frac{t-1}{t+2}\,\mathbf{j} + (t \ln t)\mathbf{k} \Rightarrow \mathbf{v}(t) = \frac{1}{t}\,\mathbf{i} + \frac{3}{(t+2)^2}\mathbf{j} + (\ln t + 1)\mathbf{k}$; $t_0 = 1 \Rightarrow \mathbf{v}(1) = \mathbf{i} + \frac{1}{3}\,\mathbf{j} + \mathbf{k}$ and $\mathbf{r}(t_0) = P_0 = (0, 0, 0) \Rightarrow x = 0 + t = t,\ y = 0 + \frac{1}{3}t = \frac{1}{3}t$, and $z = 0 + t = t$ are parametric equations of the tangent line

22. $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\,\mathbf{j} + (\sin 2t)\mathbf{k} \Rightarrow \mathbf{v}(t) = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + (2 \cos 2t)\mathbf{k}$; $t_0 = \frac{\pi}{2} \Rightarrow \mathbf{v}(t_0) = -\mathbf{i} - 2\mathbf{k}$ and $\mathbf{r}(t_0) = P_0 = (0, 1, 0) \Rightarrow x = 0 - t = -t,\ y = 1$, and $z = 0 - 2t = -2t$ are parametric equations of the tangent line

23. (a) $\mathbf{v}(t) = -(\sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow \mathbf{a}(t) = -(\cos t)\mathbf{i} - (\sin t)\mathbf{j}$;
   - (i) $|\mathbf{v}(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1 \Rightarrow$ constant speed;
   - (ii) $\mathbf{v} \cdot \mathbf{a} = (\sin t)(\cos t) - (\cos t)(\sin t) = 0 \Rightarrow$ yes, orthogonal;
   - (iii) counterclockwise movement;
   - (iv) yes, $\mathbf{r}(0) = \mathbf{i} + 0\mathbf{j}$

   (b) $\mathbf{v}(t) = -(2 \sin 2t)\mathbf{i} + (2 \cos 2t)\mathbf{j} \Rightarrow \mathbf{a}(t) = -(4 \cos 2t)\mathbf{i} - (4 \sin 2t)\mathbf{j}$;
   - (i) $|\mathbf{v}(t)| = \sqrt{4 \sin^2 2t + 4 \cos^2 2t} = 2 \Rightarrow$ constant speed;
   - (ii) $\mathbf{v} \cdot \mathbf{a} = 8 \sin 2t \cos 2t - 8 \cos 2t \sin 2t = 0 \Rightarrow$ yes, orthogonal;
   - (iii) counterclockwise movement;
   - (iv) yes, $\mathbf{r}(0) = \mathbf{i} + 0\mathbf{j}$

   (c) $\mathbf{v}(t) = -\sin\left(t - \frac{\pi}{2}\right)\mathbf{i} + \cos\left(t - \frac{\pi}{2}\right)\mathbf{j} \Rightarrow \mathbf{a}(t) = -\cos\left(t - \frac{\pi}{2}\right)\mathbf{i} - \sin\left(t - \frac{\pi}{2}\right)\mathbf{j}$;
   - (i) $|\mathbf{v}(t)| = \sqrt{\sin^2\left(t - \frac{\pi}{2}\right) + \cos^2\left(t - \frac{\pi}{2}\right)} = 1 \Rightarrow$ constant speed;
   - (ii) $\mathbf{v} \cdot \mathbf{a} = \sin\left(t - \frac{\pi}{2}\right)\cos\left(t - \frac{\pi}{2}\right) - \cos\left(t - \frac{\pi}{2}\right)\sin\left(t - \frac{\pi}{2}\right) = 0 \Rightarrow$ yes, orthogonal;
   - (iii) counterclockwise movement;
   - (iv) no, $\mathbf{r}(0) = 0\mathbf{i} - \mathbf{j}$ instead of $\mathbf{i} + 0\mathbf{j}$

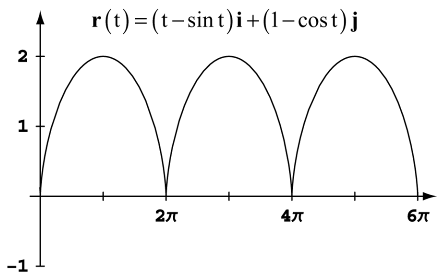
   (d) $\mathbf{v}(t) = -(\sin t)\mathbf{i} - (\cos t)\mathbf{j} \Rightarrow \mathbf{a}(t) = -(\cos t)\mathbf{i} + (\sin t)\mathbf{j}$;
   - (i) $|\mathbf{v}(t)| = \sqrt{(-\sin t)^2 + (-\cos t)^2} = 1 \Rightarrow$ constant speed;
   - (ii) $\mathbf{v} \cdot \mathbf{a} = (\sin t)(\cos t) - (\cos t)(\sin t) = 0 \Rightarrow$ yes, orthogonal;
   - (iii) clockwise movement;
   - (iv) yes, $\mathbf{r}(0) = \mathbf{i} - 0\mathbf{j}$

   (e) $\mathbf{v}(t) = -(2t \sin t)\mathbf{i} + (2t \cos t)\mathbf{j} \Rightarrow \mathbf{a}(t) = -(2 \sin t + 2t \cos t)\mathbf{i} + (2 \cos t - 2t \sin t)\mathbf{j}$;
   - (i) $|\mathbf{v}(t)| = \sqrt{[-(2t \sin t)]^2 + (2t \cos t)^2} = \sqrt{4t^2(\sin^2 t + \cos^2 t)} = 2|t| = 2t,\ t \ge 0$ $\Rightarrow$ variable speed;
   - (ii) $\mathbf{v} \cdot \mathbf{a} = 4\,(t \sin^2 t + t^2 \sin t \cos t) + 4\,(t \cos^2 t - t^2 \cos t \sin t) = 4t \ne 0$ in general $\Rightarrow$ not orthogonal in general;
   - (iii) counterclockwise movement;
   - (iv) yes, $\mathbf{r}(0) = \mathbf{i} + 0\mathbf{j}$

24. Let $\mathbf{p} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ denote the position vector of the point $(2, 2, 1)$ and let, $\mathbf{u} = \frac{1}{\sqrt{2}}\,\mathbf{i} - \frac{1}{\sqrt{2}}\,\mathbf{j}$ and $\mathbf{v} = \frac{1}{\sqrt{3}}\,\mathbf{i} + \frac{1}{\sqrt{3}}\,\mathbf{j} + \frac{1}{\sqrt{3}}\,\mathbf{k}$. Then $\mathbf{r}(t) = \mathbf{p} + (\cos t)\mathbf{u} + (\sin t)\mathbf{v}$. Note that $(2, 2, 1)$ is a point on the plane and $\mathbf{n} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$ is normal to the plane. Moreover, $\mathbf{u}$ and $\mathbf{v}$ are orthogonal unit vectors with $\mathbf{u} \cdot \mathbf{n} = \mathbf{v} \cdot \mathbf{n} = 0 \Rightarrow \mathbf{u}$ and $\mathbf{v}$ are parallel to the plane. Therefore, $\mathbf{r}(t)$ identifies a point that lies in the plane for each t. Also, for each t, $(\cos t)\mathbf{u} + (\sin t)\mathbf{v}$ is a unit vector. Starting at the point $\left(2 + \frac{1}{\sqrt{2}}, 2 - \frac{1}{\sqrt{2}}, 1\right)$ the vector $\mathbf{r}(t)$ traces out a circle of radius 1 and center $(2, 2, 1)$ in the plane $x + y - 2z = 2$.

25. The velocity vector is tangent to the graph of $y^2 = 2x$ at the point $(2, 2)$, has length 5, and a positive $\mathbf{i}$ component. Now, $y^2 = 2x \Rightarrow 2y \frac{dy}{dx} = 2 \Rightarrow \left.\frac{dy}{dx}\right|_{(2,2)} = \frac{2}{2 \cdot 2} = \frac{1}{2} \Rightarrow$ the tangent vector lies in the direction of the vector $\mathbf{i} + \frac{1}{2}\mathbf{j} \Rightarrow$ the velocity vector is $\mathbf{v} = \frac{5}{\sqrt{1+\frac{1}{4}}}\left(\mathbf{i} + \frac{1}{2}\mathbf{j}\right) = \frac{5}{\left(\frac{\sqrt{5}}{2}\right)}\left(\mathbf{i} + \frac{1}{2}\mathbf{j}\right) = 2\sqrt{5}\mathbf{i} + \sqrt{5}\mathbf{j}$

26. (a)



(b) $\mathbf{v} = (1 - \cos t)\mathbf{i} + (\sin t)\mathbf{j}$ and $\mathbf{a} = (\sin t)\mathbf{i} + (\cos t)\mathbf{j}$; $|\mathbf{v}|^2 = (1 - \cos t)^2 + \sin^2 t = 2 - 2\cos t \Rightarrow |\mathbf{v}|^2$ is at a max when $\cos t = -1 \Rightarrow t = \pi, 3\pi, 5\pi$, etc., and at these values of t, $|\mathbf{v}|^2 = 4 \Rightarrow \max |\mathbf{v}| = \sqrt{4} = 2$; $|\mathbf{v}|^2$ is at a min when $\cos t = 1 \Rightarrow t = 0, 2\pi, 4\pi$, etc., and at these values of t, $|\mathbf{v}|^2 = 0 \Rightarrow \min |\mathbf{v}| = 0$; $|\mathbf{a}|^2 = \sin^2 t + \cos^2 t = 1$ for every $t \Rightarrow \max |\mathbf{a}| = \min |\mathbf{a}| = \sqrt{1} = 1$

27. $\frac{d}{dt}(\mathbf{r} \cdot \mathbf{r}) = \mathbf{r} \cdot \frac{d\mathbf{r}}{dt} + \frac{d\mathbf{r}}{dt} \cdot \mathbf{r} = 2\mathbf{r} \cdot \frac{d\mathbf{r}}{dt} = 2 \cdot 0 = 0 \Rightarrow \mathbf{r} \cdot \mathbf{r}$ is a constant $\Rightarrow |\mathbf{r}| = \sqrt{\mathbf{r} \cdot \mathbf{r}}$ is constant

28. (a) $\frac{d}{dt}(\mathbf{u} \cdot \mathbf{v} \times \mathbf{w}) = \frac{d\mathbf{u}}{dt} \cdot (\mathbf{v} \times \mathbf{w}) + \mathbf{u} \cdot \frac{d}{dt}(\mathbf{v} \times \mathbf{w}) = \frac{d\mathbf{u}}{dt} \cdot (\mathbf{v} \times \mathbf{w}) + \mathbf{u} \cdot \left(\frac{d\mathbf{v}}{dt} \times \mathbf{w} + \mathbf{v} \times \frac{d\mathbf{w}}{dt}\right)$
$= \frac{d\mathbf{u}}{dt} \cdot (\mathbf{v} \times \mathbf{w}) + \mathbf{u} \cdot \frac{d\mathbf{v}}{dt} \times \mathbf{w} + \mathbf{u} \cdot \mathbf{v} \times \frac{d\mathbf{w}}{dt}$

(b) $\frac{d}{dt}\left[\mathbf{r} \cdot \left(\frac{d\mathbf{r}}{dt} \times \frac{d^2\mathbf{r}}{dt^2}\right)\right] = \frac{d\mathbf{r}}{dt} \cdot \left(\frac{d\mathbf{r}}{dt} \times \frac{d^2\mathbf{r}}{dt^2}\right) + \mathbf{r} \cdot \left(\frac{d^2\mathbf{r}}{dt^2} \times \frac{d^2\mathbf{r}}{dt^2}\right) + \mathbf{r} \cdot \left(\frac{d\mathbf{r}}{dt} \times \frac{d^3\mathbf{r}}{dt^3}\right) = \mathbf{r} \cdot \left(\frac{d\mathbf{r}}{dt} \times \frac{d^3\mathbf{r}}{dt^3}\right)$, since $\mathbf{A} \cdot (\mathbf{A} \times \mathbf{B}) = 0$ and $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{B}) = 0$ for any vectors $\mathbf{A}$ and $\mathbf{B}$

29. (a) $\mathbf{u} = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \Rightarrow c\mathbf{u} = cf(t)\mathbf{i} + cg(t)\mathbf{j} + ch(t)\mathbf{k} \Rightarrow \frac{d}{dt}(c\mathbf{u}) = c\frac{df}{dt}\mathbf{i} + c\frac{dg}{dt}\mathbf{j} + c\frac{dh}{dt}\mathbf{k}$
$= c\left(\frac{df}{dt}\mathbf{i} + \frac{dg}{dt}\mathbf{j} + \frac{dh}{dt}\mathbf{k}\right) = c\frac{d\mathbf{u}}{dt}$

(b) $f\mathbf{u} = ff(t)\mathbf{i} + fg(t)\mathbf{j} + fh(t)\mathbf{k} \Rightarrow \frac{d}{dt}(f\mathbf{u}) = \left[\frac{df}{dt}f(t) + f\frac{df}{dt}\right]\mathbf{i} + \left[\frac{df}{dt}g(t) + f\frac{dg}{dt}\right]\mathbf{j} + \left[\frac{df}{dt}h(t) + f\frac{dh}{dt}\right]\mathbf{k}$
$= \frac{df}{dt}[f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}] + f\left[\frac{df}{dt}\mathbf{i} + \frac{dg}{dt}\mathbf{j} + \frac{dh}{dt}\mathbf{k}\right] = \frac{df}{dt}\mathbf{u} + f\frac{d\mathbf{u}}{dt}$

30. Let $\mathbf{u} = f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k}$ and $\mathbf{v} = g_1(t)\mathbf{i} + g_2(t)\mathbf{j} + g_3(t)\mathbf{k}$. Then
$\mathbf{u} + \mathbf{v} = [f_1(t) + g_1(t)]\mathbf{i} + [f_2(t) + g_2(t)]\mathbf{j} + [f_3(t) + g_3(t)]\mathbf{k}$
$\Rightarrow \frac{d}{dt}(\mathbf{u} + \mathbf{v}) = [f_1'(t) + g_1'(t)]\mathbf{i} + [f_2'(t) + g_2'(t)]\mathbf{j} + [f_3'(t) + g_3'(t)]\mathbf{k}$
$= [f_1'(t)\mathbf{i} + f_2'(t)\mathbf{j} + f_3'(t)\mathbf{k}] + [g_1'(t)\mathbf{i} + g_2'(t)\mathbf{j} + g_3'(t)\mathbf{k}] = \frac{d\mathbf{u}}{dt} + \frac{d\mathbf{v}}{dt}$;
$\mathbf{u} - \mathbf{v} = [f_1(t) - g_1(t)]\mathbf{i} + [f_2(t) - g_2(t)]\mathbf{j} + [f_3(t) - g_3(t)]\mathbf{k}$
$\Rightarrow \frac{d}{dt}(\mathbf{u} - \mathbf{v}) = [f_1'(t) - g_1'(t)]\mathbf{i} + [f_2'(t) - g_2'(t)]\mathbf{j} + [f_3'(t) - g_3'(t)]\mathbf{k}$
$= [f_1'(t)\mathbf{i} + f_2'(t)\mathbf{j} + f_3'(t)\mathbf{k}] - [g_1'(t)\mathbf{i} + g_2'(t)\mathbf{j} + g_3'(t)\mathbf{k}] = \frac{d\mathbf{u}}{dt} - \frac{d\mathbf{v}}{dt}$

31. Suppose $\mathbf{r}$ is continuous at $t = t_0$. Then $\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{r}(t_0) \Leftrightarrow \lim_{t \to t_0} [f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}]$
$= f(t_0)\mathbf{i} + g(t_0)\mathbf{j} + h(t_0)\mathbf{k} \Leftrightarrow \lim_{t \to t_0} f(t) = f(t_0)$, $\lim_{t \to t_0} g(t) = g(t_0)$, and $\lim_{t \to t_0} h(t) = h(t_0) \Leftrightarrow$ f, g, and h are continuous at $t = t_0$.

32. $\lim_{t \to t_0} [\mathbf{r}_1(t) \times \mathbf{r}_2(t)] = \lim_{t \to t_0} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ f_1(t) & f_2(t) & f_3(t) \\ g_1(t) & g_2(t) & g_3(t) \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \lim_{t \to t_0} f_1(t) & \lim_{t \to t_0} f_2(t) & \lim_{t \to t_0} f_3(t) \\ \lim_{t \to t_0} g_1(t) & \lim_{t \to t_0} g_2(t) & \lim_{t \to t_0} g_3(t) \end{vmatrix}$

$= \lim_{t \to t_0} \mathbf{r}_1(t) \times \lim_{t \to t_0} \mathbf{r}_2(t) = \mathbf{A} \times \mathbf{B}$

33. $\mathbf{r}'(t_0)$ exists $\Rightarrow$ $f'(t_0)\mathbf{i} + g'(t_0)\mathbf{j} + h'(t_0)\mathbf{k}$ exists $\Rightarrow$ $f'(t_0)$, $g'(t_0)$, $h'(t_0)$ all exist $\Rightarrow$ f, g, and h are continuous at $t = t_0$ $\Rightarrow$ $\mathbf{r}(t)$ is continuous at $t = t_0$

34. $\mathbf{u} = \mathbf{C} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ with a, b, c real constants $\Rightarrow$ $\frac{d\mathbf{u}}{dt} = \frac{da}{dt}\mathbf{i} + \frac{db}{dt}\mathbf{j} + \frac{dc}{dt}\mathbf{k} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$

35-38. Example CAS commands:

Maple:

```
> with( plots );
r := t -> [sin(t)-t*cos(t),cos(t)+t*sin(t),t^2];
t0 := 3*Pi/2;
lo := 0;
hi := 6*Pi;
P1 := spacecurve( r(t), t=lo..hi, axes=boxed, thickness=3 ):
display( P1, title="#35(a) (Section 13.1)" );
Dr := unapply( diff(r(t),t), t );                    # (b)
Dr(t0);                                              # (c)
q1 := expand( r(t0) + Dr(t0)*(t-t0) );
T := unapply( q1, t );
P2 := spacecurve( T(t), t=lo..hi, axes=boxed, thickness=3, color=black ):
display( [P1,P2], title="#35(d) (Section 13.1)" );
```

39-40. Example CAS commands:

Maple:

```
a := 'a'; b := 'b';
r := (a,b,t) -> [cos(a*t),sin(a*t),b*t];
Dr := unapply( diff(r(a,b,t),t), (a,b,t) );
t0 := 3*Pi/2;
q1 := expand( r(a,b,t0) + Dr(a,b,t0)*(t-t0) );
T := unapply( q1, (a,b,t) );
lo := 0;
hi := 4*Pi;
P := NULL:
for a in [ 1, 2, 4, 6 ] do
  P1 := spacecurve( r(a,1,t), t=lo..hi, thickness=3 ):
  P2 := spacecurve( T(a,1,t), t=lo..hi, thickness=3, color=black ):
  P := P, display( [P1,P2], axes=boxed, title=sprintf("#39 (Section 13.1)\n a=%a",a) );
end do:
display( [P], insequence=true );
```

35-40. Example CAS commands:

Mathematica: (assigned functions, parameters, and intervals will vary)

The x-y-z components for the curve are entered as a list of functions of t. The unit vectors **i**, **j**, **k** are not inserted.

If a graph is too small, highlight it and drag out a corner or side to make it larger.

Only the components of r[t] and values for t0, tmin, and tmax require alteration for each problem.

```
Clear[r, v, t, x, y, z]
r[t_]={ Sin[t] − t Cos[t], Cos[t] + t Sin[t], t^2}
t0= 3π / 2; tmin= 0; tmax= 6π;
ParametricPlot3D[Evaluate[r[t]], {t, tmin, tmax}, AxesLabel → {x, y, z}];
v[t_]= r'[t]
tanline[t_]= v[t0] t + r[t0]
ParametricPlot3D[Evaluate[{r[t], tanline[t]}], {t, tmin, tmax}, AxesLabel → {x, y, z}];
```

For 39 and 40, the curve can be defined as a function of t, a, and b. Leave a space between a and t and b and t.

```
Clear[r, v, t, x, y, z, a, b]
r[t_,a_,b_]:={Cos[a t], Sin[a t], b t}
t0= 3π / 2; tmin= 0; tmax= 4π;
v[t_,a_,b_]= D[r[t, a, b], t]
tanline[t_,a_,b_]=v[t0, a, b] t + r[t0, a, b]
pa1=ParametricPlot3D[Evaluate[{r[t, 1, 1], tanline[t, 1, 1]}], {t,tmin, tmax}, AxesLabel → {x, y, z}];
pa2=ParametricPlot3D[Evaluate[{r[t, 2, 1], tanline[t, 2, 1]}], {t,tmin, tmax}, AxesLabel → {x, y, z}];
pa4=ParametricPlot3D[Evaluate[{r[t, 4, 1], tanline[t, 4, 1]}], {t,tmin, tmax}, AxesLabel → {x, y, z}];
pa6=ParametricPlot3D[Evaluate[{r[t, 6, 1], tanline[t, 6, 1]}], {t,tmin, tmax}, AxesLabel → {x, y, z}];
Show[GraphicsRow[{pa1, pa2, pa4, pa6}]]
```

13.2 INTEGRALS OF VECTOR FUNCTIONS; PROJECTILE MOTION

1. $\int_0^1 [t^3\mathbf{i} + 7\mathbf{j} + (t+1)\mathbf{k}]\,dt = \left[\frac{t^4}{4}\right]_0^1 \mathbf{i} + [7t]_0^1 \mathbf{j} + \left[\frac{t^2}{2} + t\right]_0^1 \mathbf{k} = \frac{1}{4}\mathbf{i} + 7\mathbf{j} + \frac{3}{2}\mathbf{k}$

2. $\int_1^2 \left[(6-6t)\mathbf{i} + 3\sqrt{t}\,\mathbf{j} + \left(\frac{4}{t^2}\right)\mathbf{k}\right] dt = [6t - 3t^2]_1^2 \mathbf{i} + \left[2t^{3/2}\right]_1^2 \mathbf{j} + [-4t^{-1}]_1^2 \mathbf{k} = -3\mathbf{i} + \left(4\sqrt{2} - 2\right)\mathbf{j} + 2\mathbf{k}$

3. $\int_{-\pi/4}^{\pi/4} [(\sin t)\mathbf{i} + (1 + \cos t)\mathbf{j} + (\sec^2 t)\,\mathbf{k}]\,dt = [-\cos t]_{-\pi/4}^{\pi/4} \mathbf{i} + [t + \sin t]_{-\pi/4}^{\pi/4} \mathbf{j} + [\tan t]_{-\pi/4}^{\pi/4} \mathbf{k} = \left(\frac{\pi + 2\sqrt{2}}{2}\right)\mathbf{j} + 2\mathbf{k}$

4. $\int_0^{\pi/3} [(\sec t \tan t)\mathbf{i} + (\tan t)\mathbf{j} + (2 \sin t \cos t)\,\mathbf{k}]\,dt = \int_0^{\pi/3} [(\sec t \tan t)\mathbf{i} + (\tan t)\mathbf{j} + (\sin 2t)\mathbf{k}]\,dt$
$= [\sec t]_0^{\pi/3} \mathbf{i} + [-\ln(\cos t)]_0^{\pi/3} \mathbf{j} + \left[-\frac{1}{2}\cos 2t\right]_0^{\pi/3} \mathbf{k} = \mathbf{i} + (\ln 2)\mathbf{j} + \frac{3}{4}\mathbf{k}$

5. $\int_1^4 \left(\frac{1}{t}\mathbf{i} + \frac{1}{5-t}\mathbf{j} + \frac{1}{2t}\mathbf{k}\right) dt = \ = [\ln t]_1^4 \mathbf{i} + [-\ln(5-t)]_1^4 \mathbf{j} + \left[\frac{1}{2}\ln t\right]_1^4 \mathbf{k} = (\ln 4)\mathbf{i} + (\ln 4)\mathbf{j} + (\ln 2)\mathbf{k}$

6. $\int_0^1 \left(\frac{2}{\sqrt{1-t^2}}\mathbf{i} + \frac{\sqrt{3}}{1+t^2}\mathbf{k}\right) dt = [2\sin^{-1} t]_0^1 \mathbf{i} + \left[\sqrt{3}\tan^{-1} t\right]_0^1 \mathbf{k} = \pi\mathbf{i} + \frac{\pi\sqrt{3}}{4}\mathbf{k}$

7. $\int_0^1 \left(te^{t^2}\mathbf{i} + e^{-t}\mathbf{j} + \mathbf{k}\right) dt = \left[\frac{1}{2}e^{t^2}\right]_0^1 \mathbf{i} - [e^{-t}]_0^1 \mathbf{j} + [t]_0^1 \mathbf{k} = \frac{e-1}{2}\mathbf{i} + \frac{e-1}{e}\mathbf{i} + \mathbf{k}$

8. $\int_1^{\ln 3} (te^t\,\mathbf{i} + e^t\,\mathbf{j} + \ln t\,\mathbf{k})\,dt = [te^t - e^t]_1^{\ln 3} \mathbf{i} - [e^t]_1^{\ln 3} \mathbf{j} + [t \ln t - t]_1^{\ln 3} \mathbf{k}$
$= 3(\ln 3 - 1)\mathbf{i} + (3 - e)\mathbf{i} + (\ln 3(\ln(\ln 3) - 1) + 1)\,\mathbf{k}$

9. $\int_0^{\pi/2} [(\cos t)\mathbf{i} - (\sin 2t)\mathbf{j} + (\sin^2 t)\,\mathbf{k}]\,dt = \int_0^{\pi/2} \left[(\cos t)\mathbf{i} - (\sin 2t)\mathbf{j} + \left(\frac{1}{2} - \frac{1}{2}\cos 2t\right)\mathbf{k}\right] dt =$
$= [\sin t]_0^{\pi/2} \mathbf{i} + \left[\frac{1}{2}\cos t\right]_0^{\pi/2} \mathbf{j} + \left[\frac{1}{2}t - \frac{1}{4}\sin 2t\right]_0^{\pi/2} \mathbf{k} = \mathbf{i} - \mathbf{j} + \frac{\pi}{4}\mathbf{k}$

10. $\int_0^{\pi/4} [(\sec t)\mathbf{i} + (\tan^2 t)\mathbf{j} - (t \sin t)\mathbf{k}]\, dt = \int_0^{\pi/4} [(\sec t)\mathbf{i} + (\sec^2 t - 1)\mathbf{j} - (t \sin t)\mathbf{k}]\, dt$
$= [\ln(\sec t + \tan t)]_0^{\pi/4}\,\mathbf{i} + [\tan t - t]_0^{\pi/4}\,\mathbf{j} + [t\cos t - \sin t]_0^{\pi/4}\,\mathbf{k} = \ln\left(1+\sqrt{2}\right)\mathbf{i} + \left(1 - \frac{\pi}{4}\right)\mathbf{j} + \left(\frac{\pi}{4\sqrt{2}} - \frac{1}{\sqrt{2}}\right)\mathbf{k}$

11. $\mathbf{r} = \int (-t\mathbf{i} - t\mathbf{j} - t\mathbf{k})\, dt = -\frac{t^2}{2}\mathbf{i} - \frac{t^2}{2}\mathbf{j} - \frac{t^2}{2}\mathbf{k} + \mathbf{C}$; $\mathbf{r}(0) = 0\mathbf{i} - 0\mathbf{j} - 0\mathbf{k} + \mathbf{C} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} \Rightarrow \mathbf{C} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$
$\Rightarrow \mathbf{r} = \left(-\frac{t^2}{2} + 1\right)\mathbf{i} + \left(-\frac{t^2}{2} + 2\right)\mathbf{j} + \left(-\frac{t^2}{2} + 3\right)\mathbf{k}$

12. $\mathbf{r} = \int [(180t)\mathbf{i} + (180t - 16t^2)\mathbf{j}]\, dt = 90t^2\mathbf{i} + \left(90t^2 - \frac{16}{3}t^3\right)\mathbf{j} + \mathbf{C}$; $\mathbf{r}(0) = 90(0)^2\mathbf{i} + \left[90(0)^2 - \frac{16}{3}(0)^3\right]\mathbf{j} + \mathbf{C}$
$= 100\mathbf{j} \Rightarrow \mathbf{C} = 100\mathbf{j} \Rightarrow \mathbf{r} = 90t^2\mathbf{i} + \left(90t^2 - \frac{16}{3}t^3 + 100\right)\mathbf{j}$

13. $\mathbf{r} = \int \left[\left(\frac{3}{2}(t+1)^{1/2}\right)\mathbf{i} + e^{-t}\mathbf{j} + \left(\frac{1}{t+1}\right)\mathbf{k}\right] dt = (t+1)^{3/2}\,\mathbf{i} - e^{-t}\,\mathbf{j} + \ln(t+1)\mathbf{k} + \mathbf{C}$;
$\mathbf{r}(0) = (0+1)^{3/2}\,\mathbf{i} - e^{-0}\,\mathbf{j} + \ln(0+1)\mathbf{k} + \mathbf{C} = \mathbf{k} \Rightarrow \mathbf{C} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$
$\Rightarrow \mathbf{r} = \left[(t+1)^{3/2} - 1\right]\mathbf{i} + (1 - e^{-t})\,\mathbf{j} + [1 + \ln(t+1)]\mathbf{k}$

14. $\mathbf{r} = \int [(t^3 + 4t)\,\mathbf{i} + t\mathbf{j} + 2t^2\mathbf{k}]\, dt = \left(\frac{t^4}{4} + 2t^2\right)\mathbf{i} + \frac{t^2}{2}\mathbf{j} + \frac{2t^3}{3}\mathbf{k} + \mathbf{C}$; $\mathbf{r}(0) = \left[\frac{0^4}{4} + 2(0)^2\right]\mathbf{i} + \frac{0^2}{2}\mathbf{j} + \frac{2(0)^3}{3}\mathbf{k} + \mathbf{C}$
$= \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{C} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{r} = \left(\frac{t^4}{4} + 2t^2 + 1\right)\mathbf{i} + \left(\frac{t^2}{2} + 1\right)\mathbf{j} + \frac{2t^3}{3}\mathbf{k}$

15. $\frac{d\mathbf{r}}{dt} = \int (-32\mathbf{k})\, dt = -32t\mathbf{k} + \mathbf{C}_1$; $\frac{d\mathbf{r}}{dt}(0) = 8\mathbf{i} + 8\mathbf{j} \Rightarrow -32(0)\mathbf{k} + \mathbf{C}_1 = 8\mathbf{i} + 8\mathbf{j} \Rightarrow \mathbf{C}_1 = 8\mathbf{i} + 8\mathbf{j}$
$\Rightarrow \frac{d\mathbf{r}}{dt} = 8\mathbf{i} + 8\mathbf{j} - 32t\mathbf{k}$; $\mathbf{r} = \int (8\mathbf{i} + 8\mathbf{j} - 32t\mathbf{k})\, dt = 8t\mathbf{i} + 8t\mathbf{j} - 16t^2\mathbf{k} + \mathbf{C}_2$; $\mathbf{r}(0) = 100\mathbf{k}$
$\Rightarrow 8(0)\mathbf{i} + 8(0)\mathbf{j} - 16(0)^2\mathbf{k} + \mathbf{C}_2 = 100\mathbf{k} \Rightarrow \mathbf{C}_2 = 100\mathbf{k} \Rightarrow \mathbf{r} = 8t\mathbf{i} + 8t\mathbf{j} + (100 - 16t^2)\,\mathbf{k}$

16. $\frac{d\mathbf{r}}{dt} = \int -(\mathbf{i} + \mathbf{j} + \mathbf{k})\, dt = -(t\mathbf{i} + t\mathbf{j} + t\mathbf{k}) + \mathbf{C}_1$; $\frac{d\mathbf{r}}{dt}(0) = \mathbf{0} \Rightarrow -(0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}) + \mathbf{C}_1 = \mathbf{0} \Rightarrow \mathbf{C}_1 = \mathbf{0}$
$\Rightarrow \frac{d\mathbf{r}}{dt} = -(t\mathbf{i} + t\mathbf{j} + t\mathbf{k})$; $\mathbf{r} = \int -(t\mathbf{i} + t\mathbf{j} + t\mathbf{k})\, dt = -\left(\frac{t^2}{2}\mathbf{i} + \frac{t^2}{2}\mathbf{j} + \frac{t^2}{2}\mathbf{k}\right) + \mathbf{C}_2$; $\mathbf{r}(0) = 10\mathbf{i} + 10\mathbf{j} + 10\mathbf{k}$
$\Rightarrow -\left(\frac{0^2}{2}\mathbf{i} + \frac{0^2}{2}\mathbf{j} + \frac{0^2}{2}\mathbf{k}\right) + \mathbf{C}_2 = 10\mathbf{i} + 10\mathbf{j} + 10\mathbf{k} \Rightarrow \mathbf{C}_2 = 10\mathbf{i} + 10\mathbf{j} + 10\mathbf{k}$
$\Rightarrow \mathbf{r} = \left(-\frac{t^2}{2} + 10\right)\mathbf{i} + \left(-\frac{t^2}{2} + 10\right)\mathbf{j} + \left(-\frac{t^2}{2} + 10\right)\mathbf{k}$

17. $\frac{d\mathbf{v}}{dt} = \mathbf{a} = 3\mathbf{i} - \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{v}(t) = 3t\mathbf{i} - t\mathbf{j} + t\mathbf{k} + \mathbf{C}_1$; the particle travels in the direction of the vector $(4-1)\mathbf{i} + (1-2)\mathbf{j} + (4-3)\mathbf{k} = 3\mathbf{i} - \mathbf{j} + \mathbf{k}$ (since it travels in a straight line), and at time $t = 0$ it has speed
$2 \Rightarrow \mathbf{v}(0) = \frac{2}{\sqrt{9+1+1}}(3\mathbf{i} - \mathbf{j} + \mathbf{k}) = \mathbf{C}_1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{v}(t) = \left(3t + \frac{6}{\sqrt{11}}\right)\mathbf{i} - \left(t + \frac{2}{\sqrt{11}}\right)\mathbf{j} + \left(t + \frac{2}{\sqrt{11}}\right)\mathbf{k}$
$\Rightarrow \mathbf{r}(t) = \left(\frac{3}{2}t^2 + \frac{6}{\sqrt{11}}t\right)\mathbf{i} - \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{11}}t\right)\mathbf{j} + \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{11}}t\right)\mathbf{k} + \mathbf{C}_2$; $\mathbf{r}(0) = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} = \mathbf{C}_2$
$\Rightarrow \mathbf{r}(t) = \left(\frac{3}{2}t^2 + \frac{6}{\sqrt{11}}t + 1\right)\mathbf{i} - \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{11}}t - 2\right)\mathbf{j} + \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{11}}t + 3\right)\mathbf{k}$
$= \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{11}}t\right)(3\mathbf{i} - \mathbf{j} + \mathbf{k}) + (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$

18. $\frac{d\mathbf{v}}{dt} = \mathbf{a} = 2\mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{v}(t) = 2t\mathbf{i} + t\mathbf{j} + t\mathbf{k} + \mathbf{C}_1$; the particle travels in the direction of the vector $(3-1)\mathbf{i} + (0-(-1))\mathbf{j} + (3-2)\mathbf{k} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ (since it travels in a straight line), and at time $t = 0$ it has speed 2
$\Rightarrow \mathbf{v}(0) = \frac{2}{\sqrt{4+1+1}}(2\mathbf{i} + \mathbf{j} + \mathbf{k}) = \mathbf{C}_1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{v}(t) = \left(2t + \frac{4}{\sqrt{6}}\right)\mathbf{i} + \left(t + \frac{2}{\sqrt{6}}\right)\mathbf{j} + \left(t + \frac{2}{\sqrt{6}}\right)\mathbf{k}$
$\Rightarrow \mathbf{r}(t) = \left(t^2 + \frac{4}{\sqrt{6}}t\right)\mathbf{i} + \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{6}}t\right)\mathbf{j} + \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{6}}t\right)\mathbf{k} + \mathbf{C}_2$; $\mathbf{r}(0) = \mathbf{i} - \mathbf{j} + 2\mathbf{k} = \mathbf{C}_2$
$\Rightarrow \mathbf{r}(t) = \left(t^2 + \frac{4}{\sqrt{6}}t + 1\right)\mathbf{i} + \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{6}}t - 1\right)\mathbf{j} + \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{6}}t + 2\right)\mathbf{k} = \left(\frac{1}{2}t^2 + \frac{2}{\sqrt{6}}t\right)(2\mathbf{i} + \mathbf{j} + \mathbf{k}) + (\mathbf{i} - \mathbf{j} + 2\mathbf{k})$

19. $x = (v_0 \cos\alpha)t \Rightarrow (21\text{ km})\left(\frac{1000\text{ m}}{1\text{ km}}\right) = (840\text{ m/s})(\cos 60°)t \Rightarrow t = \frac{21{,}000\text{ m}}{(840\text{ m/s})(\cos 60°)} = 50$ seconds

20. $R = \frac{v_0^2}{g}\sin 2\alpha$ and maximum R occurs when $\alpha = 45° \Rightarrow 24.5\text{ km} = \left(\frac{v_0^2}{9.8\text{ m/s}^2}\right)(\sin 90°)$
$\Rightarrow v_0 = \sqrt{(9.8)(24{,}500)\text{ m}^2/\text{s}^2} = 490\text{ m/s}$

21. (a) $t = \frac{2v_0\sin\alpha}{g} = \frac{2(500\text{ m/s})(\sin 45°)}{9.8\text{ m/s}^2} \approx 72.2$ seconds; $R = \frac{v_0^2}{g}\sin 2\alpha = \frac{(500\text{ m/s})^2}{9.8\text{ m/s}^2}(\sin 90°) \approx 25{,}510.2$ m
(b) $x = (v_0\cos\alpha)t \Rightarrow 5000\text{ m} = (500\text{ m/s})(\cos 45°)t \Rightarrow t = \frac{5000\text{ m}}{(500\text{ m/s})(\cos 45°)} \approx 14.14$ s; thus,
$y = (v_0\sin\alpha)t - \frac{1}{2}gt^2 \Rightarrow y \approx (500\text{ m/s})(\sin 45°)(14.14\text{ s}) - \frac{1}{2}(9.8\text{ m/s}^2)(14.14\text{ s})^2 \approx 4020$ m
(c) $y_{max} = \frac{(v_0\sin\alpha)^2}{2g} = \frac{((500\text{ m/s})(\sin 45°))^2}{2(9.8\text{ m/s}^2)} \approx 6378$ m

22. $y = y_0 + (v_0\sin\alpha)t - \frac{1}{2}gt^2 \Rightarrow y = 32\text{ ft} + (32\text{ ft/sec})(\sin 30°)t - \frac{1}{2}(32\text{ ft/sec}^2)t^2 \Rightarrow y = 32 + 16t - 16t^2$;
the ball hits the ground when $y = 0 \Rightarrow 0 = 32 + 16t - 16t^2 \Rightarrow t = -1$ or $t = 2 \Rightarrow t = 2$ sec since $t > 0$; thus,
$x = (v_0\cos\alpha)t \Rightarrow x = (32\text{ ft/sec})(\cos 30°)t = 32\left(\frac{\sqrt{3}}{2}\right)(2) \approx 55.4$ ft

23. (a) $R = \frac{v_0^2}{g}\sin 2\alpha \Rightarrow 10\text{ m} = \left(\frac{v_0^2}{9.8\text{ m/s}^2}\right)(\sin 90°) \Rightarrow v_0^2 = 98\text{ m}^2\text{s}^2 \Rightarrow v_0 \approx 9.9$ m/s;
(b) $6\text{m} \approx \frac{(9.9\text{ m/s})^2}{9.8\text{ m/s}^2}(\sin 2\alpha) \Rightarrow \sin 2\alpha \approx 0.59999 \Rightarrow 2\alpha \approx 36.87°$ or $143.12° \Rightarrow \alpha \approx 18.4°$ or $71.6°$

24. $v_0 = 5 \times 10^6$ m/s and $x = 40$ cm $= 0.4$ m; thus $x = (v_0\cos\alpha)t \Rightarrow 0.4\text{m} = (5 \times 10^6\text{ m/s})(\cos 0°)t$
$\Rightarrow t = 0.08 \times 10^{-6}\text{ s} = 8 \times 10^{-8}$ s; also, $y = y_0 + (v_0\sin\alpha)t - \frac{1}{2}gt^2$
$\Rightarrow y = (5 \times 10^6\text{ m/s})(\sin 0°)(8 \times 10^{-8}\text{ s}) - \frac{1}{2}(9.8\text{ m/s}^2)(8 \times 10^{-8}\text{ s})^2 = -3.136 \times 10^{-14}$ m or
-3.136×10^{-12} cm. Therefore, it drops 3.136×10^{-12} cm.

25. $R = \frac{v_0^2}{g}\sin 2\alpha \Rightarrow 16{,}000\text{ m} = \frac{(400\text{ m/s})^2}{9.8\text{ m/s}^2}\sin 2\alpha \Rightarrow \sin 2\alpha = 0.98 \Rightarrow 2\alpha \approx 78.5°$ or $2\alpha \approx 101.5° \Rightarrow \alpha \approx 39.3°$
or $50.7°$

26. (a) $R = \frac{(2v_0)^2}{g}\sin 2\alpha = \frac{4v_0^2}{g}\sin 2\alpha = 4\left(\frac{v_0^2}{g}\sin\alpha\right)$ or 4 times the original range.
(b) Now, let the initial range be $R = \frac{v_0^2}{g}\sin 2\alpha$. Then we want the factor p so that pv_0 will double the range
$\Rightarrow \frac{(pv_0)^2}{g}\sin 2\alpha = 2\left(\frac{v_0^2}{g}\sin 2\alpha\right) \Rightarrow p^2 = 2 \Rightarrow p = \sqrt{2}$ or about 141%. The same percentage will approximately
double the height: $\frac{(pv_0\sin\alpha)^2}{2g} = \frac{2(v_0\sin\alpha)^2}{2g} \Rightarrow p^2 = 2 \Rightarrow p = \sqrt{2}$.

27. The projectile reaches its maximum height when its vertical component of velocity is zero $\Rightarrow \frac{dy}{dt} = v_0\sin\alpha - gt = 0$
$\Rightarrow t = \frac{v_0\sin\alpha}{g} \Rightarrow y_{max} = (v_0\sin\alpha)\left(\frac{v_0\sin\alpha}{g}\right) - \frac{1}{2}g\left(\frac{v_0\sin\alpha}{g}\right)^2 = \frac{(v_0\sin\alpha)^2}{g} - \frac{(v_0\sin\alpha)^2}{2g} = \frac{(v_0\sin\alpha)^2}{2g}$. To find the flight time
we find the time when the projectile lands: $(v_0\sin\alpha)t - \frac{1}{2}gt^2 = 0 \Rightarrow t\left(v_0\sin\alpha - \frac{1}{2}gt\right) = 0 \Rightarrow t = 0$ or $t = \frac{2v_0\sin\alpha}{g}$.
$t = 0$ is the time when the projectile is fired, so $t = \frac{2v_0\sin\alpha}{g}$ is the time when the projectile strikes the ground. The range is
the value of the horizontal component when $t = \frac{2v_0\sin\alpha}{g} \Rightarrow R = x = (v_0\cos\alpha)\left(\frac{2v_0\sin\alpha}{g}\right) = \frac{v_0^2}{g}(2\sin\alpha\cos\alpha) = \frac{v_0^2}{g}\sin 2\alpha$.
The range is largest when $\sin 2\alpha = 1 \Rightarrow \alpha = 45°$.

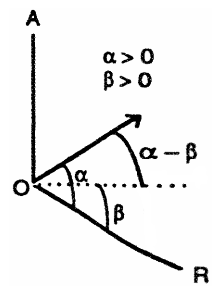
28. When marble A is located R units downrange, we have $x = (v_0\cos\alpha)t \Rightarrow R = (v_0\cos\alpha)t \Rightarrow t = \frac{R}{v_0\cos\alpha}$. At
that time the height of marble A is $y = y_0 + (v_0\sin\alpha)t - \frac{1}{2}gt^2 = (v_0\sin\alpha)\left(\frac{R}{v_0\cos\alpha}\right) - \frac{1}{2}g\left(\frac{R}{v_0\cos\alpha}\right)^2$
$\Rightarrow y = R\tan\alpha - \frac{1}{2}g\left(\frac{R^2}{v_0^2\cos^2\alpha}\right)$. The height of marble B at the same time $t = \frac{R}{v_0\cos\alpha}$ seconds is

$h = R\tan\alpha - \frac{1}{2}gt^2 = R\tan\alpha - \frac{1}{2}g\left(\frac{R^2}{v_0^2\cos^2\alpha}\right)$. Since the heights are the same, the marbles collide regardless of the initial velocity v_0.

29. $\frac{d\mathbf{r}}{dt} = \int(-g\mathbf{j})\,dt = -gt\mathbf{j} + \mathbf{C}_1$ and $\frac{d\mathbf{r}}{dt}(0) = (v_0\cos\alpha)\mathbf{i} + (v_0\sin\alpha)\mathbf{j} \Rightarrow -g(0)\mathbf{j} + \mathbf{C}_1 = (v_0\cos\alpha)\mathbf{i} + (v_0\sin\alpha)\mathbf{j}$ $\Rightarrow \mathbf{C}_1 = (v_0\cos\alpha)\mathbf{i} + (v_0\sin\alpha)\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dt} = (v_0\cos\alpha)\mathbf{i} + (v_0\sin\alpha - gt)\mathbf{j}$; $\mathbf{r} = \int[(v_0\cos\alpha)\mathbf{i} + (v_0\sin\alpha - gt)\mathbf{j}]\,dt$ $= (v_0 t\cos\alpha)\mathbf{i} + \left(v_0 t\sin\alpha - \frac{1}{2}gt^2\right)\mathbf{j} + \mathbf{C}_2$ and $\mathbf{r}(0) = x_0\mathbf{i} + y_0\mathbf{j} \Rightarrow [v_0(0)\cos\alpha]\mathbf{i} + \left[v_0(0)\sin\alpha - \frac{1}{2}g(0)^2\right]\mathbf{j} + \mathbf{C}_2$ $= x_0\mathbf{i} + y_0\mathbf{j} \Rightarrow \mathbf{C}_2 = x_0\mathbf{i} + y_0\mathbf{j} \Rightarrow \mathbf{r} = (x_0 + v_0 t\cos\alpha)\mathbf{i} + \left(y_0 + v_0 t\sin\alpha - \frac{1}{2}gt^2\right)\mathbf{j} \Rightarrow x = x_0 + v_0 t\cos\alpha$ and $y = y_0 + v_0 t\sin\alpha - \frac{1}{2}gt^2$

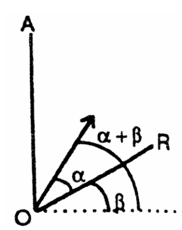
30. The maximum height is $y = \frac{(v_0\sin\alpha)^2}{2g}$ and this occurs for $x = \frac{v_0^2}{2g}\sin 2\alpha = \frac{v_0^2\sin\alpha\cos\alpha}{g}$. These equations describe parametrically the points on a curve in the xy-plane associated with the maximum heights on the parabolic trajectories in terms of the parameter (launch angle) α. Eliminating the parameter α, we have $x^2 = \frac{v_0^4\sin^2\alpha\cos^2\alpha}{g^2} = \frac{(v_0^4\sin^2\alpha)(1-\sin^2\alpha)}{g^2}$ $= \frac{v_0^4\sin^2\alpha}{g^2} - \frac{v_0^4\sin^4\alpha}{g^2} = \frac{v_0^2}{g}(2y) - (2y)^2 \Rightarrow x^2 + 4y^2 - \left(\frac{2v_0^2}{g}\right)y = 0 \Rightarrow x^2 + 4\left[y^2 - \left(\frac{v_0^2}{2g}\right)y + \frac{v_0^4}{16g^2}\right] = \frac{v_0^4}{4g^2}$ $\Rightarrow x^2 + 4\left(y - \frac{v_0^2}{4g}\right)^2 = \frac{v_0^4}{4g^2}$, where $x \ge 0$.

31. (a) At the time t when the projectile hits the line OR we have $\tan\beta = \frac{y}{x}$; $x = [v_0\cos(\alpha-\beta)]t$ and $y = [v_0\sin(\alpha-\beta)]t - \frac{1}{2}gt^2 < 0$ since R is below level ground. Therefore let $|y| = \frac{1}{2}gt^2 - [v_0\sin(\alpha-\beta)]t > 0$ so that $\tan\beta = \frac{\left[\frac{1}{2}gt^2(v_0\sin(\alpha-\beta))t\right]}{[v_0\cos(\alpha-\beta)]t} = \frac{\left[\frac{1}{2}gt - v_0\sin(\alpha-\beta)\right]}{v_0\cos(\alpha-\beta)}$ $\Rightarrow v_0\cos(\alpha-\beta)\tan\beta = \frac{1}{2}gt - v_0\sin(\alpha-\beta)$ $\Rightarrow t = \frac{2v_0\sin(\alpha-\beta) + 2v_0\cos(\alpha-\beta)\tan\beta}{g}$, which is the time when the projectile hits the downhill slope. Therefore,



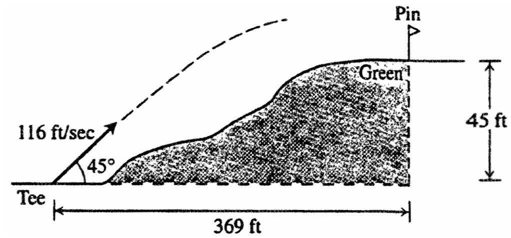
$x = [v_0\cos(\alpha-\beta)]\left[\frac{2v_0\sin(\alpha-\beta) + 2v_0\cos(\alpha-\beta)\tan\beta}{g}\right] = \frac{2v_0^2}{g}\left[\cos^2(\alpha-\beta)\tan\beta + \sin(\alpha-\beta)\cos(\alpha-\beta)\right]$. If x is maximized, then OR is maximized: $\frac{dx}{d\alpha} = \frac{2v_0^2}{g}[-\sin 2(\alpha-\beta)\tan\beta + \cos 2(\alpha-\beta)] = 0$ $\Rightarrow -\sin 2(\alpha-\beta)\tan\beta + \cos 2(\alpha-\beta) = 0 \Rightarrow \tan\beta = \cot 2(\alpha-\beta) \Rightarrow 2(\alpha-\beta) = 90^\circ - \beta$ $\Rightarrow \alpha - \beta = \frac{1}{2}(90^\circ - \beta) \Rightarrow \alpha = \frac{1}{2}(90^\circ + \beta) = \frac{1}{2}$ of $\angle AOR$.

(b) At the time t when the projectile hits OR we have $\tan\beta = \frac{y}{x}$; $x = [v_0\cos(\alpha+\beta)]t$ and $y = [v_0\sin(\alpha+\beta)]t - \frac{1}{2}gt^2$ $\Rightarrow \tan\beta = \frac{[v_0\sin(\alpha+\beta)]t - \frac{1}{2}gt^2}{[v_0\cos(\alpha+\beta)]t} = \frac{\left[v_0\sin(\alpha+\beta) - \frac{1}{2}gt\right]}{v_0\cos(\alpha+\beta)}$ $\Rightarrow v_0\cos(\alpha+\beta)\tan\beta = v_0\sin(\alpha+\beta) - \frac{1}{2}gt$ $\Rightarrow t = \frac{2v_0\sin(\alpha+\beta) - 2v_0\cos(\alpha+\beta)\tan\beta}{g}$, which is the time when the projectile hits the uphill slope. Therefore,



$x = [v_0\cos(\alpha+\beta)]\left[\frac{2v_0\sin(\alpha+\beta) - 2v_0\cos(\alpha+\beta)\tan\beta}{g}\right] = \frac{2v_0^2}{g}\left[\sin(\alpha+\beta)\cos(\alpha+\beta) - \cos^2(\alpha+\beta)\tan\beta\right]$. If x is maximized, then OR is maximized: $\frac{dx}{d\alpha} = \frac{2v_0^2}{g}[\cos 2(\alpha+\beta) + \sin 2(\alpha+\beta)\tan\beta] = 0$ $\Rightarrow \cos 2(\alpha+\beta) + \sin 2(\alpha+\beta)\tan\beta = 0 \Rightarrow \cot 2(\alpha+\beta) + \tan\beta = 0 \Rightarrow \cot 2(\alpha+\beta) = -\tan\beta$ $= \tan(-\beta) \Rightarrow 2(\alpha+\beta) = 90^\circ - (-\beta) = 90^\circ + \beta \Rightarrow \alpha = \frac{1}{2}(90^\circ - \beta) = \frac{1}{2}$ of $\angle AOR$. Therefore v_0 would bisect $\angle AOR$ for maximum range uphill.

32. $v_0 = 116$ ft/sec, $\alpha = 45°$, and $x = (v_0 \cos \alpha)t$
$\Rightarrow\ 369 = (116 \cos 45°)t \ \Rightarrow\ t \approx 4.50$ sec;
also $y = (v_0 \sin \alpha)t - \frac{1}{2}gt^2$
$\Rightarrow\ y = (116 \sin 45°)(4.50) - \frac{1}{2}(32)(4.50)^2$
≈ 45.11 ft. It will take the ball 4.50 sec to travel 369 ft. At that time the ball will be 45.11 ft in the air and will hit the green past the pin.



33. (a) (Assuming that "x" is zero at the point of impact:)
$\mathbf{r}(t) = (x(t))\mathbf{i} + (y(t))\mathbf{j}$; where $x(t) = (35 \cos 27°)t$ and $y(t) = 4 + (35 \sin 27°)t - 16t^2$.
(b) $y_{max} = \frac{(v_0 \sin \alpha)^2}{2g} + 4 = \frac{(35 \sin 27°)^2}{64} + 4 \approx 7.945$ feet, which is reached at $t = \frac{v_0 \sin \alpha}{g} = \frac{35 \sin 27°}{32} \approx 0.497$ seconds.
(c) For the time, solve $y = 4 + (35 \sin 27°)t - 16t^2 = 0$ for t, using the quadratic formula
$t = \frac{35 \sin 27° + \sqrt{(-35 \sin 27°)^2 + 256}}{32} \approx 1.201$ sec. Then the range is about $x(1.201) = (35 \cos 27°)(1.201) \approx 37.453$ feet.
(d) For the time, solve $y = 4 + (35 \sin 27°)t - 16t^2 = 7$ for t, using the quadratic formula
$t = \frac{35 \sin 27° + \sqrt{(-35 \sin 27°)^2 - 192}}{32} \approx 0.254$ and 0.740 seconds. At those times the ball is about
$x(0.254) = (35 \cos 27°)(0.254) \approx 7.921$ feet and $x(0.740) = (35 \cos 27°)(0.740) \approx 23.077$ feet the impact point, or about $37.453 - 7.921 \approx 29.532$ feet and $37.453 - 23.077 \approx 14.376$ feet from the landing spot.
(e) Yes. It changes things because the ball won't clear the net ($y_{max} \approx 7.945$).

34. $x = x_0 + (v_0 \cos \alpha)t = 0 + (v_0 \cos 40°)t \approx 0.766\, v_0 t$ and $y = y_0 + (v_0 \sin \alpha)t - \frac{1}{2}gt^2 = 6.5 + (v_0 \sin 40°)t - 16t^2$
$\approx 6.5 + 0.643\, v_0 t - 16t^2$; now the shot went 73.833 ft $\Rightarrow 73.833 = 0.766\, v_0 t \Rightarrow t \approx \frac{96.383}{v_0}$ sec; the shot lands when $y = 0$
$\Rightarrow 0 = 6.5 + (0.643)(96.383) - 16\left(\frac{96.383}{v_0}\right)^2 \Rightarrow 0 \approx 68.474 - \frac{148{,}635}{v_0^2} \Rightarrow v_0 \approx \sqrt{\frac{148{,}635}{68.474}} \approx 46.6$ ft/sec, the shot's initial speed

35. Flight time = 1 sec and the measure of the angle of elevation is about 64° (using a protractor) so that $t = \frac{2v_0 \sin \alpha}{g}$
$\Rightarrow 1 = \frac{2v_0 \sin 64°}{32} \Rightarrow v_0 \approx 17.80$ ft/sec. Then $y_{max} = \frac{(17.80 \sin 64°)^2}{2(32)} \approx 4.00$ ft and $R = \frac{v_0^2}{g}\sin 2\alpha \Rightarrow R = \frac{(17.80)^2}{32}\sin 128°$
≈ 7.80 ft $\Rightarrow$ the engine traveled about 7.80 ft in 1 sec $\Rightarrow$ the engine velocity was about 7.80 ft/sec

36. (a) $\mathbf{r}(t) = (x(t))\mathbf{i} + (y(t))\mathbf{j}$; where $x(t) = (145 \cos 23° - 14)t$ and $y(t) = 2.5 + (145 \sin 23°)t - 16t^2$.
(b) $y_{max} = \frac{(v_0 \sin \alpha)^2}{2g} + 2.5 = \frac{(145 \sin 23°)^2}{64} + 2.5 \approx 52.655$ feet, which is reached at $t = \frac{v_0 \sin \alpha}{g} = \frac{145 \sin 23°}{32} \approx 1.771$ seconds.
(c) For the time, solve $y = 2.5 + (145 \sin 23°)t - 16t^2 = 0$ for t, using the quadratic formula
$t = \frac{145 \sin 23° + \sqrt{(145 \sin 23°)^2 + 160}}{32} \approx 3.585$ sec. Then the range at $t \approx 3.585$ is about $x = (145 \cos 23° - 14)(3.585)$
≈ 428.311 feet.
(d) For the time, solve $y = 2.5 + (145 \sin 23°)t - 16t^2 = 20$ for t, using the quadratic formula
$t = \frac{145 \sin 23° + \sqrt{(145 \sin 23°)^2 - 1120}}{32} \approx 0.342$ and 3.199 seconds. At those times the ball is about
$x(0.342) = (145 \cos 23° - 14)(0.342) \approx 40.860$ feet from home plate and $x(3.199) = (145 \cos 23° - 14)(3.199)$
≈ 382.195 feet from home plate.
(e) Yes. According to part (d), the ball is still 20 feet above the ground when it is 382 feet from home plate.

37. $\frac{d^2\mathbf{r}}{dt^2} + k\frac{d\mathbf{r}}{dt} = -g\mathbf{j} \Rightarrow P(t) = k$ and $\mathbf{Q}(t) = -g\mathbf{j} \Rightarrow \int P(t)\,dt = kt \Rightarrow v(t) = e^{\int P(t)\,dt} = e^{kt} \Rightarrow \frac{d\mathbf{r}}{dt} = \frac{1}{v(t)}\int v(t)\,\mathbf{Q}(t)\,dt$
$= -ge^{-kt}\int e^{kt}\,\mathbf{j}\,dt = -ge^{-kt}\left[\frac{e^{kt}}{k}\mathbf{j} + \mathbf{C}_1\right] = -\frac{g}{k}\mathbf{j} + \mathbf{C}e^{-kt}$, where $\mathbf{C} = -g\mathbf{C}_1$; apply the initial condition:
$\left.\frac{d\mathbf{r}}{dt}\right|_{t=0} = (v_0 \cos \alpha)\mathbf{i} + (v_0 \sin \alpha)\mathbf{j} = -\frac{g}{k}\mathbf{j} + \mathbf{C} \Rightarrow \mathbf{C} = (v_0 \cos \alpha)\mathbf{i} + \left(\frac{g}{k} + v_0 \sin \alpha\right)\mathbf{j}$
$\Rightarrow \frac{d\mathbf{r}}{dt} = \left(v_0 e^{-kt} \cos \alpha\right)\mathbf{i} + \left(-\frac{g}{k} + e^{-kt}\left(\frac{g}{k} + v_0 \sin \alpha\right)\right)\mathbf{j},\ \mathbf{r} = \int\left[\left(v_0 e^{-kt} \cos \alpha\right)\mathbf{i} + \left(-\frac{g}{k} + e^{-kt}\left(\frac{g}{k} + v_0 \sin \alpha\right)\right)\mathbf{j}\right]dt$

$= \left(-\frac{v_0}{k}e^{-kt}\cos\alpha\right)\mathbf{i} + \left(-\frac{gt}{k} - \frac{e^{-kt}}{k}\left(\frac{g}{k} + v_0\sin\alpha\right)\right)\mathbf{j} + \mathbf{C}_2$; apply the initial condition:

$\mathbf{r}(0) = \mathbf{0} = \left(-\frac{v_0}{k}\cos\alpha\right)\mathbf{i} + \left(-\frac{g}{k^2} - \frac{v_0\sin\alpha}{k}\right)\mathbf{j} + \mathbf{C}_2 \Rightarrow \mathbf{C}_2 = \left(\frac{v_0}{k}\cos\alpha\right)\mathbf{i} + \left(\frac{g}{k^2} + \frac{v_0\sin\alpha}{k}\right)\mathbf{j}$

$\Rightarrow \mathbf{r}(t) = \left(\frac{v_0}{k}\left(1 - e^{-kt}\right)\cos\alpha\right)\mathbf{i} + \left(\frac{v_0}{k}\left(1 - e^{-kt}\right)\sin\alpha + \frac{g}{k^2}\left(1 - kt - e^{-kt}\right)\right)\mathbf{j}$

38. (a) $\mathbf{r}(t) = (x(t))\mathbf{i} + (y(t))\mathbf{j}$; where $x(t) = \left(\frac{152}{0.12}\right)(1 - e^{-0.12t})(\cos 20^\circ)$ and $y(t) = 3 + \left(\frac{152}{0.12}\right)(1 - e^{-0.12t})(\sin 20^\circ) + \left(\frac{32}{0.12^2}\right)(1 - 0.12t - e^{-0.12t})$

(b) Solve graphically using a calculator or CAS: At $t \approx 1.484$ seconds the ball reaches a maximum height of about 40.435 feet.

(c) Use a graphing calculator or CAS to find that $y = 0$ when the ball has traveled for ≈ 3.126 seconds. The range is about $x(3.126) = \left(\frac{152}{0.12}\right)\left(1 - e^{-0.12(3.126)}\right)(\cos 20^\circ) \approx 372.311$ feet.

(d) Use a graphing calculator or CAS to find that $y = 30$ for $t \approx 0.689$ and 2.305 seconds, at which times the ball is about $x(0.689) \approx 94.454$ feet and $x(2.305) \approx 287.621$ feet from home plate.

(e) Yes, the batter has hit a home run since a graph of the trajectory shows that the ball is more than 14 feet above the ground when it passes over the fence.

39. (a) $\int_a^b k\mathbf{r}(t)\,dt = \int_a^b [kf(t)\mathbf{i} + kg(t)\mathbf{j} + kh(t)\mathbf{k}]\,dt = \int_a^b [kf(t)]\,dt\,\mathbf{i} + \int_a^b [kg(t)]\,dt\,\mathbf{j} + \int_a^b [kh(t)]\,dt\,\mathbf{k}$

$= k\left(\int_a^b f(t)\,dt\,\mathbf{i} + \int_a^b g(t)\,dt\,\mathbf{j} + \int_a^b h(t)\,dt\,\mathbf{k}\right) = k\int_a^b \mathbf{r}(t)\,dt$

(b) $\int_a^b [\mathbf{r}_1(t) \pm \mathbf{r}_2(t)]\,dt = \int_a^b ([f_1(t)\mathbf{i} + g_1(t)\mathbf{j} + h_1(t)\mathbf{k}] \pm [f_2(t)\mathbf{i} + g_2(t)\mathbf{j} + h_2(t)\mathbf{k}])\,dt$

$= \int_a^b ([f_1(t) \pm f_2(t)]\,\mathbf{i} + [g_1(t) \pm g_2(t)]\,\mathbf{j} + [h_1(t) \pm h_2(t)]\,\mathbf{k})\,dt$

$= \int_a^b [f_1(t) \pm f_2(t)]\,dt\,\mathbf{i} + \int_a^b [g_1(t) \pm g_2(t)]\,dt\,\mathbf{j} + \int_a^b [h_1(t) \pm h_2(t)]\,dt\,\mathbf{k}$

$= \left[\int_a^b f_1(t)\,dt\,\mathbf{i} \pm \int_a^b f_2(t)\,dt\,\mathbf{i}\right] + \left[\int_a^b g_1(t)\,dt\,\mathbf{j} \pm \int_a^b g_2(t)\,dt\,\mathbf{j}\right] + \left[\int_a^b h_1(t)\,dt\,\mathbf{k} \pm \int_a^b h_2(t)\,dt\,\mathbf{k}\right]$

$= \int_a^b \mathbf{r}_1(t)\,dt \pm \int_a^b \mathbf{r}_2(t)\,dt$

(c) Let $\mathbf{C} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$. Then $\int_a^b \mathbf{C}\cdot\mathbf{r}(t)\,dt = \int_a^b [c_1f(t) + c_2g(t) + c_3h(t)]\,dt$

$= c_1\int_a^b f(t)\,dt + c_2\int_a^b g(t)\,dt + c_3\int_a^b h(t)\,dt = \mathbf{C}\cdot\int_a^b \mathbf{r}(t)\,dt$;

$\int_a^b \mathbf{C}\times\mathbf{r}(t)\,dt = \int_a^b [c_2h(t) - c_3g(t)]\,\mathbf{i} + [c_3f(t) - c_1h(t)]\,\mathbf{j} + [c_1g(t) - c_2f(t)]\,\mathbf{k}\,dt$

$= \left[c_2\int_a^b h(t)\,dt - c_3\int_a^b g(t)\,dt\right]\mathbf{i} + \left[c_3\int_a^b f(t)\,dt - c_1\int_a^b h(t)\,dt\right]\mathbf{j} + \left[c_1\int_a^b g(t)\,dt - c_2\int_a^b f(t)\,dt\right]\mathbf{k}$

$= \mathbf{C}\times\int_a^b \mathbf{r}(t)\,dt$

40. (a) Let u and $\mathbf{r}$ be continuous on $[a, b]$. Then $\lim_{t\to t_0} u(t)\mathbf{r}(t) = \lim_{t\to t_0} [u(t)f(t)\mathbf{i} + u(t)g(t)\mathbf{j} + u(t)h(t)\mathbf{k}]$

$= u(t_0)f(t_0)\mathbf{i} + u(t_0)g(t_0)\mathbf{j} + u(t_0)h(t_0)\mathbf{k} = u(t_0)\mathbf{r}(t_0) \Rightarrow u\mathbf{r}$ is continuous for every t_0 in $[a, b]$.

(b) Let u and $\mathbf{r}$ be differentiable. Then $\frac{d}{dt}(u\mathbf{r}) = \frac{d}{dt}[u(t)f(t)\mathbf{i} + u(t)g(t)\mathbf{j} + u(t)h(t)\mathbf{k}]$

$= \left(\frac{du}{dt}f(t) + u(t)\frac{df}{dt}\right)\mathbf{i} + \left(\frac{du}{dt}g(t) + u(t)\frac{dg}{dt}\right)\mathbf{j} + \left(\frac{du}{dt}h(t) + u(t)\frac{dh}{dt}\right)\mathbf{k}$

$= [f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}]\frac{du}{dt} + u(t)\left(\frac{df}{dt}\mathbf{i} + \frac{dg}{dt}\mathbf{j} + \frac{dh}{dt}\mathbf{k}\right) = \mathbf{r}\frac{du}{dt} + u\frac{d\mathbf{r}}{dt}$

41. (a) If $\mathbf{R}_1(t)$ and $\mathbf{R}_2(t)$ have identical derivatives on I, then $\frac{d\mathbf{R}_1}{dt} = \frac{df_1}{dt}\mathbf{i} + \frac{dg_1}{dt}\mathbf{j} + \frac{dh_1}{dt}\mathbf{k} = \frac{df_2}{dt}\mathbf{i} + \frac{dg_2}{dt}\mathbf{j} + \frac{dh_2}{dt}\mathbf{k}$ $= \frac{d\mathbf{R}_2}{dt} \Rightarrow \frac{df_1}{dt} = \frac{df_2}{dt}, \frac{dg_1}{dt} = \frac{dg_2}{dt}, \frac{dh_1}{dt} = \frac{dh_2}{dt} \Rightarrow f_1(t) = f_2(t) + c_1, g_1(t) = g_2(t) + c_2, h_1(t) = h_2(t) + c_3$ $\Rightarrow f_1(t)\mathbf{i} + g_1(t)\mathbf{j} + h_1(t)\mathbf{k} = [f_2(t) + c_1]\mathbf{i} + [g_2(t) + c_2]\mathbf{j} + [h_2(t) + c_3]\mathbf{k} \Rightarrow \mathbf{R}_1(t) = \mathbf{R}_2(t) + \mathbf{C}$, where $\mathbf{C} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$.

(b) Let $\mathbf{R}(t)$ be an antiderivative of $\mathbf{r}(t)$ on I. Then $\mathbf{R}'(t) = \mathbf{r}(t)$. If $\mathbf{U}(t)$ is an antiderivative of $\mathbf{r}(t)$ on I, then $\mathbf{U}'(t) = \mathbf{r}(t)$. Thus $\mathbf{U}'(t) = \mathbf{R}'(t)$ on I $\Rightarrow$ $\mathbf{U}(t) = \mathbf{R}(t) + \mathbf{C}$.

42. $\frac{d}{dt}\int_a^t \mathbf{r}(\tau)\,d\tau = \frac{d}{dt}\int_a^t [f(\tau)\mathbf{i} + g(\tau)\mathbf{j} + h(\tau)\mathbf{k}]\,d\tau = \frac{d}{dt}\int_a^t f(\tau)\,d\tau\,\mathbf{i} + \frac{d}{dt}\int_a^t g(\tau)\,d\tau\,\mathbf{j} + \frac{d}{dt}\int_a^t h(\tau)\,d\tau\,\mathbf{k}$ $= f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} = \mathbf{r}(t)$. Since $\frac{d}{dt}\int_a^t \mathbf{r}(\tau)\,d\tau = \mathbf{r}(t)$, we have that $\int_a^t \mathbf{r}(\tau)\,d\tau$ is an antiderivative of $\mathbf{r}$. If $\mathbf{R}$ is any antiderivative of $\mathbf{r}$, then $\mathbf{R}(t) = \int_a^t \mathbf{r}(\tau)\,d\tau + \mathbf{C}$ by Exercise 41(b). Then $\mathbf{R}(a) = \int_a^a \mathbf{r}(\tau)\,d\tau + \mathbf{C}$ $= \mathbf{0} + \mathbf{C} \Rightarrow \mathbf{C} = \mathbf{R}(a) \Rightarrow \int_a^t \mathbf{r}(\tau)\,d\tau = \mathbf{R}(t) - \mathbf{C} = \mathbf{R}(t) - \mathbf{R}(a) \Rightarrow \int_a^b \mathbf{r}(\tau)\,d\tau = \mathbf{R}(b) - \mathbf{R}(a)$.

43. (a) $\mathbf{r}(t) = (x(t))\mathbf{i} + (y(t))\mathbf{j}$; where $x(t) = \left(\frac{1}{0.08}\right)(1 - e^{-0.08t})(152\cos 20^\circ - 17.6)$ and $y(t) = 3 + \left(\frac{152}{0.08}\right)(1 - e^{-0.08t})(\sin 20^\circ) + \left(\frac{32}{0.08^2}\right)(1 - 0.08t - e^{-0.08t})$

(b) Solve graphically using a calculator or CAS: At $t \approx 1.527$ seconds the ball reaches a maximum height of about 41.893 feet.

(c) Use a graphing calculator or CAS to find that $y = 0$ when the ball has traveled for ≈ 3.181 seconds. The range is about $x(3.181) = \left(\frac{1}{0.08}\right)\left(1 - e^{-0.08(3.181)}\right)(152\cos 20^\circ - 17.6) \approx 351.734$ feet.

(d) Use a graphing calculator or CAS to find that $y = 35$ for $t \approx 0.877$ and 2.190 seconds, at which times the ball is about $x(0.877) \approx 106.028$ feet and $x(2.190) \approx 251.530$ feet from home plate.

(e) No; the range is less than 380 feet. To find the wind needed for a home run, first use the method of part (d) to find that $y = 20$ at $t \approx 0.376$ and 2.716 seconds. Then define $x(w) = \left(\frac{1}{0.08}\right)\left(1 - e^{-0.08(2.716)}\right)(152\cos 20^\circ + w)$, and solve $x(w) = 380$ to find $w \approx 12.846$ ft/sec.

44. $y_{max} = \frac{(v_0 \sin\alpha)^2}{2g} \Rightarrow \frac{3}{4}y_{max} = \frac{3(v_0\sin\alpha)^2}{8g}$ and $y = (v_0\sin\alpha)t - \frac{1}{2}gt^2 \Rightarrow \frac{3(v_0\sin\alpha)^2}{8g} = (v_0\sin\alpha)t - \frac{1}{2}gt^2$ $\Rightarrow 3(v_0\sin\alpha)^2 = (8gv_0\sin\alpha)t - 4g^2t^2 \Rightarrow 4g^2t^2 - (8gv_0\sin\alpha)t + 3(v_0\sin\alpha)^2 = 0 \Rightarrow 2gt - 3v_0\sin\alpha = 0$ or $2gt - v_0\sin\alpha = 0 \Rightarrow t = \frac{3v_0\sin\alpha}{2g}$ or $t = \frac{v_0\sin\alpha}{2g}$. Since the time it takes to reach y_{max} is $t_{max} = \frac{v_0\sin\alpha}{g}$, then the time it takes the projectile to reach $\frac{3}{4}$ of y_{max} is the shorter time $t = \frac{v_0\sin\alpha}{2g}$ or half the time it takes to reach the maximum height.

13.3 ARC LENGTH IN SPACE

1. $\mathbf{r} = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j} + \sqrt{5}t\mathbf{k} \Rightarrow \mathbf{v} = (-2\sin t)\mathbf{i} + (2\cos t)\mathbf{j} + \sqrt{5}\mathbf{k}$ $\Rightarrow |\mathbf{v}| = \sqrt{(-2\sin t)^2 + (2\cos t)^2 + \left(\sqrt{5}\right)^2} = \sqrt{4\sin^2 t + 4\cos^2 t + 5} = 3$; $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$ $= \left(-\frac{2}{3}\sin t\right)\mathbf{i} + \left(\frac{2}{3}\cos t\right)\mathbf{j} + \frac{\sqrt{5}}{3}\mathbf{k}$ and Length $= \int_0^\pi |\mathbf{v}|\,dt = \int_0^\pi 3\,dt = [3t]_0^\pi = 3\pi$

2. $\mathbf{r} = (6\sin 2t)\mathbf{i} + (6\cos 2t)\mathbf{j} + 5t\mathbf{k} \Rightarrow \mathbf{v} = (12\cos 2t)\mathbf{i} + (-12\sin 2t)\mathbf{j} + 5\mathbf{k}$ $\Rightarrow |\mathbf{v}| = \sqrt{(12\cos 2t)^2 + (-12\sin 2t)^2 + 5^2} = \sqrt{144\cos^2 2t + 144\sin^2 2t + 25} = 13$; $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$ $= \left(\frac{12}{13}\cos 2t\right)\mathbf{i} - \left(\frac{12}{13}\sin 2t\right)\mathbf{j} + \frac{5}{13}\mathbf{k}$ and Length $= \int_0^\pi |\mathbf{v}|\,dt = \int_0^\pi 13\,dt = [13t]_0^\pi = 13\pi$

3. $\mathbf{r} = t\mathbf{i} + \frac{2}{3}t^{3/2}\mathbf{k} \Rightarrow \mathbf{v} = \mathbf{i} + t^{1/2}\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{1^2 + \left(t^{1/2}\right)^2} = \sqrt{1+t}$; $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{1+t}}\mathbf{i} + \frac{\sqrt{t}}{\sqrt{1+t}}\mathbf{k}$ and Length $= \int_0^8 \sqrt{1+t}\,dt = \left[\frac{2}{3}(1+t)^{3/2}\right]_0^8 = \frac{52}{3}$

4. $\mathbf{r} = (2+t)\mathbf{i} - (t+1)\mathbf{j} + t\mathbf{k} \Rightarrow \mathbf{v} = \mathbf{i} - \mathbf{j} + \mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$; $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$ and Length $= \int_0^3 \sqrt{3}\,dt = \left[\sqrt{3}t\right]_0^3 = 3\sqrt{3}$

5. $\mathbf{r} = (\cos^3 t)\,\mathbf{j} + (\sin^3 t)\,\mathbf{k} \Rightarrow \mathbf{v} = (-3\cos^2 t \sin t)\,\mathbf{j} + (3\sin^2 t \cos t)\,\mathbf{k} \Rightarrow |\mathbf{v}|$
$= \sqrt{(-3\cos^2 t \sin t)^2 + (3\sin^2 t \cos t)^2} = \sqrt{(9\cos^2 t \sin^2 t)(\cos^2 t + \sin^2 t)} = 3\,|\cos t \sin t|$;
$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{-3\cos^2 t \sin t}{3\,|\cos t \sin t|}\,\mathbf{j} + \frac{3\sin^2 t \cos t}{3\,|\cos t \sin t|}\,\mathbf{k} = (-\cos t)\mathbf{j} + (\sin t)\mathbf{k}$, if $0 \le t \le \frac{\pi}{2}$, and
$\text{Length} = \int_0^{\pi/2} 3\,|\cos t \sin t|\,dt = \int_0^{\pi/2} 3\cos t \sin t\,dt = \int_0^{\pi/2} \frac{3}{2}\sin 2t\,dt = \left[-\frac{3}{4}\cos 2t\right]_0^{\pi/2} = \frac{3}{2}$

6. $\mathbf{r} = 6t^3\mathbf{i} - 2t^3\mathbf{j} - 3t^3\mathbf{k} \Rightarrow \mathbf{v} = 18t^2\mathbf{i} - 6t^2\mathbf{j} - 9t^2\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(18t^2)^2 + (-6t^2)^2 + (-9t^2)^2} = \sqrt{441t^4} = 21t^2$;
$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{18t^2}{21t^2}\,\mathbf{i} - \frac{6t^2}{21t^2}\,\mathbf{j} - \frac{9t^2}{21t^2}\,\mathbf{k} = \frac{6}{7}\,\mathbf{i} - \frac{2}{7}\,\mathbf{j} - \frac{3}{7}\,\mathbf{k}$ and $\text{Length} = \int_1^2 21t^2\,dt = [7t^3]_1^2 = 49$

7. $\mathbf{r} = (t\cos t)\mathbf{i} + (t\sin t)\mathbf{j} + \frac{2\sqrt{2}}{3}\,t^{3/2}\mathbf{k} \Rightarrow \mathbf{v} = (\cos t - t\sin t)\mathbf{i} + (\sin t + t\cos t)\mathbf{j} + \left(\sqrt{2}\,t^{1/2}\right)\mathbf{k}$
$\Rightarrow |\mathbf{v}| = \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + \left(\sqrt{2t}\right)^2} = \sqrt{1 + t^2 + 2t} = \sqrt{(t+1)^2} = |t+1| = t+1$, if $t \ge 0$;
$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{\cos t - t\sin t}{t+1}\right)\mathbf{i} + \left(\frac{\sin t + t\cos t}{t+1}\right)\mathbf{j} + \left(\frac{\sqrt{2}\,t^{1/2}}{t+1}\right)\mathbf{k}$ and $\text{Length} = \int_0^{\pi} (t+1)\,dt = \left[\frac{t^2}{2} + t\right]_0^{\pi} = \frac{\pi^2}{2} + \pi$

8. $\mathbf{r} = (t\sin t + \cos t)\mathbf{i} + (t\cos t - \sin t)\mathbf{j} \Rightarrow \mathbf{v} = (\sin t + t\cos t - \sin t)\mathbf{i} + (\cos t - t\sin t - \cos t)\mathbf{j}$
$= (t\cos t)\mathbf{i} - (t\sin t)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{(t\cos t)^2 + (-t\sin t)^2} = \sqrt{t^2} = |t| = t$ if $\sqrt{2} \le t \le 2$; $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$
$= \left(\frac{t\cos t}{t}\right)\mathbf{i} - \left(\frac{t\sin t}{t}\right)\mathbf{j} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j}$ and $\text{Length} = \int_{\sqrt{2}}^{2} t\,dt = \left[\frac{t^2}{2}\right]_{\sqrt{2}}^{2} = 1$

9. Let $P(t_0)$ denote the point. Then $\mathbf{v} = (5\cos t)\mathbf{i} - (5\sin t)\mathbf{j} + 12\mathbf{k}$ and $26\pi = \int_0^{t_0} \sqrt{25\cos^2 t + 25\sin^2 t + 144}\,dt$
$= \int_0^{t_0} 13\,dt = 13t_0 \Rightarrow t_0 = 2\pi$, and the point is $P(2\pi) = (5\sin 2\pi, 5\cos 2\pi, 24\pi) = (0, 5, 24\pi)$

10. Let $P(t_0)$ denote the point. Then $\mathbf{v} = (12\cos t)\mathbf{i} + (12\sin t)\mathbf{j} + 5\mathbf{k}$ and
$-13\pi = \int_0^{t_0} \sqrt{144\cos^2 t + 144\sin^2 t + 25}\,dt = \int_0^{t_0} 13\,dt = 13t_0 \Rightarrow t_0 = -\pi$, and the point is
$P(-\pi) = (12\sin(-\pi), -12\cos(-\pi), -5\pi) = (0, 12, -5\pi)$

11. $\mathbf{r} = (4\cos t)\mathbf{i} + (4\sin t)\mathbf{j} + 3t\mathbf{k} \Rightarrow \mathbf{v} = (-4\sin t)\mathbf{i} + (4\cos t)\mathbf{j} + 3\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(-4\sin t)^2 + (4\cos t)^2 + 3^2}$
$= \sqrt{25} = 5 \Rightarrow s(t) = \int_0^t 5\,d\tau = 5t \Rightarrow \text{Length} = s\left(\frac{\pi}{2}\right) = \frac{5\pi}{2}$

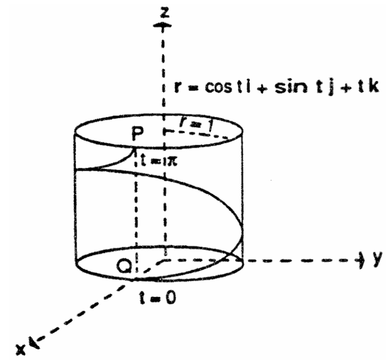
12. $\mathbf{r} = (\cos t + t\sin t)\mathbf{i} + (\sin t - t\cos t)\mathbf{j} \Rightarrow \mathbf{v} = (-\sin t + \sin t + t\cos t)\mathbf{i} + (\cos t - \cos t + t\sin t)\mathbf{j}$
$= (t\cos t)\mathbf{i} + (t\sin t)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{(t\cos t)^2 + (t\cos t)^2} = = \sqrt{t^2} = t$, since $\frac{\pi}{2} \le t \le \pi \Rightarrow s(t) = \int_0^t \tau\,d\tau = \frac{t^2}{2}$
$\Rightarrow \text{Length} = s(\pi) - s\left(\frac{\pi}{2}\right) = \frac{\pi^2}{2} - \frac{\left(\frac{\pi}{2}\right)^2}{2} = \frac{3\pi^2}{8}$

13. $\mathbf{r} = (e^t\cos t)\,\mathbf{i} + (e^t\sin t)\,\mathbf{j} + e^t\mathbf{k} \Rightarrow \mathbf{v} = (e^t\cos t - e^t\sin t)\,\mathbf{i} + (e^t\sin t + e^t\cos t)\,\mathbf{j} + e^t\mathbf{k}$
$\Rightarrow |\mathbf{v}| = \sqrt{(e^t\cos t - e^t\sin t)^2 + (e^t\sin t + e^t\cos t)^2 + (e^t)^2} = = \sqrt{3e^{2t}} = \sqrt{3}\,e^t \Rightarrow s(t) = \int_0^t \sqrt{3}\,e^{\tau}\,d\tau$
$= \sqrt{3}\,e^t - \sqrt{3} \Rightarrow \text{Length} = s(0) - s(-\ln 4) = 0 - \left(\sqrt{3}\,e^{-\ln 4} - \sqrt{3}\right) = \frac{3\sqrt{3}}{4}$

14. $\mathbf{r} = (1+2t)\mathbf{i} + (1+3t)\mathbf{j} + (6-6t)\mathbf{k} \Rightarrow \mathbf{v} = 2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{2^2 + 3^2 + (-6)^2} = 7 \Rightarrow s(t) = \int_0^t 7\,d\tau = 7t$
$\Rightarrow \text{Length} = s(0) - s(-1) = 0 - (-7) = 7$

15. $\mathbf{r} = \left(\sqrt{2}t\right)\mathbf{i} + \left(\sqrt{2}t\right)\mathbf{j} + (1-t^2)\mathbf{k} \Rightarrow \mathbf{v} = \sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j} - 2t\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{\left(\sqrt{2}\right)^2 + \left(\sqrt{2}\right)^2 + (-2t)^2} = \sqrt{4+4t^2}$ $= 2\sqrt{1+t^2} \Rightarrow \text{Length} = \int_0^1 2\sqrt{1+t^2}\,dt = \left[2\left(\frac{t}{2}\sqrt{1+t^2} + \frac{1}{2}\ln\left(t+\sqrt{1+t^2}\right)\right)\right]_0^1 = \sqrt{2} + \ln\left(1+\sqrt{2}\right)$

16. Let the helix make one complete turn from $t = 0$ to $t = 2\pi$. Note that the radius of the cylinder is $1 \Rightarrow$ the circumference of the base is 2π. When $t = 2\pi$, the point P is $(\cos 2\pi, \sin 2\pi, 2\pi) = (1, 0, 2\pi) \Rightarrow$ the cylinder is 2π units high. Cut the cylinder along PQ and flatten. The resulting rectangle has a width equal to the circumference of the cylinder $= 2\pi$ and a height equal to 2π, the height of the cylinder. Therefore, the rectangle is a square and the portion of the helix from $t = 0$ to $t = 2\pi$ is its diagonal.

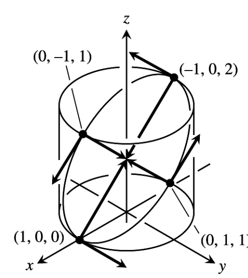


17. (a) $\mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (1-\cos t)\mathbf{k},\ 0 \le t \le 2\pi \Rightarrow x = \cos t,\ y = \sin t,\ z = 1 - \cos t \Rightarrow x^2 + y^2$ $= \cos^2 t + \sin^2 t = 1$, a right circular cylinder with the z-axis as the axis and radius $= 1$. Therefore $P(\cos t, \sin t, 1-\cos t)$ lies on the cylinder $x^2 + y^2 = 1$; $t = 0 \Rightarrow P(1, 0, 0)$ is on the curve; $t = \frac{\pi}{2} \Rightarrow Q(0, 1, 1)$ is on the curve; $t = \pi \Rightarrow R(-1, 0, 2)$ is on the curve. Then $\overrightarrow{PQ} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\overrightarrow{PR} = -2\mathbf{i} + 2\mathbf{k}$

$$\Rightarrow \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 1 \\ -2 & 0 & 2 \end{bmatrix} = 2\mathbf{i} + 2\mathbf{k}$$ is a vector normal to the plane of P, Q, and R. Then the plane containing P, Q, and R has an equation $2x + 2z = 2(1) + 2(0)$ or $x + z = 1$. Any point on the curve will satisfy this equation since $x + z = \cos t + (1 - \cos t) = 1$. Therefore, any point on the curve lies on the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x + z = 1 \Rightarrow$ the curve is an ellipse.

(b) $\mathbf{v} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + (\sin t)\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{\sin^2 t + \cos^2 t + \sin^2 t} = \sqrt{1+\sin^2 t} \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$ $= \frac{(-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + (\sin t)\mathbf{k}}{\sqrt{1+\sin^2 t}} \Rightarrow \mathbf{T}(0) = \mathbf{j},\ \mathbf{T}\left(\frac{\pi}{2}\right) = \frac{-\mathbf{i}+\mathbf{k}}{\sqrt{2}},\ \mathbf{T}(\pi) = -\mathbf{j},\ \mathbf{T}\left(\frac{3\pi}{2}\right) = \frac{\mathbf{i}-\mathbf{k}}{\sqrt{2}}$

(c) $\mathbf{a} = (-\cos t)\mathbf{i} - (\sin t)\mathbf{j} + (\cos t)\mathbf{k}$; $\mathbf{n} = \mathbf{i} + \mathbf{k}$ is normal to the plane $x + z = 1 \Rightarrow \mathbf{n} \cdot \mathbf{a} = -\cos t + \cos t$ $= 0 \Rightarrow \mathbf{a}$ is orthogonal to $\mathbf{n} \Rightarrow \mathbf{a}$ is parallel to the plane; $\mathbf{a}(0) = -\mathbf{i} + \mathbf{k},\ \mathbf{a}\left(\frac{\pi}{2}\right) = -\mathbf{j},\ \mathbf{a}(\pi) = \mathbf{i} - \mathbf{k}$, $\mathbf{a}\left(\frac{3\pi}{2}\right) = \mathbf{j}$



(d) $|\mathbf{v}| = \sqrt{1+\sin^2 t}$ (See part (b) $\Rightarrow L = \int_0^{2\pi} \sqrt{1+\sin^2 t}\,dt$

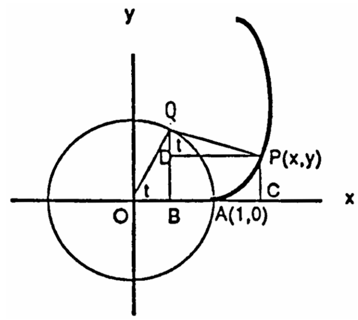
(e) $L \approx 7.64$ (by *Mathematica*)

18. (a) $\mathbf{r} = (\cos 4t)\mathbf{i} + (\sin 4t)\mathbf{j} + 4t\mathbf{k} \Rightarrow \mathbf{v} = (-4\sin 4t)\mathbf{i} + (4\cos 4t)\mathbf{j} + 4\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(-4\sin 4t)^2 + (4\cos 4t)^2 + 4^2}$ $= \sqrt{32} = 4\sqrt{2} \Rightarrow \text{Length} = \int_0^{\pi/2} 4\sqrt{2}\,dt = \left[4\sqrt{2}\,t\right]_0^{\pi/2} = 2\pi\sqrt{2}$

(b) $\mathbf{r} = \left(\cos\frac{t}{2}\right)\mathbf{i} + \left(\sin\frac{t}{2}\right)\mathbf{j} + \frac{t}{2}\mathbf{k} \Rightarrow \mathbf{v} = \left(-\frac{1}{2}\sin\frac{t}{2}\right)\mathbf{i} + \left(\frac{1}{2}\cos\frac{t}{2}\right)\mathbf{j} + \frac{1}{2}\mathbf{k}$ $\Rightarrow |\mathbf{v}| = \sqrt{\left(-\frac{1}{2}\sin\frac{t}{2}\right)^2 + \left(\frac{1}{2}\cos\frac{t}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{1}{4}+\frac{1}{4}} = \frac{\sqrt{2}}{2} \Rightarrow \text{Length} = \int_0^{4\pi} \frac{\sqrt{2}}{2}\,dt = \left[\frac{\sqrt{2}}{2}t\right]_0^{4\pi} = 2\pi\sqrt{2}$

(c) $\mathbf{r} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j} - t\mathbf{k} \Rightarrow \mathbf{v} = (-\sin t)\mathbf{i} - (\cos t)\mathbf{j} - \mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(-\sin t)^2 + (-\cos t)^2 + (-1)^2} = \sqrt{1+1}$ $= \sqrt{2} \Rightarrow \text{Length} = \int_{-2\pi}^{0} \sqrt{2}\,dt = \left[\sqrt{2}\,t\right]_{-2\pi}^{0} = 2\pi\sqrt{2}$

19. $\angle PQB = \angle QOB = t$ and $PQ = \text{arc}(AQ) = t$ since PQ = length of the unwound string = length of arc (AQ); thus $x = OB + BC = OB + DP = \cos t + t\sin t$, and $y = PC = QB - QD = \sin t - t\cos t$



20. $\mathbf{r} = (\cos t + t\sin t)\mathbf{i} + (\sin t + t\cos t)\mathbf{j} \Rightarrow \mathbf{v} = (-\sin t + t\cos t + \sin t)\mathbf{i} + (\cos t - (t(-\sin t) + \cos t))\mathbf{j}$
$= (t\cos t)\mathbf{i} + (t\sin t)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{(t\cos t)^2 + (t\sin t)^2} = \sqrt{t^2} = |t| = t,\ t \ge 0 \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{t\cos t}{t}\mathbf{i} + \frac{t\sin t}{t}\mathbf{j}$
$= \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$

21. $\mathbf{v} = \frac{d}{dt}(x_0 + t u_1)\mathbf{i} + \frac{d}{dt}(y_0 + t u_2)\mathbf{j} + \frac{d}{dt}(z_0 + t u_3)\mathbf{k} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k} = \mathbf{u}$, so $s(t) = \int_0^t |\mathbf{v}|dt = \int_0^t |\mathbf{u}|d\tau = \int_0^t 1\,d\tau = t$

22. $\mathbf{r}(t) = t\,\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k} \Rightarrow \mathbf{v}(t) = \mathbf{i} + 2t\,\mathbf{j} + 3t^2\mathbf{k} \Rightarrow |\mathbf{v}(t)| = \sqrt{(1)^2 + (2t)^2 + (3t^2)^2} = \sqrt{1 + 4t^2 + 9t^4}$. $(0, 0, 0) \Rightarrow t = 0$ and $(2, 4, 8) \Rightarrow t = 2$. Thus $L = \int_0^2 |\mathbf{v}(t)|\,dt = \int_0^2 \sqrt{1 + 4t^2 + 9t^4}\,dt$. Using Simpson's rule with $n = 10$ and $\Delta x = \frac{2-0}{10} = 0.2 \Rightarrow L \approx \frac{0.2}{3}\Big(|\mathbf{v}(0)| + 4|\mathbf{v}(0.2)| + 2|\mathbf{v}(0.4)| + 4|\mathbf{v}(0.6)| + 2|\mathbf{v}(0.8)| + 4|\mathbf{v}(1)| + 2|\mathbf{v}(1.2)| + 4|\mathbf{v}(1.4)|$
$+ 2|\mathbf{v}(1.6)| + 4|\mathbf{v}(1.8)| + |\mathbf{v}(2)|\Big) \approx \frac{0.2}{3}\Big(1 + 4(1.0837) + 2(1.3676) + 4(1.8991) + 2(2.6919) + 4(3.7417)$
$+ 2(5.0421) + 4(6.5890) + 2(8.3800) + 4(10.4134) + 12.6886\Big) = \frac{0.2}{3}(143.5594) \approx 9.5706$

13.4 CURVATURE AND NORMAL VECTORS OF A CURVE

1. $\mathbf{r} = t\mathbf{i} + \ln(\cos t)\mathbf{j} \Rightarrow \mathbf{v} = \mathbf{i} + \left(\frac{-\sin t}{\cos t}\right)\mathbf{j} = \mathbf{i} - (\tan t)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{1^2 + (-\tan t)^2} = \sqrt{\sec^2 t} = |\sec t| = \sec t$, since $-\frac{\pi}{2} < t < \frac{\pi}{2} \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{1}{\sec t}\right)\mathbf{i} - \left(\frac{\tan t}{\sec t}\right)\mathbf{j} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j}$; $\frac{d\mathbf{T}}{dt} = (-\sin t)\mathbf{i} - (\cos t)\mathbf{j}$
$\Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{(-\sin t)^2 + (-\cos t)^2} = 1 \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = (-\sin t)\mathbf{i} - (\cos t)\mathbf{j}$; $\kappa = \frac{1}{|\mathbf{v}|} \cdot \left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{\sec t} \cdot 1 = \cos t$.

2. $\mathbf{r} = \ln(\sec t)\mathbf{i} + t\mathbf{j} \Rightarrow \mathbf{v} = \left(\frac{\sec t\tan t}{\sec t}\right)\mathbf{i} + \mathbf{j} = (\tan t)\mathbf{i} + \mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{(\tan t)^2 + 1^2} = \sqrt{\sec^2 t} = |\sec t| = \sec t$, since $-\frac{\pi}{2} < t < \frac{\pi}{2} \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{\tan t}{\sec t}\right)\mathbf{i} - \left(\frac{1}{\sec t}\right)\mathbf{j} = (\sin t)\mathbf{i} + (\cos t)\mathbf{j}$; $\frac{d\mathbf{T}}{dt} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j}$
$\Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{(\cos t)^2 + (-\sin t)^2} = 1 \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j}$; $\kappa = \frac{1}{|\mathbf{v}|} \cdot \left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{\sec t} \cdot 1 = \cos t$.

3. $\mathbf{r} = (2t + 3)\mathbf{i} + (5 - t^2)\mathbf{j} \Rightarrow \mathbf{v} = 2\mathbf{i} - 2t\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{2^2 + (-2t)^2} = 2\sqrt{1 + t^2} \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{2\sqrt{1+t^2}}\mathbf{i} + \frac{-2t}{2\sqrt{1+t^2}}\mathbf{j}$
$= \frac{1}{\sqrt{1+t^2}}\mathbf{i} - \frac{t}{\sqrt{1+t^2}}\mathbf{j}$; $\frac{d\mathbf{T}}{dt} = \frac{-t}{\left(\sqrt{1+t^2}\right)^3}\mathbf{i} - \frac{1}{\left(\sqrt{1+t^2}\right)^3}\mathbf{j} \Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\left(\frac{-t}{\left(\sqrt{1+t^2}\right)^3}\right)^2 + \left(-\frac{1}{\left(\sqrt{1+t^2}\right)^3}\right)^2}$
$= \sqrt{\frac{1}{(1+t^2)^2}} = \frac{1}{1+t^2} \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = \frac{-t}{\sqrt{1+t^2}}\mathbf{i} - \frac{1}{\sqrt{1+t^2}}\mathbf{j}$; $\kappa = \frac{1}{|\mathbf{v}|} \cdot \left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{2\sqrt{1+t^2}} \cdot \frac{1}{1+t^2} = \frac{1}{2(1+t^2)^{3/2}}$

4. $\mathbf{r} = (\cos t + t\sin t)\mathbf{i} + (\sin t - t\cos t)\mathbf{j} \Rightarrow \mathbf{v} = (t\cos t)\mathbf{i} + (t\sin t)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{(t\cos t)^2 + (t\sin t)^2} = \sqrt{t^2} = |t| = t$, since $t > 0 \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{(t\cos t)\mathbf{i} + (t\sin t)\mathbf{j}}{t} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$; $\frac{d\mathbf{T}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{(-\sin t)^2 + (\cos t)^2}$
$= 1 \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$; $\kappa = \frac{1}{|\mathbf{v}|} \cdot \left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{t} \cdot 1 = \frac{1}{t}$

5. (a) $\kappa(x) = \frac{1}{|\mathbf{v}(x)|} \cdot \left|\frac{dT(x)}{dt}\right|$. Now, $\mathbf{v} = \mathbf{i} + f'(x)\mathbf{j} \Rightarrow |\mathbf{v}(x)| = \sqrt{1 + [f'(x)]^2} \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$

$= \left(1 + [f'(x)]^2\right)^{-1/2}\mathbf{i} + f'(x)\left(1 + [f'(x)]^2\right)^{-1/2}\mathbf{j}$. Thus $\frac{d\mathbf{T}}{dt}(x) = \frac{-f'(x)f''(x)}{\left(1+[f'(x)]^2\right)^{3/2}}\mathbf{i} + \frac{f''(x)}{\left(1+[f'(x)]^2\right)^{3/2}}\mathbf{j}$

$\Rightarrow \left|\frac{d\mathbf{T}(x)}{dt}\right| = \sqrt{\left[\frac{-f'(x)f''(x)}{\left(1+[f'(x)]^2\right)^{3/2}}\right]^2 + \left(\frac{f''(x)}{\left(1+[f'(x)]^2\right)^{3/2}}\right)^2} = \sqrt{\frac{[f''(x)]^2\left(1+[f'(x)]^2\right)}{\left(1+[f'(x)]^2\right)^3}} = \frac{|f''(x)|}{\left|1+[f'(x)]^2\right|}$

Thus $\kappa(x) = \frac{1}{(1+[f'(x)]^2)^{1/2}} \cdot \frac{|f''(x)|}{|1+[f'(x)]^2|} = \frac{|f''(x)|}{\left(1+[f'(x)]^2\right)^{3/2}}$

(b) $y = \ln(\cos x) \Rightarrow \frac{dy}{dx} = \left(\frac{1}{\cos x}\right)(-\sin x) = -\tan x \Rightarrow \frac{d^2y}{dx^2} = -\sec^2 x \Rightarrow \kappa = \frac{|-\sec^2 x|}{[1+(-\tan x)^2]^{3/2}} = \frac{\sec^2 x}{|\sec^3 x|}$

$= \frac{1}{\sec x} = \cos x$, since $-\frac{\pi}{2} < x < \frac{\pi}{2}$

(c) Note that $f''(x) = 0$ at an inflection point.

6. (a) $\mathbf{r} = f(t)\mathbf{i} + g(t)\mathbf{j} = x\mathbf{i} + y\mathbf{j} \Rightarrow \mathbf{v} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{\dot{x}^2 + \dot{y}^2} \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}\mathbf{i} + \frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}\mathbf{j}$

$\frac{d\mathbf{T}}{dt} = \frac{\dot{y}(\dot{y}\ddot{x} - \dot{x}\ddot{y})}{(\dot{x}^2+\dot{y}^2)^{3/2}}\mathbf{i} + \frac{\dot{x}(\dot{x}\ddot{y} - \dot{y}\ddot{x})}{(\dot{x}^2+\dot{y}^2)^{3/2}}\mathbf{j} \Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\left[\frac{\dot{y}(\dot{y}\ddot{x} - \dot{x}\ddot{y})}{(\dot{x}^2+\dot{y}^2)^{3/2}}\right]^2 + \left[\frac{\dot{x}(\dot{x}\ddot{y} - \dot{y}\ddot{x})}{(\dot{x}^2+\dot{y}^2)^{3/2}}\right]^2} = \sqrt{\frac{(\dot{y}^2+\dot{x}^2)(\dot{y}\ddot{x} - \dot{x}\ddot{y})^2}{(\dot{x}^2+\dot{y}^2)^3}}$

$= \frac{|\dot{y}\ddot{x} - \dot{x}\ddot{y}|}{|\dot{x}^2+\dot{y}^2|}$; $\kappa = \frac{1}{|\mathbf{v}|} \cdot \left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{\sqrt{\dot{x}^2+\dot{y}^2}} \cdot \frac{|\dot{y}\ddot{x} - \dot{x}\ddot{y}|}{|\dot{x}^2+\dot{y}^2|} = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2+\dot{y}^2)^{3/2}}$.

(b) $\mathbf{r}(t) = t\mathbf{i} + \ln(\sin t)\mathbf{j}$, $0 < t < \pi \Rightarrow x = t$ and $y = \ln(\sin t) \Rightarrow \dot{x} = 1, \ddot{x} = 0; \dot{y} = \frac{\cos t}{\sin t} = \cot t, \ddot{y} = -\csc^2 t$

$\Rightarrow \kappa = \frac{|-\csc^2 t - 0|}{(1+\cot^2 t))^{3/2}} = \frac{\csc^2 t}{\csc^3 t} = \sin t$

(c) $\mathbf{r}(t) = \tan^{-1}(\sinh t)\mathbf{i} + \ln(\cosh t)\mathbf{j} \Rightarrow x = \tan^{-1}(\sinh t)$ and $y = \ln(\cosh t) \Rightarrow \dot{x} = \frac{\cosh t}{1+\sinh^2 t} = \frac{1}{\cosh t}$

$= \operatorname{sech} t, \ddot{x} = -\operatorname{sech} t \tanh t; \dot{y} = \frac{\sinh t}{\cosh t} = \tanh t, \ddot{y} = \operatorname{sech}^2 t \Rightarrow \kappa = \frac{|\operatorname{sech}^3 t + \operatorname{sech} t \tanh^2 t|}{(\operatorname{sech}^2 t + \tanh^2 t)} = |\operatorname{sech} t| = \operatorname{sech} t$

7. (a) $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} \Rightarrow \mathbf{v} = f'(t)\mathbf{i} + g'(t)\mathbf{j}$ is tangent to the curve at the point $(f(t), g(t))$;
$\mathbf{n} \cdot \mathbf{v} = [-g'(t)\mathbf{i} + f'(t)\mathbf{j}] \cdot [f'(t)\mathbf{i} + g'(t)\mathbf{j}] = -g'(t)f'(t) + f'(t)g'(t) = 0$; $-\mathbf{n} \cdot \mathbf{v} = -(\mathbf{n} \cdot \mathbf{v}) = 0$; thus, $\mathbf{n}$ and $-\mathbf{n}$ are both normal to the curve at the point

(b) $\mathbf{r}(t) = t\mathbf{i} + e^{2t}\mathbf{j} \Rightarrow \mathbf{v} = \mathbf{i} + 2e^{2t}\mathbf{j} \Rightarrow \mathbf{n} = -2e^{2t}\mathbf{i} + \mathbf{j}$ points toward the concave side of the curve; $\mathbf{N} = \frac{\mathbf{n}}{|\mathbf{n}|}$ and $|\mathbf{n}| = \sqrt{4e^{4t}+1} \Rightarrow \mathbf{N} = \frac{-2e^{2t}}{\sqrt{1+4e^{4t}}}\mathbf{i} + \frac{1}{\sqrt{1+4e^{4t}}}\mathbf{j}$

(c) $\mathbf{r}(t) = \sqrt{4-t^2}\,\mathbf{i} + t\mathbf{j} \Rightarrow \mathbf{v} = \frac{-t}{\sqrt{4-t^2}}\mathbf{i} + \mathbf{j} \Rightarrow \mathbf{n} = -\mathbf{i} - \frac{t}{\sqrt{4-t^2}}\mathbf{j}$ points toward the concave side of the curve;
$\mathbf{N} = \frac{\mathbf{n}}{|\mathbf{n}|}$ and $|\mathbf{n}| = \sqrt{1 + \frac{t^2}{4-t^2}} = \frac{2}{\sqrt{4-t^2}} \Rightarrow \mathbf{N} = -\frac{1}{2}\left(\sqrt{4-t^2}\,\mathbf{i} + t\mathbf{j}\right)$

8. (a) $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{3}t^3\mathbf{j} \Rightarrow \mathbf{v} = \mathbf{i} + t^2\mathbf{j} \Rightarrow \mathbf{n} = t^2\mathbf{i} - \mathbf{j}$ points toward the concave side of the curve when $t < 0$ and $-\mathbf{n} = -t^2\mathbf{i} + \mathbf{j}$ points toward the concave side when $t > 0 \Rightarrow \mathbf{N} = \frac{1}{\sqrt{1+t^4}}(t^2\mathbf{i} - \mathbf{j})$ for $t < 0$ and
$\mathbf{N} = \frac{1}{\sqrt{1+t^4}}(-t^2\mathbf{i} + \mathbf{j})$ for $t > 0$

(b) From part (a), $|\mathbf{v}| = \sqrt{1+t^4} \Rightarrow \mathbf{T} = \frac{1}{\sqrt{1+t^4}}\mathbf{i} + \frac{t^2}{\sqrt{1+t^4}}\mathbf{j} \Rightarrow \frac{d\mathbf{T}}{dt} = \frac{-2t^3}{(1+t^4)^{3/2}}\mathbf{i} + \frac{2t}{(1+t^4)^{3/2}}\mathbf{j} \Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\frac{4t^6+4t^2}{(1+t^4)^3}}$

$= \frac{2|t|}{1+t^4}$; $\mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = \frac{1+t^4}{2|t|}\left(\frac{-2t^3}{(1+t^4)^{3/2}}\mathbf{i} + \frac{2t}{(1+t^4)^{3/2}}\mathbf{j}\right) = \frac{-t^3}{|t|\sqrt{1+t^4}}\mathbf{i} + \frac{t}{|t|\sqrt{1+t^4}}\mathbf{j}$; $t \neq 0$. $\mathbf{N}$ does not exist at $t = 0$, where the curve has a point of inflection; $\left.\frac{d\mathbf{T}}{dt}\right|_{t=0} = 0$ so the curvature $\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \left|\frac{d\mathbf{T}}{dt} \cdot \frac{dt}{ds}\right| = 0$ at $t = 0 \Rightarrow \mathbf{N} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds}$ is undefined. Since $x = t$ and $y = \frac{1}{3}t^3 \Rightarrow y = \frac{1}{3}x^3$, the curve is the cubic power curve which is concave down for $x = t < 0$ and concave up for $x = t > 0$.

9. $\mathbf{r} = (3\sin t)\mathbf{i} + (3\cos t)\mathbf{j} + 4t\mathbf{k} \Rightarrow \mathbf{v} = (3\cos t)\mathbf{i} + (-3\sin t)\mathbf{j} + 4\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(3\cos t)^2 + (-3\sin t)^2 + 4^2} = \sqrt{25}$
$= 5 \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{3}{5}\cos t\right)\mathbf{i} - \left(\frac{3}{5}\sin t\right)\mathbf{j} + \frac{4}{5}\mathbf{k} \Rightarrow \frac{d\mathbf{T}}{dt} = \left(-\frac{3}{5}\sin t\right)\mathbf{i} - \left(\frac{3}{5}\cos t\right)\mathbf{j}$

$\Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\left(-\frac{3}{5}\sin t\right)^2 + \left(-\frac{3}{5}\cos t\right)^2} = \frac{3}{5} \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = (-\sin t)\mathbf{i} - (\cos t)\mathbf{j}\,;\ \kappa = \frac{1}{5}\cdot\frac{3}{5} = \frac{3}{25}$

10. $\mathbf{r} = (\cos t + t\sin t)\mathbf{i} + (\sin t - t\cos t)\mathbf{j} + 3\mathbf{k} \Rightarrow \mathbf{v} = (t\cos t)\mathbf{i} + (t\sin t)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{(t\cos t)^2 + (t\sin t)^2} = \sqrt{t^2}$
$= |t| = t$, if $t > 0 \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j}$, $t > 0 \Rightarrow \frac{d\mathbf{T}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$
$\Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1 \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}\,;\ \kappa = \frac{1}{t}\cdot 1 = \frac{1}{t}$

11. $\mathbf{r} = (e^t\cos t)\,\mathbf{i} + (e^t\sin t)\,\mathbf{j} + 2\mathbf{k} \Rightarrow \mathbf{v} = (e^t\cos t - e^t\sin t)\,\mathbf{i} + (e^t\sin t + e^t\cos t)\,\mathbf{j} \Rightarrow$
$|\mathbf{v}| = \sqrt{(e^t\cos t - e^t\sin t)^2 + (e^t\sin t + e^t\cos t)^2} = \sqrt{2e^{2t}} = e^t\sqrt{2}\,;$
$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{\cos t - \sin t}{\sqrt{2}}\right)\mathbf{i} + \left(\frac{\sin t + \cos t}{\sqrt{2}}\right)\mathbf{j} \Rightarrow \frac{d\mathbf{T}}{dt} = \left(\frac{-\sin t - \cos t}{\sqrt{2}}\right)\mathbf{i} + \left(\frac{\cos t - \sin t}{\sqrt{2}}\right)\mathbf{j}$
$\Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\left(\frac{-\sin t - \cos t}{\sqrt{2}}\right)^2 + \left(\frac{\cos t - \sin t}{\sqrt{2}}\right)^2} = 1 \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = \left(\frac{-\cos t - \sin t}{\sqrt{2}}\right)\mathbf{i} + \left(\frac{-\sin t + \cos t}{\sqrt{2}}\right)\mathbf{j}\,;$
$\kappa = \frac{1}{|\mathbf{v}|}\cdot\left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{e^t\sqrt{2}}\cdot 1 = \frac{1}{e^t\sqrt{2}}$

12. $\mathbf{r} = (6\sin 2t)\mathbf{i} + (6\cos 2t)\mathbf{j} + 5t\mathbf{k} \Rightarrow \mathbf{v} = (12\cos 2t)\mathbf{i} - (12\sin 2t)\mathbf{j} + 5\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(12\cos 2t)^2 + (-12\sin 2t)^2 + 5^2}$
$= \sqrt{169} = 13 \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{12}{13}\cos 2t\right)\mathbf{i} - \left(\frac{12}{13}\sin 2t\right)\mathbf{j} + \frac{5}{13}\mathbf{k} \Rightarrow \frac{d\mathbf{T}}{dt} = \left(-\frac{24}{13}\sin 2t\right)\mathbf{i} - \left(\frac{24}{13}\cos 2t\right)\mathbf{j}$
$\Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\left(-\frac{24}{13}\sin 2t\right)^2 + \left(-\frac{24}{13}\cos 2t\right)^2} = \frac{24}{13} \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = (-\sin 2t)\mathbf{i} - (\cos 2t)\mathbf{j}\,;$
$\kappa = \frac{1}{|\mathbf{v}|}\cdot\left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{13}\cdot\frac{24}{13} = \frac{24}{169}.$

13. $\mathbf{r} = \left(\frac{t^3}{3}\right)\mathbf{i} + \left(\frac{t^2}{2}\right)\mathbf{j}$, $t > 0 \Rightarrow \mathbf{v} = t^2\mathbf{i} + t\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{t^4 + t^2} = t\sqrt{t^2+1}$, since $t > 0 \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$
$= \frac{t}{\sqrt{t^2+t}}\mathbf{i} + \frac{1}{\sqrt{t^2+1}}\mathbf{j} \Rightarrow \frac{d\mathbf{T}}{dt} = \frac{1}{(t^2+1)^{3/2}}\mathbf{i} - \frac{t}{(t^2+1)^{3/2}}\mathbf{j} \Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\left(\frac{1}{(t^2+1)^{3/2}}\right)^2 + \left(\frac{-t}{(t^2+1)^{3/2}}\right)^2}$
$= \sqrt{\frac{1+t^2}{(t^2+1)^3}} = \frac{1}{t^2+1} \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = \frac{1}{\sqrt{t^2+1}}\mathbf{i} - \frac{t}{\sqrt{t^2+1}}\mathbf{j}\,;\ \kappa = \frac{1}{|\mathbf{v}|}\cdot\left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{t\sqrt{t^2+1}}\cdot\frac{1}{t^2+1} = \frac{1}{t\,(t^2+1)^{3/2}}.$

14. $\mathbf{r} = (\cos^3 t)\,\mathbf{i} + (\sin^3 t)\,\mathbf{j}$, $0 < t < \frac{\pi}{2} \Rightarrow \mathbf{v} = (-3\cos^2 t\sin t)\,\mathbf{i} + (3\sin^2 t\cos t)\,\mathbf{j}$
$\Rightarrow |\mathbf{v}| = \sqrt{(-3\cos^2 t\sin t)^2 + (3\sin^2 t\cos t)^2} = \sqrt{9\cos^4 t\sin^2 t + 9\sin^4 t\cos^2 t} = 3\cos t\sin t$, since $0 < t < \frac{\pi}{2}$
$\Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = (-\cos t)\mathbf{i} + (\sin t)\mathbf{j} \Rightarrow \frac{d\mathbf{T}}{dt} = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\sin^2 t + \cos^2 t} = 1 \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|}$
$= (\sin t)\mathbf{i} + (\cos t)\mathbf{j}$; $\kappa = \frac{1}{|\mathbf{v}|}\cdot\left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{3\cos t\sin t}\cdot 1 = \frac{1}{3\cos t\sin t}.$

15. $\mathbf{r} = t\mathbf{i} + \left(a\cosh\frac{t}{a}\right)\mathbf{j}$, $a > 0 \Rightarrow \mathbf{v} = \mathbf{i} + \left(\sinh\frac{t}{a}\right)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{1 + \sinh^2\left(\frac{t}{a}\right)} = \sqrt{\cosh^2\left(\frac{t}{a}\right)} = \cosh\frac{t}{a}$
$\Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\operatorname{sech}\frac{t}{a}\right)\mathbf{i} + \left(\tanh\frac{t}{a}\right)\mathbf{j} \Rightarrow \frac{d\mathbf{T}}{dt} = \left(-\frac{1}{a}\operatorname{sech}\frac{t}{a}\tanh\frac{t}{a}\right)\mathbf{i} + \left(\frac{1}{a}\operatorname{sech}^2\frac{t}{a}\right)\mathbf{j}$
$\Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\frac{1}{a^2}\operatorname{sech}^2\left(\frac{t}{a}\right)\tanh^2\left(\frac{t}{a}\right) + \frac{1}{a^2}\operatorname{sech}^4\left(\frac{t}{a}\right)} = \frac{1}{a}\operatorname{sech}\left(\frac{t}{a}\right) \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = \left(-\tanh\frac{t}{a}\right)\mathbf{i} + \left(\operatorname{sech}\frac{t}{a}\right)\mathbf{j}\,;$
$\kappa = \frac{1}{|\mathbf{v}|}\cdot\left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{\cosh\frac{t}{a}}\cdot\frac{1}{a}\operatorname{sech}\left(\frac{t}{a}\right) = \frac{1}{a}\operatorname{sech}^2\left(\frac{t}{a}\right).$

16. $\mathbf{r} = (\cosh t)\mathbf{i} - (\sinh t)\mathbf{j} + t\mathbf{k} \Rightarrow \mathbf{v} = (\sinh t)\mathbf{i} - (\cosh t)\mathbf{j} + \mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{\sinh^2 t + (-\cosh t)^2 + 1} = \sqrt{2}\cosh t$
$\Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{1}{\sqrt{2}}\tanh t\right)\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j} + \left(\frac{1}{\sqrt{2}}\operatorname{sech} t\right)\mathbf{k} \Rightarrow \frac{d\mathbf{T}}{dt} = \left(\frac{1}{\sqrt{2}}\operatorname{sech}^2 t\right)\mathbf{i} - \left(\frac{1}{\sqrt{2}}\operatorname{sech} t\tanh t\right)\mathbf{k}$
$\Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\frac{1}{2}\operatorname{sech}^4 t + \frac{1}{2}\operatorname{sech}^2 t\tanh^2 t} = \frac{1}{\sqrt{2}}\operatorname{sech} t \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = (\operatorname{sech} t)\mathbf{i} - (\tanh t)\mathbf{k}\,;$
$\kappa = \frac{1}{|\mathbf{v}|}\cdot\left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{\sqrt{2}\cosh t}\cdot\frac{1}{\sqrt{2}}\operatorname{sech} t = \frac{1}{2}\operatorname{sech}^2 t.$

17. $y = ax^2 \Rightarrow y' = 2ax \Rightarrow y'' = 2a$; from Exercise 5(a), $\kappa(x) = \frac{|2a|}{(1+4a^2x^2)^{3/2}} = |2a|\left(1+4a^2x^2\right)^{-3/2}$
$\Rightarrow \kappa'(x) = -\frac{3}{2}|2a|\left(1+4a^2x^2\right)^{-5/2}(8a^2x)$; thus, $\kappa'(x) = 0 \Rightarrow x = 0$. Now, $\kappa'(x) > 0$ for $x < 0$ and $\kappa'(x) < 0$ for $x > 0$ so that $\kappa(x)$ has an absolute maximum at $x = 0$ which is the vertex of the parabola. Since $x = 0$ is the only critical point for $\kappa(x)$, the curvature has no minimum value.

18. $\mathbf{r} = (a\cos t)\mathbf{i} + (b\sin t)\mathbf{j} \Rightarrow \mathbf{v} = (-a\sin t)\mathbf{i} + (b\cos t)\mathbf{j} \Rightarrow \mathbf{a} = (-a\cos t)\mathbf{i} - (b\sin t)\mathbf{j} \Rightarrow \mathbf{v}\times\mathbf{a}$
$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin t & b\cos t & 0 \\ -a\cos t & -b\sin t & 0 \end{vmatrix} = ab\mathbf{k} \Rightarrow |\mathbf{v}\times\mathbf{a}| = |ab| = ab$, since $a > b > 0$; $\kappa(t) = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3}$
$= ab\left(a^2\sin^2 t + b^2\cos^2 t\right)^{-3/2}$; $\kappa'(t) = -\frac{3}{2}(ab)\left(a^2\sin^2 t + b^2\cos^2 t\right)^{-5/2}\left(2a^2\sin t\cos t - 2b^2\sin t\cos t\right)$
$= -\frac{3}{2}(ab)\left(a^2 - b^2\right)(\sin 2t)\left(a^2\sin^2 t + b^2\cos^2 t\right)^{-5/2}$; thus, $\kappa'(t) = 0 \Rightarrow \sin 2t = 0 \Rightarrow t = 0, \pi$ identifying points on the major axis, or $t = \frac{\pi}{2}, \frac{3\pi}{2}$ identifying points on the minor axis. Furthermore, $\kappa'(t) < 0$ for $0 < t < \frac{\pi}{2}$ and for $\pi < t < \frac{3\pi}{2}$; $\kappa'(t) > 0$ for $\frac{\pi}{2} < t < \pi$ and $\frac{3\pi}{2} < t < 2\pi$. Therefore, the points associated with $t = 0$ and $t = \pi$ on the major axis give absolute maximum curvature and the points associated with $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$ on the minor axis give absolute minimum curvature.

19. $\kappa = \frac{a}{a^2+b^2} \Rightarrow \frac{d\kappa}{da} = \frac{-a^2+b^2}{(a^2+b^2)^2}$; $\frac{d\kappa}{da} = 0 \Rightarrow -a^2 + b^2 = 0 \Rightarrow a = \pm b \Rightarrow a = b$ since $a, b \ge 0$. Now, $\frac{d\kappa}{da} > 0$ if $a < b$ and $\frac{d\kappa}{da} < 0$ if $a > b \Rightarrow \kappa$ is at a maximum for $a = b$ and $\kappa(b) = \frac{b}{b^2+b^2} = \frac{1}{2b}$ is the maximum value of κ.

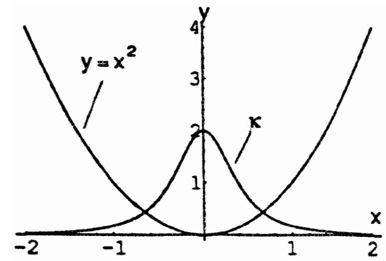
20. (a) From Example 5, the curvature of the helix $\mathbf{r}(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j} + bt\mathbf{k}$, $a, b \ge 0$ is $\kappa = \frac{a}{a^2+b^2}$; also $|\mathbf{v}| = \sqrt{a^2+b^2}$. For the helix $\mathbf{r}(t) = (3\cos t)\mathbf{i} + (3\sin t)\mathbf{j} + t\mathbf{k}$, $0 \le t \le 4\pi$, $a = 3$ and $b = 1 \Rightarrow \kappa = \frac{3}{3^2+1^2} = \frac{3}{10}$ and $|\mathbf{v}| = \sqrt{10} \Rightarrow K = \int_0^{4\pi} \frac{3}{10}\sqrt{10}\,dt = \left[\frac{3}{\sqrt{10}}t\right]_0^{4\pi} = \frac{12\pi}{\sqrt{10}}$

(b) $y = x^2 \Rightarrow x = t$ and $y = t^2$, $-\infty < t < \infty \Rightarrow \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} \Rightarrow \mathbf{v} = \mathbf{i} + 2t\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{1+4t^2}$;
$\mathbf{T} = \frac{1}{\sqrt{1+4t^2}}\mathbf{i} + \frac{2t}{\sqrt{1+4t^2}}\mathbf{j}$; $\frac{d\mathbf{T}}{dt} = \frac{-4t}{(1+4t^2)^{3/2}}\mathbf{i} + \frac{2}{(1+4t^2)^{3/2}}\mathbf{j}$; $\left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\frac{16t^2+4}{(1+4t^2)^3}} = \frac{2}{1+4t^2}$. Thus
$\kappa = \frac{1}{\sqrt{1+4t^2}} \cdot \frac{2}{1+4t^2} = \frac{2}{\left(\sqrt{1+4t^2}\right)^3}$. Then $K = \int_{-\infty}^{\infty} \frac{2}{\left(\sqrt{1+4t^2}\right)^3}\left(\sqrt{1+4t^2}\right)dt = \int_{-\infty}^{\infty} \frac{2}{1+4t^2}\,dt$
$= \lim_{a\to-\infty} \int_a^0 \frac{2}{1+4t^2}\,dt + \lim_{b\to\infty} \int_0^b \frac{2}{1+4t^2}\,dt = \lim_{a\to-\infty} \left[\tan^{-1} 2t\right]_a^0 + \lim_{b\to\infty} \left[\tan^{-1} 2t\right]_0^b$
$= \lim_{a\to-\infty} \left(-\tan^{-1} 2a\right) + \lim_{b\to\infty} \left(\tan^{-1} 2b\right) = \frac{\pi}{2} + \frac{\pi}{2} = \pi$

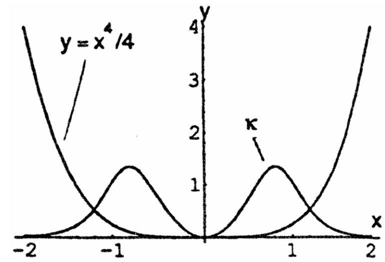
21. $\mathbf{r} = t\mathbf{i} + (\sin t)\mathbf{j} \Rightarrow \mathbf{v} = \mathbf{i} + (\cos t)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{1^2 + (\cos t)^2} = \sqrt{1+\cos^2 t} \Rightarrow \left|\mathbf{v}\left(\frac{\pi}{2}\right)\right| = \sqrt{1+\cos^2\left(\frac{\pi}{2}\right)} = 1$; $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$
$= \frac{\mathbf{i} + \cos t\,\mathbf{j}}{\sqrt{1+\cos^2 t}} \Rightarrow \frac{d\mathbf{T}}{dt} = \frac{\sin t\cos t}{(1+\cos^2 t)^{3/2}}\mathbf{i} + \frac{-\sin t}{(1+\cos^2 t)^{3/2}}\mathbf{j} \Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \frac{|\sin t|}{1+\cos^2 t}$; $\left|\frac{d\mathbf{T}}{dt}\right|_{t=\frac{\pi}{2}} = \frac{\left|\sin\frac{\pi}{2}\right|}{1+\cos^2\left(\frac{\pi}{2}\right)} = \frac{1}{1} = 1$. Thus $\kappa\left(\frac{\pi}{2}\right) = \frac{1}{1}\cdot 1 = 1$
$\Rightarrow \rho = \frac{1}{1} = 1$ and the center is $\left(\frac{\pi}{2}, 0\right) \Rightarrow \left(x - \frac{\pi}{2}\right)^2 + y^2 = 1$

22. $\mathbf{r} = (2\ln t)\mathbf{i} - \left(t + \frac{1}{t}\right)\mathbf{j} \Rightarrow \mathbf{v} = \left(\frac{2}{t}\right)\mathbf{i} - \left(1 - \frac{1}{t^2}\right)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{\frac{4}{t^2} + \left(1 - \frac{1}{t^2}\right)^2} = \frac{t^2+1}{t^2} \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2t}{t^2+1}\mathbf{i} - \frac{t^2-1}{t^2+1}\mathbf{j}$;
$\frac{d\mathbf{T}}{dt} = \frac{-2(t^2-1)}{(t^2+1)^2}\mathbf{i} - \frac{4t}{(t^2+1)^2}\mathbf{j} \Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\frac{4(t^2-1)^2 + 16t^2}{(t^2+1)^4}} = \frac{2}{t^2+1}$. Thus $\kappa = \frac{1}{|\mathbf{v}|}\cdot\left|\frac{d\mathbf{T}}{dt}\right| = \frac{t^2}{t^2+1}\cdot\frac{2}{t^2+1} = \frac{2t^2}{(t^2+1)^2} \Rightarrow \kappa(1) = \frac{2}{2^2}$
$= \frac{1}{2} \Rightarrow \rho = \frac{1}{\kappa} = 2$. The circle of curvature is tangent to the curve at $P(0, -2) \Rightarrow$ circle has same tangent as the curve $\Rightarrow \mathbf{v}(1) = 2\mathbf{i}$ is tangent to the circle $\Rightarrow$ the center lies on the y-axis. If $t \ne 1$ $(t > 0)$, then $(t-1)^2 > 0$
$\Rightarrow t^2 - 2t + 1 > 0 \Rightarrow t^2 + 1 > 2t \Rightarrow \frac{t^2+1}{t} > 2$ since $t > 0 \Rightarrow t + \frac{1}{t} > 2 \Rightarrow -\left(t + \frac{1}{t}\right) < -2 \Rightarrow y < -2$ on both sides of $(0, -2) \Rightarrow$ the curve is concave down $\Rightarrow$ center of circle of curvature is $(0, -4) \Rightarrow x^2 + (y+4)^2 = 4$ is an equation of the circle of curvature

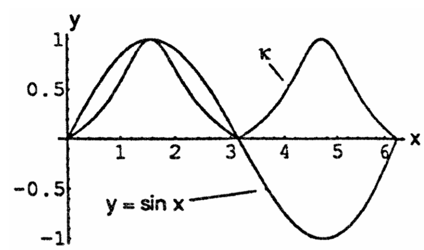
23. $y = x^2 \Rightarrow f'(x) = 2x$ and $f''(x) = 2$
$\Rightarrow \kappa = \frac{|2|}{(1+(2x)^2)^{3/2}} = \frac{2}{(1+4x^2)^{3/2}}$



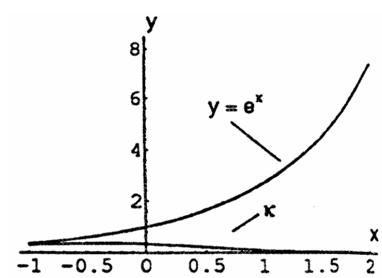
24. $y = \frac{x^4}{4} \Rightarrow f'(x) = x^3$ and $f''(x) = 3x^2$
$\Rightarrow \kappa = \frac{|3x^2|}{\left(1+(x^3)^2\right)^{3/2}} = \frac{3x^2}{(1+x^6)^{3/2}}$



25. $y = \sin x \Rightarrow f'(x) = \cos x$ and $f''(x) = -\sin x$
$\Rightarrow \kappa = \frac{|-\sin x|}{(1+\cos^2 x)^{3/2}} = \frac{|\sin x|}{(1+\cos^2 x)^{3/2}}$



26. $y = e^x \Rightarrow f'(x) = e^x$ and $f''(x) = e^x$
$\Rightarrow \kappa = \frac{|e^x|}{\left(1+(e^x)^2\right)^{3/2}} = \frac{e^x}{(1+e^{2x})^{3/2}}$



27-34. Example CAS commands:

Maple:

```
with( plots );
r := t -> [3*cos(t),5*sin(t)];
lo := 0;
hi := 2*Pi;
t0 := Pi/4;
P1 := plot( [r(t)[], t=lo..hi] ):
display( P1, scaling=constrained, title="#27(a) (Section 13.4)" );
CURVATURE := (x,y,t) ->simplify(abs(diff(x,t)*diff(y,t,t)-diff(y,t)*diff(x,t,t))/(diff(x,t)^2+diff(y,t)^2)^(3/2));
kappa := eval(CURVATURE(r(t)[],t),t=t0);
UnitNormal := (x,y,t) ->expand( [-diff(y,t),diff(x,t)]/sqrt(diff(x,t)^2+diff(y,t)^2) );
N := eval( UnitNormal(r(t)[],t), t=t0 );
C := expand( r(t0) + N/kappa );
OscCircle := (x-C[1])^2+(y-C[2])^2 = 1/kappa^2;
evalf( OscCircle );
```

```
P2 := implicitplot( (x-C[1])^2+(y-C[2])^2 = 1/kappa^2, x=-7..4, y=-4..6, color=blue ):
display( [P1,P2], scaling=constrained, title="#27(e) (Section 13.4)" );
```

Mathematica: (assigned functions and parameters may vary)

In Mathematica, the dot product can be applied either with a period "." or with the word, "Dot".

Similarly, the cross product can be applied either with a very small "x" (in the palette next to the arrow) or with the word, "Cross". However, the Cross command assumes the vectors are in three dimensions

For the purposes of applying the cross product command, we will define the position vector r as a three dimensional vector with zero for its z-component. For graphing, we will use only the first two components.

```
Clear[r, t, x, y]
r[t_]={3 Cos[t], 5 Sin[t] }
t0= π /4;  tmin= 0;  tmax= 2π;
r2[t_]= {r[t][[1]], r[t][[2]]}
pp=ParametricPlot[r2[t], {t, tmin, tmax}];
mag[v_]=Sqrt[v.v]
vel[t_]= r'[t]
speed[t_]=mag[vel[t]]
acc[t_]= vel'[t]
curv[t_]= mag[Cross[vel[t],acc[t]]]/speed[t]^3//Simplify
unittan[t_]= vel[t]/speed[t]//Simplify
unitnorm[t_]= unittan'[t] / mag[unittan'[t]]
ctr= r[t0] + (1 / curv[t0]) unitnorm[t0] //Simplify
{a,b}= {ctr[[1]], ctr[[2]]}
```

To plot the osculating circle, load a graphics package and then plot it, and show it together with the original curve.

```
<<Graphics`ImplicitPlot`
pc=ImplicitPlot[(x − a)2 + (y − b)2 == 1/curv[t0]^2 , {x, −8, 8},{y, −8, 8}]
radius=Graphics[Line[{{a, b}, r2[t0]}]]
Show[pp, pc, radius, AspectRatio → 1]
```

13.5 TANGENTIAL AND NORMAL COMPONENTS OF ACCELERATION

1. $\mathbf{r} = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j} + bt\mathbf{k} \Rightarrow \mathbf{v} = (-a\sin t)\mathbf{i} + (a\cos t)\mathbf{j} + b\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(-a\sin t)^2 + (a\cos t)^2 + b^2}$ $= \sqrt{a^2+b^2} \Rightarrow a_T = \frac{d}{dt}|\mathbf{v}| = 0; \mathbf{a} = (-a\cos t)\mathbf{i} + (-a\sin t)\mathbf{j} \Rightarrow |\mathbf{a}| = \sqrt{(-a\cos t)^2 + (-a\sin t)^2} = \sqrt{a^2} = |a|$ $\Rightarrow a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = \sqrt{|\mathbf{a}|^2 - 0^2} = |\mathbf{a}| = |a| \Rightarrow \mathbf{a} = (0)\mathbf{T} + |a|\,\mathbf{N} = |a|\,\mathbf{N}$

2. $\mathbf{r} = (1+3t)\mathbf{i} + (t-2)\mathbf{j} - 3t\mathbf{k} \Rightarrow \mathbf{v} = 3\mathbf{i} + \mathbf{j} - 3\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{3^2+1^2+(-3)^2} = \sqrt{19} \Rightarrow a_T = \frac{d}{dt}|\mathbf{v}| = 0; \mathbf{a} = \mathbf{0}$ $\Rightarrow a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = 0 \Rightarrow \mathbf{a} = (0)\mathbf{T} + (0)\mathbf{N} = \mathbf{0}$

3. $\mathbf{r} = (t+1)\mathbf{i} + 2t\mathbf{j} + t^2\mathbf{k} \Rightarrow \mathbf{v} = \mathbf{i} + 2\mathbf{j} + 2t\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{1^2+2^2+(2t)^2} = \sqrt{5+4t^2} \Rightarrow a_T = \frac{1}{2}(5+4t^2)^{-1/2}(8t)$ $= 4t(5+4t^2)^{-1/2} \Rightarrow a_T(1) = \frac{4}{\sqrt{9}} = \frac{4}{3}; \mathbf{a} = 2\mathbf{k} \Rightarrow \mathbf{a}(1) = 2\mathbf{k} \Rightarrow |\mathbf{a}(1)| = 2 \Rightarrow a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = \sqrt{2^2 - \left(\frac{4}{3}\right)^2}$ $= \sqrt{\frac{20}{9}} = \frac{2\sqrt{5}}{3} \Rightarrow \mathbf{a}(1) = \frac{4}{3}\mathbf{T} + \frac{2\sqrt{5}}{3}\mathbf{N}$

4. $\mathbf{r} = (t\cos t)\mathbf{i} + (t\sin t)\mathbf{j} + t^2\mathbf{k} \Rightarrow \mathbf{v} = (\cos t - t\sin t)\mathbf{i} + (\sin t + t\cos t)\mathbf{j} + 2t\mathbf{k}$ $\Rightarrow |\mathbf{v}| = \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + (2t)^2} = \sqrt{5t^2+1} \Rightarrow a_T = \frac{1}{2}(5t^2+1)^{-1/2}(10t)$

$= \frac{5t}{\sqrt{5t^2+1}} \Rightarrow a_T(0) = 0$; $\mathbf{a} = (-2\sin t - t\cos t)\mathbf{i} + (2\cos t - t\sin t)\mathbf{j} + 2\mathbf{k} \Rightarrow \mathbf{a}(0) = 2\mathbf{j} + 2\mathbf{k} \Rightarrow |\mathbf{a}(0)|$
$= \sqrt{2^2+2^2} = 2\sqrt{2} \Rightarrow a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = \sqrt{\left(2\sqrt{2}\right)^2 - 0^2} = 2\sqrt{2} \Rightarrow \mathbf{a}(0) = (0)\mathbf{T} + 2\sqrt{2}\mathbf{N} = 2\sqrt{2}\mathbf{N}$

5. $\mathbf{r} = t^2\mathbf{i} + \left(t + \frac{1}{3}t^3\right)\mathbf{j} + \left(t - \frac{1}{3}t^3\right)\mathbf{k} \Rightarrow \mathbf{v} = 2t\mathbf{i} + \left(1+t^2\right)\mathbf{j} + \left(1-t^2\right)\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(2t)^2 + \left(1+t^2\right)^2 + \left(1-t^2\right)^2}$
$= \sqrt{2\left(t^4+2t^2+1\right)} = \sqrt{2}\left(1+t^2\right) \Rightarrow a_T = 2t\sqrt{2} \Rightarrow a_T(0) = 0$; $\mathbf{a} = 2\mathbf{i} + 2t\mathbf{j} - 2t\mathbf{k} \Rightarrow \mathbf{a}(0) = 2\mathbf{i} \Rightarrow |\mathbf{a}(0)| = 2$
$\Rightarrow a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = \sqrt{2^2 - 0^2} = 2 \Rightarrow \mathbf{a}(0) = (0)\mathbf{T} + 2\mathbf{N} = 2\mathbf{N}$

6. $\mathbf{r} = \left(e^t\cos t\right)\mathbf{i} + \left(e^t\sin t\right)\mathbf{j} + \sqrt{2}e^t\mathbf{k} \Rightarrow \mathbf{v} = \left(e^t\cos t - e^t\sin t\right)\mathbf{i} + \left(e^t\sin t + e^t\cos t\right)\mathbf{j} + \sqrt{2}e^t\mathbf{k}$
$\Rightarrow |\mathbf{v}| = \sqrt{\left(e^t\cos t - e^t\sin t\right)^2 + \left(e^t\sin t + e^t\cos t\right)^2 + \left(\sqrt{2}e^t\right)^2} = \sqrt{4e^{2t}} = 2e^t \Rightarrow a_T = 2e^t \Rightarrow a_T(0) = 2$;
$\mathbf{a} = \left(e^t\cos t - e^t\sin t - e^t\sin t - e^t\cos t\right)\mathbf{i} + \left(e^t\sin t + e^t\cos t + e^t\cos t - e^t\sin t\right)\mathbf{j} + \sqrt{2}e^t\mathbf{k}$
$= \left(-2e^t\sin t\right)\mathbf{i} + \left(2e^t\cos t\right)\mathbf{j} + \sqrt{2}e^t\mathbf{k} \Rightarrow \mathbf{a}(0) = 2\mathbf{j} + \sqrt{2}\mathbf{k} \Rightarrow |\mathbf{a}(0)| = \sqrt{2^2 + \left(\sqrt{2}\right)^2} = \sqrt{6}$
$\Rightarrow a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = \sqrt{\left(\sqrt{6}\right)^2 - 2^2} = \sqrt{2} \Rightarrow \mathbf{a}(0) = 2\mathbf{T} + \sqrt{2}\mathbf{N}$

7. $\mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} - \mathbf{k} \Rightarrow \mathbf{v} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1 \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$
$= (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow \mathbf{T}\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$; $\frac{d\mathbf{T}}{dt} = (-\cos t)\mathbf{i} - (\sin t)\mathbf{j} \Rightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{(-\cos t)^2 + (-\sin t)^2}$
$= 1 \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = (-\cos t)\mathbf{i} - (\sin t)\mathbf{j} \Rightarrow \mathbf{N}\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$; $\mathbf{B} = \mathbf{T}\times\mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin t & \cos t & 0 \\ -\cos t & -\sin t & 0 \end{vmatrix} = \mathbf{k}$
$\Rightarrow \mathbf{B}\left(\frac{\pi}{4}\right) = \mathbf{k}$, the normal to the osculating plane; $\mathbf{r}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j} - \mathbf{k} \Rightarrow P = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, -1\right)$ lies on the osculating plane $\Rightarrow 0\left(x - \frac{\sqrt{2}}{2}\right) + 0\left(y - \frac{\sqrt{2}}{2}\right) + (z - (-1)) = 0 \Rightarrow z = -1$ is the osculating plane; $\mathbf{T}$ is normal to the normal plane $\Rightarrow \left(-\frac{\sqrt{2}}{2}\right)\left(x - \frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(y - \frac{\sqrt{2}}{2}\right) + 0(z - (-1)) = 0 \Rightarrow -\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y = 0$
$\Rightarrow -x + y = 0$ is the normal plane; $\mathbf{N}$ is normal to the rectifying plane
$\Rightarrow \left(-\frac{\sqrt{2}}{2}\right)\left(x - \frac{\sqrt{2}}{2}\right) + \left(-\frac{\sqrt{2}}{2}\right)\left(y - \frac{\sqrt{2}}{2}\right) + 0(z - (-1)) = 0 \Rightarrow -\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y = -1 \Rightarrow x + y = \sqrt{2}$ is the rectifying plane

8. $\mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k} \Rightarrow \mathbf{v} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2} \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$
$= \left(-\frac{1}{\sqrt{2}}\sin t\right)\mathbf{i} + \left(\frac{1}{\sqrt{2}}\cos t\right)\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k} \Rightarrow \frac{d\mathbf{T}}{dt} = \left(-\frac{1}{\sqrt{2}}\cos t\right)\mathbf{i} + \left(-\frac{1}{\sqrt{2}}\sin t\right)\mathbf{j} \Rightarrow \left|\frac{d\mathbf{T}}{dt}\right|$
$= \sqrt{\frac{1}{2}\cos^2 t + \frac{1}{2}\sin^2 t} = \frac{1}{\sqrt{2}} \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = (-\cos t)\mathbf{i} - (\sin t)\mathbf{j}$; thus $\mathbf{T}(0) = \frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}$ and $\mathbf{N}(0) = -\mathbf{i}$
$\Rightarrow \mathbf{B}(0) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -1 & 0 & 0 \end{vmatrix} = -\frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}$, the normal to the osculating plane; $\mathbf{r}(0) = \mathbf{i} \Rightarrow P(1,0,0)$ lies on the osculating plane $\Rightarrow 0(x-1) - \frac{1}{\sqrt{2}}(y-0) + \frac{1}{\sqrt{2}}(z-0) = 0 \Rightarrow y - z = 0$ is the osculating plane; $\mathbf{T}$ is normal to the normal plane $\Rightarrow 0(x-1) + \frac{1}{\sqrt{2}}(y-0) + \frac{1}{\sqrt{2}}(z-0) = 0 \Rightarrow y + z = 0$ is the normal plane; $\mathbf{N}$ is normal to the rectifying plane $\Rightarrow -1(x-1) + 0(y-0) + 0(z-0) = 0 \Rightarrow x = 1$ is the rectifying plane.

9. By Exercise 9 in Section 13.4, $\mathbf{T} = \left(\frac{3}{5}\cos t\right)\mathbf{i} + \left(-\frac{3}{5}\sin t\right)\mathbf{j} + \frac{4}{5}\mathbf{k}$ and $\mathbf{N} = (-\sin t)\mathbf{i} - (\cos t)\mathbf{j}$ so that $\mathbf{B} = \mathbf{T}\times\mathbf{N}$ $= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{3}{5}\cos t & -\frac{3}{5}\sin t & \frac{4}{5} \\ -\sin t & -\cos t & 0 \end{vmatrix} = \left(\frac{4}{5}\cos t\right)\mathbf{i} - \left(\frac{4}{5}\sin t\right)\mathbf{j} - \frac{3}{5}\mathbf{k}$. Also $\mathbf{v} = (3\cos t)\mathbf{i} + (-3\sin t)\mathbf{j} + 4\mathbf{k}$ $\Rightarrow \mathbf{a} = (-3\sin t)\mathbf{i} + (-3\cos t)\mathbf{j} \Rightarrow \frac{d\mathbf{a}}{dt} = (-3\cos t)\mathbf{i} + (3\sin t)\mathbf{j}$ and $\mathbf{v}\times\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3\cos t & -3\sin t & 4 \\ -3\sin t & -3\cos t & 0 \end{vmatrix}$ $= (12\cos t)\mathbf{i} - (12\sin t)\mathbf{j} - 9\mathbf{k} \Rightarrow |\mathbf{v}\times\mathbf{a}|^2 = (12\cos t)^2 + (-12\sin t)^2 + (-9)^2 = 225$. Thus
$$\tau = \frac{\begin{vmatrix} 3\cos t & -3\sin t & 4 \\ -3\sin t & -3\sin t & 0 \\ -3\cos t & 3\sin t & 0 \end{vmatrix}}{225} = \frac{4\cdot(-9\sin^2 t - 9\cos^2 t)}{225} = \frac{-36}{225} = -\frac{4}{25}$$

10. By Exercise 10 in Section 13.4, $\mathbf{T} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$ and $\mathbf{N} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$; thus $\mathbf{B} = \mathbf{T}\times\mathbf{N}$ $= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \end{vmatrix} = (\cos^2 t + \sin^2 t)\mathbf{k} = \mathbf{k}$. Also $\mathbf{v} = (t\cos t)\mathbf{i} + (t\sin t)\mathbf{j}$ $\Rightarrow \mathbf{a} = (t(-\sin t) + \cos t)\mathbf{i} + (t\cos t + \sin t)\mathbf{j} \Rightarrow \frac{d\mathbf{a}}{dt} = (-t\cos t - \sin t - \sin t)\mathbf{i} + (-t\sin t + \cos t + \cos t)\mathbf{j}$ $= (-t\cos t - 2\sin t)\mathbf{i} + (2\cos t - t\sin t)\mathbf{j}$. Thus $\mathbf{v}\times\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t\cos t & t\sin t & 0 \\ (-t\sin t + \cos t) & (t\cos t + \sin t) & 0 \end{vmatrix}$ $= [(t\cos t)(t\cos t + \sin t) - (t\sin t)(-t\sin t + \cos t)]\mathbf{k} = t^2\mathbf{k} \Rightarrow |\mathbf{v}\times\mathbf{a}|^2 = (t^2)^2 = t^4$. Thus
$$\tau = \frac{\begin{vmatrix} t\cos t & t\sin t & 0 \\ \cos t - t\sin t & \sin t + t\cos t & 0 \\ -2\sin t - t\cos t & 2\cos t - t\sin t & 0 \end{vmatrix}}{t^4} = \frac{0}{t^4} = 0$$

11. By Exercise 11 in Section 13.4, $\mathbf{T} = \left(\frac{\cos t - \sin t}{\sqrt{2}}\right)\mathbf{i} + \left(\frac{\sin t + \cos t}{\sqrt{2}}\right)\mathbf{j}$ and $\mathbf{N} = \left(\frac{-\cos t - \sin t}{\sqrt{2}}\right)\mathbf{i} + \left(\frac{-\sin t + \cos t}{\sqrt{2}}\right)\mathbf{j}$; Thus $\mathbf{B} = \mathbf{T}\times\mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\cos t - \sin t}{\sqrt{2}} & \frac{\sin t + \cos t}{\sqrt{2}} & 0 \\ \frac{-\cos t - \sin t}{\sqrt{2}} & \frac{-\sin t + \cos t}{\sqrt{2}} & 0 \end{vmatrix} = \left[\left(\frac{\cos^2 t - 2\cos t\sin t + \sin^2 t}{2}\right) + \left(\frac{\sin^2 t + 2\sin t\cos t + \cos^2 t}{2}\right)\right]\mathbf{k}$ $= \left[\left(\frac{1 - \sin(2t)}{2}\right) + \left(\frac{1 + \sin(2t)}{2}\right)\right]\mathbf{k} = \mathbf{k}$. Also, $\mathbf{v} = (e^t\cos t - e^t\sin t)\mathbf{i} + (e^t\sin t + e^t\cos t)\mathbf{j}$ $\Rightarrow \mathbf{a} = [e^t(-\sin t - \cos t) + e^t(\cos t - \sin t)]\mathbf{i} + [e^t(\cos t - \sin t) + e^t(\sin t + \cos t)]\mathbf{j} = (-2e^t\sin t)\mathbf{i} + (2e^t\cos t)\mathbf{j}$ $\Rightarrow \frac{d\mathbf{a}}{dt} = -2e^t(\cos t + \sin t)\mathbf{i} + 2e^t(-\sin t + \cos t)\mathbf{j}$. Thus $\mathbf{v}\times\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ e^t(\cos t - \sin t) & e^t(\sin t + \cos t) & 0 \\ -2e^t\sin t & 2e^t\cos t & 0 \end{vmatrix} = 2e^{2t}\mathbf{k}$ $\Rightarrow |\mathbf{v}\times\mathbf{a}|^2 = (2e^{2t})^2 = 4e^{4t}$. Thus $\tau = \frac{\begin{vmatrix} e^t(\cos t - \sin t) & e^t(\sin t + \cos t) & 0 \\ -2e^t\sin t & 2e^t\cos t & 0 \\ -2e^t(\cos t + \sin t) & 2e^t(-\sin t + \cos t) & 0 \end{vmatrix}}{4e^{4t}} = 0$

12. By Exercise 12 in Section 13.4, $\mathbf{T} = \left(\frac{12}{13}\cos 2t\right)\mathbf{i} - \left(\frac{12}{13}\sin 2t\right)\mathbf{j} + \frac{5}{13}\mathbf{k}$ and $\mathbf{N} = (-\sin 2t)\mathbf{i} - (\cos 2t)\mathbf{j}$ so $\mathbf{B} = \mathbf{T}\times\mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \left(\frac{12}{13}\cos 2t\right) & \left(-\frac{12}{13}\sin 2t\right) & \frac{5}{13} \\ (-\sin 2t) & (-\cos 2t) & 0 \end{vmatrix} = \left(\frac{5}{13}\cos 2t\right)\mathbf{i} - \left(\frac{5}{13}\sin 2t\right)\mathbf{j} - \frac{12}{13}\mathbf{k}$. Also, $\mathbf{v} = (12\cos 2t)\mathbf{i} - (12\sin 2t)\mathbf{j} + 5\mathbf{k} \Rightarrow \mathbf{a} = (-24\sin 2t)\mathbf{i} - (24\cos 2t)\mathbf{j}$ and $\frac{d\mathbf{a}}{dt} = (-48\cos 2t)\mathbf{i} + (48\sin 2t)\mathbf{j}$ $\mathbf{v}\times\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 12\cos 2t & -12\sin 2t & 5 \\ -24\sin 2t & -24\cos 2t & 0 \end{vmatrix} = (120\cos 2t)\mathbf{i} - (120\sin 2t)\mathbf{j} - 288\mathbf{k} \Rightarrow |\mathbf{v}\times\mathbf{a}|^2$ $= (120\cos 2t)^2 + (-120\sin 2t)^2 + (-288)^2 = 120^2(\cos^2 2t + \sin^2 2t) + 288^2 = 97344$. Thus

$\tau = \frac{\begin{vmatrix} 12\cos 2t & -12\sin 2t & 5 \\ -24\sin 2t & -24\cos 2t & 0 \\ -48\cos 2t & 48\sin 2t & 0 \end{vmatrix}}{97344} = \frac{5\cdot(-24\cdot 48)}{97344} = -\frac{10}{169}$

13. By Exercise 13 in Section 13.4, $\mathbf{T} = \frac{t}{(t^2+1)^{1/2}}\,\mathbf{i} + \frac{1}{(t^2+1)^{1/2}}\,\mathbf{j}$ and $\mathbf{N} = \frac{1}{\sqrt{t^2+1}}\,\mathbf{i} - \frac{t}{\sqrt{t^2+1}}\,\mathbf{j}$ so that $\mathbf{B} = \mathbf{T} \times \mathbf{N}$ $= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{t}{\sqrt{t^2+1}} & \frac{1}{\sqrt{t^2+1}} & 0 \\ \frac{1}{\sqrt{t^2+1}} & \frac{-t}{\sqrt{t^2+1}} & 0 \end{vmatrix} = -\mathbf{k}$. Also, $\mathbf{v} = t^2\mathbf{i} + t\mathbf{j} \Rightarrow \mathbf{a} = 2t\mathbf{i} + \mathbf{j} \Rightarrow \frac{d\mathbf{a}}{dt} = 2\mathbf{i}$ so that $\begin{vmatrix} t^2 & t & 0 \\ 2t & 1 & 0 \\ 2 & 0 & 0 \end{vmatrix} = 0 \Rightarrow \tau = 0$

14. By Exercise 14 in Section 13.4, $\mathbf{T} = (-\cos t)\mathbf{i} + (\sin t)\mathbf{j}$ and $\mathbf{N} = (\sin t)\mathbf{i} + (\cos t)\mathbf{j}$ so that $\mathbf{B} = \mathbf{T} \times \mathbf{N}$ $= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\cos t & \sin t & 0 \\ \sin t & \cos t & 0 \end{vmatrix} = -\mathbf{k}$. Also, $\mathbf{v} = (-3\cos^2 t \sin t)\,\mathbf{i} + (3\sin^2 t \cos t)\,\mathbf{j}$ $\Rightarrow \mathbf{a} = \frac{d}{dt}(-3\cos^2 t\sin t)\,\mathbf{i} + \frac{d}{dt}(3\sin^2 t\cos t)\,\mathbf{j} \Rightarrow \frac{d\mathbf{a}}{dt} = \frac{d}{dt}\left(\frac{d}{dt}(-3\cos^2 t\sin t)\right)\mathbf{i} + \frac{d}{dt}\left(\frac{d}{dt}(3\sin^2 t\cos t)\right)\mathbf{j}$ $\Rightarrow \begin{vmatrix} -3\cos^2 t\sin t & 3\sin^2 t\cos t & 0 \\ \frac{d}{dt}(-3\cos^2 t\sin t) & \frac{d}{dt}(3\sin^2 t\cos t) & 0 \\ \frac{d}{dt}\left(\frac{d}{dt}(-3\cos^2 t\sin t)\right) & \frac{d}{dt}\left(\frac{d}{dt}(3\sin^2 t\cos t)\right) & 0 \end{vmatrix} = 0 \Rightarrow \tau = 0$

15. By Exercise 15 in Section 13.4, $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\operatorname{sech}\frac{t}{a}\right)\mathbf{i} + \left(\tanh\frac{t}{a}\right)\mathbf{j}$ and $\mathbf{N} = \left(-\tanh\frac{t}{a}\right)\mathbf{i} + \left(\operatorname{sech}\frac{t}{a}\right)\mathbf{j}$ so that $\mathbf{B} = \mathbf{T} \times \mathbf{N}$ $= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \operatorname{sech}\left(\frac{t}{a}\right) & \tanh\left(\frac{t}{a}\right) & 0 \\ -\tanh\left(\frac{t}{a}\right) & \operatorname{sech}\left(\frac{t}{a}\right) & 0 \end{vmatrix} = \mathbf{k}$. Also, $\mathbf{v} = \mathbf{i} + \left(\sinh\frac{t}{a}\right)\mathbf{j} \Rightarrow \mathbf{a} = \left(\frac{1}{a}\cosh\frac{t}{a}\right)\mathbf{j} \Rightarrow \frac{d\mathbf{a}}{dt} = \frac{1}{a^2}\sinh\left(\frac{t}{a}\right)\mathbf{j}$ so that $\begin{vmatrix} 1 & \sinh\left(\frac{t}{a}\right) & 0 \\ 0 & \frac{1}{a}\cosh\left(\frac{t}{a}\right) & 0 \\ 0 & \frac{1}{a^2}\sinh\left(\frac{t}{a}\right) & 0 \end{vmatrix} = 0 \Rightarrow \tau = 0$

16. By Exercise 16 in Section 13.4, $\mathbf{T} = \left(\frac{1}{\sqrt{2}}\tanh t\right)\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j} + \left(\frac{1}{\sqrt{2}}\operatorname{sech} t\right)\mathbf{k}$ and $\mathbf{N} = (\operatorname{sech} t)\mathbf{i} - (\tanh t)\mathbf{k}$ so that $\mathbf{B} = \mathbf{T} \times \mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{1}{\sqrt{2}}\tanh t & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\operatorname{sech} t \\ \operatorname{sech} t & 0 & -\tanh t \end{vmatrix} = \left(\frac{1}{\sqrt{2}}\tanh t\right)\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} + \left(\frac{1}{\sqrt{2}}\operatorname{sech} t\right)\mathbf{k}$. Also, $\mathbf{v} = (\sinh t)\mathbf{i} - (\cosh t)\mathbf{j} + \mathbf{k}$ $\mathbf{a} = (\cosh t)\mathbf{i} - (\sinh t)\mathbf{j} \Rightarrow \frac{d\mathbf{a}}{dt} = (\sinh t)\mathbf{i} - (\cosh t)\mathbf{j}$ and $\mathbf{v} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sinh t & -\cosh t & 1 \\ \cosh t & -\sinh t & 0 \end{vmatrix}$ $= (\sinh t)\mathbf{i} + (\cosh t)\mathbf{j} + (\cosh^2 t - \sinh^2 t)\mathbf{k} = (\sinh t)\mathbf{i} + (\cosh t)\mathbf{j} + \mathbf{k} \Rightarrow |\mathbf{v} \times \mathbf{a}|^2 = \sinh^2 t + \cosh^2 t + 1$. Thus $\tau = \frac{\begin{vmatrix} \sinh t & -\cosh t & 1 \\ \cosh t & -\sinh t & 0 \\ \sinh t & -\cosh t & 0 \end{vmatrix}}{\sinh^2 t + \cosh^2 t + 1} = \frac{-1}{\sinh^2 t + \cosh^2 t + 1} = \frac{-1}{2\cosh^2 t}$.

17. Yes. If the car is moving along a curved path, then $\kappa \neq 0$ and $a_N = \kappa\,|\mathbf{v}|^2 \neq 0 \Rightarrow \mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N} \neq \mathbf{0}$.

18. $|\mathbf{v}|$ constant $\Rightarrow a_T = \frac{d}{dt}|\mathbf{v}| = 0 \Rightarrow \mathbf{a} = a_N\mathbf{N}$ is orthogonal to $\mathbf{T} \Rightarrow$ the acceleration is normal to the path

19. $\mathbf{a} \perp \mathbf{v} \Rightarrow \mathbf{a} \perp \mathbf{T} \Rightarrow a_T = 0 \Rightarrow \frac{d}{dt}|\mathbf{v}| = 0 \Rightarrow |\mathbf{v}|$ is constant

20. $\mathbf{a}(t) = a_T\mathbf{T} + a_N\mathbf{N}$, where $a_T = \frac{d}{dt}|\mathbf{v}| = \frac{d}{dt}(10) = 0$ and $a_N = \kappa\,|\mathbf{v}|^2 = 100\kappa \Rightarrow \mathbf{a} = 0\mathbf{T} + 100\kappa\mathbf{N}$. Now, from Exercise 5(a) Section 12.4, we find for $y = f(x) = x^2$ that $\kappa = \frac{|f''(x)|}{\left[1 + (f'(x))^2\right]^{3/2}} = \frac{2}{\left[1 + (2x)^2\right]^{3/2}} = \frac{2}{(1 + 4x^2)^{3/2}}$; also,

$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$ is the position vector of the moving mass $\Rightarrow \mathbf{v} = \mathbf{i} + 2t\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{1+4t^2}$
$\Rightarrow \mathbf{T} = \frac{1}{\sqrt{1+4t^2}}(\mathbf{i} + 2t\mathbf{j})$. At $(0,0)$: $\mathbf{T}(0) = \mathbf{i}, \mathbf{N}(0) = \mathbf{j}$ and $\kappa(0) = 2 \Rightarrow \mathbf{F} = m\mathbf{a} = m(100\kappa)\mathbf{N} = 200m\,\mathbf{j}$;
At $\left(\sqrt{2}, 2\right)$: $\mathbf{T}\left(\sqrt{2}\right) = \frac{1}{3}\left(\mathbf{i} + 2\sqrt{2}\mathbf{j}\right) = \frac{1}{3}\mathbf{i} + \frac{2\sqrt{2}}{3}\mathbf{j}$, $\mathbf{N}\left(\sqrt{2}\right) = -\frac{2\sqrt{2}}{3}\mathbf{i} + \frac{1}{3}\mathbf{j}$, and $\kappa\left(\sqrt{2}\right) = \frac{2}{27} \Rightarrow \mathbf{F} = m\mathbf{a}$
$= m(100\kappa)\mathbf{N} = \left(\frac{200}{27}m\right)\left(-\frac{2\sqrt{2}}{3}\mathbf{i} + \frac{1}{3}\mathbf{j}\right) = -\frac{400\sqrt{2}}{81}m\,\mathbf{i} + \frac{200}{81}m\,\mathbf{j}$

21. By $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$ we have $\mathbf{v} \times \mathbf{a} = \left(\frac{ds}{dt}\mathbf{T}\right) \times \left[\frac{d^2s}{dt^2}\mathbf{T} + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N}\right] = \left(\frac{ds}{dt}\frac{d^2s}{dt^2}\right)(\mathbf{T}\times\mathbf{T}) + \kappa\left(\frac{ds}{dt}\right)^3(\mathbf{T}\times\mathbf{N})$
$= \kappa\left(\frac{ds}{dt}\right)^3\mathbf{B}$. It follows that $|\mathbf{v}\times\mathbf{a}| = \kappa\left|\frac{ds}{dt}\right|^3|\mathbf{B}| = \kappa\,|\mathbf{v}|^3 \Rightarrow \kappa = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3}$

22. $a_N = 0 \Rightarrow \kappa\,|\mathbf{v}|^2 = 0 \Rightarrow \kappa = 0$ (since the particle is moving, we cannot have zero speed) $\Rightarrow$ the curvature is zero so the particle is moving along a straight line

23. From Example 1, $|\mathbf{v}| = t$ and $a_N = t$ so that $a_N = \kappa\,|\mathbf{v}|^2 \Rightarrow \kappa = \frac{a_N}{|\mathbf{v}|^2} = \frac{t}{t^2} = \frac{1}{t}, t \neq 0 \Rightarrow \rho = \frac{1}{\kappa} = t$

24. $\mathbf{r} = (x_0 + At)\mathbf{i} + (y_0 + Bt)\mathbf{j} + (z_0 + Ct)\mathbf{k} \Rightarrow \mathbf{v} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k} \Rightarrow \mathbf{a} = \mathbf{0} \Rightarrow \mathbf{v}\times\mathbf{a} = \mathbf{0} \Rightarrow \kappa = 0$. Since the curve is a plane curve, $\tau = 0$.

25. If a plane curve is sufficiently differentiable the torsion is zero as the following argument shows:
$\mathbf{r} = f(t)\mathbf{i} + g(t)\mathbf{j} \Rightarrow \mathbf{v} = f'(t)\mathbf{i} + g'(t)\mathbf{j} \Rightarrow \mathbf{a} = f''(t)\mathbf{i} + g''(t)\mathbf{j} \Rightarrow \frac{d\mathbf{a}}{dt} = f'''(t)\mathbf{i} + g'''(t)\mathbf{j}$

$$\Rightarrow \tau = \frac{\begin{vmatrix} f'(t) & g'(t) & 0 \\ f''(t) & g''(t) & 0 \\ f'''(t) & g'''(t) & 0 \end{vmatrix}}{|\mathbf{v}\times\mathbf{a}|^2} = 0$$

26. $\mathbf{v} = -(a\sin t)\mathbf{i} + (a\cos t)\mathbf{j} + b\,\mathbf{k}$ and $\mathbf{a} = -(a\cos t)\mathbf{i} - (a\sin t)\mathbf{j}$

To find the torsion: $$\tau = \frac{\begin{vmatrix} -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \\ a\sin t & -a\cos t & 0 \end{vmatrix}}{\left(a\sqrt{a^2+b^2}\right)^2} = \frac{b\left(a^2\cos^2 t + a^2\sin^2 t\right)}{a^2(a^2+b^2)} = \frac{a^2 b\left(\cos^2 t + \sin^2 t\right)}{a^2(a^2+b^2)} = \frac{b}{a^2+b^2} \Rightarrow \tau'(b) = \frac{a^2-b^2}{(a^2+b^2)^2};$$
$\tau'(b) = 0 \Rightarrow \frac{a^2-b^2}{(a^2+b^2)^2} = 0 \Rightarrow a^2 - b^2 = 0 \Rightarrow b = \pm a \Rightarrow b = a$ since $a, b > 0$. Also $b < a \Rightarrow \tau' > 0$ and $b > a$
$\Rightarrow \tau' < 0$ so τ_{max} occurs when $b = a \Rightarrow \tau_{max} = \frac{a}{a^2+a^2} = \frac{1}{2a}$

27. $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \Rightarrow \mathbf{v} = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$; $\mathbf{v}\cdot\mathbf{k} = 0 \Rightarrow h'(t) = 0 \Rightarrow h(t) = C$
$\Rightarrow \mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + C\mathbf{k}$ and $\mathbf{r}(a) = f(a)\mathbf{i} + g(a)\mathbf{j} + C\mathbf{k} = \mathbf{0} \Rightarrow f(a) = 0, g(a) = 0$ and $C = 0 \Rightarrow h(t) = 0$.

28. From Exercise 26, $\mathbf{v} = -(a\sin t)\mathbf{i} + (a\cos t)\mathbf{j} + b\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{a^2+b^2} \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$
$= \frac{1}{\sqrt{a^2+b^2}}\left[-(a\sin t)\mathbf{i} + (a\cos t)\mathbf{j} + b\mathbf{k}\right]$; $\frac{d\mathbf{T}}{dt} = \frac{1}{\sqrt{a^2+b^2}}\left[-(a\cos t)\mathbf{i} - (a\sin t)\mathbf{j}\right] \Rightarrow \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|}$

$$= -(\cos t)\mathbf{i} - (\sin t)\mathbf{j};\ \mathbf{B} = \mathbf{T}\times\mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\frac{a\sin t}{\sqrt{a^2+b^2}} & \frac{a\cos t}{\sqrt{a^2+b^2}} & \frac{b}{\sqrt{a^2+b^2}} \\ -\cos t & -\sin t & 0 \end{vmatrix}$$

$= \frac{b\sin t}{\sqrt{a^2+b^2}}\mathbf{i} - \frac{b\cos t}{\sqrt{a^2+b^2}}\mathbf{j} + \frac{a}{\sqrt{a^2+b^2}}\mathbf{k} \Rightarrow \frac{d\mathbf{B}}{dt} = \frac{1}{\sqrt{a^2+b^2}}\left[(b\cos t)\mathbf{i} + (b\sin t)\mathbf{j}\right] \Rightarrow \frac{d\mathbf{B}}{dt}\cdot\mathbf{N} = -\frac{b}{\sqrt{a^2+b^2}}$
$\Rightarrow \tau = -\frac{1}{|\mathbf{v}|}\left(\frac{d\mathbf{B}}{dt}\cdot\mathbf{N}\right) = \left(-\frac{1}{\sqrt{a^2+b^2}}\right)\left(-\frac{b}{\sqrt{a^2+b^2}}\right) = \frac{b}{a^2+b^2}$, which is consistent with the result in Exercise 26.

29-32. Example CAS commands:

Maple:

```
with( LinearAlgebra );
r := < t*cos(t) | t*sin(t) | t >;
t0 := sqrt(3);
rr := eval( r, t=t0 );
v := map( diff, r, t );
vv := eval( v, t=t0 );
a := map( diff, v, t );
aa := eval( a, t=t0 );
s := simplify(Norm( v, 2 )) assuming t::real;
ss := eval( s, t=t0 );
T := v/s;
TT := vv/ss ;
q1 := map( diff, simplify(T), t ):
NN := simplify(eval( q1/Norm(q1,2), t=t0 ));
BB := CrossProduct( TT, NN );
kappa := Norm(CrossProduct(vv,aa),2)/ss^3;
tau := simplify( Determinant(< vv, aa, eval(map(diff,a,t),t=t0) >)/Norm(CrossProduct(vv,aa),2)^3 );
a_t := eval( diff( s, t ), t=t0 );
a_n := evalf[4]( kappa*ss^2 );
```

Mathematica: (assigned functions and value for t0 will vary)

```
Clear[t, v, a, t]
mag[vector_]:=Sqrt[vector.vector]
Print["The position vector is ", r[t_]={t Cos[t], t Sin[t], t}]
Print["The velocity vector is ", v[t_]= r'[t]]
Print["The acceleration vector is ", a[t_]= v'[t]]
Print["The speed is ", speed[t_]= mag[v[t]]//Simplify]
Print["The unit tangent vector is ", utan[t_]= v[t]/speed[t] //Simplify]
Print["The curvature is ", curv[t_]= mag[Cross[v[t],a[t]]] / speed[t]^3 //Simplify]
Print["The torsion is ", torsion[t_]= Det[{v[t], a[t], a'[t]}] / mag[Cross[v[t],a[t]]]^2 //Simplify]
Print["The unit normal vector is ", unorm[t_]= utan'[t] / mag[utan'[t]] //Simplify]
Print["The unit binormal vector is ", ubinorm[t_]= Cross[utan[t],unorm[t]] //Simplify]
Print["The tangential component of the acceleration is ", at[t_]=a[t].utan[t] //Simplify]
Print["The normal component of the acceleration is ", an[t_]=a[t].unorm[t] //Simplify]
```

You can evaluate any of these functions at a specified value of t.

```
t0= Sqrt[3]
{utan[t0], unorm[t0], ubinorm[t0]}
N[{utan[t0], unorm[t0], ubinorm[t0]}]
{curv[t0], torsion[t0]}
N[{curv[t0], torsion[t0]}]
{at[t0], an[t0]}
N[{at[t0], an[t0]}]
```

To verify that the tangential and normal components of the acceleration agree with the formulas in the book:

```
at[t]== speed'[t] //Simplify
an[t]==curv [t] speed[t]^2 //Simplify
```

13.6 VELOCITY AND ACCELERATION IN POLAR COORDINATES

1. $\frac{d\theta}{dt} = 3 = \dot{\theta} \Rightarrow \ddot{\theta} = 0$, $r = a(1 - \cos\theta) \Rightarrow \dot{r} = a\sin\theta\frac{d\theta}{dt} = 3a\sin\theta \Rightarrow \ddot{r} = 3a\cos\theta\frac{d\theta}{dt} = 9a\cos\theta$
$\mathbf{v} = (3a\sin\theta)\mathbf{u}_r + (a(1-\cos\theta))(3)\mathbf{u}_\theta = (3a\sin\theta)\mathbf{u}_r + 3a(1-\cos\theta)\mathbf{u}_\theta$
$\mathbf{a} = \left(9a\cos\theta - a(1-\cos\theta)(3)^2\right)\mathbf{u}_r + (a(1-\cos\theta)\cdot 0 + 2(3a\sin\theta)(3))\mathbf{u}_\theta$
$= (9a\cos\theta - 9a + 9a\cos\theta)\mathbf{u}_r + (18a\sin\theta)\mathbf{u}_\theta = 9a(2\cos\theta - 1)\mathbf{u}_r + (18a\sin\theta)\mathbf{u}_\theta$

2. $\frac{d\theta}{dt} = 2t = \dot{\theta} \Rightarrow \ddot{\theta} = 2$, $r = a\sin 2\theta \Rightarrow \dot{r} = a\cos 2\theta \cdot 2\frac{d\theta}{dt} = 4ta\cos 2\theta \Rightarrow \ddot{r} = 4ta\left(-\sin 2\theta \cdot 2\frac{d\theta}{dt}\right) + 4a\cos 2\theta$
$= -16t^2a\sin 2\theta + 4a\cos 2\theta$
$\mathbf{v} = (4ta\cos 2\theta)\mathbf{u}_r + (a\sin 2\theta)(2t)\mathbf{u}_\theta = (4ta\cos 2\theta)\mathbf{u}_r + (2ta\sin 2\theta)\mathbf{u}_\theta$
$\mathbf{a} = \left[(-16t^2a\sin 2\theta + 4a\cos 2\theta) - (a\sin 2\theta)(2t)^2\right]\mathbf{u}_r + \left[(a\sin 2\theta)(2) + 2(4ta\cos 2\theta)(2t)\right]\mathbf{u}_\theta$
$= \left[-16t^2a\sin 2\theta + 4a\cos 2\theta - 4t^2a\sin 2\theta\right]\mathbf{u}_r + \left[2a\sin 2\theta + 16t^2a\cos 2\theta\right]\mathbf{u}_\theta$
$= \left[-20t^2a\sin 2\theta + 4a\cos 2\theta\right]\mathbf{u}_r + \left[2a\sin 2\theta + 16t^2a\cos 2\theta\right]\mathbf{u}_\theta = 4a(\cos 2\theta - 5t^2\sin 2\theta)\mathbf{u}_r + 2a(\sin 2\theta + 8t^2\cos 2\theta)\mathbf{u}_\theta$

3. $\frac{d\theta}{dt} = 2 = \dot{\theta} \Rightarrow \ddot{\theta} = 0$, $r = e^{a\theta} \Rightarrow \dot{r} = e^{a\theta}\cdot a\frac{d\theta}{dt} = 2a\,e^{a\theta} \Rightarrow \ddot{r} = 2a\,e^{a\theta}\cdot a\frac{d\theta}{dt} = 4a^2\,e^{a\theta}$
$\mathbf{v} = \left(2a\,e^{a\theta}\right)\mathbf{u}_r + \left(e^{a\theta}\right)(2)\mathbf{u}_\theta = \left(2a\,e^{a\theta}\right)\mathbf{u}_r + \left(2e^{a\theta}\right)\mathbf{u}_\theta$
$\mathbf{a} = \left[\left(4a^2\,e^{a\theta}\right) - \left(e^{a\theta}\right)(2)^2\right]\mathbf{u}_r + \left[\left(e^{a\theta}\right)(0) + 2\left(2a\,e^{a\theta}\right)(2)\right]\mathbf{u}_\theta = \left[4a^2\,e^{a\theta} - 4e^{a\theta}\right]\mathbf{u}_r + \left[0 + 8a\,e^{a\theta}\right]\mathbf{u}_\theta$
$= 4e^{a\theta}(a^2 - 1)\mathbf{u}_r + \left(8a\,e^{a\theta}\right)\mathbf{u}_\theta$

4. $\theta = 1 - e^{-t} \Rightarrow \dot{\theta} = e^{-t} \Rightarrow \ddot{\theta} = -e^{-t}$, $r = a(1 + \sin t) \Rightarrow \dot{r} = a\cos t \Rightarrow \ddot{r} = -a\sin t$
$\mathbf{v} = (a\cos t)\mathbf{u}_r + (a(1+\sin t))(e^{-t})\mathbf{u}_\theta = (a\cos t)\mathbf{u}_r + a\,e^{-t}(1+\sin t)\mathbf{u}_\theta$
$\mathbf{a} = \left[(-a\sin t) - (a(1+\sin t))(e^{-t})^2\right]\mathbf{u}_r + \left[(a(1+\sin t))(-e^{-t}) + 2(a\cos t)(e^{-t})\right]\mathbf{u}_\theta$
$= \left[-a\sin t - a\,e^{-2t}(1+\sin t)\right]\mathbf{u}_r + \left[-a\,e^{-t}(1+\sin t) + 2a\,e^{-t}\cos t\right]\mathbf{u}_\theta$
$= -a(\sin t + e^{-2t}(1+\sin t))\mathbf{u}_r + a\,e^{-t}(-(1+\sin t) + 2\cos t)\mathbf{u}_\theta$
$= -a(\sin t + e^{-2t}(1+\sin t))\mathbf{u}_r + a\,e^{-t}(2\cos t - 1 - \sin t)\mathbf{u}_\theta$

5. $\theta = 2t \Rightarrow \dot{\theta} = 2 \Rightarrow \ddot{\theta} = 0$, $r = 2\cos 4t \Rightarrow \dot{r} = -8\sin 4t \Rightarrow \ddot{r} = -32\cos 4t$
$\mathbf{v} = (-8\sin 4t)\mathbf{u}_r + (2\cos 4t)(2)\mathbf{u}_\theta = -8(\sin 4t)\mathbf{u}_r + 4(\cos 4t)\mathbf{u}_\theta$
$\mathbf{a} = \left((-32\cos 4t) - (2\cos 4t)(2)^2\right)\mathbf{u}_r + ((2\cos 4t)\cdot 0 + 2(-8\sin 4t)(2))\mathbf{u}_\theta$
$= (-32\cos 4t - 8\cos 4t)\mathbf{u}_r + (0 - 32\sin 4t)\mathbf{u}_\theta = -40(\cos 4t)\mathbf{u}_r - 32(\sin 4t)\mathbf{u}_\theta$

6. $e = \frac{r_0v_0^2}{GM} - 1 \;\Rightarrow\; v_0^2 = \frac{GM(e+1)}{r_0} \;\Rightarrow\; v_0 = \sqrt{\frac{GM(e+1)}{r_0}}$;
Circle: $e = 0 \;\Rightarrow\; v_0 = \sqrt{\frac{GM}{r_0}}$
Ellipse: $0 < e < 1 \;\Rightarrow\; \sqrt{\frac{GM}{r_0}} < v_0 < \sqrt{\frac{2GM}{r_0}}$
Parabola: $e = 1 \;\Rightarrow\; v_0 = \sqrt{\frac{2GM}{r_0}}$
Hyperbola: $e > 1 \;\Rightarrow\; v_0 > \sqrt{\frac{2GM}{r_0}}$

7. $r = \frac{GM}{v^2} \;\Rightarrow\; v^2 = \frac{GM}{r} \;\Rightarrow\; v = \sqrt{\frac{GM}{r}}$ which is constant since G, M, and r (the radius of orbit) are constant

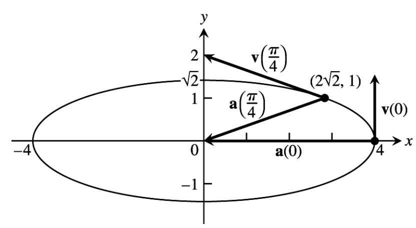
8. $\Delta A = \frac{1}{2}\,|\mathbf{r}(t+\Delta t)\times\mathbf{r}(t)| \Rightarrow \frac{\Delta A}{\Delta t} = \frac{1}{2}\left|\frac{\mathbf{r}(t+\Delta t)}{\Delta t}\times\mathbf{r}(t)\right| = \frac{1}{2}\left|\frac{\mathbf{r}(t+\Delta t)-\mathbf{r}(t)+\mathbf{r}(t)}{\Delta t}\times\mathbf{r}(t)\right|$
$= \frac{1}{2}\left|\frac{\mathbf{r}(t+\Delta t)-\mathbf{r}(t)}{\Delta t}\times\mathbf{r}(t)+\frac{1}{\Delta t}\,\mathbf{r}(t)\times\mathbf{r}(t)\right| = \frac{1}{2}\left|\frac{\mathbf{r}(t+\Delta t)-\mathbf{r}(t)}{\Delta t}\times\mathbf{r}(t)\right| \Rightarrow \frac{dA}{dt} = \lim_{\Delta t\to 0}\frac{1}{2}\left|\frac{\mathbf{r}(t+\Delta t)-\mathbf{r}(t)}{\Delta t}\times\mathbf{r}(t)\right|$
$= \frac{1}{2}\left|\frac{d\mathbf{r}}{dt}\times\mathbf{r}(t)\right| = \frac{1}{2}\left|\mathbf{r}(t)\times\frac{d\mathbf{r}}{dt}\right| = \frac{1}{2}\,|\mathbf{r}\times\dot{\mathbf{r}}|$

9. $T = \left(\frac{2\pi a^2}{r_0 v_0}\right)\sqrt{1-e^2} \Rightarrow T^2 = \left(\frac{4\pi^2 a^4}{r_0^2 v_0^2}\right)(1-e^2) = \left(\frac{4\pi^2 a^4}{r_0^2 v_0^2}\right)\left[1-\left(\frac{r_0 v_0^2}{GM}-1\right)^2\right]$ (from Equation 5)
$= \left(\frac{4\pi^2 a^4}{r_0^2 v_0^2}\right)\left[-\frac{r_0^2 v_0^4}{G^2M^2}+2\left(\frac{r_0 v_0^2}{GM}\right)\right] = \left(\frac{4\pi^2 a^4}{r_0^2 v_0^2}\right)\left[\frac{2GMr_0 v_0^2 - r_0^2 v_0^4}{G^2M^2}\right] = \frac{(4\pi^2 a^4)\,(2GM - r_0 v_0^2)}{r_0 G^2 M^2}$
$= (4\pi^2 a^4)\left(\frac{2GM - r_0 v_0^2}{2r_0 GM}\right)\left(\frac{2}{GM}\right) = (4\pi^2 a^4)\left(\frac{1}{2a}\right)\left(\frac{2}{GM}\right)$ (from Equation 10) $\Rightarrow T^2 = \frac{4\pi^2 a^3}{GM} \Rightarrow \frac{T^2}{a^3} = \frac{4\pi^2}{GM}$

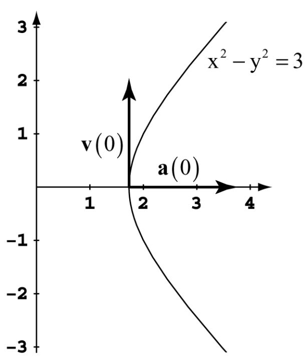
10. $r = 365.256$ days $= 365.256$ days $\times\, 24\,\frac{\text{hours}}{\text{day}} \times 60\,\frac{\text{minutes}}{\text{hour}} \times 60\,\frac{\text{seconds}}{\text{minute}} = 31{,}558{,}118.4$ seconds $\approx 3.16\times 10^7$,
$G = 6.6726\times 10^{-11}\,\frac{N\cdot m^2}{kg^2}$, and the mass of the sun $M = 1.99\times 10^{30}$ kg. $\frac{T^2}{a^3} = \frac{4\pi^2}{GM} \Rightarrow a^3 = T^2\,\frac{GM}{4\pi^2}$
$\Rightarrow a^3 = (3.16\times 10^7)^2\,\frac{(6.6726\times 10^{-11})(1.99\times 10^{30})}{4\pi^2} \approx 3.35863335\times 10^{33} \Rightarrow a = \sqrt[3]{3.35863335\times 10^{33}}$
≈ 149757138111 m ≈ 149.757 billion km

CHAPTER 13 PRACTICE EXERCISES

1. $\mathbf{r}(t) = (4\cos t)\mathbf{i} + \left(\sqrt{2}\sin t\right)\mathbf{j} \Rightarrow x = 4\cos t$
and $y = \sqrt{2}\sin t \Rightarrow \frac{x^2}{16} + \frac{y^2}{2} = 1$;
$\mathbf{v} = (-4\sin t)\mathbf{i} + \left(\sqrt{2}\cos t\right)\mathbf{j}$ and
$\mathbf{a} = (-4\cos t)\mathbf{i} - \left(\sqrt{2}\sin t\right)\mathbf{j}$; $\mathbf{r}(0) = 4\,\mathbf{i}$, $\mathbf{v}(0) = \sqrt{2}\mathbf{j}$,
$\mathbf{a}(0) = -4\mathbf{i}$; $\mathbf{r}\left(\frac{\pi}{4}\right) = 2\sqrt{2}\mathbf{i} + \mathbf{j}$, $\mathbf{v}\left(\frac{\pi}{4}\right) = -2\sqrt{2}\mathbf{i} + \mathbf{j}$,
$\mathbf{a}\left(\frac{\pi}{4}\right) = -2\sqrt{2}\mathbf{i} - \mathbf{j}$; $|\mathbf{v}| = \sqrt{16\sin^2 t + 2\cos^2 t}$
$\Rightarrow a_T = \frac{d}{dt}\,|\mathbf{v}| = \frac{14\sin t\cos t}{\sqrt{16\sin^2 t + 2\cos^2 t}}$; at $t = 0$: $a_T = 0$, $a_N = \sqrt{|\mathbf{a}|^2 - 0} = 4$, $\mathbf{a} = 0\mathbf{T} + 4\mathbf{N} = 4\mathbf{N}$, $\kappa = \frac{a_N}{|\mathbf{v}|^2} = \frac{4}{2} = 2$;
at $t = \frac{\pi}{4}$: $a_T = \frac{7}{\sqrt{8+1}} = \frac{7}{3}$, $a_N = \sqrt{9 - \frac{49}{9}} = \frac{4\sqrt{2}}{3}$, $\mathbf{a} = \frac{7}{3}\mathbf{T} + \frac{4\sqrt{2}}{3}\mathbf{N}$, $\kappa = \frac{a_N}{|\mathbf{v}|^2} = \frac{4\sqrt{2}}{27}$



2. $\mathbf{r}(t) = \left(\sqrt{3}\sec t\right)\mathbf{i} + \left(\sqrt{3}\tan t\right)\mathbf{j} \Rightarrow x = \sqrt{3}\sec t$ and $y = \sqrt{3}\tan t \Rightarrow \frac{x^2}{3} - \frac{y^2}{3} = \sec^2 t - \tan^2 t = 1$;
$\Rightarrow x^2 - y^2 = 3$; $\mathbf{v} = \left(\sqrt{3}\sec t\tan t\right)\mathbf{i} + \left(\sqrt{3}\sec^2 t\right)\mathbf{j}$
and
$\mathbf{a} = \left(\sqrt{3}\sec t\tan^2 t + \sqrt{3}\sec^3 t\right)\mathbf{i} - \left(2\sqrt{3}\sec^2 t\tan t\right)\mathbf{j}$;
$\mathbf{r}(0) = \sqrt{3}\mathbf{i}$, $\mathbf{v}(0) = \sqrt{3}\mathbf{j}$, $\mathbf{a}(0) = \sqrt{3}\mathbf{i}$;
$|\mathbf{v}| = \sqrt{3\sec^2 t\tan^2 t + 3\sec^4 t}$
$\Rightarrow a_T = \frac{d}{dt}\,|\mathbf{v}| = \frac{6\sec^2 t\tan^3 t + 18\sec^4 t\tan t}{2\sqrt{3\sec^2 t\tan^2 t + 3\sec^4 t}}$;
at $t = 0$: $a_T = 0$, $a_N = \sqrt{|\mathbf{a}|^2 - 0} = \sqrt{3}$,
$\mathbf{a} = 0\mathbf{T} + \sqrt{3}\mathbf{N} = \sqrt{3}\mathbf{N}$, $\kappa = \frac{a_N}{|\mathbf{v}|^2} = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$



3. $\mathbf{r} = \frac{1}{\sqrt{1+t^2}}\,\mathbf{i} + \frac{t}{\sqrt{1+t^2}}\,\mathbf{j} \Rightarrow \mathbf{v} = -t(1+t^2)^{-3/2}\mathbf{i} + (1+t^2)^{-3/2}\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{\left[-t(1+t^2)^{-3/2}\right]^2 + \left[(1+t^2)^{-3/2}\right]^2}$
$= \frac{1}{1+t^2}$. We want to maximize $|\mathbf{v}|$: $\frac{d|\mathbf{v}|}{dt} = \frac{-2t}{(1+t^2)^2}$ and $\frac{d|\mathbf{v}|}{dt} = 0 \Rightarrow \frac{-2t}{(1+t^2)^2} = 0 \Rightarrow t = 0$. For $t < 0$, $\frac{-2t}{(1+t^2)^2} > 0$; for $t > 0$, $\frac{-2t}{(1+t^2)^2} < 0 \Rightarrow |\mathbf{v}|_{\max}$ occurs when $t = 0 \Rightarrow |\mathbf{v}|_{\max} = 1$

4. $\mathbf{r} = (e^t \cos t)\,\mathbf{i} + (e^t \sin t)\,\mathbf{j} \Rightarrow \mathbf{v} = (e^t \cos t - e^t \sin t)\,\mathbf{i} + (e^t \sin t + e^t \cos t)\,\mathbf{j}$
$\Rightarrow \mathbf{a} = (e^t \cos t - e^t \sin t - e^t \sin t - e^t \cos t)\,\mathbf{i} + (e^t \sin t + e^t \cos t + e^t \cos t - e^t \sin t)\,\mathbf{j}$
$= (-2e^t \sin t)\,\mathbf{i} + (2e^t \cos t)\,\mathbf{j}$. Let θ be the angle between $\mathbf{r}$ and $\mathbf{a}$. Then $\theta = \cos^{-1}\left(\frac{\mathbf{r}\cdot\mathbf{a}}{|\mathbf{r}|\,|\mathbf{a}|}\right)$
$= \cos^{-1}\left(\frac{-2e^{2t}\sin t\cos t + 2e^{2t}\sin t\cos t}{\sqrt{(e^t\cos t)^2 + (e^t \sin t)^2}\,\sqrt{(-2e^t\sin t)^2 + (2e^t\cos t)^2}}\right) = \cos^{-1}\left(\frac{0}{2e^{2t}}\right) = \cos^{-1} 0 = \frac{\pi}{2}$ for all t

5. $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{a} = 5\mathbf{i} + 15\mathbf{j} \Rightarrow \mathbf{v}\times\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 4 & 0 \\ 5 & 15 & 0 \end{vmatrix} = 25\mathbf{k} \Rightarrow |\mathbf{v}\times\mathbf{a}| = 25;\ |\mathbf{v}| = \sqrt{3^2+4^2} = 5$
$\Rightarrow \kappa = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3} = \frac{25}{5^3} = \frac{1}{5}$

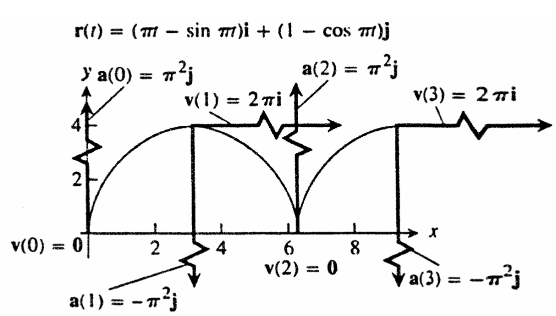
6. $\kappa = \frac{|y''|}{\left[1+(y')^2\right]^{3/2}} = e^x\left(1+e^{2x}\right)^{-3/2} \Rightarrow \frac{d\kappa}{dx} = e^x\left(1+e^{2x}\right)^{-3/2} + e^x\left[-\frac{3}{2}\left(1+e^{2x}\right)^{-5/2}\left(2e^{2x}\right)\right]$
$= e^x\left(1+e^{2x}\right)^{-3/2} - 3e^{3x}\left(1+e^{2x}\right)^{-5/2} = e^x\left(1+e^{2x}\right)^{-5/2}\left[\left(1+e^{2x}\right) - 3e^{2x}\right] = e^x\left(1+e^{2x}\right)^{-5/2}\left(1-2e^{2x}\right)$;
$\frac{d\kappa}{dx} = 0 \Rightarrow \left(1-2e^{2x}\right) = 0 \Rightarrow e^{2x} = \frac{1}{2} \Rightarrow 2x = -\ln 2 \Rightarrow x = -\frac{1}{2}\ln 2 = -\ln\sqrt{2} \Rightarrow y = \frac{1}{\sqrt{2}}$; therefore κ is at a maximum at the point $\left(-\ln\sqrt{2},\ \frac{1}{\sqrt{2}}\right)$

7. $\mathbf{r} = x\mathbf{i} + y\mathbf{j} \Rightarrow \mathbf{v} = \frac{dx}{dt}\,\mathbf{i} + \frac{dy}{dt}\,\mathbf{j}$ and $\mathbf{v}\cdot\mathbf{i} = y \Rightarrow \frac{dx}{dt} = y$. Since the particle moves around the unit circle $x^2 + y^2 = 1$, $2x\,\frac{dx}{dt} + 2y\,\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -\frac{x}{y}\,\frac{dx}{dt} \Rightarrow \frac{dy}{dt} = -\frac{x}{y}(y) = -x$. Since $\frac{dx}{dt} = y$ and $\frac{dy}{dt} = -x$, we have $\mathbf{v} = y\mathbf{i} - x\mathbf{j} \Rightarrow$ at $(1,0)$, $\mathbf{v} = -\mathbf{j}$ and the motion is clockwise.

8. $9y = x^3 \Rightarrow 9\,\frac{dy}{dt} = 3x^2\,\frac{dx}{dt} \Rightarrow \frac{dy}{dt} = \frac{1}{3}x^2\,\frac{dx}{dt}$. If $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, where x and y are differentiable functions of t, then $\mathbf{v} = \frac{dx}{dt}\,\mathbf{i} + \frac{dy}{dt}\,\mathbf{j}$. Hence $\mathbf{v}\cdot\mathbf{i} = 4 \Rightarrow \frac{dx}{dt} = 4$ and $\mathbf{v}\cdot\mathbf{j} = \frac{dy}{dt} = \frac{1}{3}x^2\,\frac{dx}{dt} = \frac{1}{3}(3)^2(4) = 12$ at $(3,3)$. Also, $\mathbf{a} = \frac{d^2x}{dt^2}\,\mathbf{i} + \frac{d^2y}{dt^2}\,\mathbf{j}$ and $\frac{d^2y}{dt^2} = \left(\frac{2}{3}x\right)\left(\frac{dx}{dt}\right)^2 + \left(\frac{1}{3}x^2\right)\frac{d^2x}{dt^2}$. Hence $\mathbf{a}\cdot\mathbf{i} = -2 \Rightarrow \frac{d^2x}{dt^2} = -2$ and $\mathbf{a}\cdot\mathbf{j} = \frac{d^2y}{dt^2} = \frac{2}{3}(3)(4)^2 + \frac{1}{3}(3)^2(-2) = 26$ at the point $(x,y) = (3,3)$.

9. $\frac{d\mathbf{r}}{dt}$ orthogonal to $\mathbf{r} \Rightarrow 0 = \frac{d\mathbf{r}}{dt}\cdot\mathbf{r} = \frac{1}{2}\,\frac{d\mathbf{r}}{dt}\cdot\mathbf{r} + \frac{1}{2}\,\mathbf{r}\cdot\frac{d\mathbf{r}}{dt} = \frac{1}{2}\,\frac{d}{dt}(\mathbf{r}\cdot\mathbf{r}) \Rightarrow \mathbf{r}\cdot\mathbf{r} = K$, a constant. If $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, where x and y are differentiable functions of t, then $\mathbf{r}\cdot\mathbf{r} = x^2 + y^2 \Rightarrow x^2 + y^2 = K$, which is the equation of a circle centered at the origin.

10. (a)



(b) $\mathbf{v} = (\pi - \pi\cos\pi t)\mathbf{i} + (\pi\sin\pi t)\mathbf{j}$
$\Rightarrow \mathbf{a} = (\pi^2 \sin\pi t)\,\mathbf{i} + (\pi^2\cos\pi t)\,\mathbf{j}$;
$\mathbf{v}(0) = \mathbf{0}$ and $\mathbf{a}(0) = \pi^2\mathbf{j}$;
$\mathbf{v}(1) = 2\pi\mathbf{i}$ and $\mathbf{a}(1) = -\pi^2\mathbf{j}$;
$\mathbf{v}(2) = \mathbf{0}$ and $\mathbf{a}(2) = \pi^2\mathbf{j}$;
$\mathbf{v}(3) = 2\pi\mathbf{i}$ and $\mathbf{a}(3) = -\pi^2\mathbf{j}$

(c) Forward speed at the topmost point is $|\mathbf{v}(1)| = |\mathbf{v}(3)| = 2\pi$ ft/sec; since the circle makes $\frac{1}{2}$ revolution per second, the center moves π ft parallel to the x-axis each second $\Rightarrow$ the forward speed of C is π ft/sec.

11. $y = y_0 + (v_0 \sin\alpha)t - \frac{1}{2}gt^2 \Rightarrow y = 6.5 + (44 \text{ ft/sec})(\sin 45°)(3 \text{ sec}) - \frac{1}{2}(32 \text{ ft/sec}^2)(3 \text{ sec})^2 = 6.5 + 66\sqrt{2} - 144$ ≈ -44.16 ft $\Rightarrow$ the shot put is on the ground. Now, $y = 0 \Rightarrow 6.5 + 22\sqrt{2}t - 16t^2 = 0 \Rightarrow t \approx 2.13$ sec (the positive root) $\Rightarrow x \approx (44 \text{ ft/sec})(\cos 45°)(2.13 \text{ sec}) \approx 66.27$ ft or about 66 ft, 3 in. from the stopboard

12. $y_{max} = y_0 + \frac{(v_0 \sin\alpha)^2}{2g} = 7 \text{ ft} + \frac{[(80 \text{ ft/sec})(\sin 45°)]^2}{(2)(32 \text{ ft/sec}^2)} \approx 57$ ft

13. $x = (v_0 \cos\alpha)t$ and $y = (v_0 \sin\alpha)t - \frac{1}{2}gt^2 \Rightarrow \tan\phi = \frac{y}{x} = \frac{(v_0 \sin\alpha)t - \frac{1}{2}gt^2}{(v_0 \cos\alpha)t} = \frac{(v_0 \sin\alpha) - \frac{1}{2}gt}{v_0 \cos\alpha}$

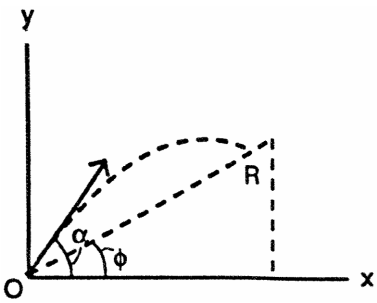
$\Rightarrow v_0 \cos\alpha \tan\phi = v_0 \sin\alpha - \frac{1}{2}gt \Rightarrow t = \frac{2v_0 \sin\alpha - 2v_0 \cos\alpha \tan\phi}{g}$, which is the time when the golf ball hits the upward slope. At this time $x = (v_0 \cos\alpha)\left(\frac{2v_0 \sin\alpha - 2v_0 \cos\alpha \tan\phi}{g}\right) = \left(\frac{2}{g}\right)\left(v_0^2 \sin\alpha \cos\alpha - v_0^2 \cos^2\alpha \tan\phi\right)$.

Now $OR = \frac{x}{\cos\phi} \Rightarrow OR = \left(\frac{2}{g}\right)\left(\frac{v_0^2 \sin\alpha \cos\alpha - v_0^2 \cos^2\alpha \tan\phi}{\cos\phi}\right)$

$= \left(\frac{2v_0^2 \cos\alpha}{g}\right)\left(\frac{\sin\alpha}{\cos\phi} - \frac{\cos\alpha \tan\phi}{\cos\phi}\right)$

$= \left(\frac{2v_0^2 \cos\alpha}{g}\right)\left(\frac{\sin\alpha \cos\phi - \cos\alpha \sin\phi}{\cos^2\phi}\right)$

$= \left(\frac{2v_0^2 \cos\alpha}{g \cos^2\phi}\right)[\sin(\alpha - \phi)]$. The distance OR is maximized when x is maximized:



$\frac{dx}{d\alpha} = \left(\frac{2v_0^2}{g}\right)(\cos 2\alpha + \sin 2\alpha \tan\phi) = 0$

$\Rightarrow (\cos 2\alpha + \sin 2\alpha \tan\phi) = 0 \Rightarrow \cot 2\alpha + \tan\phi = 0 \Rightarrow \cot 2\alpha = \tan(-\phi) \Rightarrow 2\alpha = \frac{\pi}{2} + \phi \Rightarrow \alpha = \frac{\phi}{2} + \frac{\pi}{4}$

14. (a) $x = v_0(\cos 40°)t$ and $y = 6.5 + v_0(\sin 40°)t - \frac{1}{2}gt^2 = 6.5 + v_0(\sin 40°)t - 16t^2$; $x = 262\frac{5}{12}$ ft and $y = 0$ ft

$\Rightarrow 262\frac{5}{12} = v_0(\cos 40°)t$ or $v_0 = \frac{262.4167}{(\cos 40°)t}$ and $0 = 6.5 + \left[\frac{262.4167}{(\cos 40°)t}\right](\sin 40°)t - 16t^2 \Rightarrow t^2 = 14.1684$

$\Rightarrow t \approx 3.764$ sec. Therefore, $262.4167 \approx v_0(\cos 40°)(3.764 \text{ sec}) \Rightarrow v_0 \approx \frac{262.4167}{(\cos 40°)(3.764 \text{ sec})} \Rightarrow v_0 \approx 91$ ft/sec

(b) $y_{max} = y_0 + \frac{(v_0 \sin\alpha)^2}{2g} \approx 6.5 + \frac{((91)(\sin 40°))^2}{(2)(32)} \approx 60$ ft

15. $\mathbf{r} = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j} + t^2\mathbf{k} \Rightarrow \mathbf{v} = (-2\sin t)\mathbf{i} + (2\cos t)\mathbf{j} + 2t\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(-2\sin t)^2 + (2\cos t)^2 + (2t)^2}$ $= 2\sqrt{1+t^2} \Rightarrow \text{Length} = \int_0^{\pi/4} 2\sqrt{1+t^2}\,dt = \left[t\sqrt{1+t^2} + \ln\left|t + \sqrt{1+t^2}\right|\right]_0^{\pi/4} = \frac{\pi}{4}\sqrt{1+\frac{\pi^2}{16}} + \ln\left(\frac{\pi}{4} + \sqrt{1+\frac{\pi^2}{16}}\right)$

16. $\mathbf{r} = (3\cos t)\mathbf{i} + (3\sin t)\mathbf{j} + 2t^{3/2}\mathbf{k} \Rightarrow \mathbf{v} = (-3\sin t)\mathbf{i} + (3\cos t)\mathbf{j} + 3t^{1/2}\mathbf{k} \Rightarrow |\mathbf{v}| = \sqrt{(-3\sin t)^2 + (3\cos t)^2 + (3t^{1/2})^2}$ $= \sqrt{9+9t} = 3\sqrt{1+t} \Rightarrow \text{Length} = \int_0^3 3\sqrt{1+t}\,dt = \left[2(1+t)^{3/2}\right]_0^3 = 14$

17. $\mathbf{r} = \frac{4}{9}(1+t)^{3/2}\mathbf{i} + \frac{4}{9}(1-t)^{3/2}\mathbf{j} + \frac{1}{3}t\mathbf{k} \Rightarrow \mathbf{v} = \frac{2}{3}(1+t)^{1/2}\mathbf{i} - \frac{2}{3}(1-t)^{1/2}\mathbf{j} + \frac{1}{3}\mathbf{k}$

$\Rightarrow |\mathbf{v}| = \sqrt{\left[\frac{2}{3}(1+t)^{1/2}\right]^2 + \left[-\frac{2}{3}(1-t)^{1/2}\right]^2 + \left(\frac{1}{3}\right)^2} = 1 \Rightarrow \mathbf{T} = \frac{2}{3}(1+t)^{1/2}\mathbf{i} - \frac{2}{3}(1-t)^{1/2}\mathbf{j} + \frac{1}{3}\mathbf{k}$

$\Rightarrow \mathbf{T}(0) = \frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}$; $\frac{d\mathbf{T}}{dt} = \frac{1}{3}(1+t)^{-1/2}\mathbf{i} + \frac{1}{3}(1-t)^{-1/2}\mathbf{j} \Rightarrow \frac{d\mathbf{T}}{dt}(0) = \frac{1}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} \Rightarrow \left|\frac{d\mathbf{T}}{dt}(0)\right| = \frac{\sqrt{2}}{3}$

$\Rightarrow \mathbf{N}(0) = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$; $\mathbf{B}(0) = \mathbf{T}(0) \times \mathbf{N}(0) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{vmatrix} = -\frac{1}{3\sqrt{2}}\mathbf{i} + \frac{1}{3\sqrt{2}}\mathbf{j} + \frac{4}{3\sqrt{2}}\mathbf{k}$;

$\mathbf{a} = \frac{1}{3}(1+t)^{-1/2}\mathbf{i} + \frac{1}{3}(1-t)^{-1/2}\mathbf{j} \Rightarrow \mathbf{a}(0) = \frac{1}{3}\mathbf{i} + \frac{1}{3}\mathbf{j}$ and $\mathbf{v}(0) = \frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k} \Rightarrow \mathbf{v}(0) \times \mathbf{a}(0)$

$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 \end{vmatrix} = -\frac{1}{9}\mathbf{i} + \frac{1}{9}\mathbf{j} + \frac{4}{9}\mathbf{k} \Rightarrow |\mathbf{v}\times\mathbf{a}| = \frac{\sqrt{2}}{3} \Rightarrow \kappa(0) = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3} = \frac{\left(\frac{\sqrt{2}}{3}\right)}{1^3} = \frac{\sqrt{2}}{3}$;

$\dot{\mathbf{a}} = -\frac{1}{6}(1+t)^{-3/2}\mathbf{i} + \frac{1}{6}(1-t)^{-3/2}\mathbf{j} \Rightarrow \dot{\mathbf{a}}(0) = -\frac{1}{6}\mathbf{i} + \frac{1}{6}\mathbf{j} \Rightarrow \tau(0) = \frac{\begin{vmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 \\ -\frac{1}{6} & \frac{1}{6} & 0 \end{vmatrix}}{|\mathbf{v}\times\mathbf{a}|^2} = \frac{\left(\frac{1}{3}\right)\left(\frac{2}{18}\right)}{\left(\frac{\sqrt{2}}{3}\right)^2} = \frac{1}{6}$

18. $\mathbf{r} = (e^t \sin 2t)\mathbf{i} + (e^t \cos 2t)\mathbf{j} + 2e^t\mathbf{k} \Rightarrow \mathbf{v} = (e^t \sin 2t + 2e^t \cos 2t)\mathbf{i} + (e^t \cos 2t - 2e^t \sin 2t)\mathbf{j} + 2e^t\mathbf{k}$

$\Rightarrow |\mathbf{v}| = \sqrt{(e^t \sin 2t + 2e^t \cos 2t)^2 + (e^t \cos 2t - 2e^t \sin 2t)^2 + (2e^t)^2} = 3e^t \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$

$= \left(\frac{1}{3}\sin 2t + \frac{2}{3}\cos 2t\right)\mathbf{i} + \left(\frac{1}{3}\cos 2t - \frac{2}{3}\sin 2t\right)\mathbf{j} + \frac{2}{3}\mathbf{k} \Rightarrow \mathbf{T}(0) = \frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$;

$\frac{d\mathbf{T}}{dt} = \left(\frac{2}{3}\cos 2t - \frac{4}{3}\sin 2t\right)\mathbf{i} + \left(-\frac{2}{3}\sin 2t - \frac{4}{3}\cos 2t\right)\mathbf{j} \Rightarrow \frac{d\mathbf{T}}{dt}(0) = \frac{2}{3}\mathbf{i} - \frac{4}{3}\mathbf{j} \Rightarrow \left|\frac{d\mathbf{T}}{dt}(0)\right| = \frac{2}{3}\sqrt{5}$

$\Rightarrow \mathbf{N}(0) = \frac{\left(\frac{2}{3}\mathbf{i} - \frac{4}{3}\mathbf{j}\right)}{\left(\frac{2\sqrt{5}}{3}\right)} = \frac{1}{\sqrt{5}}\mathbf{i} - \frac{2}{\sqrt{5}}\mathbf{j}$; $\mathbf{B}(0) = \mathbf{T}(0)\times\mathbf{N}(0) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} & 0 \end{vmatrix} = \frac{4}{3\sqrt{5}}\mathbf{i} + \frac{2}{3\sqrt{5}}\mathbf{j} - \frac{5}{3\sqrt{5}}\mathbf{k}$;

$\mathbf{a} = (4e^t \cos 2t - 3e^t \sin 2t)\mathbf{i} + (-3e^t \cos 2t - 4e^t \sin 2t)\mathbf{j} + 2e^t\mathbf{k} \Rightarrow \mathbf{a}(0) = 4\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ and $\mathbf{v}(0) = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

$\Rightarrow \mathbf{v}(0)\times\mathbf{a}(0) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 2 \\ 4 & -3 & 2 \end{vmatrix} = 8\mathbf{i} + 4\mathbf{j} - 10\mathbf{k} \Rightarrow |\mathbf{v}\times\mathbf{a}| = \sqrt{64+16+100} = 6\sqrt{5}$ and $|\mathbf{v}(0)| = 3$

$\Rightarrow \kappa(0) = \frac{6\sqrt{5}}{3^3} = \frac{2\sqrt{5}}{9}$;

$\dot{\mathbf{a}} = (4e^t \cos 2t - 8e^t \sin 2t - 3e^t \sin 2t - 6e^t \cos 2t)\mathbf{i} + (-3e^t \cos 2t + 6e^t \sin 2t - 4e^t \sin 2t - 8e^t \cos 2t)\mathbf{j} + 2e^t\mathbf{k}$

$= (-2e^t \cos 2t - 11e^t \sin 2t)\mathbf{i} + (-11\,e^t \cos 2t + 2e^t \sin 2t)\mathbf{j} + 2e^t\mathbf{k} \Rightarrow \dot{\mathbf{a}}(0) = -2\mathbf{i} - 11\mathbf{j} + 2\mathbf{k}$

$\Rightarrow \tau(0) = \frac{\begin{vmatrix} 2 & 1 & 2 \\ 4 & -3 & 2 \\ -2 & -11 & 2 \end{vmatrix}}{|\mathbf{v}\times\mathbf{a}|^2} = \frac{-80}{180} = -\frac{4}{9}$

19. $\mathbf{r} = t\mathbf{i} + \frac{1}{2}e^{2t}\mathbf{j} \Rightarrow \mathbf{v} = \mathbf{i} + e^{2t}\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{1+e^{4t}} \Rightarrow \mathbf{T} = \frac{1}{\sqrt{1+e^{4t}}}\mathbf{i} + \frac{e^{2t}}{\sqrt{1+e^{4t}}}\mathbf{j} \Rightarrow \mathbf{T}(\ln 2) = \frac{1}{\sqrt{17}}\mathbf{i} + \frac{4}{\sqrt{17}}\mathbf{j}$;

$\frac{d\mathbf{T}}{dt} = \frac{-2e^{4t}}{(1+e^{4t})^{3/2}}\mathbf{i} + \frac{2e^{2t}}{(1+e^{4t})^{3/2}}\mathbf{j} \Rightarrow \frac{d\mathbf{T}}{dt}(\ln 2) = \frac{-32}{17\sqrt{17}}\mathbf{i} + \frac{8}{17\sqrt{17}}\mathbf{j} \Rightarrow \mathbf{N}(\ln 2) = -\frac{4}{\sqrt{17}}\mathbf{i} + \frac{1}{\sqrt{17}}\mathbf{j}$;

$\mathbf{B}(\ln 2) = \mathbf{T}(\ln 2)\times\mathbf{N}(\ln 2) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{1}{\sqrt{17}} & \frac{4}{\sqrt{17}} & 0 \\ -\frac{4}{\sqrt{17}} & \frac{1}{\sqrt{17}} & 0 \end{vmatrix} = \mathbf{k}$; $\mathbf{a} = 2e^{2t}\mathbf{j} \Rightarrow \mathbf{a}(\ln 2) = 8\mathbf{j}$ and $\mathbf{v}(\ln 2) = \mathbf{i} + 4\mathbf{j}$

$\Rightarrow \mathbf{v}(\ln 2)\times\mathbf{a}(\ln 2) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & 0 \\ 0 & 8 & 0 \end{vmatrix} = 8\mathbf{k} \Rightarrow |\mathbf{v}\times\mathbf{a}| = 8$ and $|\mathbf{v}(\ln 2)| = \sqrt{17} \Rightarrow \kappa(\ln 2) = \frac{8}{17\sqrt{17}}$; $\dot{\mathbf{a}} = 4e^{2t}\mathbf{j}$

$\Rightarrow \dot{\mathbf{a}}(\ln 2) = 16\mathbf{j} \Rightarrow \tau(\ln 2) = \frac{\begin{vmatrix} 1 & 4 & 0 \\ 0 & 8 & 0 \\ 0 & 16 & 0 \end{vmatrix}}{|\mathbf{v}\times\mathbf{a}|^2} = 0$

20. $\mathbf{r} = (3\cosh 2t)\mathbf{i} + (3\sinh 2t)\mathbf{j} + 6t\mathbf{k} \Rightarrow \mathbf{v} = (6\sinh 2t)\mathbf{i} + (6\cosh 2t)\mathbf{j} + 6\mathbf{k}$

$\Rightarrow |\mathbf{v}| = \sqrt{36\sinh^2 2t + 36\cosh^2 2t + 36} = 6\sqrt{2}\cosh 2t \Rightarrow \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{1}{\sqrt{2}}\tanh 2t\right)\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} + \left(\frac{1}{\sqrt{2}}\operatorname{sech} 2t\right)\mathbf{k}$

$\Rightarrow \mathbf{T}(\ln 2) = \frac{15}{17\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} + \frac{8}{17\sqrt{2}}\mathbf{k}$; $\frac{d\mathbf{T}}{dt} = \left(\frac{2}{\sqrt{2}}\operatorname{sech}^2 2t\right)\mathbf{i} - \left(\frac{2}{\sqrt{2}}\operatorname{sech} 2t \tanh 2t\right)\mathbf{k} \Rightarrow \frac{d\mathbf{T}}{dt}(\ln 2)$

$= \left(\frac{2}{\sqrt{2}}\right)\left(\frac{8}{17}\right)^2\mathbf{i} - \left(\frac{2}{\sqrt{2}}\right)\left(\frac{8}{17}\right)\left(\frac{15}{17}\right)\mathbf{k} = \frac{128}{289\sqrt{2}}\mathbf{i} - \frac{240}{289\sqrt{2}}\mathbf{k} \Rightarrow \left|\frac{d\mathbf{T}}{dt}(\ln 2)\right| = \sqrt{\left(\frac{128}{289\sqrt{2}}\right)^2 + \left(-\frac{240}{289\sqrt{2}}\right)^2} = \frac{8\sqrt{2}}{17}$

$\Rightarrow\ \mathbf{N}(\ln 2) = \frac{8}{17}\,\mathbf{i} - \frac{15}{17}\,\mathbf{k}\,;\ \mathbf{B}(\ln 2) = \mathbf{T}(\ln 2) \times \mathbf{N}(\ln 2) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{15}{17\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{8}{17\sqrt{2}} \\ \frac{8}{17} & 0 & -\frac{15}{17} \end{vmatrix} = -\frac{15}{17\sqrt{2}}\,\mathbf{i} + \frac{1}{\sqrt{2}}\,\mathbf{j} - \frac{8}{17\sqrt{2}}\,\mathbf{k}\,;$

$\mathbf{a} = (12 \cosh 2t)\mathbf{i} + (12 \sinh 2t)\mathbf{j}\ \Rightarrow\ \mathbf{a}(\ln 2) = 12\left(\frac{17}{8}\right)\mathbf{i} + 12\left(\frac{15}{8}\right)\mathbf{j} = \frac{51}{2}\,\mathbf{i} + \frac{45}{2}\,\mathbf{j}$ and

$\mathbf{v}(\ln 2) = 6\left(\frac{15}{8}\right)\mathbf{i} + 6\left(\frac{17}{8}\right)\mathbf{j} + 6\mathbf{k} = \frac{45}{4}\,\mathbf{i} + \frac{51}{4}\,\mathbf{j} + 6\mathbf{k}\ \Rightarrow\ \mathbf{v}(\ln 2) \times \mathbf{a}(\ln 2) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{45}{4} & \frac{51}{4} & 6 \\ \frac{51}{2} & \frac{45}{2} & 0 \end{vmatrix}$

$= -135\mathbf{i} + 153\mathbf{j} - 72\mathbf{k}\ \Rightarrow\ |\mathbf{v} \times \mathbf{a}| = 153\sqrt{2}$ and $|\mathbf{v}(\ln 2)| = \frac{51}{4}\sqrt{2}\ \Rightarrow\ \kappa(\ln 2) = \frac{153\sqrt{2}}{\left(\frac{51}{4}\sqrt{2}\right)^3} = \frac{32}{867}\,;$

$\dot{\mathbf{a}} = (24 \sinh 2t)\mathbf{i} + (24 \cosh 2t)\mathbf{j}\ \Rightarrow\ \dot{\mathbf{a}}(\ln 2) = 45\mathbf{i} + 51\mathbf{j}\ \Rightarrow\ \tau(\ln 2) = \frac{\begin{vmatrix} \frac{45}{4} & \frac{51}{4} & 6 \\ \frac{51}{2} & \frac{45}{2} & 0 \\ 45 & 51 & 0 \end{vmatrix}}{|\mathbf{v}\times\mathbf{a}|^2} = \frac{32}{867}$

21. $\mathbf{r} = \left(2 + 3t + 3t^2\right)\mathbf{i} + \left(4t + 4t^2\right)\mathbf{j} - (6 \cos t)\mathbf{k}\ \Rightarrow\ \mathbf{v} = (3 + 6t)\mathbf{i} + (4 + 8t)\mathbf{j} + (6 \sin t)\mathbf{k}$
$\Rightarrow\ |\mathbf{v}| = \sqrt{(3 + 6t)^2 + (4 + 8t)^2 + (6 \sin t)^2} = \sqrt{25 + 100t + 100t^2 + 36 \sin^2 t}$
$\Rightarrow\ \frac{d\,|\mathbf{v}|}{dt} = \frac{1}{2}\left(25 + 100t + 100t^2 + 36 \sin^2 t\right)^{-1/2}(100 + 200t + 72 \sin t \cos t)\ \Rightarrow\ a_T(0) = \frac{d\,|\mathbf{v}|}{dt}(0) = 10;$
$\mathbf{a} = 6\mathbf{i} + 8\mathbf{j} + (6 \cos t)\mathbf{k}\ \Rightarrow\ |\mathbf{a}| = \sqrt{6^2 + 8^2 + (6 \cos t)^2} = \sqrt{100 + 36 \cos^2 t}\ \Rightarrow\ |\mathbf{a}(0)| = \sqrt{136}$
$\Rightarrow\ a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = \sqrt{136 - 10^2} = \sqrt{36} = 6\ \Rightarrow\ \mathbf{a}(0) = 10\mathbf{T} + 6\mathbf{N}$

22. $\mathbf{r} = (2 + t)\mathbf{i} + \left(t + 2t^2\right)\mathbf{j} + \left(1 + t^2\right)\mathbf{k}\ \Rightarrow\ \mathbf{v} = \mathbf{i} + (1 + 4t)\mathbf{j} + 2t\mathbf{k}\ \Rightarrow\ |\mathbf{v}| = \sqrt{1^2 + (1 + 4t)^2 + (2t)^2}$
$= \sqrt{2 + 8t + 20t^2}\ \Rightarrow\ \frac{d\,|\mathbf{v}|}{dt} = \frac{1}{2}\left(2 + 8t + 20t^2\right)^{-1/2}(8 + 40t)\ \Rightarrow\ a_T = \frac{d\,|\mathbf{v}|}{dt}(0) = 2\sqrt{2};\ \mathbf{a} = 4\mathbf{j} + 2\mathbf{k}$
$\Rightarrow\ |\mathbf{a}| = \sqrt{4^2 + 2^2} = \sqrt{20}\ \Rightarrow\ a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = \sqrt{20 - \left(2\sqrt{2}\right)^2} = \sqrt{12} = 2\sqrt{3}\ \Rightarrow\ \mathbf{a}(0) = 2\sqrt{2}\mathbf{T} + 2\sqrt{3}\mathbf{N}$

23. $\mathbf{r} = (\sin t)\mathbf{i} + \left(\sqrt{2} \cos t\right)\mathbf{j} + (\sin t)\mathbf{k}\ \Rightarrow\ \mathbf{v} = (\cos t)\mathbf{i} - \left(\sqrt{2} \sin t\right)\mathbf{j} + (\cos t)\mathbf{k}$
$\Rightarrow\ |\mathbf{v}| = \sqrt{(\cos t)^2 + \left(-\sqrt{2} \sin t\right)^2 + (\cos t)^2} = \sqrt{2}\ \Rightarrow\ \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{1}{\sqrt{2}} \cos t\right)\mathbf{i} - (\sin t)\mathbf{j} + \left(\frac{1}{\sqrt{2}} \cos t\right)\mathbf{k}\,;$
$\frac{d\mathbf{T}}{dt} = \left(-\frac{1}{\sqrt{2}} \sin t\right)\mathbf{i} - (\cos t)\mathbf{j} - \left(\frac{1}{\sqrt{2}} \sin t\right)\mathbf{k}\ \Rightarrow\ \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\left(-\frac{1}{\sqrt{2}} \sin t\right)^2 + (-\cos t)^2 + \left(-\frac{1}{\sqrt{2}} \sin t\right)^2} = 1$
$\Rightarrow\ \mathbf{N} = \frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|} = \left(-\frac{1}{\sqrt{2}} \sin t\right)\mathbf{i} - (\cos t)\mathbf{j} - \left(\frac{1}{\sqrt{2}} \sin t\right)\mathbf{k}\,;\ \mathbf{B} = \mathbf{T} \times \mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{1}{\sqrt{2}} \cos t & -\sin t & \frac{1}{\sqrt{2}} \cos t \\ -\frac{1}{\sqrt{2}} \sin t & -\cos t & -\frac{1}{\sqrt{2}} \sin t \end{vmatrix}$
$= \frac{1}{\sqrt{2}}\,\mathbf{i} - \frac{1}{\sqrt{2}}\,\mathbf{k}\,;\ \mathbf{a} = (-\sin t)\mathbf{i} - \left(\sqrt{2} \cos t\right)\mathbf{j} - (\sin t)\mathbf{k}\ \Rightarrow\ \mathbf{v} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & -\sqrt{2} \sin t & \cos t \\ -\sin t & -\sqrt{2} \cos t & -\sin t \end{vmatrix}$
$= \sqrt{2}\,\mathbf{i} - \sqrt{2}\,\mathbf{k}\ \Rightarrow\ |\mathbf{v} \times \mathbf{a}| = \sqrt{4} = 2\ \Rightarrow\ \kappa = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3} = \frac{2}{\left(\sqrt{2}\right)^3} = \frac{1}{\sqrt{2}}\,;\ \dot{\mathbf{a}} = (-\cos t)\mathbf{i} + \left(\sqrt{2} \sin t\right)\mathbf{j} - (\cos t)\mathbf{k}$
$\Rightarrow\ \tau = \frac{\begin{vmatrix} \cos t & -\sqrt{2} \sin t & \cos t \\ -\sin t & -\sqrt{2} \cos t & -\sin t \\ -\cos t & \sqrt{2} \sin t & -\cos t \end{vmatrix}}{|\mathbf{v}\times\mathbf{a}|^2} = \frac{(\cos t)\left(\sqrt{2}\right) - \left(\sqrt{2} \sin t\right)(0) + (\cos t)\left(-\sqrt{2}\right)}{4} = 0$

24. $\mathbf{r} = \mathbf{i} + (5 \cos t)\mathbf{j} + (3 \sin t)\mathbf{k}\ \Rightarrow \mathbf{v} = (-5 \sin t)\mathbf{j} + (3 \cos t)\mathbf{k}\ \Rightarrow\ \mathbf{a} = (-5 \cos t)\mathbf{j} - (3 \sin t)\mathbf{k}$
$\Rightarrow\ \mathbf{v} \cdot \mathbf{a} = 25 \sin t \cos t - 9 \sin t \cos t = 16 \sin t \cos t;\ \mathbf{v} \cdot \mathbf{a} = 0\ \Rightarrow\ 16 \sin t \cos t = 0\ \Rightarrow\ \sin t = 0 \text{ or } \cos t = 0$
$\Rightarrow\ t = 0,\ \frac{\pi}{2} \text{ or } \pi$

25. $\mathbf{r} = 2\mathbf{i} + \left(4 \sin \frac{t}{2}\right)\mathbf{j} + \left(3 - \frac{t}{\pi}\right)\mathbf{k} \Rightarrow 0 = \mathbf{r}\cdot(\mathbf{i}-\mathbf{j}) = 2(1) + \left(4 \sin \frac{t}{2}\right)(-1) \Rightarrow 0 = 2 - 4\sin\frac{t}{2} \Rightarrow \sin\frac{t}{2} = \frac{1}{2} \Rightarrow \frac{t}{2} = \frac{\pi}{6}$ $\Rightarrow\ t = \frac{\pi}{3}$ (for the first time)

26. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k} \ \Rightarrow\ \mathbf{v} = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k} \ \Rightarrow\ |\mathbf{v}| = \sqrt{1+4t^2+9t^4} \ \Rightarrow\ |\mathbf{v}(1)| = \sqrt{14}$ $\Rightarrow\ \mathbf{T}(1) = \frac{1}{\sqrt{14}}\mathbf{i} + \frac{2}{\sqrt{14}}\mathbf{j} + \frac{3}{\sqrt{14}}\mathbf{k}$, which is normal to the normal plane $\Rightarrow\ \frac{1}{\sqrt{14}}(x-1) + \frac{2}{\sqrt{14}}(y-1) + \frac{3}{\sqrt{14}}(z-1) = 0$ or $x + 2y + 3z = 6$ is an equation of the normal plane. Next we calculate $\mathbf{N}(1)$ which is normal to the rectifying plane. Now, $\mathbf{a} = 2\mathbf{j} + 6t\mathbf{k} \ \Rightarrow\ \mathbf{a}(1) = 2\mathbf{j} + 6\mathbf{k} \ \Rightarrow\ \mathbf{v}(1)\times\mathbf{a}(1)$ $= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 0 & 2 & 6 \end{vmatrix} = 6\mathbf{i} - 6\mathbf{j} + 2\mathbf{k} \ \Rightarrow\ |\mathbf{v}(1)\times\mathbf{a}(1)| = \sqrt{76} \ \Rightarrow\ \kappa(1) = \frac{\sqrt{76}}{\left(\sqrt{14}\right)^3} = \frac{\sqrt{19}}{7\sqrt{14}}$; $\frac{ds}{dt} = |\mathbf{v}(t)| \ \Rightarrow\ \left.\frac{d^2s}{dt^2}\right|_{t=1}$ $= \frac{1}{2}\left(1+4t^2+9t^4\right)^{-1/2}\left(8t+36t^3\right)\Big|_{t=1} = \frac{22}{\sqrt{14}}$, so $\mathbf{a} = \frac{d^2s}{dt^2}\mathbf{T} + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N} \ \Rightarrow\ 2\mathbf{j} + 6\mathbf{k}$ $= \frac{22}{\sqrt{14}}\left(\frac{\mathbf{i}+2\mathbf{j}+3\mathbf{k}}{\sqrt{14}}\right) + \frac{\sqrt{19}}{7\sqrt{14}}\left(\sqrt{14}\right)^2\mathbf{N} \ \Rightarrow\ \mathbf{N} = \frac{\sqrt{14}}{2\sqrt{19}}\left(-\frac{11}{7}\mathbf{i} - \frac{8}{7}\mathbf{j} + \frac{9}{7}\mathbf{k}\right) \ \Rightarrow\ -\frac{11}{7}(x-1) - \frac{8}{7}(y-1) + \frac{9}{7}(z-1)$ $= 0$ or $11x + 8y - 9z = 10$ is an equation of the rectifying plane. Finally, $\mathbf{B}(1) = \mathbf{T}(1)\times\mathbf{N}(1)$ $= \left(\frac{\sqrt{14}}{2\sqrt{19}}\right)\left(\frac{1}{\sqrt{14}}\right)\left(\frac{1}{7}\right)\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -11 & -8 & 9 \end{vmatrix} = \frac{1}{\sqrt{19}}(3\mathbf{i} - 3\mathbf{j} + \mathbf{k}) \ \Rightarrow\ 3(x-1) - 3(y-1) + (z-1) = 0$ or $3x - 3y + z$ $= 1$ is an equation of the osculating plane.

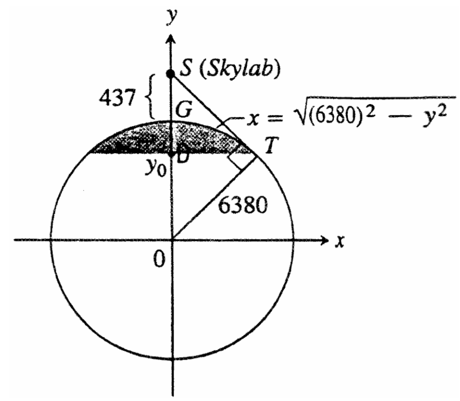
27. $\mathbf{r} = e^t\mathbf{i} + (\sin t)\mathbf{j} + \ln(1-t)\mathbf{k} \ \Rightarrow\ \mathbf{v} = e^t\mathbf{i} + (\cos t)\mathbf{j} - \left(\frac{1}{1-t}\right)\mathbf{k} \ \Rightarrow\ \mathbf{v}(0) = \mathbf{i} + \mathbf{j} - \mathbf{k}$; $\mathbf{r}(0) = \mathbf{i} \ \Rightarrow\ (1,0,0)$ is on the line $\Rightarrow\ x = 1 + t$, $y = t$, and $z = -t$ are parametric equations of the line

28. $\mathbf{r} = \left(\sqrt{2}\cos t\right)\mathbf{i} + \left(\sqrt{2}\sin t\right)\mathbf{j} + t\mathbf{k} \ \Rightarrow\ \mathbf{v} = \left(-\sqrt{2}\sin t\right)\mathbf{i} + \left(\sqrt{2}\cos t\right)\mathbf{j} + \mathbf{k} \ \Rightarrow\ \mathbf{v}\left(\frac{\pi}{4}\right)$ $= \left(-\sqrt{2}\sin\frac{\pi}{4}\right)\mathbf{i} + \left(\sqrt{2}\cos\frac{\pi}{4}\right)\mathbf{j} + \mathbf{k} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$ is a vector tangent to the helix when $t = \frac{\pi}{4} \ \Rightarrow$ the tangent line is parallel to $\mathbf{v}\left(\frac{\pi}{4}\right)$; also $\mathbf{r}\left(\frac{\pi}{4}\right) = \left(\sqrt{2}\cos\frac{\pi}{4}\right)\mathbf{i} + \left(\sqrt{2}\sin\frac{\pi}{4}\right)\mathbf{j} + \frac{\pi}{4}\mathbf{k} \ \Rightarrow$ the point $\left(1, 1, \frac{\pi}{4}\right)$ is on the line $\Rightarrow\ x = 1 - t$, $y = 1 + t$, and $z = \frac{\pi}{4} + t$ are parametric equations of the line

29. $x^2 = \left(v_0^2\cos^2\alpha\right)t^2$ and $\left(y + \frac{1}{2}gt^2\right)^2 = \left(v_0^2\sin^2\alpha\right)t^2 \ \Rightarrow\ x^2 + \left(y + \frac{1}{2}gt^2\right)^2 = v_0^2t^2$

30. $\ddot{s} = \frac{d}{dt}\sqrt{\dot{x}^2+\dot{y}^2} = \frac{\dot{x}\ddot{x}+\dot{y}\ddot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}} \ \Rightarrow\ \ddot{x}^2 + \ddot{y}^2 - \ddot{s}^2 = \ddot{x}^2 + \ddot{y}^2 - \frac{(\dot{x}\ddot{x}+\dot{y}\ddot{y})^2}{\dot{x}^2+\dot{y}^2}$ $= \frac{(\ddot{x}^2+\ddot{y}^2)(\dot{x}^2+\dot{y}^2) - (\dot{x}^2\ddot{x}^2 + 2\dot{x}\ddot{x}\dot{y}\ddot{y} + \dot{y}^2\ddot{y}^2)}{\dot{x}^2+\dot{y}^2} = \frac{\dot{x}^2\ddot{y}^2 + \dot{y}^2\ddot{x}^2 - 2\dot{x}\ddot{x}\dot{y}\ddot{y}}{\dot{x}^2+\dot{y}^2} = \frac{(\dot{x}\ddot{y}-\dot{y}\ddot{x})^2}{\dot{x}^2+\dot{y}^2}$ $\Rightarrow\ \sqrt{\ddot{x}^2+\ddot{y}^2-\ddot{s}^2} = \frac{|\dot{x}\ddot{y}-\dot{y}\ddot{x}|}{\sqrt{\dot{x}^2+\dot{y}^2}} \ \Rightarrow\ \frac{\dot{x}^2+\dot{y}^2}{\sqrt{\ddot{x}^2+\ddot{y}^2-\ddot{s}^2}} = \frac{(\dot{x}^2+\dot{y}^2)^{3/2}}{|\dot{x}\ddot{y}-\dot{y}\ddot{x}|} = \frac{1}{\kappa} = \rho$

31. $s = a\theta \ \Rightarrow\ \theta = \frac{s}{a} \ \Rightarrow\ \phi = \frac{s}{a} + \frac{\pi}{2} \ \Rightarrow\ \frac{d\phi}{ds} = \frac{1}{a} \ \Rightarrow\ \kappa = \left|\frac{1}{a}\right| = \frac{1}{a}$ since $a > 0$



32. (1) $\Delta SOT \approx \Delta TOD \Rightarrow \frac{DO}{OT} = \frac{OT}{SO} \Rightarrow \frac{y_0}{6380} = \frac{6380}{6380+437}$

$\Rightarrow y_0 = \frac{6380^2}{6817} \Rightarrow y_0 \approx 5971$ km;

(2) $VA = \int_{5971}^{6380} 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$

$= 2\pi \int_{5971}^{6817} \sqrt{6380^2 - y^2}\left(\frac{6380}{\sqrt{6380^2 - y^2}}\right) dy$

$= 2\pi \int_{5971}^{6817} 6380\, dy = 2\pi\, [6380y]_{5971}^{6817}$

$= 16{,}395{,}469 \text{ km}^2 \approx 1.639 \times 10^7 \text{ km}^2$;

(3) percentage visible $\approx \frac{16{,}395{,}469 \text{ km}^2}{4\pi(6380 \text{ km})^2} \approx 3.21\%$

CHAPTER 13 ADDITIONAL AND ADVANCED EXERCISES

1. (a) $\mathbf{r}(\theta) = (a\cos\theta)\mathbf{i} + (a\sin\theta)\mathbf{j} + b\theta\mathbf{k} \Rightarrow \frac{d\mathbf{r}}{dt} = [(-a\sin\theta)\mathbf{i} + (a\cos\theta)\mathbf{j} + b\mathbf{k}]\frac{d\theta}{dt}$; $|\mathbf{v}| = \sqrt{2gz} = \left|\frac{d\mathbf{r}}{dt}\right|$

$= \sqrt{a^2+b^2}\,\frac{d\theta}{dt} \Rightarrow \frac{d\theta}{dt} = \sqrt{\frac{2gz}{a^2+b^2}} = \sqrt{\frac{2gb\theta}{a^2+b^2}} \Rightarrow \left.\frac{d\theta}{dt}\right|_{\theta=2\pi} = \sqrt{\frac{4\pi gb}{a^2+b^2}} = 2\sqrt{\frac{\pi gb}{a^2+b^2}}$

(b) $\frac{d\theta}{dt} = \sqrt{\frac{2gb\theta}{a^2+b^2}} \Rightarrow \frac{d\theta}{\sqrt{\theta}} = \sqrt{\frac{2gb}{a^2+b^2}}\,dt \Rightarrow 2\theta^{1/2} = \sqrt{\frac{2gb}{a^2+b^2}}\,t + C$; $t = 0 \Rightarrow \theta = 0 \Rightarrow C = 0$

$\Rightarrow 2\theta^{1/2} = \sqrt{\frac{2gb}{a^2+b^2}}\,t \Rightarrow \theta = \frac{gbt^2}{2(a^2+b^2)}$; $z = b\theta \Rightarrow z = \frac{gb^2t^2}{2(a^2+b^2)}$

(c) $\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = [(-a\sin\theta)\mathbf{i} + (a\cos\theta)\mathbf{j} + b\mathbf{k}]\frac{d\theta}{dt} = [(-a\sin\theta)\mathbf{i} + (a\cos\theta)\mathbf{j} + b\mathbf{k}]\left(\frac{gbt}{a^2+b^2}\right)$, from part (b)

$\Rightarrow \mathbf{v}(t) = \left[\frac{(-a\sin\theta)\mathbf{i} + (a\cos\theta)\mathbf{j} + b\mathbf{k}}{\sqrt{a^2+b^2}}\right]\left(\frac{gbt}{\sqrt{a^2+b^2}}\right) = \frac{gbt}{\sqrt{a^2+b^2}}\,\mathbf{T}$;

$\frac{d^2\mathbf{r}}{dt^2} = [(-a\cos\theta)\mathbf{i} - (a\sin\theta)\mathbf{j}]\left(\frac{d\theta}{dt}\right)^2 + [(-a\sin\theta)\mathbf{i} + (a\cos\theta)\mathbf{j} + b\mathbf{k}]\frac{d^2\theta}{dt^2}$

$= \left(\frac{gbt}{a^2+b^2}\right)^2 [(-a\cos\theta)\mathbf{i} - (a\sin\theta)\mathbf{j}] + [(-a\sin\theta)\mathbf{i} + (a\cos\theta)\mathbf{j} + b\mathbf{k}]\left(\frac{gb}{a^2+b^2}\right)$

$= \left[\frac{(-a\sin\theta)\mathbf{i} + (a\cos\theta)\mathbf{j} + b\mathbf{k}}{\sqrt{a^2+b^2}}\right]\left(\frac{gb}{\sqrt{a^2+b^2}}\right) + a\left(\frac{gbt}{a^2+b^2}\right)^2 [(-\cos\theta)\mathbf{i} - (\sin\theta)\mathbf{j}]$

$= \frac{gb}{\sqrt{a^2+b^2}}\,\mathbf{T} + a\left(\frac{gbt}{a^2+b^2}\right)^2 \mathbf{N}$ (there is no component in the direction of **B**).

2. (a) $\mathbf{r}(\theta) = (a\theta\cos\theta)\mathbf{i} + (a\theta\sin\theta)\mathbf{j} + b\theta\mathbf{k} \Rightarrow \frac{d\mathbf{r}}{dt} = [(a\cos\theta - a\theta\sin\theta)\mathbf{i} + (a\sin\theta + a\theta\cos\theta)\mathbf{j} + b\mathbf{k}]\frac{d\theta}{dt}$;

$|\mathbf{v}| = \sqrt{2gz} = \left|\frac{d\mathbf{r}}{dt}\right| = (a^2 + a^2\theta^2 + b^2)^{1/2}\left(\frac{d\theta}{dt}\right) \Rightarrow \frac{d\theta}{dt} = \frac{\sqrt{2gb\theta}}{\sqrt{a^2 + a^2\theta^2 + b^2}}$

(b) $s = \int_0^t |\mathbf{v}|\,dt = \int_0^t (a^2 + a^2\theta^2 + b^2)^{1/2}\frac{d\theta}{dt}\,dt = \int_0^t (a^2 + a^2\theta^2 + b^2)^{1/2}\,d\theta = \int_0^\theta (a^2 + a^2u^2 + b^2)^{1/2}\,du$

$= \int_0^\theta a\sqrt{\frac{a^2+b^2}{a^2} + u^2}\,du = a\int_0^\theta \sqrt{c^2+u^2}\,du$, where $c = \frac{\sqrt{a^2+b^2}}{|a|}$

$\Rightarrow s = a\left[\frac{u}{2}\sqrt{c^2+u^2} + \frac{c^2}{2}\ln\left|u + \sqrt{c^2+u^2}\right|\right]_0^\theta = \frac{a}{2}\left(\theta\sqrt{c^2+\theta^2} + c^2\ln\left|\theta + \sqrt{c^2+\theta^2}\right| - c^2\ln c\right)$

3. $r = \frac{(1+e)r_0}{1+e\cos\theta} \Rightarrow \frac{dr}{d\theta} = \frac{(1+e)r_0(e\sin\theta)}{(1+e\cos\theta)^2}$; $\frac{dr}{d\theta} = 0 \Rightarrow \frac{(1+e)r_0(e\sin\theta)}{(1+e\cos\theta)^2} = 0 \Rightarrow (1+e)r_0(e\sin\theta) = 0$

$\Rightarrow \sin\theta = 0 \Rightarrow \theta = 0$ or π. Note that $\frac{dr}{d\theta} > 0$ when $\sin\theta > 0$ and $\frac{dr}{d\theta} < 0$ when $\sin\theta < 0$. Since $\sin\theta < 0$ on $-\pi < \theta < 0$ and $\sin\theta > 0$ on $0 < \theta < \pi$, r is a minimum when $\theta = 0$ and $r(0) = \frac{(1+e)r_0}{1+e\cos 0} = r_0$

4. (a) $f(x) = x - 1 - \frac{1}{2}\sin x = 0 \Rightarrow f(0) = -1$ and $f(2) = 2 - 1 - \frac{1}{2}\sin 2 \geq \frac{1}{2}$ since $|\sin 2| \leq 1$; since f is continuous on $[0, 2]$, the Intermediate Value Theorem implies there is a root between 0 and 2

(b) Root ≈ 1.4987011335179

5. (a) $\mathbf{v} = \dot{x}\,\mathbf{i} + \dot{y}\,\mathbf{j}$ and $\mathbf{v} = \dot{r}\,\mathbf{u}_r + r\,\dot{\theta}\,\mathbf{u}_\theta = (\dot{r})\,[(\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}] + \left(r\,\dot{\theta}\right)[(-\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j}] \Rightarrow \mathbf{v}\cdot\mathbf{i} = \dot{x}$ and $\mathbf{v}\cdot\mathbf{i} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta \Rightarrow \dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$; $\mathbf{v}\cdot\mathbf{j} = \dot{y}$ and $\mathbf{v}\cdot\mathbf{j} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$
$\Rightarrow \dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$

(b) $\mathbf{u}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} \Rightarrow \mathbf{v}\cdot\mathbf{u}_r = \dot{x}\cos\theta + \dot{y}\sin\theta$
$= \left(\dot{r}\cos\theta - r\dot{\theta}\sin\theta\right)(\cos\theta) + \left(\dot{r}\sin\theta + r\dot{\theta}\cos\theta\right)(\sin\theta)$ by part (a),
$\Rightarrow \mathbf{v}\cdot\mathbf{u}_r = \dot{r}$; therefore, $\dot{r} = \dot{x}\cos\theta + \dot{y}\sin\theta$;
$\mathbf{u}_\theta = -(\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j} \Rightarrow \mathbf{v}\cdot\mathbf{u}_\theta = -\dot{x}\sin\theta + \dot{y}\cos\theta$
$= \left(\dot{r}\cos\theta - r\dot{\theta}\sin\theta\right)(-\sin\theta) + \left(\dot{r}\sin\theta + r\dot{\theta}\cos\theta\right)(\cos\theta)$ by part (a) $\Rightarrow \mathbf{v}\cdot\mathbf{u}_\theta = r\dot{\theta}$;
therefore, $r\dot{\theta} = -\dot{x}\sin\theta + \dot{y}\cos\theta$

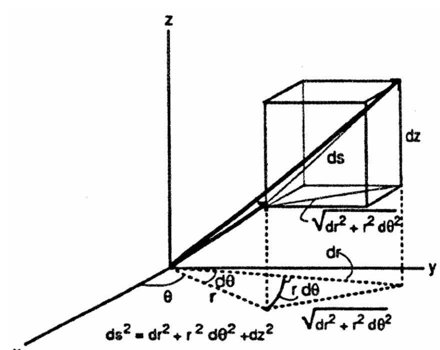
6. $\mathbf{r} = f(\theta) \Rightarrow \frac{dr}{dt} = f'(\theta)\frac{d\theta}{dt} \Rightarrow \frac{d^2r}{dt^2} = f''(\theta)\left(\frac{d\theta}{dt}\right)^2 + f'(\theta)\frac{d^2\theta}{dt^2}$; $\mathbf{v} = \frac{dr}{dt}\mathbf{u}_r + r\frac{d\theta}{dt}\mathbf{u}_\theta$
$= \left(\cos\theta\frac{dr}{dt} - r\sin\theta\frac{d\theta}{dt}\right)\mathbf{i} + \left(\sin\theta\frac{dr}{dt} + r\cos\theta\frac{d\theta}{dt}\right)\mathbf{j} \Rightarrow |\mathbf{v}| = \left[\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right]^{1/2} = \left[(f')^2 + f^2\right]^{1/2}\left(\frac{d\theta}{dt}\right)$;
$|\mathbf{v}\times\mathbf{a}| = |\dot{x}\,\ddot{y} - \dot{y}\,\ddot{x}|$, where $x = r\cos\theta$ and $y = r\sin\theta$. Then $\frac{dx}{dt} = (-r\sin\theta)\frac{d\theta}{dt} + (\cos\theta)\frac{dr}{dt}$
$\Rightarrow \frac{d^2x}{dt^2} = (-2\sin\theta)\frac{d\theta}{dt}\frac{dr}{dt} - (r\cos\theta)\left(\frac{d\theta}{dt}\right)^2 - (r\sin\theta)\frac{d^2\theta}{dt^2} + (\cos\theta)\frac{d^2r}{dt^2}$; $\frac{dy}{dt} = (r\cos\theta)\frac{d\theta}{dt} + (\sin\theta)\frac{dr}{dt}$
$\Rightarrow \frac{d^2y}{dt^2} = (2\cos\theta)\frac{d\theta}{dt}\frac{dr}{dt} - (r\sin\theta)\left(\frac{d\theta}{dt}\right)^2 + (r\cos\theta)\frac{d^2\theta}{dt^2} + (\sin\theta)\frac{d^2r}{dt^2}$. Then $|\mathbf{v}\times\mathbf{a}|$
$=$ (after <u>much</u> algebra) $r^2\left(\frac{d\theta}{dt}\right)^3 + r\frac{d^2\theta}{dt^2}\frac{dr}{dt} - r\frac{d\theta}{dt}\frac{d^2r}{dt^2} + 2\frac{d\theta}{dt}\left(\frac{dr}{dt}\right)^2 = \left(\frac{d\theta}{dt}\right)^3\left(f^2 - f\cdot f'' + 2(f')^2\right)$
$\Rightarrow \kappa = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|} = \frac{f^2 - f\cdot f'' + 2(f')^2}{\left[(f')^2 + f^2\right]^{3/2}}$

7. (a) Let $r = 2 - t$ and $\theta = 3t \Rightarrow \frac{dr}{dt} = -1$ and $\frac{d\theta}{dt} = 3 \Rightarrow \frac{d^2r}{dt^2} = \frac{d^2\theta}{dt^2} = 0$. The halfway point is $(1, 3) \Rightarrow t = 1$;
$\mathbf{v} = \frac{dr}{dt}\mathbf{u}_r + r\frac{d\theta}{dt}\mathbf{u}_\theta \Rightarrow \mathbf{v}(1) = -\mathbf{u}_r + 3\mathbf{u}_\theta$; $\mathbf{a} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\mathbf{u}_r + \left[r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right]\mathbf{u}_\theta \Rightarrow \mathbf{a}(1) = -9\mathbf{u}_r - 6\mathbf{u}_\theta$

(b) It takes the beetle 2 min to crawl to the origin $\Rightarrow$ the rod has revolved 6 radians
$\Rightarrow L = \int_0^6 \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\,d\theta = \int_0^6 \sqrt{\left(2 - \frac{\theta}{3}\right)^2 + \left(-\frac{1}{3}\right)^2}\,d\theta = \int_0^6 \sqrt{4 - \frac{4\theta}{3} + \frac{\theta^2}{9} + \frac{1}{9}}\,d\theta$
$= \int_0^6 \sqrt{\frac{37 - 12\theta + \theta^2}{9}}\,d\theta = \frac{1}{3}\int_0^6 \sqrt{(\theta - 6)^2 + 1}\,d\theta = \frac{1}{3}\left[\frac{(\theta-6)}{2}\sqrt{(\theta - 6)^2 + 1} + \frac{1}{2}\ln\left|\theta - 6 + \sqrt{(\theta - 6)^2 + 1}\right|\right]_0^6$
$= \sqrt{37} - \frac{1}{6}\ln\left(\sqrt{37} - 6\right) \approx 6.5$ in.

8. (a) $x = r\cos\theta \Rightarrow dx = \cos\theta\,dr - r\sin\theta\,d\theta$; $y = r\sin\theta \Rightarrow dy = \sin\theta\,dr + r\cos\theta\,d\theta$; thus
$dx^2 = \cos^2\theta\,dr^2 - 2r\sin\theta\cos\theta\,dr\,d\theta + r^2\sin^2\theta\,d\theta^2$ and
$dy^2 = \sin^2\theta\,dr^2 + 2r\sin\theta\cos\theta\,dr\,d\theta + r^2\cos^2\theta\,d\theta^2 \Rightarrow ds^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2\,d\theta^2 + dz^2$

(b)



(c) $r = e^\theta \Rightarrow dr = e^\theta\,d\theta$
$\Rightarrow L = \int_0^{\ln 8} \sqrt{dr^2 + r^2\,d\theta^2 + dz^2}$
$= \int_0^{\ln 8} \sqrt{e^{2\theta} + e^{2\theta} + e^{2\theta}}\,d\theta$
$= \int_0^{\ln 8} \sqrt{3}e^\theta\,d\theta = \left[\sqrt{3}\,e^\theta\right]_0^{\ln 8}$
$= 8\sqrt{3} - \sqrt{3} = 7\sqrt{3}$

9. (a) $\mathbf{u}_r \times \mathbf{u}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \end{vmatrix} = \mathbf{k} \Rightarrow$ a right-handed frame of unit vectors

(b) $\frac{d\mathbf{u}_r}{d\theta} = (-\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j} = \mathbf{u}_\theta$ and $\frac{d\mathbf{u}_\theta}{d\theta} = (-\cos\theta)\mathbf{i} - (\sin\theta)\mathbf{j} = -\mathbf{u}_r$

(c) From Eq. (7), $\mathbf{v} = \dot{r}\mathbf{u}_r + r\dot{\theta}\mathbf{u}_\theta + \dot{z}\mathbf{k} \Rightarrow \mathbf{a} = \dot{\mathbf{v}} = (\ddot{r}\mathbf{u}_r + \dot{r}\,\dot{\mathbf{u}}_r) + \left(\dot{r}\,\dot{\theta}\mathbf{u}_\theta + r\ddot{\theta}\mathbf{u}_\theta + r\dot{\theta}\,\dot{\mathbf{u}}_\theta\right) + \ddot{z}\,\mathbf{k}$
$= \left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{u}_r + \left(r\ddot{\theta} + 2\dot{r}\,\dot{\theta}\right)\mathbf{u}_\theta + \ddot{z}\,\mathbf{k}$

10. $\mathbf{L}(t) = \mathbf{r}(t) \times m\mathbf{v}(t) \Rightarrow \frac{d\mathbf{L}}{dt} = \left(\frac{d\mathbf{r}}{dt} \times m\mathbf{v}\right) + \left(\mathbf{r} \times m\,\frac{d^2\mathbf{r}}{dt^2}\right) \Rightarrow \frac{d\mathbf{L}}{dt} = (\mathbf{v} \times m\mathbf{v}) + (\mathbf{r} \times m\mathbf{a}) = \mathbf{r} \times m\mathbf{a}\,;\, \mathbf{F} = m\mathbf{a} \Rightarrow -\frac{c}{|\mathbf{r}|^3}\,\mathbf{r}$
$= m\mathbf{a} \Rightarrow \frac{d\mathbf{L}}{dt} = \mathbf{r} \times m\mathbf{a} = \mathbf{r} \times \left(-\frac{c}{|\mathbf{r}|^3}\,\mathbf{r}\right) = -\frac{c}{|\mathbf{r}|^3}\,(\mathbf{r} \times \mathbf{r}) = \mathbf{0} \Rightarrow \mathbf{L} =$ constant vector

NOTES:

CHAPTER 14 PARTIAL DERIVATIVES

14.1 FUNCTIONS OF SEVERAL VARIABLES

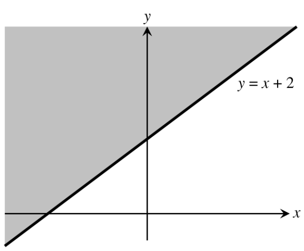
1. (a) $f(0,0) = 0$ (b) $f(-1,1) = 0$ (c) $f(2,3) = 58$
   (d) $f(-3,-2) = 33$

2. (a) $f\left(2, \frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ (b) $f\left(-3, \frac{\pi}{12}\right) = -\frac{1}{\sqrt{2}}$ (c) $f\left(\pi, \frac{1}{4}\right) = \frac{1}{\sqrt{2}}$
   (d) $f\left(-\frac{\pi}{2}, -7\right) = -1$

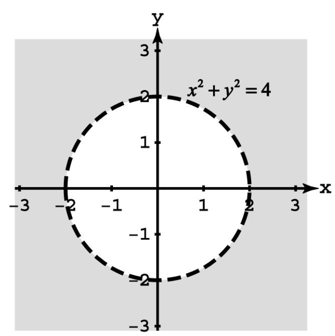
3. (a) $f(3,-1,2) = \frac{4}{5}$ (b) $f\left(1, \frac{1}{2}, -\frac{1}{4}\right) = \frac{8}{5}$ (c) $f\left(0, -\frac{1}{3}, 0\right) = 3$
   (d) $f(2,2,100) = 0$

4. (a) $f(0,0,0) = 7$ (b) $f(2,-3,6) = 0$ (c) $f(-1,2,3) = \sqrt{35}$
   (d) $f\left(\frac{4}{\sqrt{2}}, \frac{5}{\sqrt{2}}, \frac{6}{\sqrt{2}}\right) = \sqrt{\frac{21}{2}}$

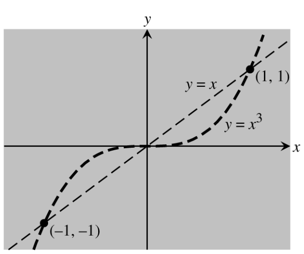
5. Domain: all points (x, y) on or above the line $y = x + 2$



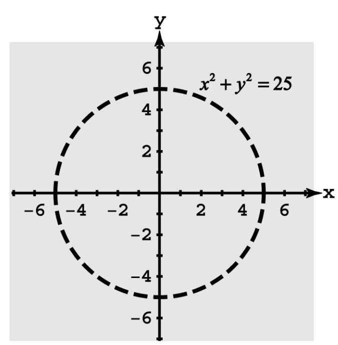
6. Domain: all points (x, y) outside the circle $x^2 + y^2 = 4$



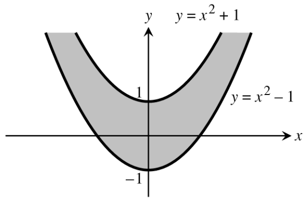
7. Domain: all points (x, y) not liying on the graph of $y = x$ or $y = x^3$



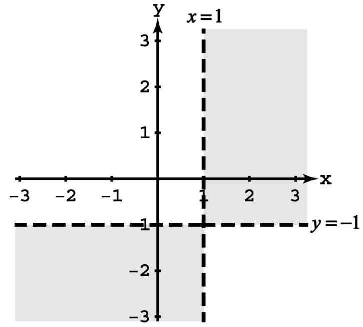
8. Domain: all points (x, y) not liying on the graph of $x^2 + y^2 = 25$



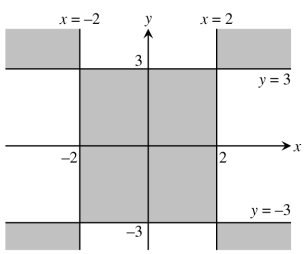
9. Domain: all points (x, y) satisfying $x^2 - 1 \le y \le x^2 + 1$



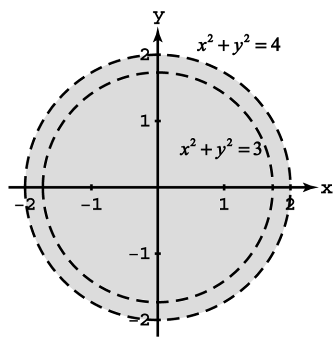
10. Domain: all points (x, y) satisfying $(x - 1)(y + 1) > 0$



11. Domain: all points (x, y) satisfying $(x - 2)(x + 2)(y - 3)(y + 3) \ge 0$



12. Domain: all points (x, y) inside the circle $x^2 + y^2 = 4$ such that $x^2 + y^2 \ne 3$



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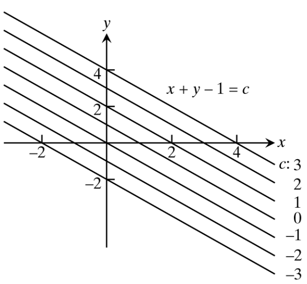

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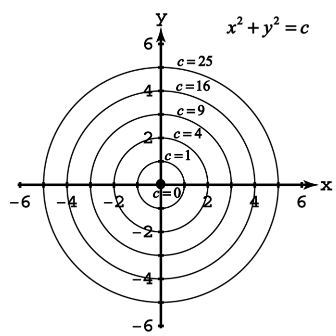

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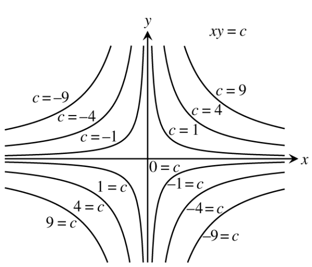
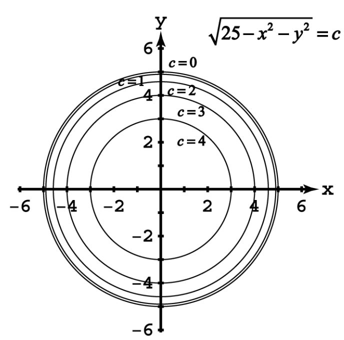


16.



17. (a) Domain: all points in the xy-plane
(b) Range: all real numbers

(c) level curves are straight lines $y - x = c$ parallel to the line $y = x$
(d) no boundary points
(e) both open and closed
(f) unbounded

18. (a) Domain: set of all (x, y) so that $y - x \geq 0 \Rightarrow y \geq x$
(b) Range: $z \geq 0$
(c) level curves are straight lines of the form $y - x = c$ where $c \geq 0$
(d) boundary is $\sqrt{y - x} = 0 \Rightarrow y = x$, a straight line
(e) closed
(f) unbounded

19. (a) Domain: all points in the xy-plane
(b) Range: $z \geq 0$
(c) level curves: for $f(x, y) = 0$, the origin; for $f(x, y) = c > 0$, ellipses with center $(0, 0)$ and major and minor axes along the x- and y-axes, respectively
(d) no boundary points
(e) both open and closed
(f) unbounded

20. (a) Domain: all points in the xy-plane
(b) Range: all real numbers
(c) level curves: for $f(x, y) = 0$, the union of the lines $y = \pm x$; for $f(x, y) = c \neq 0$, hyperbolas centered at $(0, 0)$ with foci on the x-axis if $c > 0$ and on the y-axis if $c < 0$
(d) no boundary points
(e) both open and closed
(f) unbounded

21. (a) Domain: all points in the xy-plane
(b) Range: all real numbers
(c) level curves are hyperbolas with the x- and y-axes as asymptotes when $f(x, y) \neq 0$, and the x- and y-axes when $f(x, y) = 0$
(d) no boundary points
(e) both open and closed
(f) unbounded

22. (a) Domain: all $(x, y) \neq (0, y)$
(b) Range: all real numbers
(c) level curves: for $f(x, y) = 0$, the x-axis minus the origin; for $f(x, y) = c \neq 0$, the parabolas $y = c\,x^2$ minus the origin
(d) boundary is the line $x = 0$
(e) open
(f) unbounded

23. (a) Domain: all (x, y) satisfying $x^2 + y^2 < 16$
(b) Range: $z \geq \frac{1}{4}$
(c) level curves are circles centered at the origin with radii $r < 4$
(d) boundary is the circle $x^2 + y^2 = 16$

(e) open
(f) bounded

24. (a) Domain: all (x, y) satisfying $x^2 + y^2 \leq 9$
(b) Range: $0 \leq z \leq 3$
(c) level curves are circles centered at the origin with radii $r \leq 3$
(d) boundary is the circle $x^2 + y^2 = 9$
(e) closed
(f) bounded

25. (a) Domain: $(x, y) \neq (0, 0)$
(b) Range: all real numbers
(c) level curves are circles with center $(0, 0)$ and radii $r > 0$
(d) boundary is the single point $(0, 0)$
(e) open
(f) unbounded

26. (a) Domain: all points in the xy-plane
(b) Range: $0 < z \leq 1$
(c) level curves are the origin itself and the circles with center $(0, 0)$ and radii $r > 0$
(d) no boundary points
(e) both open and closed
(f) unbounded

27. (a) Domain: all (x, y) satisfying $-1 \leq y - x \leq 1$
(b) Range: $-\frac{\pi}{2} \leq z \leq \frac{\pi}{2}$
(c) level curves are straight lines of the form $y - x = c$ where $-1 \leq c \leq 1$
(d) boundary is the two straight lines $y = 1 + x$ and $y = -1 + x$
(e) closed
(f) unbounded

28. (a) Domain: all (x, y), $x \neq 0$
(b) Range: $-\frac{\pi}{2} < z < \frac{\pi}{2}$
(c) level curves are the straight lines of the form $y = c\,x$, c any real number and $x \neq 0$
(d) boundary is the line $x = 0$
(e) open
(f) unbounded

29. (a) Domain: all points (x, y) outside the circle $x^2 + y^2 = 1$
(b) Range: all reals
(c) Circles centered ar the origin with radii $r > 1$
(d) Boundary: the cricle $x^2 + y^2 = 1$
(e) open
(f) unbounded

30. (a) Domain: all points (x, y) inside the circle $x^2 + y^2 = 9$
(b) Range: $z < \ln 9$
(c) Circles centered ar the origin with radii $r < 9$
(d) Boundary: the cricle $x^2 + y^2 = 9$

(e) open
(f) bounded

31. f

32. e

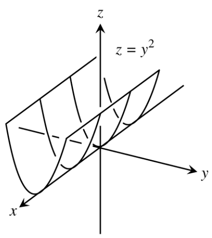
33. a

34. c

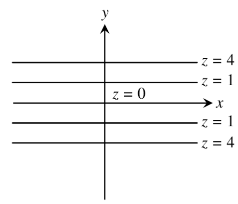
35. d

36. b

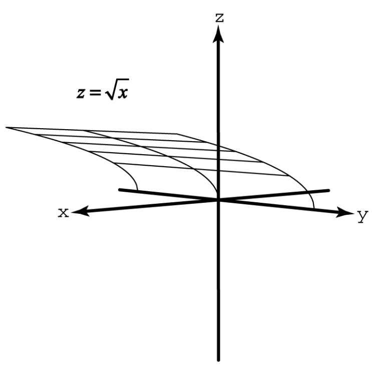
37. (a)



(b)



38. (a)



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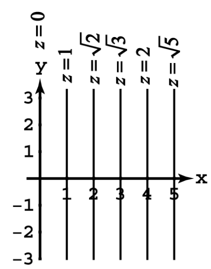
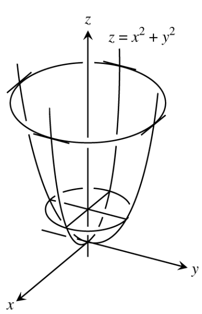
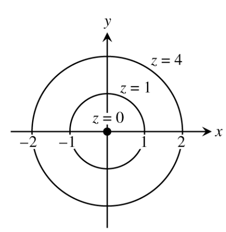


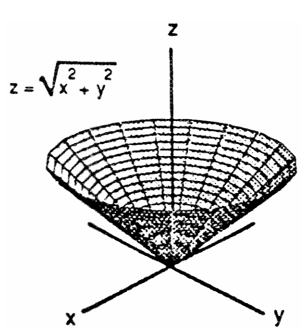
39. (a)



(b)



40. (a)



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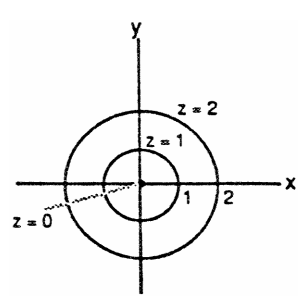
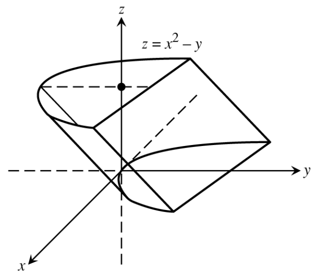
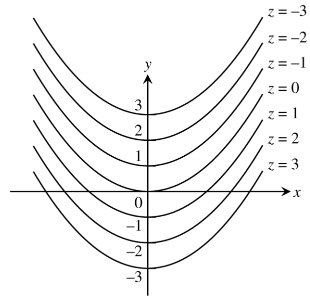


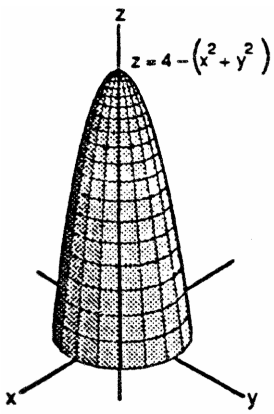
41. (a)



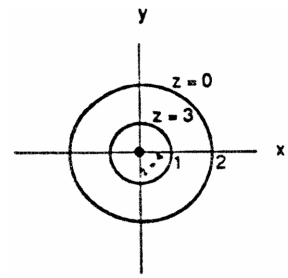
(b)



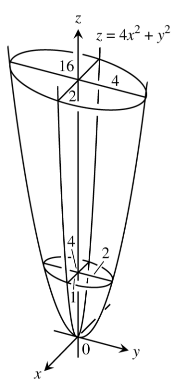
42. (a)



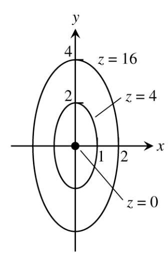
(b)



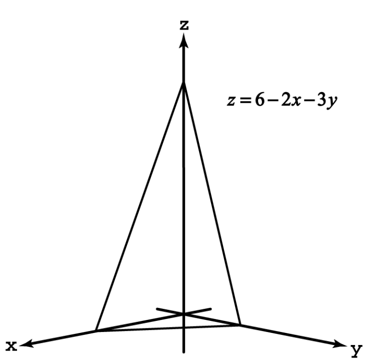
43. (a)



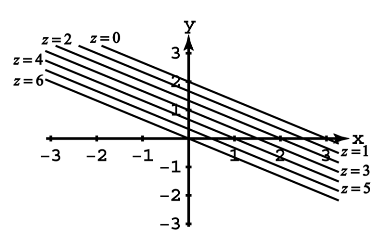
(b)



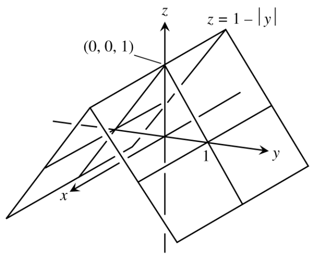
44. (a)



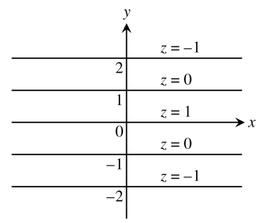
(b)



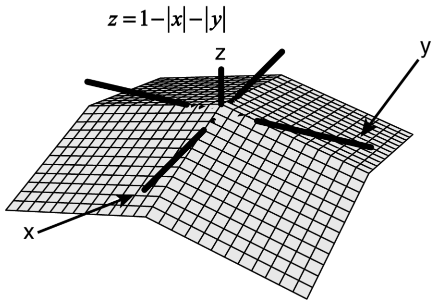
45. (a)



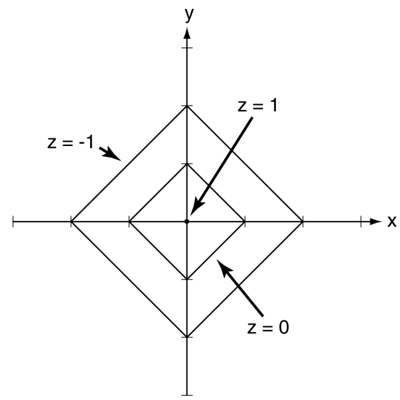
(b)



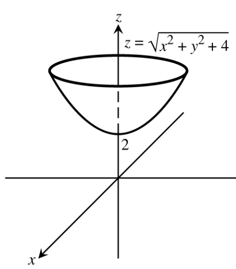
46. (a)



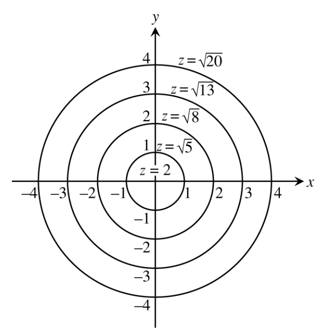
(b)



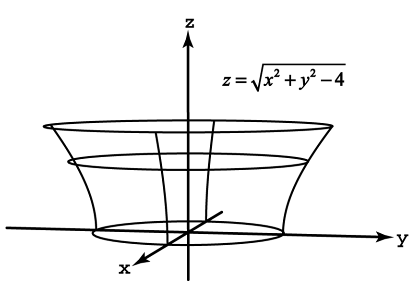
47. (a)



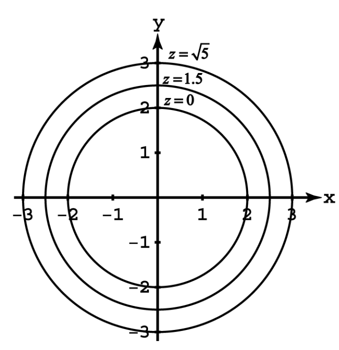
(b)



48. (a)



(b)



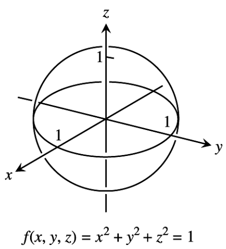
49. $f(x, y) = 16 - x^2 - y^2$ and $\left(2\sqrt{2}, \sqrt{2}\right) \Rightarrow z = 16 - \left(2\sqrt{2}\right)^2 - \left(\sqrt{2}\right)^2 = 6 \Rightarrow 6 = 16 - x^2 - y^2 \Rightarrow x^2 + y^2 = 10$

50. $f(x, y) = \sqrt{x^2 - 1}$ and $(1, 0) \Rightarrow z = \sqrt{1^2 - 1} = 0 \Rightarrow x^2 - 1 = 0 \Rightarrow x = 1$ or $x = -1$

51. $f(x, y) = \sqrt{x + y^2 - 3}$ and $(3, -1) \Rightarrow z = \sqrt{3 + (-1)^2 - 3} = 1 \Rightarrow x + y^2 - 3 = 1 \Rightarrow x + y^2 = 4$

52. $f(x, y) = \frac{2y - x}{x + y + 1}$ and $(-1, 1) \Rightarrow z = \frac{2(1) - (-1)}{(-1) + 1 + 1} = 3 \Rightarrow 3 = \frac{2y - x}{x + y + 1} \Rightarrow y = -4x - 3$

53.



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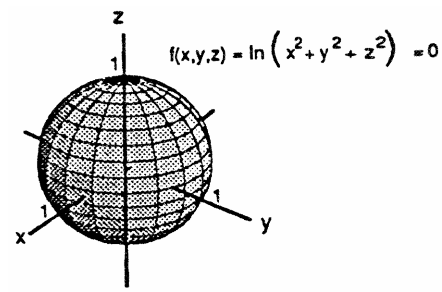
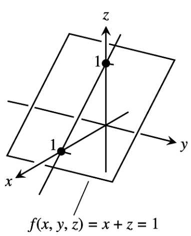
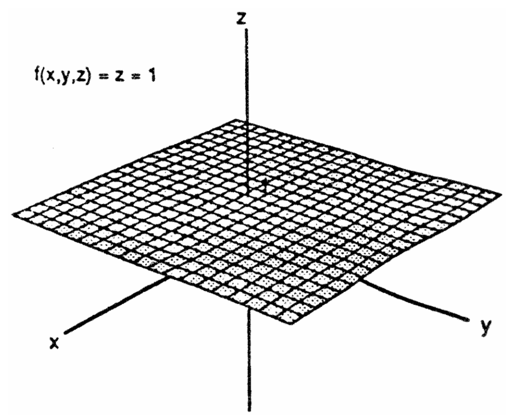


55.



56.



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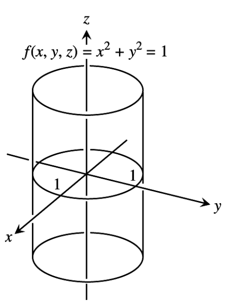
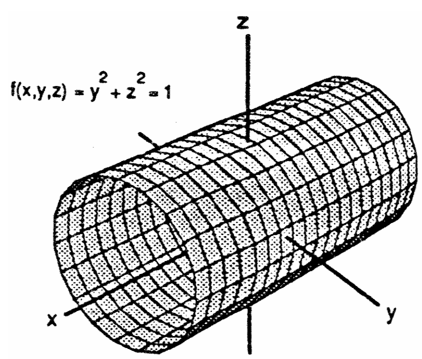
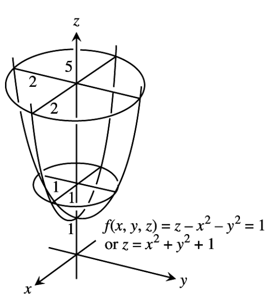


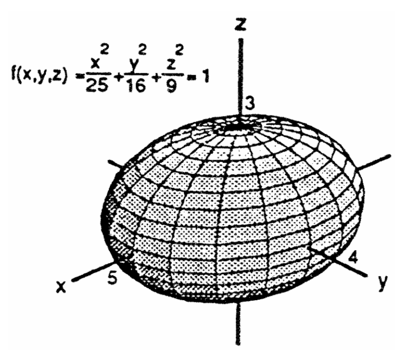
58.



59.



60.



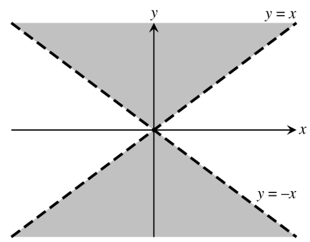
61. $f(x,y,z) = \sqrt{x-y} - \ln z$ at $(3,-1,1) \Rightarrow w = \sqrt{x-y} - \ln z$; at $(3,-1,1) \Rightarrow w = \sqrt{3-(-1)} - \ln 1 = 2$
$\Rightarrow \sqrt{x-y} - \ln z = 2$

62. $f(x,y,z) = \ln(x^2+y+z^2)$ at $(-1,2,1) \Rightarrow w = \ln(x^2+y+z^2)$; at $(-1,2,1) \Rightarrow w = \ln(1+2+1) = \ln 4$
$\Rightarrow \ln 4 = \ln(x^2+y+z^2) \Rightarrow x^2+y+z^2 = 4$

63. $g(x,y,z) = \sqrt{x^2+y^2+z^2}$ at $\left(1,-1,\sqrt{2}\right) \Rightarrow w = \sqrt{x^2+y^2+z^2}$; at $\left(1,-1,\sqrt{2}\right) \Rightarrow w = \sqrt{1^2+(-1)^2+\left(\sqrt{2}\right)^2}$
$= 2 \Rightarrow 2 = \sqrt{x^2+y^2+z^2} \Rightarrow x^2+y^2+z^2 = 4$

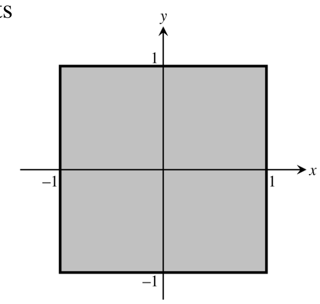
64. $g(x,y,z) = \frac{x-y+z}{2x+y-z}$ at $(1,0,-2) \Rightarrow w = \frac{x-y+z}{2x+y-z}$; at $(1,0,-2) \Rightarrow w = \frac{1-0+(-2)}{2(1)+0-(-2)} = -\frac{1}{4} \Rightarrow -\frac{1}{4} = \frac{x-y+z}{2x+y-z}$
$\Rightarrow 2x - y + z = 0$

65. $f(x,y) = \sum_{n=0}^{\infty} \left(\frac{x}{y}\right)^n = \frac{1}{1-\left(\frac{x}{y}\right)} = \frac{y}{y-x}$ for
$\left|\frac{x}{y}\right| < 1 \Rightarrow$ Domain: all points (x, y) satisfying $|x| < |y|$;
at $(1, 2) \Rightarrow$ since $\left|\frac{1}{2}\right| < 1 \Rightarrow z = \frac{2}{2-1} = 2$
$\Rightarrow \frac{y}{y-x} = 2 \Rightarrow y = 2x$



66. $g(x,y,z) = \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!\,z^n} = e^{(x+y)/z} \Rightarrow$ Domain: all points (x, y, z) satisfying $z \neq 0$; at $(\ln 4, \ln 9, 2)$
$\Rightarrow w = e^{(\ln 4 + \ln 9)/2} = e^{(\ln 36)/2} = e^{\ln 6} = 6 \Rightarrow 6 = e^{(x+y)/z} \Rightarrow \frac{x+y}{z} = \ln 6$

67. $f(x,y) = \int_x^y \frac{d\theta}{\sqrt{1-\theta^2}} = \sin^{-1}y - \sin^{-1}x \Rightarrow$ Domain: all points
(x, y) satisfying $-1 \le x \le 1$ and $-1 \le y \le 1$;
at $(0, 1) \Rightarrow \sin^{-1}1 - \sin^{-1}0 = \frac{\pi}{2} \Rightarrow \sin^{-1}y - \sin^{-1}x$
$= \frac{\pi}{2}$. Since $-\frac{\pi}{2} \le \sin^{-1}y \le \frac{\pi}{2}$ and $-\frac{\pi}{2} \le \sin^{-1}x \le \frac{\pi}{2}$, in
order for $\sin^{-1}y - \sin^{-1}x$ to equal $\frac{\pi}{2}$, $0 \le \sin^{-1}y \le \frac{\pi}{2}$ and
$-\frac{\pi}{2} \le \sin^{-1}x \le 0$; that is $0 \le y \le 1$ and $-1 \le x \le 0$. Thus
$y = \sin\left(\frac{\pi}{2} + \sin^{-1}x\right) = \sqrt{1-x^2}$, $x \le 0$



68. $g(x,y,z) = \int_x^y \frac{dt}{1+t^2} + \int_0^z \frac{d\theta}{\sqrt{4-\theta^2}} = \tan^{-1}y - \tan^{-1}x + \sin^{-1}\left(\frac{z}{2}\right) \Rightarrow$ Domain: all points (x, y, z) satisfying $-2 \le z \le 2$;
at $\left(0, 1, \sqrt{3}\right) \Rightarrow \tan^{-1}1 - \tan^{-1}0 + \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{7\pi}{12} \Rightarrow \tan^{-1}y - \tan^{-1}x + \sin^{-1}\left(\frac{z}{2}\right) = \frac{7\pi}{12}$. Since $-\frac{\pi}{2} \le \sin^{-1}\left(\frac{z}{2}\right) \le \frac{\pi}{2}$,
$\frac{\pi}{12} \le \tan^{-1}y - \tan^{-1}x \le \frac{13\pi}{12} \Rightarrow z = 2\sin\left(\frac{7\pi}{12} - \tan^{-1}y + \tan^{-1}x\right)$, $\frac{\pi}{12} \le \tan^{-1}y - \tan^{-1}x \le \frac{13\pi}{12}$

69-72. Example CAS commands:

Maple:

```
with( plots );
f := (x,y) -> x*sin(y/2) + y*sin(2*x);
xdomain := x=0..5*Pi;
ydomain := y=0..5*Pi;
x0,y0 := 3*Pi,3*Pi;
plot3d( f(x,y), xdomain, ydomain, axes=boxed, style=patch, shading=zhue, title="#69(a) (Section 14.1)" );
```

```
plot3d( f(x,y), xdomain, ydomain, grid=[50,50], axes=boxed, shading=zhue, style=patchcontour, orientation=[-90,0],
    title="#69(b) (Section 14.1)" );                                  # (b)
L := evalf( f(x0,y0) );                                  # (c)
plot3d( f(x,y), xdomain, ydomain, grid=[50,50], axes=boxed, shading=zhue, style=patchcontour, contours=[L],
    orientation=[-90,0], title="#45(c) (Section 13.1)" );
```

73-76. Example CAS commands:

Maple:

```
eq := 4*ln(x^2+y^2+z^2)=1;
implicitplot3d( eq, x=-2..2, y=-2..2, z=-2..2, grid=[30,30,30], axes=boxed, title="#73 (Section 14.1)" );
```

77-80. Example CAS commands:

Maple:

```
x := (u,v) -> u*cos(v);
y := (u,v) ->u*sin(v);
z := (u,v) -> u;
plot3d( [x(u,v),y(u,v),z(u,v)], u=0..2, v=0..2*Pi, axes=boxed, style=patchcontour, contours=[($0..4)/2], shading=zhue,
    title="#77 (Section 14.1)" );
```

69-60. Example CAS commands:

Mathematica: (assigned functions and bounds will vary)

For 69 - 72, the command **ContourPlot** draws 2-dimensional contours that are z-level curves of surfaces z = f(x,y).

```
Clear[x, y, f]
f[x_, y_]:= x Sin[y/2] + y Sin[2x]
xmin= 0; xmax= 5π; ymin= 0; ymax= 5π; {x0, y0}={3π, 3π};
cp= ContourPlot[f[x,y], {x, xmin, xmax}, {y, ymin, ymax}, ContourShading → False];
cp0= ContourPlot[[f[x,y], {x, xmin, xmax}, {y, ymin, ymax}, Contours → {f[x0,y0]}, ContourShading → False,
        PlotStyle → {RGBColor[1,0,0]}];
Show[cp, cp0]
```

For 73 - 76, the command **ContourPlot3D** will be used. Write the function f[x, y, z] so that when it is equated to zero, it represents the level surface given.

For 73, the problem associated with Log[0] can be avoided by rewriting the function as x2 + y2 +z2 - e1/4

```
Clear[x, y, z, f]
f[x_, y_, z_]:= x^2 + y^2 + z^2 − Exp[1/4]
ContourPlot3D[f[x, y, z], {x, −5, 5}, {y, −5, 5}, {z, −5, 5}, PlotPoints → {7, 7}];
```

For 77 - 80, the command ParametricPlot3D will be used. To get the z-level curves here, we solve x and y in terms of z and either u or v (v here), create a table of level curves, then plot that table.

```
Clear[x, y, z, u, v]
ParametricPlot3D[{u Cos[v], u Sin[v], u}, {u, 0, 2}, {v, 0, 2p}];
zlevel= Table[{z Cos[v], z sin[v]}, {z, 0, 2, .1}];
ParametricPlot[Evaluate[zlevel],{v, 0, 2π}];
```

14.2 LIMITS AND CONTINUITY IN HIGHER DIMENSIONS

1. $\lim\limits_{(x,y)\to(0,0)} \frac{3x^2-y^2+5}{x^2+y^2+2} = \frac{3(0)^2-0^2+5}{0^2+0^2+2} = \frac{5}{2}$

2. $\lim\limits_{(x,y)\to(0,4)} \frac{x}{\sqrt{y}} = \frac{0}{\sqrt{4}} = 0$

3. $\lim\limits_{(x,y)\to(3,4)} \sqrt{x^2+y^2-1} = \sqrt{3^2+4^2-1} = \sqrt{24} = 2\sqrt{6}$

4. $\lim\limits_{(x,y)\to(2,-3)} \left(\frac{1}{x}+\frac{1}{y}\right)^2 = \left[\frac{1}{2}+\left(\frac{1}{-3}\right)\right]^2 = \left(\frac{1}{6}\right)^2 = \frac{1}{36}$

5. $\lim\limits_{(x,y)\to\left(0,\frac{\pi}{4}\right)} \sec x \tan y = (\sec 0)\left(\tan\frac{\pi}{4}\right) = (1)(1) = 1$

6. $\lim\limits_{(x,y)\to(0,0)} \cos\left(\frac{x^2+y^3}{x+y+1}\right) = \cos\left(\frac{0^2+0^3}{0+0+1}\right) = \cos 0 = 1$

7. $\lim\limits_{(x,y)\to(0,\ln 2)} e^{x-y} = e^{0-\ln 2} = e^{\ln\left(\frac{1}{2}\right)} = \frac{1}{2}$

8. $\lim\limits_{(x,y)\to(1,1)} \ln\left|1+x^2y^2\right| = \ln\left|1+(1)^2(1)^2\right| = \ln 2$

9. $\lim\limits_{(x,y)\to(0,0)} \frac{e^y \sin x}{x} = \lim\limits_{(x,y)\to(0,0)} (e^y)\left(\frac{\sin x}{x}\right) = e^0 \cdot \lim\limits_{x\to 0}\left(\frac{\sin x}{x}\right) = 1\cdot 1 = 1$

10. $\lim\limits_{(x,y)\to(1/27,\pi^3)} \cos\sqrt[3]{xy} = \cos\sqrt[3]{\left(\frac{1}{27}\right)\pi^3} = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

11. $\lim\limits_{(x,y)\to(1,\pi/6)} \frac{x\sin y}{x^2+1} = \frac{1\cdot\sin\left(\frac{\pi}{6}\right)}{1^2+1} = \frac{1/2}{2} = \frac{1}{4}$

12. $\lim\limits_{(x,y)\to\left(\frac{\pi}{2},0\right)} \frac{\cos y+1}{y-\sin x} = \frac{(\cos 0)+1}{0-\sin\left(\frac{\pi}{2}\right)} = \frac{1+1}{-1} = -2$

13. $\lim\limits_{\substack{(x,y)\to(1,1)\\ x\neq y}} \frac{x^2-2xy+y^2}{x-y} = \lim\limits_{(x,y)\to(1,1)} \frac{(x-y)^2}{x-y} = \lim\limits_{(x,y)\to(1,1)} (x-y) = (1-1) = 0$

14. $\lim\limits_{\substack{(x,y)\to(1,1)\\ x\neq y}} \frac{x^2-y^2}{x-y} = \lim\limits_{(x,y)\to(1,1)} \frac{(x+y)(x-y)}{x-y} = \lim\limits_{(x,y)\to(1,1)} (x+y) = (1+1) = 2$

15. $\lim\limits_{\substack{(x,y)\to(1,1)\\ x\neq 1}} \frac{xy-y-2x+2}{x-1} = \lim\limits_{\substack{(x,y)\to(1,1)\\ x\neq 1}} \frac{(x-1)(y-2)}{x-1} = \lim\limits_{(x,y)\to(1,1)} (y-2) = (1-2) = -1$

16. $\lim\limits_{\substack{(x,y)\to(2,-4)\\ y\neq -4,\, x\neq x^2}} \frac{y+4}{x^2y-xy+4x^2-4x} = \lim\limits_{\substack{(x,y)\to(2,-4)\\ y\neq -4,\, x\neq x^2}} \frac{y+4}{x(x-1)(y+4)} = \lim\limits_{\substack{(x,y)\to(2,-4)\\ x\neq x^2}} \frac{1}{x(x-1)} = \frac{1}{2(2-1)} = \frac{1}{2}$

17. $\lim\limits_{\substack{(x,y)\to(0,0)\\ x\neq y}} \frac{x-y+2\sqrt{x}-2\sqrt{y}}{\sqrt{x}-\sqrt{y}} = \lim\limits_{\substack{(x,y)\to(0,0)\\ x\neq y}} \frac{\left(\sqrt{x}-\sqrt{y}\right)\left(\sqrt{x}+\sqrt{y}+2\right)}{\sqrt{x}-\sqrt{y}} = \lim\limits_{(x,y)\to(0,0)} \left(\sqrt{x}+\sqrt{y}+2\right)$

$= \left(\sqrt{0}+\sqrt{0}+2\right) = 2$

Note: (x, y) must approach $(0, 0)$ through the first quadrant only with $x \neq y$.

18. $\lim\limits_{\substack{(x,y)\to(2,2)\\ x+y\neq 4}} \frac{x+y-4}{\sqrt{x+y}-2} = \lim\limits_{\substack{(x,y)\to(2,2)\\ x+y\neq 4}} \frac{\left(\sqrt{x+y}+2\right)\left(\sqrt{x+y}-2\right)}{\sqrt{x+y}-2} = \lim\limits_{\substack{(x,y)\to(2,2)\\ x+y\neq 4}} \left(\sqrt{x+y}+2\right)$

$= \left(\sqrt{2+2}+2\right) = 2+2 = 4$

19. $\lim\limits_{\substack{(x,y)\to(2,0)\\ 2x-y\neq 4}} \frac{\sqrt{2x-y}-2}{2x-y-4} = \lim\limits_{\substack{(x,y)\to(2,0)\\ 2x-y\neq 4}} \frac{\sqrt{2x-y}-2}{\left(\sqrt{2x-y}+2\right)\left(\sqrt{2x-y}-2\right)} = \lim\limits_{(x,y)\to(2,0)} \frac{1}{\sqrt{2x-y}+2}$
$= \frac{1}{\sqrt{(2)(2)-0}+2} = \frac{1}{2+2} = \frac{1}{4}$

20. $\lim\limits_{\substack{(x,y)\to(4,3)\\ x-y\neq 1}} \frac{\sqrt{x}-\sqrt{y+1}}{x-y-1} = \lim\limits_{\substack{(x,y)\to(4,3)\\ x-y\neq 1}} \frac{\sqrt{x}-\sqrt{y+1}}{\left(\sqrt{x}+\sqrt{y+1}\right)\left(\sqrt{x}-\sqrt{y+1}\right)} = \lim\limits_{(x,y)\to(4,3)} \frac{1}{\sqrt{x}+\sqrt{y+1}}$
$= \frac{1}{\sqrt{4}+\sqrt{3+1}} = \frac{1}{2+2} = \frac{1}{4}$

21. $\lim\limits_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = \lim\limits_{r\to 0} \frac{\sin(r^2)}{r^2} = \lim\limits_{r\to 0} \frac{2r\cdot\cos(r^2)}{2r} = \lim\limits_{r\to 0} \cos(r^2) = 1$

22. $\lim\limits_{(x,y)\to(0,0)} \frac{1-\cos(xy)}{xy} = \lim\limits_{u\to 0} \frac{1-\cos u}{u} = \lim\limits_{u\to 0} \frac{\sin u}{1} = 0$

23. $\lim\limits_{(x,y)\to(1,-1)} \frac{x^3+y^3}{x+y} = \lim\limits_{(x,y)\to(1,-1)} \frac{(x+y)(x^2-xy+y^2)}{x+y} = \lim\limits_{(x,y)\to(1,-1)} (x^2-xy+y^2) = \left(1^2-(1)(-1)+(-1)^2\right) = 3$

24. $\lim\limits_{(x,y)\to(2,2)} \frac{x-y}{x^4-y^4} = \lim\limits_{(x,y)\to(2,2)} \frac{x-y}{(x+y)(x-y)(x^2+y^2)} = \lim\limits_{(x,y)\to(2,2)} \frac{1}{(x+y)(x^2+y^2)} = \frac{1}{(2+2)(2^2+2^2)} = \frac{1}{32}$

25. $\lim\limits_{P\to(1,3,4)} \left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) = \frac{1}{1}+\frac{1}{3}+\frac{1}{4} = \frac{12+4+3}{12} = \frac{19}{12}$

26. $\lim\limits_{P\to(1,-1,-1)} \frac{2xy+yz}{x^2+z^2} = \frac{2(1)(-1)+(-1)(-1)}{1^2+(-1)^2} = \frac{-2+1}{1+1} = -\frac{1}{2}$

27. $\lim\limits_{P\to(3,3,0)} (\sin^2 x+\cos^2 y+\sec^2 z) = (\sin^2 3+\cos^2 3)+\sec^2 0 = 1+1^2 = 2$

28. $\lim\limits_{P\to\left(-\frac{1}{4},\frac{\pi}{2},2\right)} \tan^{-1}(xyz) = \tan^{-1}\left(-\frac{1}{4}\cdot\frac{\pi}{2}\cdot 2\right) = \tan^{-1}\left(-\frac{\pi}{4}\right)$

29. $\lim\limits_{P\to(\pi,0,3)} ze^{-2y}\cos 2x = 3e^{-2(0)}\cos 2\pi = (3)(1)(1) = 3$

30. $\lim\limits_{P\to(2,-3,6)} \ln\sqrt{x^2+y^2+z^2} = \ln\sqrt{2^2+(-3)^2+6^2} = \ln\sqrt{49} = \ln 7$

31. (a) All (x, y) (b) All (x, y) except $(0, 0)$

32. (a) All (x, y) so that $x \neq y$ (b) All (x, y)

33. (a) All (x, y) except where $x = 0$ or $y = 0$ (b) All (x, y)

34. (a) All (x, y) so that $x^2 - 3x + 2 \neq 0 \Rightarrow (x-2)(x-1) \neq 0 \Rightarrow x \neq 2$ and $x \neq 1$
(b) All (x, y) so that $y \neq x^2$

35. (a) All (x, y, z) (b) All (x, y, z) except the interior of the cylinder $x^2 + y^2 = 1$

36. (a) All (x, y, z) so that $xyz > 0$ (b) All (x, y, z)

37. (a) All (x, y, z) with $z \neq 0$ (b) All (x, y, z) with $x^2 + z^2 \neq 1$

38. (a) All (x, y, z) except $(x, 0, 0)$ (b) All (x, y, z) except $(0, y, 0)$ or $(x, 0, 0)$

39. (a) All (x, y, z) such that $z > x^2 + y^2 + 1$ (b) All (x, y, z) such that $z \neq \sqrt{x^2 + y^2}$

40. (a) All (x, y, z) such that $x^2 + y^2 + z^2 \leq 4$
(b) All (x, y, z) such that $x^2 + y^2 + z^2 \geq 9$ except when $x^2 + y^2 + z^2 = 25$

41. $\lim\limits_{\substack{(x,y)\to(0,0) \\ \text{along } y = x \\ x > 0}} -\frac{x}{\sqrt{x^2+y^2}} = \lim\limits_{x \to 0^+} -\frac{x}{\sqrt{x^2+x^2}} = \lim\limits_{x \to 0^+} -\frac{x}{\sqrt{2}\,|x|} = \lim\limits_{x \to 0^+} -\frac{x}{\sqrt{2}\,x} = \lim\limits_{x \to 0^+} -\frac{1}{\sqrt{2}} = -\frac{1}{\sqrt{2}}$;
$\lim\limits_{\substack{(x,y)\to(0,0) \\ \text{along } y = x \\ x < 0}} -\frac{x}{\sqrt{x^2+y^2}} = \lim\limits_{x \to 0^-} -\frac{x}{\sqrt{2}\,|x|} = \lim\limits_{x \to 0^-} -\frac{x}{\sqrt{2}(-x)} = \lim\limits_{x \to 0^-} \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$

42. $\lim\limits_{\substack{(x,y)\to(0,0) \\ \text{along } y = 0}} \frac{x^4}{x^4+y^2} = \lim\limits_{x \to 0} \frac{x^4}{x^4+0^2} = 1$; $\lim\limits_{\substack{(x,y)\to(0,0) \\ \text{along } y = x^2}} \frac{x^4}{x^4+y^2} = \lim\limits_{x \to 0} \frac{x^4}{x^4+(x^2)^2} = \lim\limits_{x \to 0} \frac{x^4}{2x^4} = \frac{1}{2}$

43. $\lim\limits_{\substack{(x,y)\to(0,0) \\ \text{along } y = kx^2}} \frac{x^4-y^2}{x^4+y^2} = \lim\limits_{x \to 0} \frac{x^4-(kx^2)^2}{x^4+(kx^2)^2} = \lim\limits_{x \to 0} \frac{x^4-k^2x^4}{x^4+k^2x^4} = \frac{1-k^2}{1+k^2} \Rightarrow$ different limits for different values of k

44. $\lim\limits_{\substack{(x,y)\to(0,0) \\ \text{along } y = kx \\ k \neq 0}} \frac{xy}{|xy|} = \lim\limits_{x \to 0} \frac{x(kx)}{|x(kx)|} = \lim\limits_{x \to 0} \frac{kx^2}{|kx^2|} = \lim\limits_{x \to 0} \frac{k}{|k|}$; if $k > 0$, the limit is 1; but if $k < 0$, the limit is -1

45. $\lim\limits_{\substack{(x,y)\to(0,0) \\ \text{along } y = kx \\ k \neq -1}} \frac{x-y}{x+y} = \lim\limits_{x \to 0} \frac{x-kx}{x+kx} = \frac{1-k}{1+k} \Rightarrow$ different limits for different values of k, $k \neq -1$

46. $\lim\limits_{\substack{(x,y)\to(0,0) \\ \text{along } y = kx \\ k \neq 1}} \frac{x^2-y}{x-y} = \lim\limits_{x \to 0} \frac{x^2-kx}{x-kx} = \lim\limits_{x \to 0} \frac{x-k}{1-k} = \frac{-k}{1-k} \Rightarrow$ different limits for different values of k, $k \neq 1$

47. $\lim\limits_{\substack{(x,y)\to(0,0) \\ \text{along } y = kx^2 \\ k \neq 0}} \frac{x^2+y}{y} = \lim\limits_{x \to 0} \frac{x^2+kx^2}{kx^2} = \frac{1+k}{k} \Rightarrow$ different limits for different values of k, $k \neq 0$

48. $\lim\limits_{\substack{(x,y)\to(0,0) \\ \text{along } y = kx^2}} \frac{x^2y}{x^4+y^2} = \lim\limits_{x \to 0} \frac{kx^4}{x^4+k^2x^4} = \frac{k}{1+k^2} \Rightarrow$ different limits for different values of k

49. $\lim\limits_{\substack{(x,y)\to(1,1) \\ \text{along } x = 1}} \frac{xy^2-1}{y-1} = \lim\limits_{y \to 1} \frac{y^2-1}{y-1} = \lim\limits_{y \to 1} (y+1) = 2$; $\lim\limits_{\substack{(x,y)\to(1,1) \\ \text{along } y = x}} \frac{xy^2-1}{y-1} = \lim\limits_{y \to 1} \frac{y^3-1}{y-1} = \lim\limits_{y \to 1} (y^2+y+1) = 3$

50. $\lim\limits_{\substack{(x,y)\to(1,-1) \\ \text{along } y = -1}} \frac{xy+1}{x^2-y^2} = \lim\limits_{x \to 1} \frac{-x+1}{x^2-1} = \lim\limits_{x \to 1} \frac{-1}{x+1} = -\frac{1}{2}$; $\lim\limits_{\substack{(x,y)\to(1,-1) \\ \text{along } y = -x^2}} \frac{xy+1}{x^2-y^2} = \lim\limits_{x \to 1} \frac{-x^3+1}{x^2-x^4} = \lim\limits_{x \to 1} \frac{x^2+x+1}{(x+1)(x^2+1)}$
$= \frac{3}{2}$

51. $f(x, y) = \begin{cases} 1 & \text{if } y \geq x^4 \\ 1 & \text{if } y \leq 0 \\ 0 & \text{otherwise} \end{cases}$

(a) $\lim_{(x,y) \to (0,1)} f(x, y) = 1$ since any path through $(0, 1)$ that is close to $(0, 1)$ satisfies $y \geq x^4$

(b) $\lim_{(x,y) \to (2,3)} f(x, y) = 0$ since any path through $(2, 3)$ that is close to $(2, 3)$ does not satisfiy either $y \geq x^4$ or $y \leq 0$

(c) $\lim_{\substack{(x,y) \to (0,0) \\ \text{along } x = 0}} f(x, y) = 1$ and $\lim_{\substack{(x,y) \to (0,0) \\ \text{along } y = x^2}} f(x, y) = 0 \Rightarrow \lim_{(x,y) \to (0,0)} f(x, y)$ does not exist

52. $f(x, y) = \begin{cases} x^2 & \text{if } x \geq 0 \\ x^3 & \text{if } x < 0 \end{cases}$

(a) $\lim_{(x,y) \to (3,-2)} f(x, y) = 3^2 = 9$ since any path through $(3, -2)$ that is close to $(3, -2)$ satisfies $x \geq 0$

(b) $\lim_{(x,y) \to (-2,1)} f(x, y) = (-2)^3 = -8$ since any path through $(-2, 1)$ that is close to $(-2, 1)$ satisfies $x < 0$

(c) $\lim_{(x,y) \to (0,0)} f(x, y) = 0$ since the limit is 0 along any path through $(0, 0)$ with $x < 0$ and the limit is also zero along any path through $(0, 0)$ with $x \geq 0$

53. First consider the vertical line $x = 0 \Rightarrow \lim_{\substack{(x,y) \to (0,0) \\ \text{along } x = 0}} \frac{2x^2y}{x^4 + y^2} = \lim_{y \to 0} \frac{2(0)^2y}{(0)^4 + y^2} = \lim_{y \to 0} 0 = 0$. Now consider any nonvertical through $(0, 0)$. The equation of any line through $(0, 0)$ is of the form $y = mx \Rightarrow \lim_{\substack{(x,y) \to (0,0) \\ \text{along } y = mx}} f(x, y) = \lim_{\substack{(x,y) \to (0,0) \\ \text{along } y = mx}} \frac{2x^2y}{x^4 + y^2}$ $= \lim_{x \to 0} \frac{2x^2(mx)}{x^4 + (mx)^2} = \lim_{x \to 0} \frac{2mx^3}{x^4 + m^2x^2} = \lim_{x \to 0} \frac{2mx^3}{x^2(x^2 + m^2)} = \lim_{x \to 0} \frac{2mx}{(x^2 + m^2)} = 0$. Thus $\lim_{\substack{(x,y) \to (0,0) \\ \text{any line though } (0,0)}} \frac{2x^2y}{x^4 + y^2} = 0$.

54. If f is continuous at (x_0, y_0), then $\lim_{(x,y) \to (x_0,y_0)} f(x, y)$ must equal $f(x_0, y_0) = 3$. If f is not continuous at (x_0, y_0), the limit could have any value different from 3, and need not even exist.

55. $\lim_{(x,y) \to (0,0)} \left(1 - \frac{x^2y^2}{3}\right) = 1$ and $\lim_{(x,y) \to (0,0)} 1 = 1 \Rightarrow \lim_{(x,y) \to (0,0)} \frac{\tan^{-1} xy}{xy} = 1$, by the Sandwich Theorem

56. If $xy > 0$, $\lim_{(x,y) \to (0,0)} \frac{2|xy| - \left(\frac{x^2y^2}{6}\right)}{|xy|} = \lim_{(x,y) \to (0,0)} \frac{2xy - \left(\frac{x^2y^2}{6}\right)}{xy} = \lim_{(x,y) \to (0,0)} \left(2 - \frac{xy}{6}\right) = 2$ and $\lim_{(x,y) \to (0,0)} \frac{2|xy|}{|xy|} = \lim_{(x,y) \to (0,0)} 2 = 2$; if $xy < 0$, $\lim_{(x,y) \to (0,0)} \frac{2|xy| - \left(\frac{x^2y^2}{6}\right)}{|xy|} = \lim_{(x,y) \to (0,0)} \frac{-2xy - \left(\frac{x^2y^2}{6}\right)}{-xy}$ $= \lim_{(x,y) \to (0,0)} \left(2 + \frac{xy}{6}\right) = 2$ and $\lim_{(x,y) \to (0,0)} \frac{2|xy|}{|xy|} = 2 \Rightarrow \lim_{(x,y) \to (0,0)} \frac{4 - 4\cos\sqrt{|xy|}}{|xy|} = 2$, by the Sandwich Theorem

57. The limit is 0 since $\left|\sin\left(\frac{1}{x}\right)\right| \leq 1 \Rightarrow -1 \leq \sin\left(\frac{1}{x}\right) \leq 1 \Rightarrow -y \leq y\sin\left(\frac{1}{x}\right) \leq y$ for $y \geq 0$, and $-y \geq y\sin\left(\frac{1}{x}\right) \geq y$ for $y \leq 0$. Thus as $(x, y) \to (0, 0)$, both $-y$ and y approach $0 \Rightarrow y\sin\left(\frac{1}{x}\right) \to 0$, by the Sandwich Theorem.

58. The limit is 0 since $\left|\cos\left(\frac{1}{y}\right)\right| \leq 1 \Rightarrow -1 \leq \cos\left(\frac{1}{y}\right) \leq 1 \Rightarrow -x \leq x\cos\left(\frac{1}{y}\right) \leq x$ for $x \geq 0$, and $-x \geq x\cos\left(\frac{1}{y}\right) \geq x$ for $x \leq 0$. Thus as $(x, y) \to (0, 0)$, both $-x$ and x approach $0 \Rightarrow x\cos\left(\frac{1}{y}\right) \to 0$, by the Sandwich Theorem.

59. (a) $f(x, y)\big|_{y=mx} = \frac{2m}{1 + m^2} = \frac{2\tan\theta}{1 + \tan^2\theta} = \sin 2\theta$. The value of $f(x, y) = \sin 2\theta$ varies with θ, which is the line's angle of inclination.

(b) Since $f(x,y)|_{y=mx} = \sin 2\theta$ and since $-1 \le \sin 2\theta \le 1$ for every θ, $\lim_{(x,y)\to(0,0)} f(x,y)$ varies from -1 to 1 along $y = mx$.

60. $|xy\,(x^2-y^2)| = |xy|\,|x^2-y^2| \le |x|\,|y|\,|x^2+y^2| = \sqrt{x^2}\,\sqrt{y^2}\,|x^2+y^2| \le \sqrt{x^2+y^2}\,\sqrt{x^2+y^2}\,|x^2+y^2|$ $= (x^2+y^2)^2 \Rightarrow \left|\frac{xy\,(x^2-y^2)}{x^2+y^2}\right| \le \frac{(x^2+y^2)^2}{x^2+y^2} = x^2+y^2 \Rightarrow -(x^2+y^2) \le \frac{xy\,(x^2-y^2)}{x^2+y^2} \le (x^2+y^2)$ $\Rightarrow \lim_{(x,y)\to(0,0)} \left(xy\,\frac{x^2-y^2}{x^2+y^2}\right) = 0$ by the Sandwich Theorem, since $\lim_{(x,y)\to(0,0)} \pm(x^2+y^2) = 0$; thus, define $f(0,0) = 0$

61. $\lim_{(x,y)\to(0,0)} \frac{x^3-xy^2}{x^2+y^2} = \lim_{r\to 0} \frac{r^3\cos^3\theta - (r\cos\theta)(r^2\sin^2\theta)}{r^2\cos^2\theta + r^2\sin^2\theta} = \lim_{r\to 0} \frac{r\,(\cos^3\theta - \cos\theta\sin^2\theta)}{1} = 0$

62. $\lim_{(x,y)\to(0,0)} \cos\left(\frac{x^3-y^3}{x^2+y^2}\right) = \lim_{r\to 0} \cos\left(\frac{r^3\cos^3\theta - r^3\sin^3\theta}{r^2\cos^2\theta + r^2\sin^2\theta}\right) = \lim_{r\to 0} \cos\left[\frac{r\,(\cos^3\theta - \sin^3\theta)}{1}\right] = \cos 0 = 1$

63. $\lim_{(x,y)\to(0,0)} \frac{y^2}{x^2+y^2} = \lim_{r\to 0} \frac{r^2\sin^2\theta}{r^2} = \lim_{r\to 0} (\sin^2\theta) = \sin^2\theta$; the limit does not exist since $\sin^2\theta$ is between 0 and 1 depending on θ

64. $\lim_{(x,y)\to(0,0)} \frac{2x}{x^2+x+y^2} = \lim_{r\to 0} \frac{2r\cos\theta}{r^2+r\cos\theta} = \lim_{r\to 0} \frac{2\cos\theta}{r+\cos\theta} = \frac{2\cos\theta}{\cos\theta}$; the limit does not exist for $\cos\theta = 0$

65. $\lim_{(x,y)\to(0,0)} \tan^{-1}\left[\frac{|x|+|y|}{x^2+y^2}\right] = \lim_{r\to 0} \tan^{-1}\left[\frac{|r\cos\theta|+|r\sin\theta|}{r^2}\right] = \lim_{r\to 0} \tan^{-1}\left[\frac{|r|\,(|\cos\theta|+|\sin\theta|)}{r^2}\right]$; if $r \to 0^+$, then $\lim_{r\to 0^+} \tan^{-1}\left[\frac{|r|\,(|\cos\theta|+|\sin\theta|)}{r^2}\right] = \lim_{r\to 0^+} \tan^{-1}\left[\frac{|\cos\theta|+|\sin\theta|}{r}\right] = \frac{\pi}{2}$; if $r \to 0^-$, then $\lim_{r\to 0^-} \tan^{-1}\left[\frac{|r|\,(|\cos\theta|+|\sin\theta|)}{r^2}\right] = \lim_{r\to 0^-} \tan^{-1}\left(\frac{|\cos\theta|+|\sin\theta|}{-r}\right) = \frac{\pi}{2} \Rightarrow$ the limit is $\frac{\pi}{2}$

66. $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \lim_{r\to 0} \frac{r^2\cos^2\theta - r^2\sin^2\theta}{r^2} = \lim_{r\to 0} (\cos^2\theta - \sin^2\theta) = \lim_{r\to 0} (\cos 2\theta)$ which ranges between -1 and 1 depending on $\theta \Rightarrow$ the limit does not exist

67. $\lim_{(x,y)\to(0,0)} \ln\left(\frac{3x^2 - x^2y^2 + 3y^2}{x^2+y^2}\right) = \lim_{r\to 0} \ln\left(\frac{3r^2\cos^2\theta - r^4\cos^2\theta\sin^2\theta + 3r^2\sin^2\theta}{r^2}\right)$ $= \lim_{r\to 0} \ln(3 - r^2\cos^2\theta\sin^2\theta) = \ln 3 \Rightarrow$ define $f(0,0) = \ln 3$

68. $\lim_{(x,y)\to(0,0)} \frac{3xy^2}{x^2+y^2} = \lim_{r\to 0} \frac{(3r\cos\theta)(r^2\sin^2\theta)}{r^2} = \lim_{r\to 0} 3r\cos\theta\sin^2\theta = 0 \Rightarrow$ define $f(0,0) = 0$

69. Let $\delta = 0.1$. Then $\sqrt{x^2+y^2} < \delta \Rightarrow \sqrt{x^2+y^2} < 0.1 \Rightarrow x^2+y^2 < 0.01 \Rightarrow |x^2+y^2-0| < 0.01$ $\Rightarrow |f(x,y) - f(0,0)| < 0.01 = \epsilon$.

70. Let $\delta = 0.05$. Then $|x| < \delta$ and $|y| < \delta \Rightarrow |f(x,y) - f(0,0)| = \left|\frac{y}{x^2+1} - 0\right| = \left|\frac{y}{x^2+1}\right| \le |y| < 0.05 = \epsilon$.

71. Let $\delta = 0.005$. Then $|x| < \delta$ and $|y| < \delta \Rightarrow |f(x,y) - f(0,0)| = \left|\frac{x+y}{x^2+1} - 0\right| = \left|\frac{x+y}{x^2+1}\right| \le |x+y| < |x|+|y|$ $< 0.005 + 0.005 = 0.01 = \epsilon$.

72. Let $\delta = 0.01$. Since $-1 \le \cos x \le 1 \Rightarrow 1 \le 2+\cos x \le 3 \Rightarrow \frac{1}{3} \le \frac{1}{2+\cos x} \le 1 \Rightarrow \frac{|x+y|}{3} \le \left|\frac{x+y}{2+\cos x}\right| \le |x+y|$ $\le |x|+|y|$. Then $|x| < \delta$ and $|y| < \delta \Rightarrow |f(x,y) - f(0,0)| = \left|\frac{x+y}{2+\cos x} - 0\right| = \left|\frac{x+y}{2+\cos x}\right| \le |x|+|y| < 0.01 + 0.01$ $= 0.02 = \epsilon$.

73. Let $\delta = 0.04$. Since $y^2 \le x^2 + y^2 \Rightarrow \frac{y^2}{x^2+y^2} \le 1 \Rightarrow \frac{|x|y^2}{x^2+y^2} \le |x| = \sqrt{x^2} \le \sqrt{x^2+y^2} < \delta \Rightarrow |f(x,y) - f(0,0)|$
$= \left|\frac{x\,y^2}{x^2+y^2} - 0\right| < 0.04 = \epsilon$.

74. Let $\delta = 0.01$. If $|y| \le 1$, then $y^2 \le |y| = \sqrt{y^2} \le \sqrt{x^2+y^2}$, so $|x| = \sqrt{x^2} \le \sqrt{x^2+y^2} \Rightarrow |x| + y^2 \le 2\sqrt{x^2+y^2}$. Since $x^2 \le x^2 + y^2 \Rightarrow \frac{x^2}{x^2+y^2} \le 1$ and $y^2 \le x^2 + y^2 \Rightarrow \frac{y^2}{x^2+y^2} \le 1$. Then $\frac{|x^3+y^4|}{x^2+y^2} \le \frac{x^2}{x^2+y^2}|x| + \frac{y^2}{x^2+y^2}y^2 \le |x| + y^2 < 2\delta$
$\Rightarrow |f(x,y) - f(0,0)| = \left|\frac{x^3+y^4}{x^2+y^2} - 0\right| < 2(0.01) = 0.002 = \epsilon$.

75. Let $\delta = \sqrt{0.015}$. Then $\sqrt{x^2+y^2+z^2} < \delta \Rightarrow |f(x,y,z) - f(0,0,0)| = |x^2+y^2+z^2 - 0| = |x^2+y^2+z^2|$
$= \left(\sqrt{x^2+t^2+x^2}\right)^2 < \left(\sqrt{0.015}\right)^2 = 0.015 = \epsilon$.

76. Let $\delta = 0.2$. Then $|x| < \delta$, $|y| < \delta$, and $|z| < \delta \Rightarrow |f(x,y,z) - f(0,0,0)| = |xyz - 0| = |xyz| = |x|\,|y|\,|z| < (0.2)^3$
$= 0.008 = \epsilon$.

77. Let $\delta = 0.005$. Then $|x| < \delta$, $|y| < \delta$, and $|z| < \delta \Rightarrow |f(x,y,z) - f(0,0,0)| = \left|\frac{x+y+z}{x^2+y^2+z^2+1} - 0\right|$
$= \left|\frac{x+y+z}{x^2+y^2+z^2+1}\right| \le |x+y+z| \le |x| + |y| + |z| < 0.005 + 0.005 + 0.005 = 0.015 = \epsilon$.

78. Let $\delta = \tan^{-1}(0.1)$. Then $|x| < \delta$, $|y| < \delta$, and $|z| < \delta \Rightarrow |f(x,y,z) - f(0,0,0)| = |\tan^2 x + \tan^2 y + \tan^2 z|$
$\le |\tan^2 x| + |\tan^2 y| + |\tan^2 z| = \tan^2 x + \tan^2 y + \tan^2 z < \tan^2\delta + \tan^2\delta + \tan^2\delta = 0.01 + 0.01 + 0.01 = 0.03 = \epsilon$.

79. $\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z) = \lim_{(x,y,z)\to(x_0,y_0,z_0)} (x + y - z) = x_0 + y_0 - z_0 = f(x_0,y_0,z_0) \Rightarrow$ f is continuous at every (x_0,y_0,z_0)

80. $\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z) = \lim_{(x,y,z)\to(x_0,y_0,z_0)} (x^2+y^2+z^2) = x_0^2 + y_0^2 + z_0^2 = f(x_0,y_0,z_0) \Rightarrow$ f is continuous at every point (x_0,y_0,z_0)

14.3 PARTIAL DERIVATIVES

1. $\frac{\partial f}{\partial x} = 4x$, $\frac{\partial f}{\partial y} = -3$

2. $\frac{\partial f}{\partial x} = 2x - y$, $\frac{\partial f}{\partial y} = -x + 2y$

3. $\frac{\partial f}{\partial x} = 2x(y+2)$, $\frac{\partial f}{\partial y} = x^2 - 1$

4. $\frac{\partial f}{\partial x} = 5y - 14x + 3$, $\frac{\partial f}{\partial y} = 5x - 2y - 6$

5. $\frac{\partial f}{\partial x} = 2y(xy-1)$, $\frac{\partial f}{\partial y} = 2x(xy-1)$

6. $\frac{\partial f}{\partial x} = 6(2x-3y)^2$, $\frac{\partial f}{\partial y} = -9(2x-3y)^2$

7. $\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}$, $\frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$

8. $\frac{\partial f}{\partial x} = \frac{2x^2}{\sqrt[3]{x^3+\left(\frac{y}{2}\right)}}$, $\frac{\partial f}{\partial y} = \frac{1}{3\sqrt[3]{x^3+\left(\frac{y}{2}\right)}}$

9. $\frac{\partial f}{\partial x} = -\frac{1}{(x+y)^2} \cdot \frac{\partial}{\partial x}(x+y) = -\frac{1}{(x+y)^2}$, $\frac{\partial f}{\partial y} = -\frac{1}{(x+y)^2} \cdot \frac{\partial}{\partial y}(x+y) = -\frac{1}{(x+y)^2}$

10. $\frac{\partial f}{\partial x} = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$, $\frac{\partial f}{\partial y} = \frac{(x^2+y^2)(0) - x(2y)}{(x^2+y^2)^2} = -\frac{2xy}{(x^2+y^2)^2}$

11. $\frac{\partial f}{\partial x} = = \frac{(xy-1)(1) - (x+y)(y)}{(xy-1)^2} = \frac{-y^2-1}{(xy-1)^2}$, $\frac{\partial f}{\partial y} = \frac{(xy-1)(1) - (x+y)(x)}{(xy-1)^2} = \frac{-x^2-1}{(xy-1)^2}$

12. $\frac{\partial f}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{\partial}{\partial x}\left(\frac{y}{x}\right) = -\frac{y}{x^2\left[1+\left(\frac{y}{x}\right)^2\right]} = -\frac{y}{x^2+y^2}$, $\frac{\partial f}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{\partial}{\partial y}\left(\frac{y}{x}\right) = \frac{1}{x\left[1+\left(\frac{y}{x}\right)^2\right]} = \frac{x}{x^2+y^2}$

13. $\frac{\partial f}{\partial x} = e^{(x+y+1)} \cdot \frac{\partial}{\partial x}(x+y+1) = e^{(x+y+1)}$, $\frac{\partial f}{\partial y} = e^{(x+y+1)} \cdot \frac{\partial}{\partial y}(x+y+1) = e^{(x+y+1)}$

14. $\frac{\partial f}{\partial x} = -e^{-x}\sin(x+y) + e^{-x}\cos(x+y)$, $\frac{\partial f}{\partial y} = e^{-x}\cos(x+y)$

15. $\frac{\partial f}{\partial x} = \frac{1}{x+y} \cdot \frac{\partial}{\partial x}(x+y) = \frac{1}{x+y}$, $\frac{\partial f}{\partial y} = \frac{1}{x+y} \cdot \frac{\partial}{\partial y}(x+y) = \frac{1}{x+y}$

16. $\frac{\partial f}{\partial x} = e^{xy} \cdot \frac{\partial}{\partial x}(xy) \cdot \ln y = ye^{xy}\ln y$, $\frac{\partial f}{\partial y} = e^{xy} \cdot \frac{\partial}{\partial y}(xy) \cdot \ln y + e^{xy} \cdot \frac{1}{y} = xe^{xy}\ln y + \frac{e^{xy}}{y}$

17. $\frac{\partial f}{\partial x} = 2\sin(x-3y) \cdot \frac{\partial}{\partial x}\sin(x-3y) = 2\sin(x-3y)\cos(x-3y) \cdot \frac{\partial}{\partial x}(x-3y) = 2\sin(x-3y)\cos(x-3y)$,
$\frac{\partial f}{\partial y} = 2\sin(x-3y) \cdot \frac{\partial}{\partial y}\sin(x-3y) = 2\sin(x-3y)\cos(x-3y) \cdot \frac{\partial}{\partial y}(x-3y) = -6\sin(x-3y)\cos(x-3y)$

18. $\frac{\partial f}{\partial x} = 2\cos(3x-y^2) \cdot \frac{\partial}{\partial x}\cos(3x-y^2) = -2\cos(3x-y^2)\sin(3x-y^2) \cdot \frac{\partial}{\partial x}(3x-y^2)$
$= -6\cos(3x-y^2)\sin(3x-y^2)$,
$\frac{\partial f}{\partial y} = 2\cos(3x-y^2) \cdot \frac{\partial}{\partial y}\cos(3x-y^2) = -2\cos(3x-y^2)\sin(3x-y^2) \cdot \frac{\partial}{\partial y}(3x-y^2)$
$= 4y\cos(3x-y^2)\sin(3x-y^2)$

19. $\frac{\partial f}{\partial x} = yx^{y-1}$, $\frac{\partial f}{\partial y} = x^y \ln x$

20. $f(x,y) = \frac{\ln x}{\ln y} \Rightarrow \frac{\partial f}{\partial x} = \frac{1}{x\ln y}$ and $\frac{\partial f}{\partial y} = \frac{-\ln x}{y(\ln y)^2}$

21. $\frac{\partial f}{\partial x} = -g(x)$, $\frac{\partial f}{\partial y} = g(y)$

22. $f(x,y) = \sum_{n=0}^{\infty}(xy)^n$, $|xy| < 1 \Rightarrow f(x,y) = \frac{1}{1-xy} \Rightarrow \frac{\partial f}{\partial x} = -\frac{1}{(1-xy)^2} \cdot \frac{\partial}{\partial x}(1-xy) = \frac{y}{(1-xy)^2}$ and
$\frac{\partial f}{\partial y} = -\frac{1}{(1-xy)^2} \cdot \frac{\partial}{\partial y}(1-xy) = \frac{x}{(1-xy)^2}$

23. $f_x = y^2$, $f_y = 2xy$, $f_z = -4z$

24. $f_x = y+z$, $f_y = x+z$, $f_z = y+x$

25. $f_x = 1$, $f_y = -\frac{y}{\sqrt{y^2+z^2}}$, $f_z = -\frac{z}{\sqrt{y^2+z^2}}$

26. $f_x = -x(x^2+y^2+z^2)^{-3/2}$, $f_y = -y(x^2+y^2+z^2)^{-3/2}$, $f_z = -z(x^2+y^2+z^2)^{-3/2}$

27. $f_x = \frac{yz}{\sqrt{1-x^2y^2z^2}}$, $f_y = \frac{xz}{\sqrt{1-x^2y^2z^2}}$, $f_z = \frac{xy}{\sqrt{1-x^2y^2z^2}}$

28. $f_x = \frac{1}{|x+yz|\sqrt{(x+yz)^2-1}}$, $f_y = \frac{z}{|x+yz|\sqrt{(x+yz)^2-1}}$, $f_z = \frac{y}{|x+yz|\sqrt{(x+yz)^2-1}}$

29. $f_x = \frac{1}{x+2y+3z}$, $f_y = \frac{2}{x+2y+3z}$, $f_z = \frac{3}{x+2y+3z}$

30. $f_x = yz \cdot \frac{1}{xy} \cdot \frac{\partial}{\partial x}(xy) = \frac{(yz)(y)}{xy} = \frac{yz}{x}$, $f_y = z\ln(xy) + yz \cdot \frac{\partial}{\partial y}\ln(xy) = z\ln(xy) + \frac{yz}{xy} \cdot \frac{\partial}{\partial y}(xy) = z\ln(xy) + z$,
$f_z = y\ln(xy) + yz \cdot \frac{\partial}{\partial z}\ln(xy) = y\ln(xy)$

31. $f_x = -2xe^{-(x^2+y^2+z^2)}$, $f_y = -2ye^{-(x^2+y^2+z^2)}$, $f_z = -2ze^{-(x^2+y^2+z^2)}$

32. $f_x = -yze^{-xyz}$, $f_y = -xze^{-xyz}$, $f_z = -xye^{-xyz}$

33. $f_x = \text{sech}^2(x + 2y + 3z)$, $f_y = 2\,\text{sech}^2(x + 2y + 3z)$, $f_z = 3\,\text{sech}^2(x + 2y + 3z)$

34. $f_x = y\cosh(xy - z^2)$, $f_y = x\cosh(xy - z^2)$, $f_z = -2z\cosh(xy - z^2)$

35. $\frac{\partial f}{\partial t} = -2\pi\sin(2\pi t - \alpha)$, $\frac{\partial f}{\partial \alpha} = \sin(2\pi t - \alpha)$

36. $\frac{\partial g}{\partial u} = v^2 e^{(2u/v)} \cdot \frac{\partial}{\partial u}\left(\frac{2u}{v}\right) = 2ve^{(2u/v)}$, $\frac{\partial g}{\partial v} = 2ve^{(2u/v)} + v^2 e^{(2u/v)} \cdot \frac{\partial}{\partial v}\left(\frac{2u}{v}\right) = 2ve^{(2u/v)} - 2ue^{(2u/v)}$

37. $\frac{\partial h}{\partial \rho} = \sin\phi\cos\theta$, $\frac{\partial h}{\partial \phi} = \rho\cos\phi\cos\theta$, $\frac{\partial h}{\partial \theta} = -\rho\sin\phi\sin\theta$

38. $\frac{\partial g}{\partial r} = 1 - \cos\theta$, $\frac{\partial g}{\partial \theta} = r\sin\theta$, $\frac{\partial g}{\partial z} = -1$

39. $W_p = V$, $W_v = P + \frac{\delta v^2}{2g}$, $W_\delta = \frac{Vv^2}{2g}$, $W_v = \frac{2V\delta v}{2g} = \frac{V\delta v}{g}$, $W_g = -\frac{V\delta v^2}{2g^2}$

40. $\frac{\partial A}{\partial c} = m$, $\frac{\partial A}{\partial h} = \frac{q}{2}$, $\frac{\partial A}{\partial k} = \frac{m}{q}$, $\frac{\partial A}{\partial m} = \frac{k}{q} + c$, $\frac{\partial A}{\partial q} = -\frac{km}{q^2} + \frac{h}{2}$

41. $\frac{\partial f}{\partial x} = 1 + y$, $\frac{\partial f}{\partial y} = 1 + x$, $\frac{\partial^2 f}{\partial x^2} = 0$, $\frac{\partial^2 f}{\partial y^2} = 0$, $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = 1$

42. $\frac{\partial f}{\partial x} = y\cos xy$, $\frac{\partial f}{\partial y} = x\cos xy$, $\frac{\partial^2 f}{\partial x^2} = -y^2\sin xy$, $\frac{\partial^2 f}{\partial y^2} = -x^2\sin xy$, $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \cos xy - xy\sin xy$

43. $\frac{\partial g}{\partial x} = 2xy + y\cos x$, $\frac{\partial g}{\partial y} = x^2 - \sin y + \sin x$, $\frac{\partial^2 g}{\partial x^2} = 2y - y\sin x$, $\frac{\partial^2 g}{\partial y^2} = -\cos y$, $\frac{\partial^2 g}{\partial y \partial x} = \frac{\partial^2 g}{\partial x \partial y} = 2x + \cos x$

44. $\frac{\partial h}{\partial x} = e^y$, $\frac{\partial h}{\partial y} = xe^y + 1$, $\frac{\partial^2 h}{\partial x^2} = 0$, $\frac{\partial^2 h}{\partial y^2} = xe^y$, $\frac{\partial^2 h}{\partial y \partial x} = \frac{\partial^2 h}{\partial x \partial y} = e^y$

45. $\frac{\partial r}{\partial x} = \frac{1}{x+y}$, $\frac{\partial r}{\partial y} = \frac{1}{x+y}$, $\frac{\partial^2 r}{\partial x^2} = \frac{-1}{(x+y)^2}$, $\frac{\partial^2 r}{\partial y^2} = \frac{-1}{(x+y)^2}$, $\frac{\partial^2 r}{\partial y \partial x} = \frac{\partial^2 r}{\partial x \partial y} = \frac{-1}{(x+y)^2}$

46. $\frac{\partial s}{\partial x} = \left[\frac{1}{1+\left(\frac{y}{x}\right)^2}\right] \cdot \frac{\partial}{\partial x}\left(\frac{y}{x}\right) = \left(-\frac{y}{x^2}\right)\left[\frac{1}{1+\left(\frac{y}{x}\right)^2}\right] = \frac{-y}{x^2+y^2}$, $\frac{\partial s}{\partial y} = \left[\frac{1}{1+\left(\frac{y}{x}\right)^2}\right] \cdot \frac{\partial}{\partial y}\left(\frac{y}{x}\right) = \left(\frac{1}{x}\right)\left[\frac{1}{1+\left(\frac{y}{x}\right)^2}\right] = \frac{x}{x^2+y^2}$,
$\frac{\partial^2 s}{\partial x^2} = \frac{y(2x)}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2}$, $\frac{\partial^2 s}{\partial y^2} = \frac{-x(2y)}{(x^2+y^2)^2} = -\frac{2xy}{(x^2+y^2)^2}$, $\frac{\partial^2 s}{\partial y \partial x} = \frac{\partial^2 s}{\partial x \partial y} = \frac{(x^2+y^2)(-1)+y(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$

47. $\frac{\partial w}{\partial x} = 2x\tan(xy) + x^2\sec^2(xy) \cdot y = 2x\tan(xy) + x^2y\sec^2(xy)$, $\frac{\partial w}{\partial y} = x^2\sec^2(xy) \cdot x = x^3\sec^2(xy)$,
$\frac{\partial^2 w}{\partial x^2} = 2\tan(xy) + 2x\sec^2(xy) \cdot y + 2xy\sec^2(xy) + x^2y\,(2\sec(xy)\sec(xy)\tan(xy) \cdot y)$
$= 2\tan(xy) + 4xy\sec^2(xy) + 2x^2y^2\sec^2(xy)\tan(xy)$, $\frac{\partial^2 w}{\partial y^2} = x^3(2\sec(xy)\sec(xy)\tan(xy) \cdot x) = 2x^4\sec^2(xy)\tan(xy)$
$\frac{\partial^2 w}{\partial y \partial x} = \frac{\partial^2 w}{\partial x \partial y} = 3x^2\sec^2(xy) + x^3(2\sec(xy)\sec(xy)\tan(xy) \cdot y) = 3x^2\sec^2(xy) + x^3y\sec^2(xy)\tan(xy)$

48. $\frac{\partial w}{\partial x} = ye^{x^2-y} \cdot 2x = 2xy\,e^{x^2-y}$, $\frac{\partial w}{\partial y} = (1)e^{x^2-y} + ye^{x^2-y} \cdot (-1) = e^{x^2-y}(1-y)$,
$\frac{\partial^2 w}{\partial x^2} = 2y\,e^{x^2-y} + 2xy\left(e^{x^2-y} \cdot 2x\right) = 2ye^{x^2-y}(1+2x^2)$, $\frac{\partial^2 w}{\partial y^2} = \left(e^{x^2-y} \cdot (-1)\right)(1-y) + e^{x^2-y}(-1)$
$= e^{x^2-y}(y-2)$, $\frac{\partial^2 w}{\partial y \partial x} = \frac{\partial^2 w}{\partial x \partial y} = \left(e^{x^2-y} \cdot 2x\right)(1-y) = 2x\,e^{x^2-y}(1-y)$

49. $\frac{\partial w}{\partial x} = \sin(x^2y) + x\cos(x^2y) \cdot 2xy = \sin(x^2y) + 2x^2y\cos(x^2y)$, $\frac{\partial w}{\partial y} = x\cos(x^2y) \cdot x^2 = x^3\cos(x^2y)$,
$\frac{\partial^2 w}{\partial x^2} = \cos(x^2y) \cdot 2xy + 4xy\cos(x^2y) - 2x^2y\sin(x^2y) \cdot 2xy = 6xy\cos(x^2y) - 4x^3y^2\sin(x^2y)$,
$\frac{\partial^2 w}{\partial y^2} = -x^3\sin(x^2y) \cdot x^2 = -x^5\sin(x^2y)$, $\frac{\partial^2 w}{\partial y \partial x} = \frac{\partial^2 w}{\partial x \partial y} = 3x^2\cos(x^2y) - x^3\sin(x^2y) \cdot 2xy = 3x^2\cos(x^2y) - 2x^4y\sin(x^2y)$

50. $\frac{\partial w}{\partial x} = \frac{(x^2+y)-(x-y)(2x)}{(x^2+y)^2} = \frac{-x^2+2xy+y}{(x^2+y)^2}$, $\frac{\partial w}{\partial y} = \frac{(x^2+y)(-1)-(x-y)}{(x^2+y)^2} = \frac{-x^2-x}{(x^2+y)^2}$,
$\frac{\partial^2 w}{\partial x^2} = \frac{(x^2+y)^2(-2x+2y)-(-x^2+2xy+y)2(x^2+y)(2x)}{\left[(x^2+y)^2\right]^2} = \frac{2(x^3-3x^2y-3xy+y^2)}{(x^2+y)^3}$,
$\frac{\partial^2 w}{\partial y^2} = \frac{(x^2+y)^2\cdot 0-(-x^2-x)2(x^2+y)\cdot 1}{\left[(x^2+y)^2\right]^2} = \frac{2x^2+2x}{(x^2+y)^3}$, $\frac{\partial^2 w}{\partial y \partial x} = \frac{\partial^2 w}{\partial x \partial y} = \frac{(x^2+y)^2(2x+1)-(-x^2+2xy+y)2(x^2+y)\cdot 1}{\left[(x^2+y)^2\right]^2}$
$= \frac{2x^3+3x^2-2xy-y}{(x^2+y)^3}$

51. $\frac{\partial w}{\partial x} = \frac{2}{2x+3y}$, $\frac{\partial w}{\partial y} = \frac{3}{2x+3y}$, $\frac{\partial^2 w}{\partial y \partial x} = \frac{-6}{(2x+3y)^2}$, and $\frac{\partial^2 w}{\partial x \partial y} = \frac{-6}{(2x+3y)^2}$

52. $\frac{\partial w}{\partial x} = e^x + \ln y + \frac{y}{x}$, $\frac{\partial w}{\partial y} = \frac{x}{y} + \ln x$, $\frac{\partial^2 w}{\partial y \partial x} = \; = \frac{1}{y} + \frac{1}{x}$, and $\frac{\partial^2 w}{\partial x \partial y} = \frac{1}{y} + \frac{1}{x}$

53. $\frac{\partial w}{\partial x} = y^2 + 2xy^3 + 3x^2y^4$, $\frac{\partial w}{\partial y} = 2xy + 3x^2y^2 + 4x^3y^3$, $\frac{\partial^2 w}{\partial y \partial x} = 2y + 6xy^2 + 12x^2y^3$, and $\frac{\partial^2 w}{\partial x \partial y} = 2y + 6xy^2 + 12x^2y^3$

54. $\frac{\partial w}{\partial x} = \sin y + y \cos x + y$, $\frac{\partial w}{\partial y} = x \cos y + \sin x + x$, $\frac{\partial^2 w}{\partial y \partial x} = \cos y + \cos x + 1$, and $\frac{\partial^2 w}{\partial x \partial y} = \cos y + \cos x + 1$

55. (a) x first (b) y first (c) x first (d) x first (e) y first (f) y first

56. (a) y first three times (b) y first three times (c) y first twice (d) x first twice

57. $f_x(1,2) = \lim_{h \to 0} \frac{f(1+h,2)-f(1,2)}{h} = \lim_{h \to 0} \frac{[1-(1+h)+2-6(1+h)^2]-(2-6)}{h} = \lim_{h \to 0} \frac{-h-6(1+2h+h^2)+6}{h}$
$= \lim_{h \to 0} \frac{-13h-6h^2}{h} = \lim_{h \to 0} (-13-6h) = -13$,
$f_y(1,2) = \lim_{h \to 0} \frac{f(1,2+h)-f(1,2)}{h} = \lim_{h \to 0} \frac{[1-1+(2+h)-3(2+h)]-(2-6)}{h} = \lim_{h \to 0} \frac{(2-6-2h)-(2-6)}{h}$
$= \lim_{h \to 0} (-2) = -2$

58. $f_x(-2,1) = \lim_{h \to 0} \frac{f(-2+h,1)-f(-2,1)}{h} = \lim_{h \to 0} \frac{[4+2(-2+h)-3-(-2+h)]-(-3+2)}{h}$
$= \lim_{h \to 0} \frac{(2h-1-h)+1}{h} = \lim_{h \to 0} 1 = 1$,
$f_y(-2,1) = \lim_{h \to 0} \frac{f(-2,1+h)-f(-2,1)}{h} = \lim_{h \to 0} \frac{[4-4-3(1+h)+2(1+h)^2]-(-3+2)}{h}$
$= \lim_{h \to 0} \frac{(-3-3h+2+4h+2h^2)+1}{h} = \lim_{h \to 0} \frac{h+2h^2}{h} = \lim_{h \to 0} (1+2h) = 1$

59. $f_x(-2,3) = \lim_{h \to 0} \frac{f(-2+h,3)-f(-2,3)}{h} = \lim_{h \to 0} \frac{\sqrt{2(-2+h)+9-1}-\sqrt{-4+9-1}}{h}$
$= \lim_{h \to 0} \frac{\sqrt{2h+4}-2}{h} = \lim_{h \to 0} \left(\frac{\sqrt{2h+4}-2}{h} \cdot \frac{\sqrt{2h+4}+2}{\sqrt{2h+4}+2}\right) = \lim_{h \to 0} \frac{2}{\sqrt{2h+4}+2} = \frac{1}{2}$,
$f_y(-2,3) = \lim_{h \to 0} \frac{f(-2,3+h)-f(-2,3)}{h} = \lim_{h \to 0} \frac{\sqrt{-4+3(3+h)-1}-\sqrt{-4+9-1}}{h}$
$= \lim_{h \to 0} \frac{\sqrt{3h+4}-2}{h} = \lim_{h \to 0} \left(\frac{\sqrt{3h+4}-2}{h} \cdot \frac{\sqrt{3h+4}+2}{\sqrt{3h+4}+2}\right) = \lim_{h \to 0} \frac{3}{\sqrt{2h+4}+2} = \frac{3}{4}$

60. $f_x(0,0) = \lim_{h \to 0} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h \to 0} \frac{\frac{\sin(h^3+0)}{h^2+0}-0}{h} = \lim_{h \to 0} \frac{\sin h^3}{h^3} = 1$
$f_y(0,0) = \lim_{h \to 0} \frac{f(0,0+h)-f(0,0)}{h} = \lim_{h \to 0} \frac{\frac{\sin(0+h^4)}{0+h^2}-0}{h} = \lim_{h \to 0} \frac{\sin h^4}{h^3} = \lim_{h \to 0} \left(h \cdot \frac{\sin h^4}{h^4}\right) = 0 \cdot 1 = 0$

61. (a) In the plane $x = 2 \Rightarrow f_y(x,y) = 3 \Rightarrow f_y(2,-1) = 3 \Rightarrow m = 3$
(b) In the plane $y = -1 \Rightarrow f_x(x,y) = 2 \Rightarrow f_y(2,-1) = 2 \Rightarrow m = 2$

62. (a) In the plane $x = -1 \Rightarrow f_y(x,y) = 3y^2 \Rightarrow f_y(-1,1) = 3(1)^2 = 3 \Rightarrow m = 3$
(b) In the plane $y = 1 \Rightarrow f_x(x,y) = 2x \Rightarrow f_y(-1,1) = 2(-1) = -2 \Rightarrow m = -2$

63. $f_z(x_0, y_0, z_0) = \lim_{h \to 0} \frac{f(x_0, y_0, z_0 + h) - f(x_0, y_0, z_0)}{h}$;
$f_z(1,2,3) = \lim_{h \to 0} \frac{f(1,2,3+h) - f(1,2,3)}{h} = \lim_{h \to 0} \frac{2(3+h)^2 - 2(9)}{h} = \lim_{h \to 0} \frac{12h + 2h^2}{h} = \lim_{h \to 0} (12 + 2h) = 12$

64. $f_y(x_0, y_0, z_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h, z_0) - f(x_0, y_0, z_0)}{h}$;
$f_y(-1,0,3) = \lim_{h \to 0} \frac{f(-1,h,3) - f(-1,0,3)}{h} = \lim_{h \to 0} \frac{(2h^2 + 9h) - 0}{h} = \lim_{h \to 0} (2h + 9) = 9$

65. $y + \left(3z^2 \frac{\partial z}{\partial x}\right) x + z^3 - 2y \frac{\partial z}{\partial x} = 0 \Rightarrow (3xz^2 - 2y) \frac{\partial z}{\partial x} = -y - z^3 \Rightarrow$ at $(1,1,1)$ we have $(3-2) \frac{\partial z}{\partial x} = -1 - 1$ or $\frac{\partial z}{\partial x} = -2$

66. $\left(\frac{\partial x}{\partial z}\right) z + x + \left(\frac{y}{x}\right) \frac{\partial x}{\partial z} - 2x \frac{\partial x}{\partial z} = 0 \Rightarrow \left(z + \frac{y}{x} - 2x\right) \frac{\partial x}{\partial z} = -x \Rightarrow$ at $(1,-1,-3)$ we have $(-3 - 1 - 2) \frac{\partial x}{\partial z} = -1$ or $\frac{\partial x}{\partial z} = \frac{1}{6}$

67. $a^2 = b^2 + c^2 - 2bc \cos A \Rightarrow 2a = (2bc \sin A) \frac{\partial A}{\partial a} \Rightarrow \frac{\partial A}{\partial a} = \frac{a}{bc \sin A}$; also $0 = 2b - 2c \cos A + (2bc \sin A) \frac{\partial A}{\partial b}$
$\Rightarrow 2c \cos A - 2b = (2bc \sin A) \frac{\partial A}{\partial b} \Rightarrow \frac{\partial A}{\partial b} = \frac{c \cos A - b}{bc \sin A}$

68. $\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{(\sin A) \frac{\partial a}{\partial A} - a \cos A}{\sin^2 A} = 0 \Rightarrow (\sin A) \frac{\partial a}{\partial x} - a \cos A = 0 \Rightarrow \frac{\partial a}{\partial A} = \frac{a \cos A}{\sin A}$; also
$\left(\frac{1}{\sin A}\right) \frac{\partial a}{\partial B} = b(-\csc B \cot B) \Rightarrow \frac{\partial a}{\partial B} = -b \csc B \cot B \sin A$

69. Differentiating each equation implicitly gives $1 = v_x \ln u + \left(\frac{v}{u}\right) u_x$ and $0 = u_x \ln v + \left(\frac{u}{v}\right) v_x$ or
$$\left.\begin{array}{l} (\ln u)\, v_x + \left(\frac{v}{u}\right) u_x = 1 \\ \left(\frac{u}{v}\right) v_x + (\ln v)\, u_x = 0 \end{array}\right\} \Rightarrow v_x = \frac{\begin{vmatrix} 1 & \frac{v}{u} \\ 0 & \ln v \end{vmatrix}}{\begin{vmatrix} \ln u & \frac{v}{u} \\ \frac{u}{v} & \ln v \end{vmatrix}} = \frac{\ln v}{(\ln u)(\ln v) - 1}$$

70. Differentiating each equation implicitly gives $1 = (2x)x_u - (2y)y_u$ and $0 = (2x)x_u - y_u$ or
$$\left.\begin{array}{l} (2x)x_u - (2y)y_u = 1 \\ (2x)x_u - \quad y_u = 0 \end{array}\right\} \Rightarrow x_u = \frac{\begin{vmatrix} 1 & -2y \\ 0 & -1 \end{vmatrix}}{\begin{vmatrix} 2x & -2y \\ 2x & -1 \end{vmatrix}} = \frac{-1}{-2x + 4xy} = \frac{1}{2x - 4xy} \text{ and}$$
$y_u = \frac{\begin{vmatrix} 2x & 1 \\ 2x & 0 \end{vmatrix}}{-2x + 4xy} = \frac{-2x}{-2x + 4xy} = \frac{2x}{2x - 4xy} = \frac{1}{1 - 2y}$; next $s = x^2 + y^2 \Rightarrow \frac{\partial s}{\partial u} = 2x \frac{\partial x}{\partial u} + 2y \frac{\partial y}{\partial u}$
$= 2x \left(\frac{1}{2x - 4xy}\right) + 2y \left(\frac{1}{1 - 2y}\right) = \frac{1}{1 - 2y} + \frac{2y}{1 - 2y} = \frac{1 + 2y}{1 - 2y}$

71. $f_x(x,y) = \begin{cases} 0 & \text{if } y \ge 0 \\ 0 & \text{if } y < 0 \end{cases} \Rightarrow f_x(x,y) = 0$ for all points (x,y); at $y = 0$, $f_y(x,0) = \lim_{h \to 0} \frac{f(x, 0+h) - f(x,0)}{h} = \lim_{h \to 0} \frac{f(x,h) - 0}{h}$
$= \lim_{h \to 0} \frac{f(x,h)}{h} = 0$ because $\lim_{h \to 0^-} \frac{f(x,h)}{h} = \lim_{h \to 0^-} \frac{h^3}{h} = 0$ and $\lim_{h \to 0^+} \frac{f(x,h)}{h} = \lim_{h \to 0^+} \frac{h^2}{h} = 0 \Rightarrow f_y(x,y) = \begin{cases} 3y^2 & \text{if } y \ge 0 \\ -2y & \text{if } y < 0 \end{cases}$;
$f_{yx}(x,y) = f_{xy}(x,y) = 0$ for all points (x,y)

72. At $x = 0$, $f_x(0,y) = \lim_{h \to 0} \frac{f(0+h, y) - f(0,y)}{h} = \lim_{h \to 0} \frac{f(h,y) - 0}{h} = \lim_{h \to 0} \frac{f(h,y)}{h}$ which does not exist because $\lim_{h \to 0^-} \frac{f(h,y)}{h}$
$= \lim_{h \to 0^-} \frac{h^2}{h} = 0$ and $\lim_{h \to 0^+} \frac{f(h,y)}{h} = \lim_{h \to 0^+} \frac{\sqrt{h}}{h} = \lim_{h \to 0^+} \frac{1}{\sqrt{h}} = +\infty \Rightarrow f_x(x,y) = \begin{cases} \frac{1}{2\sqrt{x}} & \text{if } x > 0 \\ 2x & \text{if } x < 0 \end{cases}$;
$f_y(x,y) = \begin{cases} 0 & \text{if } x \ge 0 \\ 0 & \text{if } x < 0 \end{cases} \Rightarrow f_y(x,y) = 0$ for all points (x,y); $f_{yx}(x,y) = 0$ for all points (x,y), while $f_{xy}(x,y) = 0$ for all points (x,y) such that $x \ne 0$.

73. $\frac{\partial f}{\partial x} = 2x$, $\frac{\partial f}{\partial y} = 2y$, $\frac{\partial f}{\partial z} = -4z \Rightarrow \frac{\partial^2 f}{\partial x^2} = 2$, $\frac{\partial^2 f}{\partial y^2} = 2$, $\frac{\partial^2 f}{\partial z^2} = -4 \Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 2 + 2 + (-4) = 0$

74. $\frac{\partial f}{\partial x} = -6xz$, $\frac{\partial f}{\partial y} = -6yz$, $\frac{\partial f}{\partial z} = 6z^2 - 3(x^2 + y^2)$, $\frac{\partial^2 f}{\partial x^2} = -6z$, $\frac{\partial^2 f}{\partial y^2} = -6z$, $\frac{\partial^2 f}{\partial z^2} = 12z \Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$
$= -6z - 6z + 12z = 0$

75. $\frac{\partial f}{\partial x} = -2e^{-2y} \sin 2x$, $\frac{\partial f}{\partial y} = -2e^{-2y} \cos 2x$, $\frac{\partial^2 f}{\partial x^2} = -4e^{-2y} \cos 2x$, $\frac{\partial^2 f}{\partial y^2} = 4e^{-2y} \cos 2x \Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$
$= -4e^{-2y} \cos 2x + 4e^{-2y} \cos 2x = 0$

76. $\frac{\partial f}{\partial x} = \frac{x}{x^2+y^2}$, $\frac{\partial f}{\partial y} = \frac{y}{x^2+y^2}$, $\frac{\partial^2 f}{\partial x^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$, $\frac{\partial^2 f}{\partial y^2} = \frac{x^2-y^2}{(x^2+y^2)^2} \Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2} = 0$

77. $\frac{\partial f}{\partial x} = 3$, $\frac{\partial f}{\partial y} = 2$, $\frac{\partial^2 f}{\partial x^2} = 0$, $\frac{\partial^2 f}{\partial y^2} = 0 \Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 + 0 = 0$

78. $\frac{\partial f}{\partial x} = \frac{1/y}{1+\left(\frac{x}{y}\right)^2} = \frac{y}{y^2+x^2}$, $\frac{\partial f}{\partial y} = \frac{-x/y^2}{1+\left(\frac{x}{y}\right)^2} = \frac{-x}{y^2+x^2}$, $\frac{\partial^2 f}{\partial x^2} = \frac{(y^2+x^2)\cdot 0 - y\cdot 2x}{(y^2+x^2)^2} = \frac{-2xy}{(y^2+x^2)^2}$, $\frac{\partial^2 f}{\partial y^2} = \frac{(y^2+x^2)\cdot 0 - (-x)\cdot 2y}{(y^2+x^2)^2} = \frac{2xy}{(y^2+x^2)^2}$
$\Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{-2xy}{(y^2+x^2)^2} + \frac{2xy}{(y^2+x^2)^2} = 0$

79. $\frac{\partial f}{\partial x} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2x) = -x(x^2+y^2+z^2)^{-3/2}$, $\frac{\partial f}{\partial y} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2y)$
$= -y(x^2+y^2+z^2)^{-3/2}$, $\frac{\partial f}{\partial z} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2z) = -z(x^2+y^2+z^2)^{-3/2}$;
$\frac{\partial^2 f}{\partial x^2} = -(x^2+y^2+z^2)^{-3/2} + 3x^2(x^2+y^2+z^2)^{-5/2}$, $\frac{\partial^2 f}{\partial y^2} = -(x^2+y^2+z^2)^{-3/2} + 3y^2(x^2+y^2+z^2)^{-5/2}$,
$\frac{\partial^2 f}{\partial z^2} = -(x^2+y^2+z^2)^{-3/2} + 3z^2(x^2+y^2+z^2)^{-5/2} \Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$
$= \left[-(x^2+y^2+z^2)^{-3/2} + 3x^2(x^2+y^2+z^2)^{-5/2}\right] + \left[-(x^2+y^2+z^2)^{-3/2} + 3y^2(x^2+y^2+z^2)^{-5/2}\right]$
$+ \left[-(x^2+y^2+z^2)^{-3/2} + 3z^2(x^2+y^2+z^2)^{-5/2}\right] = -3(x^2+y^2+z^2)^{-3/2} + (3x^2+3y^2+3z^2)(x^2+y^2+z^2)^{-5/2} = 0$

80. $\frac{\partial f}{\partial x} = 3e^{3x+4y} \cos 5z$, $\frac{\partial f}{\partial y} = 4e^{3x+4y} \cos 5z$, $\frac{\partial f}{\partial z} = -5e^{3x+4y} \sin 5z$; $\frac{\partial^2 f}{\partial x^2} = 9e^{3x+4y} \cos 5z$, $\frac{\partial^2 f}{\partial y^2} = 16e^{3x+4y} \cos 5z$,
$\frac{\partial^2 f}{\partial z^2} = -25e^{3x+4y} \cos 5z \Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 9e^{3x+4y} \cos 5z + 16e^{3x+4y} \cos 5z - 25e^{3x+4y} \cos 5z = 0$

81. $\frac{\partial w}{\partial x} = \cos(x+ct)$, $\frac{\partial w}{\partial t} = c\cos(x+ct)$; $\frac{\partial^2 w}{\partial x^2} = -\sin(x+ct)$, $\frac{\partial^2 w}{\partial t^2} = -c^2 \sin(x+ct) \Rightarrow \frac{\partial^2 w}{\partial t^2} = c^2[-\sin(x+ct)] = c^2 \frac{\partial^2 w}{\partial x^2}$

82. $\frac{\partial w}{\partial x} = -2\sin(2x+2ct)$, $\frac{\partial w}{\partial t} = -2c\sin(2x+2ct)$; $\frac{\partial^2 w}{\partial x^2} = -4\cos(2x+2ct)$, $\frac{\partial^2 w}{\partial t^2} = -4c^2\cos(2x+2ct)$
$\Rightarrow \frac{\partial^2 w}{\partial t^2} = c^2[-4\cos(2x+2ct)] = c^2 \frac{\partial^2 w}{\partial x^2}$

83. $\frac{\partial w}{\partial x} = \cos(x+ct) - 2\sin(2x+2ct)$, $\frac{\partial w}{\partial t} = c\cos(x+ct) - 2c\sin(2x+2ct)$;
$\frac{\partial^2 w}{\partial x^2} = -\sin(x+ct) - 4\cos(2x+2ct)$, $\frac{\partial^2 w}{\partial t^2} = -c^2\sin(x+ct) - 4c^2\cos(2x+2ct)$
$\Rightarrow \frac{\partial^2 w}{\partial t^2} = c^2[-\sin(x+ct) - 4\cos(2x+2ct)] = c^2 \frac{\partial^2 w}{\partial x^2}$

84. $\frac{\partial w}{\partial x} = \frac{1}{x+ct}$, $\frac{\partial w}{\partial t} = \frac{c}{x+ct}$; $\frac{\partial^2 w}{\partial x^2} = \frac{-1}{(x+ct)^2}$, $\frac{\partial^2 w}{\partial t^2} = \frac{-c^2}{(x+ct)^2} \Rightarrow \frac{\partial^2 w}{\partial t^2} = c^2\left[\frac{-1}{(x+ct)^2}\right] = c^2 \frac{\partial^2 w}{\partial x^2}$

85. $\frac{\partial w}{\partial x} = 2\sec^2(2x-2ct)$, $\frac{\partial w}{\partial t} = -2c\sec^2(2x-2ct)$; $\frac{\partial^2 w}{\partial x^2} = 8\sec^2(2x-2ct)\tan(2x-2ct)$,
$\frac{\partial^2 w}{\partial t^2} = 8c^2\sec^2(2x-2ct)\tan(2x-2ct) \Rightarrow ux\frac{\partial^2 w}{\partial t^2} = c^2[8\sec^2(2x-2ct)\tan(2x-2ct)] = c^2 \frac{\partial^2 w}{\partial x^2}$

86. $\frac{\partial w}{\partial x} = -15 \sin(3x + 3ct) + e^{x+ct}$, $\frac{\partial w}{\partial t} = -15c \sin(3x + 3ct) + ce^{x+ct}$; $\frac{\partial^2 w}{\partial x^2} = -45 \cos(3x + 3ct) + e^{x+ct}$, $\frac{\partial^2 w}{\partial t^2} = -45c^2 \cos(3x + 3ct) + c^2 e^{x+ct} \Rightarrow \frac{\partial^2 w}{\partial t^2} = c^2\left[-45 \cos(3x + 3ct) + e^{x+ct}\right] = c^2 \frac{\partial^2 w}{\partial x^2}$

87. $\frac{\partial w}{\partial t} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial t} = \frac{\partial f}{\partial u}(ac) \Rightarrow \frac{\partial^2 w}{\partial t^2} = (ac)\left(\frac{\partial^2 f}{\partial u^2}\right)(ac) = a^2c^2 \frac{\partial^2 f}{\partial u^2}$; $\frac{\partial w}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} = \frac{\partial f}{\partial u} \cdot a \Rightarrow \frac{\partial^2 w}{\partial x^2} = \left(a\frac{\partial^2 f}{\partial u^2}\right) \cdot a$ $= a^2 \frac{\partial^2 f}{\partial u^2} \Rightarrow \frac{\partial^2 w}{\partial t^2} = a^2c^2 \frac{\partial^2 f}{\partial u^2} = c^2\left(a^2 \frac{\partial^2 f}{\partial u^2}\right) = c^2 \frac{\partial^2 w}{\partial x^2}$

88. If the first partial derivatives are continuous throughout an open region R, then by Theorem 3 in this section of the text, $f(x, y) = f(x_0, y_0) + f_x(x_0, y_0)\,\Delta x + f_y(x_0, y_0)\,\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y$, where $\epsilon_1, \epsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$. Then as $(x, y) \to (x_0, y_0)$, $\Delta x \to 0$ and $\Delta y \to 0 \Rightarrow \lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0) \Rightarrow$ f is continuous at every point (x_0, y_0) in R.

89. Yes, since f_{xx}, f_{yy}, f_{xy}, and f_{yx} are all continuous on R, use the same reasoning as in Exercise 76 with $f_x(x, y) = f_x(x_0, y_0) + f_{xx}(x_0, y_0)\,\Delta x + f_{xy}(x_0, y_0)\,\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y$ and $f_y(x, y) = f_y(x_0, y_0) + f_{yx}(x_0, y_0)\,\Delta x + f_{yy}(x_0, y_0)\,\Delta y + \hat{\epsilon}_1\Delta x + \hat{\epsilon}_2\Delta y$. Then $\lim_{(x,y)\to(x_0,y_0)} f_x(x, y) = f_x(x_0, y_0)$ and $\lim_{(x,y)\to(x_0,y_0)} f_y(x, y) = f_y(x_0, y_0)$.

90. To find α and β so that $u_t = u_{xx} \Rightarrow u_t = -\beta \sin(\alpha x)e^{-\beta t}$ and $u_x = \alpha \cos(\alpha x)e^{-\beta t} \Rightarrow u_{xx} = -\alpha^2 \sin(\alpha x)e^{-\beta t}$; then $u_t = u_{xx} \Rightarrow -\beta \sin(\alpha x)e^{-\beta t} = -\alpha^2 \sin(\alpha x)e^{-\beta t}$, thus $u_t = u_{xx}$ only if $\beta = \alpha^2$

91. $f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h \cdot 0^2}{h^2+0^4} - 0}{h} = \lim_{h\to 0} \frac{0}{h} = 0$; $f_y(0,0) = \lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{0 \cdot h^2}{0^2+h^4} - 0}{h} = \lim_{h\to 0} \frac{0}{h} = 0$; $\lim_{\substack{(x,y)\to(0,0) \\ \text{along } x = ky^2}} f(x, y) = \lim_{y\to 0} \frac{(ky^2)y^2}{(ky^2)^2 + y^4} = \lim_{y\to 0} \frac{ky^4}{k^2y^4 + y^4} = \lim_{y\to 0} \frac{k}{k^2+1} = \frac{k}{k^2+1} \Rightarrow$ different limits for different values of $k \Rightarrow \lim_{(x,y)\to(0,0)} f(x, y)$ does not exist $\Rightarrow f(x, y)$ is not continuous at $(0,0) \Rightarrow$ by Theorem 4, $f(x, y)$ is not differentiable at $(0,0)$.

92. $f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{f(h,0) - 1}{h} = \lim_{h\to 0} \frac{1-1}{h} = 0$; $f_y(0,0) = \lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h} = \lim_{h\to 0} \frac{f(0,h) - 1}{h} = \lim_{h\to 0} \frac{1-1}{h} = 0$; $\lim_{\substack{(x,y)\to(0,0) \\ \text{along } y = x^2}} f(x, y) = \lim_{y\to 0} 0 = 0$ but $\lim_{\substack{(x,y)\to(0,0) \\ \text{along } y = 1.5x^2}} f(x, y) = \lim_{y\to 0} 1 = 1 \Rightarrow \lim_{(x,y)\to(0,0)} f(x, y)$ does not exist $\Rightarrow f(x, y)$ is not continuous at $(0,0) \Rightarrow$ by Theorem 4, $f(x, y)$ is not differentiable at $(0,0)$.

14.4 THE CHAIN RULE

1. (a) $\frac{\partial w}{\partial x} = 2x$, $\frac{\partial w}{\partial y} = 2y$, $\frac{dx}{dt} = -\sin t$, $\frac{dy}{dt} = \cos t \Rightarrow \frac{dw}{dt} = -2x \sin t + 2y \cos t = -2 \cos t \sin t + 2 \sin t \cos t$ $= 0$; $w = x^2 + y^2 = \cos^2 t + \sin^2 t = 1 \Rightarrow \frac{dw}{dt} = 0$
   (b) $\frac{dw}{dt}(\pi) = 0$

2. (a) $\frac{\partial w}{\partial x} = 2x$, $\frac{\partial w}{\partial y} = 2y$, $\frac{dx}{dt} = -\sin t + \cos t$, $\frac{dy}{dt} = -\sin t - \cos t \Rightarrow \frac{dw}{dt}$
   $= (2x)(-\sin t + \cos t) + (2y)(-\sin t - \cos t)$
   $= 2(\cos t + \sin t)(\cos t - \sin t) - 2(\cos t - \sin t)(\sin t + \cos t) = (2\cos^2 t - 2\sin^2 t) - (2\cos^2 t - 2\sin^2 t)$
   $= 0$; $w = x^2 + y^2 = (\cos t + \sin t)^2 + (\cos t - \sin t)^2 = 2\cos^2 t + 2\sin^2 t = 2 \Rightarrow \frac{dw}{dt} = 0$
   (b) $\frac{dw}{dt}(0) = 0$

3. (a) $\frac{\partial w}{\partial x} = \frac{1}{z}$, $\frac{\partial w}{\partial y} = \frac{1}{z}$, $\frac{\partial w}{\partial z} = \frac{-(x+y)}{z^2}$, $\frac{dx}{dt} = -2\cos t \sin t$, $\frac{dy}{dt} = 2\sin t \cos t$, $\frac{dz}{dt} = -\frac{1}{t^2}$

$\Rightarrow \frac{dw}{dt} = -\frac{2}{z}\cos t \sin t + \frac{2}{z}\sin t \cos t + \frac{x+y}{z^2 t^2} = \frac{\cos^2 t + \sin^2 t}{\left(\frac{1}{t^2}\right)(t^2)} = 1$; $w = \frac{x}{z} + \frac{y}{z} = \frac{\cos^2 t}{\left(\frac{1}{t}\right)} + \frac{\sin^2 t}{\left(\frac{1}{t}\right)} = t \Rightarrow \frac{dw}{dt} = 1$

(b) $\frac{dw}{dt}(3) = 1$

4. (a) $\frac{\partial w}{\partial x} = \frac{2x}{x^2+y^2+z^2}$, $\frac{\partial w}{\partial y} = \frac{2y}{x^2+y^2+z^2}$, $\frac{\partial w}{\partial z} = \frac{2z}{x^2+y^2+z^2}$, $\frac{dx}{dt} = -\sin t$, $\frac{dy}{dt} = \cos t$, $\frac{dz}{dt} = 2t^{-1/2}$

$\Rightarrow \frac{dw}{dt} = \frac{-2x\sin t}{x^2+y^2+z^2} + \frac{2y\cos t}{x^2+y^2+z^2} + \frac{4zt^{-1/2}}{x^2+y^2+z^2} = \frac{-2\cos t \sin t + 2\sin t \cos t + 4\left(4t^{1/2}\right)t^{-1/2}}{\cos^2 t + \sin^2 t + 16t}$

$= \frac{16}{1+16t}$; $w = \ln\left(x^2+y^2+z^2\right) = \ln\left(\cos^2 t + \sin^2 t + 16t\right) = \ln(1+16t) \Rightarrow \frac{dw}{dt} = \frac{16}{1+16t}$

(b) $\frac{dw}{dt}(3) = \frac{16}{49}$

5. (a) $\frac{\partial w}{\partial x} = 2ye^x$, $\frac{\partial w}{\partial y} = 2e^x$, $\frac{\partial w}{\partial z} = -\frac{1}{z}$, $\frac{dx}{dt} = \frac{2t}{t^2+1}$, $\frac{dy}{dt} = \frac{1}{t^2+1}$, $\frac{dz}{dt} = e^t \Rightarrow \frac{dw}{dt} = \frac{4yte^x}{t^2+1} + \frac{2e^x}{t^2+1} - \frac{e^t}{z}$

$= \frac{(4t)\left(\tan^{-1} t\right)\left(t^2+1\right)}{t^2+1} + \frac{2\left(t^2+1\right)}{t^2+1} - \frac{e^t}{e^t} = 4t\tan^{-1} t + 1$; $w = 2ye^x - \ln z = \left(2\tan^{-1} t\right)\left(t^2+1\right) - t$

$\Rightarrow \frac{dw}{dt} = \left(\frac{2}{t^2+1}\right)\left(t^2+1\right) + \left(2\tan^{-1} t\right)(2t) - 1 = 4t\tan^{-1} t + 1$

(b) $\frac{dw}{dt}(1) = (4)(1)\left(\frac{\pi}{4}\right) + 1 = \pi + 1$

6. (a) $\frac{\partial w}{\partial x} = -y\cos xy$, $\frac{\partial w}{\partial y} = -x\cos xy$, $\frac{\partial w}{\partial z} = 1$, $\frac{dx}{dt} = 1$, $\frac{dy}{dt} = \frac{1}{t}$, $\frac{dz}{dt} = e^{t-1} \Rightarrow \frac{dw}{dt} = -y\cos xy - \frac{x\cos xy}{t} + e^{t-1}$

$= -(\ln t)[\cos(t\ln t)] - \frac{t\cos(t\ln t)}{t} + e^{t-1} = -(\ln t)[\cos(t\ln t)] - \cos(t\ln t) + e^{t-1}$; $w = z - \sin xy$

$= e^{t-1} - \sin(t\ln t) \Rightarrow \frac{dw}{dt} = e^{t-1} - [\cos(t\ln t)]\left[\ln t + t\left(\frac{1}{t}\right)\right] = e^{t-1} - (1+\ln t)\cos(t\ln t)$

(b) $\frac{dw}{dt}(1) = 1 - (1+0)(1) = 0$

7. (a) $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = \left(4e^x \ln y\right)\left(\frac{\cos v}{u\cos v}\right) + \left(\frac{4e^x}{y}\right)(\sin v) = \frac{4e^x \ln y}{u} + \frac{4e^x \sin v}{y}$

$= \frac{4(u\cos v)\ln(u\sin v)}{u} + \frac{4(u\cos v)(\sin v)}{u\sin v} = (4\cos v)\ln(u\sin v) + 4\cos v$;

$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = \left(4e^x \ln y\right)\left(\frac{-u\sin v}{u\cos v}\right) + \left(\frac{4e^x}{y}\right)(u\cos v) = -\left(4e^x \ln y\right)(\tan v) + \frac{4e^x u\cos v}{y}$

$= [-4(u\cos v)\ln(u\sin v)](\tan v) + \frac{4(u\cos v)(u\cos v)}{u\sin v} = (-4u\sin v)\ln(u\sin v) + \frac{4u\cos^2 v}{\sin v}$;

$z = 4e^x \ln y = 4(u\cos v)\ln(u\sin v) \Rightarrow \frac{\partial z}{\partial u} = (4\cos v)\ln(u\sin v) + 4(u\cos v)\left(\frac{\sin v}{u\sin v}\right)$

$= (4\cos v)\ln(u\sin v) + 4\cos v$; also $\frac{\partial z}{\partial v} = (-4u\sin v)\ln(u\sin v) + 4(u\cos v)\left(\frac{u\cos v}{u\sin v}\right)$

$= (-4u\sin v)\ln(u\sin v) + \frac{4u\cos^2 v}{\sin v}$

(b) At $\left(2, \frac{\pi}{4}\right)$: $\frac{\partial z}{\partial u} = 4\cos\frac{\pi}{4}\ln\left(2\sin\frac{\pi}{4}\right) + 4\cos\frac{\pi}{4} = 2\sqrt{2}\ln\sqrt{2} + 2\sqrt{2} = \sqrt{2}(\ln 2 + 2)$;

$\frac{\partial z}{\partial v} = (-4)(2)\sin\frac{\pi}{4}\ln\left(2\sin\frac{\pi}{4}\right) + \frac{(4)(2)\left(\cos^2\frac{\pi}{4}\right)}{\left(\sin\frac{\pi}{4}\right)} = -4\sqrt{2}\ln\sqrt{2} + 4\sqrt{2} = -2\sqrt{2}\ln 2 + 4\sqrt{2}$

8. (a) $\frac{\partial z}{\partial u} = \left[\frac{\left(\frac{1}{y}\right)}{\left(\frac{x}{y}\right)^2+1}\right]\cos v + \left[\frac{\left(\frac{-x}{y^2}\right)}{\left(\frac{x}{y}\right)^2+1}\right]\sin v = \frac{y\cos v}{x^2+y^2} - \frac{x\sin v}{x^2+y^2} = \frac{(u\sin v)(\cos v) - (u\cos v)(\sin v)}{u^2} = 0$;

$\frac{\partial z}{\partial v} = \left[\frac{\left(\frac{1}{y}\right)}{\left(\frac{x}{y}\right)^2+1}\right](-u\sin v) + \left[\frac{\left(\frac{-x}{y^2}\right)}{\left(\frac{x}{y}\right)^2+1}\right]u\cos v = -\frac{yu\sin v}{x^2+y^2} - \frac{xu\cos v}{x^2+y^2} = \frac{-(u\sin v)(u\sin v) - (u\cos v)(u\cos v)}{u^2}$

$= -\sin^2 v - \cos^2 v = -1$; $z = \tan^{-1}\left(\frac{x}{y}\right) = \tan^{-1}(\cot v) \Rightarrow \frac{\partial z}{\partial u} = 0$ and $\frac{\partial z}{\partial v} = \left(\frac{1}{1+\cot^2 v}\right)\left(-\csc^2 v\right)$

$= \frac{-1}{\sin^2 v + \cos^2 v} = -1$

(b) At $\left(1.3, \frac{\pi}{6}\right)$: $\frac{\partial z}{\partial u} = 0$ and $\frac{\partial z}{\partial v} = -1$

9. (a) $\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} = (y+z)(1) + (x+z)(1) + (y+x)(v) = x + y + 2z + v(y+x)$
$= (u+v) + (u-v) + 2uv + v(2u) = 2u + 4uv$; $\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$
$= (y+z)(1) + (x+z)(-1) + (y+x)(u) = y - x + (y+x)u = -2v + (2u)u = -2v + 2u^2$;
$w = xy + yz + xz = (u^2 - v^2) + (u^2 v - uv^2) + (u^2 v + uv^2) = u^2 - v^2 + 2u^2 v \Rightarrow \frac{\partial w}{\partial u} = 2u + 4uv$ and
$\frac{\partial w}{\partial v} = -2v + 2u^2$

(b) At $\left(\frac{1}{2}, 1\right)$: $\frac{\partial w}{\partial u} = 2\left(\frac{1}{2}\right) + 4\left(\frac{1}{2}\right)(1) = 3$ and $\frac{\partial w}{\partial v} = -2(1) + 2\left(\frac{1}{2}\right)^2 = -\frac{3}{2}$

10. (a) $\frac{\partial w}{\partial u} = \left(\frac{2x}{x^2+y^2+z^2}\right)(e^v \sin u + ue^v \cos u) + \left(\frac{2y}{x^2+y^2+z^2}\right)(e^v \cos u - ue^v \sin u) + \left(\frac{2z}{x^2+y^2+z^2}\right)(e^v)$
$= \left(\frac{2ue^v \sin u}{u^2e^{2v}\sin^2 u + u^2e^{2v}\cos^2 u + u^2e^{2v}}\right)(e^v \sin u + ue^v \cos u)$
$+ \left(\frac{2ue^v \cos u}{u^2e^{2v}\sin^2 u + u^2e^{2v}\cos^2 u + u^2e^{2v}}\right)(e^v \cos u - ue^v \sin u)$
$+ \left(\frac{2ue^v}{u^2e^{2v}\sin^2 u + u^2e^{2v}\cos^2 u + u^2e^{2v}}\right)(e^v) = \frac{2}{u}$;
$\frac{\partial w}{\partial v} = \left(\frac{2x}{x^2+y^2+z^2}\right)(ue^v \sin u) + \left(\frac{2y}{x^2+y^2+z^2}\right)(ue^v \cos u) + \left(\frac{2z}{x^2+y^2+z^2}\right)(ue^v)$
$= \left(\frac{2ue^v \sin u}{u^2e^{2v}\sin^2 u + u^2e^{2v}\cos^2 u + u^2e^{2v}}\right)(ue^v \sin u)$
$+ \left(\frac{2ue^v \cos u}{u^2e^{2v}\sin^2 u + u^2e^{2v}\cos^2 u + u^2e^{2v}}\right)(ue^v \cos u)$
$+ \left(\frac{2ue^v}{u^2e^{2v}\sin^2 u + u^2e^{2v}\cos^2 u + u^2e^{2v}}\right)(ue^v) = 2$; $w = \ln(u^2e^{2v}\sin^2 u + u^2e^{2v}\cos^2 u + u^2e^{2v}) = \ln(2u^2e^{2v})$
$= \ln 2 + 2\ln u + 2v \Rightarrow \frac{\partial w}{\partial u} = \frac{2}{u}$ and $\frac{\partial w}{\partial v} = 2$

(b) At $(-2, 0)$: $\frac{\partial w}{\partial u} = \frac{2}{-2} = -1$ and $\frac{\partial w}{\partial v} = 2$

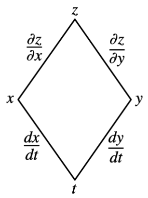
11. (a) $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} = \frac{1}{q-r} + \frac{r-p}{(q-r)^2} + \frac{p-q}{(q-r)^2} = \frac{q-r+r-p+p-q}{(q-r)^2} = 0$;
$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial y} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} = \frac{1}{q-r} - \frac{r-p}{(q-r)^2} + \frac{p-q}{(q-r)^2} = \frac{q-r-r+p+p-q}{(q-r)^2} = \frac{2p-2r}{(q-r)^2}$
$= \frac{(2x+2y+2z)-(2x+2y-2z)}{(2z-2y)^2} = \frac{z}{(z-y)^2}$; $\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial z} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial z}$
$= \frac{1}{q-r} + \frac{r-p}{(q-r)^2} - \frac{p-q}{(q-r)^2} = \frac{q-r+r-p-p+q}{(q-r)^2} = \frac{2q-2p}{(q-r)^2} = \frac{-4y}{(2z-2y)^2} = -\frac{y}{(z-y)^2}$;
$u = \frac{p-q}{q-r} = \frac{2y}{2z-2y} = \frac{y}{z-y} \Rightarrow \frac{\partial u}{\partial x} = 0$, $\frac{\partial u}{\partial y} = \frac{(z-y)-y(-1)}{(z-y)^2} = \frac{z}{(z-y)^2}$, and $\frac{\partial u}{\partial z} = \frac{(z-y)(0)-y(1)}{(z-y)^2}$
$= -\frac{y}{(z-y)^2}$

(b) At $\left(\sqrt{3}, 2, 1\right)$: $\frac{\partial u}{\partial x} = 0$, $\frac{\partial u}{\partial y} = \frac{1}{(1-2)^2} = 1$, and $\frac{\partial u}{\partial z} = \frac{-2}{(1-2)^2} = -2$

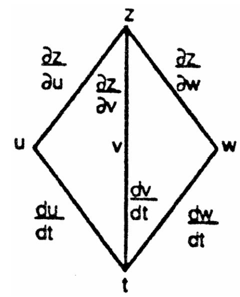
12. (a) $\frac{\partial u}{\partial x} = \frac{e^{qr}}{\sqrt{1-p^2}}(\cos x) + (re^{qr}\sin^{-1} p)(0) + (qe^{qr}\sin^{-1} p)(0) = \frac{e^{qr}\cos x}{\sqrt{1-p^2}} = \frac{e^{z\ln y}\cos x}{\sqrt{1-\sin^2 x}} = y^z$ if $-\frac{\pi}{2} < x < \frac{\pi}{2}$;
$\frac{\partial u}{\partial y} = \frac{e^{qr}}{\sqrt{1-p^2}}(0) + (re^{qr}\sin^{-1} p)\left(\frac{z^2}{y}\right) + (qe^{qr}\sin^{-1} p)(0) = \frac{z^2 re^{qr}\sin^{-1} p}{y} = \frac{z^2\left(\frac{1}{z}\right)y^z x}{y} = xzy^{z-1}$;
$\frac{\partial u}{\partial z} = \frac{e^{qr}}{\sqrt{1-p^2}}(0) + (re^{qr}\sin^{-1} p)(2z\ln y) + (qe^{qr}\sin^{-1} p)\left(-\frac{1}{z^2}\right) = (2zre^{qr}\sin^{-1} p)(\ln y) - \frac{qe^{qr}\sin^{-1} p}{z^2}$
$= (2z)\left(\frac{1}{z}\right)(y^z x\ln y) - \frac{(z^2\ln y)(y^z)x}{z^2} = xy^z\ln y$; $u = e^{z\ln y}\sin^{-1}(\sin x) = xy^z$ if $-\frac{\pi}{2} \le x \le \frac{\pi}{2} \Rightarrow \frac{\partial u}{\partial x} = y^z$,
$\frac{\partial u}{\partial y} = xzy^{z-1}$, and $\frac{\partial u}{\partial z} = \; = xy^z\ln y$ from direct calculations

(b) At $\left(\frac{\pi}{4}, \frac{1}{2}, -\frac{1}{2}\right)$: $\frac{\partial u}{\partial x} = \left(\frac{1}{2}\right)^{-1/2} = \sqrt{2}$, $\frac{\partial u}{\partial y} = \left(\frac{\pi}{4}\right)\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)^{(-1/2)-1} = -\frac{\pi\sqrt{2}}{4}$, $\frac{\partial u}{\partial z} = \left(\frac{\pi}{4}\right)\left(\frac{1}{2}\right)^{-1/2}\ln\left(\frac{1}{2}\right) = -\frac{\pi\sqrt{2}\ln 2}{4}$

13. $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$

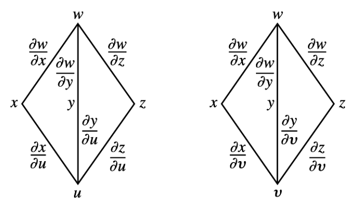


14. $\frac{dz}{dt} = \frac{\partial z}{\partial u}\frac{du}{dt} + \frac{\partial z}{\partial v}\frac{dv}{dt} + \frac{\partial x}{\partial w}\frac{dw}{dt}$

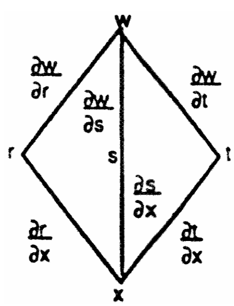


15. $\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$

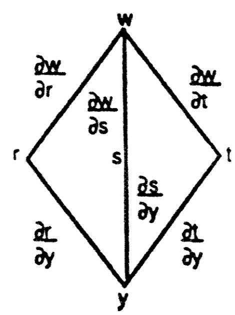
$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$



16. $\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial w}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial x}$

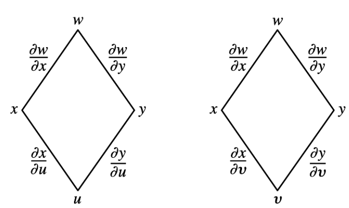


$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial w}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial y}$



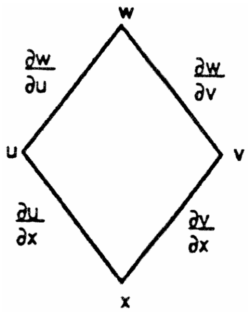
17. $\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u}$

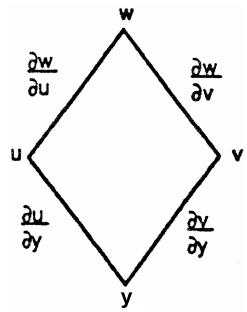
$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v}$



18. $\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x}$

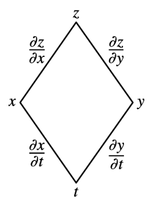
$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y}$

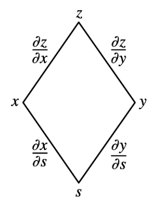




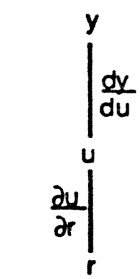
19. $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$

$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$

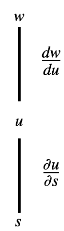


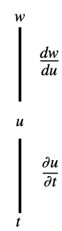


20. $\frac{\partial y}{\partial r} = \frac{dy}{du}\frac{\partial u}{\partial r}$

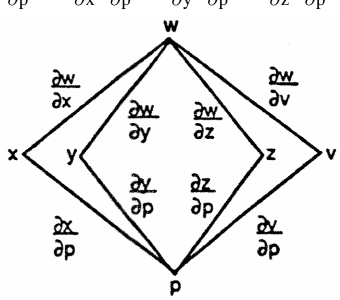


21. $\frac{\partial w}{\partial s} = \frac{dw}{du}\frac{\partial u}{\partial s}$ $\frac{\partial w}{\partial t} = \frac{dw}{du}\frac{\partial u}{\partial t}$



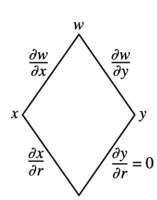


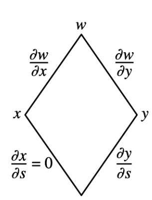
22. $\frac{\partial w}{\partial p} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial p} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial p} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial p}$



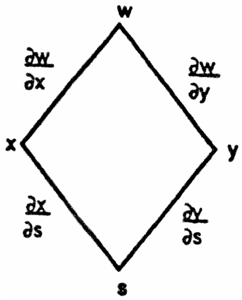
23. $\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{dx}{dr} + \frac{\partial w}{\partial y}\frac{dy}{dr} = \frac{\partial w}{\partial x}\frac{dx}{dr}$ since $\frac{dy}{dr} = 0$

$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{dx}{ds} + \frac{\partial w}{\partial y}\frac{dy}{ds} = \frac{\partial w}{\partial y}\frac{dy}{ds}$ since $\frac{dx}{ds} = 0$





24. $\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$



25. Let $F(x, y) = x^3 - 2y^2 + xy = 0 \Rightarrow F_x(x, y) = 3x^2 + y$ and $F_y(x, y) = -4y + x \Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{3x^2+y}{(-4y+x)}$ $\Rightarrow \frac{dy}{dx}(1, 1) = \frac{4}{3}$

26. Let $F(x, y) = xy + y^2 - 3x - 3 = 0 \Rightarrow F_x(x, y) = y - 3$ and $F_y(x, y) = x + 2y \Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{y-3}{x+2y}$ $\Rightarrow \frac{dy}{dx}(-1, 1) = 2$

27. Let $F(x, y) = x^2 + xy + y^2 - 7 = 0 \Rightarrow F_x(x, y) = 2x + y$ and $F_y(x, y) = x + 2y \Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2x+y}{x+2y}$ $\Rightarrow \frac{dy}{dx}(1, 2) = -\frac{4}{5}$

28. Let $F(x, y) = xe^y + \sin xy + y - \ln 2 = 0 \Rightarrow F_x(x, y) = e^y + y\cos xy$ and $F_y(x, y) = xe^y + x\sin xy + 1$ $\Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{e^y + y\cos xy}{xe^y + x\sin xy + 1} \Rightarrow \frac{dy}{dx}(0, \ln 2) = -(2 + \ln 2)$

29. Let $F(x, y, z) = z^3 - xy + yz + y^3 - 2 = 0 \Rightarrow F_x(x, y, z) = -y, F_y(x, y, z) = -x + z + 3y^2, F_z(x, y, z) = 3z^2 + y$ $\Rightarrow \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{-y}{3z^2+y} = \frac{y}{3z^2+y} \Rightarrow \frac{\partial z}{\partial x}(1, 1, 1) = \frac{1}{4}$; $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{-x+z+3y^2}{3z^2+y} = \frac{x-z-3y^2}{3z^2+y}$ $\Rightarrow \frac{\partial z}{\partial y}(1, 1, 1) = -\frac{3}{4}$

30. Let $F(x, y, z) = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1 = 0 \Rightarrow F_x(x, y, z) = -\frac{1}{x^2}, F_y(x, y, z) = -\frac{1}{y^2}, F_z(x, y, z) = -\frac{1}{z^2}$ $\Rightarrow \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{\left(-\frac{1}{x^2}\right)}{\left(-\frac{1}{z^2}\right)} = -\frac{z^2}{x^2} \Rightarrow \frac{\partial z}{\partial x}(2, 3, 6) = -9$; $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{\left(-\frac{1}{y^2}\right)}{\left(-\frac{1}{z^2}\right)} = -\frac{z^2}{y^2} \Rightarrow \frac{\partial z}{\partial y}(2, 3, 6) = -4$

31. Let $F(x, y, z) = \sin(x + y) + \sin(y + z) + \sin(x + z) = 0 \Rightarrow F_x(x, y, z) = \cos(x + y) + \cos(x + z)$, $F_y(x, y, z) = \cos(x + y) + \cos(y + z), F_z(x, y, z) = \cos(y + z) + \cos(x + z) \Rightarrow \frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$ $= -\frac{\cos(x+y)+\cos(x+z)}{\cos(y+z)+\cos(x+z)} \Rightarrow \frac{\partial z}{\partial x}(\pi, \pi, \pi) = -1$; $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{\cos(x+y)+\cos(y+z)}{\cos(y+z)+\cos(x+z)} \Rightarrow \frac{\partial z}{\partial y}(\pi, \pi, \pi) = -1$

32. Let $F(x, y, z) = xe^y + ye^z + 2\ln x - 2 - 3\ln 2 = 0 \Rightarrow F_x(x, y, z) = e^y + \frac{2}{x}, F_y(x, y, z) = xe^y + e^z, F_z(x, y, z) = ye^z$ $\Rightarrow \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{\left(e^y + \frac{2}{x}\right)}{ye^z} \Rightarrow \frac{\partial z}{\partial x}(1, \ln 2, \ln 3) = -\frac{4}{3\ln 2}$; $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{xe^y+e^z}{ye^z} \Rightarrow \frac{\partial z}{\partial y}(1, \ln 2, \ln 3) = -\frac{5}{3\ln 2}$

33. $\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} = 2(x + y + z)(1) + 2(x + y + z)[-\sin(r + s)] + 2(x + y + z)[\cos(r + s)]$ $= 2(x + y + z)[1 - \sin(r + s) + \cos(r + s)] = 2[r - s + \cos(r + s) + \sin(r + s)][1 - \sin(r + s) + \cos(r + s)]$ $\Rightarrow \left.\frac{\partial w}{\partial r}\right|_{r=1,s=-1} = 2(3)(2) = 12$

34. $\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v} = y\left(\frac{2v}{u}\right) + x(1) + \left(\frac{1}{z}\right)(0) = (u + v)\left(\frac{2v}{u}\right) + \frac{v^2}{u} \Rightarrow \left.\frac{\partial w}{\partial v}\right|_{u=-1,v=2} = (1)\left(\frac{4}{-1}\right) + \left(\frac{4}{-1}\right) = -8$

35. $\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} = \left(2x - \frac{y}{x^2}\right)(-2) + \left(\frac{1}{x}\right)(1) = \left[2(u - 2v + 1) - \frac{2u+v-2}{(u-2v+1)^2}\right](-2) + \frac{1}{u-2v+1}$ $\Rightarrow \left.\frac{\partial w}{\partial v}\right|_{u=0,v=0} = -7$

36. $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = (y\cos xy + \sin y)(2u) + (x\cos xy + x\cos y)(v)$
$= [uv\cos(u^3v + uv^3) + \sin uv](2u) + [(u^2+v^2)\cos(u^3v+uv^3) + (u^2+v^2)\cos uv](v)$
$\Rightarrow \left.\frac{\partial z}{\partial u}\right|_{u=0,v=1} = 0 + (\cos 0 + \cos 0)(1) = 2$

37. $\frac{\partial z}{\partial u} = \frac{dz}{dx}\frac{\partial x}{\partial u} = \left(\frac{5}{1+x^2}\right)e^u = \left[\frac{5}{1+(e^u+\ln v)^2}\right]e^u \Rightarrow \left.\frac{\partial z}{\partial u}\right|_{u=\ln 2,v=1} = \left[\frac{5}{1+(2)^2}\right](2) = 2;$
$\frac{\partial z}{\partial v} = \frac{dz}{dx}\frac{\partial x}{\partial v} = \left(\frac{5}{1+x^2}\right)\left(\frac{1}{v}\right) = \left[\frac{5}{1+(e^u+\ln v)^2}\right]\left(\frac{1}{v}\right) \Rightarrow \left.\frac{\partial z}{\partial v}\right|_{u=\ln 2,v=1} = \left[\frac{5}{1+(2)^2}\right](1) = 1$

38. $\frac{\partial z}{\partial u} = \frac{dz}{dq}\frac{\partial q}{\partial u} = \left(\frac{1}{q}\right)\left(\frac{\sqrt{v+3}}{1+u^2}\right) = \left(\frac{1}{\sqrt{v+3}\tan^{-1}u}\right)\left(\frac{\sqrt{v+3}}{1+u^2}\right) = \frac{1}{(\tan^{-1}u)(1+u^2)} \Rightarrow \left.\frac{\partial z}{\partial u}\right|_{u=1,v=-2} = \frac{1}{(\tan^{-1}1)(1+1^2)} = \frac{2}{\pi};$
$\frac{\partial z}{\partial v} = \frac{dz}{dq}\frac{\partial q}{\partial v} = \left(\frac{1}{q}\right)\left(\frac{\tan^{-1}u}{2\sqrt{v+3}}\right) = \left(\frac{1}{\sqrt{v+3}\tan^{-1}u}\right)\left(\frac{\tan^{-1}u}{2\sqrt{v+3}}\right) = \frac{1}{2(v+3)} \Rightarrow \left.\frac{\partial z}{\partial v}\right|_{u=1,v=-2} = \frac{1}{2}$

39. Let $x = s^3 + t^2 \Rightarrow w = f(s^3+t^2) = f(x) \Rightarrow \frac{\partial w}{\partial s} = \frac{dw}{dx}\frac{\partial x}{\partial s} = f'(x)\cdot 3s^2 = 3s^2e^{s^3+t^2}$, $\frac{\partial w}{\partial t} = \frac{dw}{dx}\frac{\partial x}{\partial t} = f'(x)\cdot 2t = 2t\,e^{s^3+t^2}$

40. Let $x = ts^2$ and $y = \frac{s}{t} \Rightarrow w = f\left(ts^2, \frac{s}{t}\right) = f(x,y) \Rightarrow \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} = f_x(x,y)\cdot 2ts + f_y(x,y)\cdot\frac{1}{t}$
$= (ts^2)\left(\frac{s}{t}\right)\cdot 2ts + \frac{(ts^2)^2}{2}\cdot\frac{1}{t} = 2s^4t + \frac{s^4t}{2} = \frac{5s^4t}{2}$; $\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} = f_x(x,y)\cdot s^2 + f_y(x,y)\cdot\frac{-s}{t^2}$
$= (ts^2)\left(\frac{s}{t}\right)\cdot s^2 + \frac{(ts^2)^2}{2}\cdot\left(-\frac{s}{t^2}\right) = s^5 - \frac{s^5}{2} = \frac{s^5}{2}$

41. $V = IR \Rightarrow \frac{\partial V}{\partial I} = R$ and $\frac{\partial V}{\partial R} = I$; $\frac{dV}{dt} = \frac{\partial V}{\partial I}\frac{dI}{dt} + \frac{\partial V}{\partial R}\frac{dR}{dt} = R\frac{dI}{dt} + I\frac{dR}{dt} \Rightarrow -0.01$ volts/sec
$= (600 \text{ ohms})\frac{dI}{dt} + (0.04 \text{ amps})(0.5 \text{ ohms/sec}) \Rightarrow \frac{dI}{dt} = -0.00005$ amps/sec

42. $V = abc \Rightarrow \frac{dV}{dt} = \frac{\partial V}{\partial a}\frac{da}{dt} + \frac{\partial V}{\partial b}\frac{db}{dt} + \frac{\partial V}{\partial c}\frac{dc}{dt} = (bc)\frac{da}{dt} + (ac)\frac{db}{dt} + (ab)\frac{dc}{dt}$
$\Rightarrow \left.\frac{dV}{dt}\right|_{a=1,b=2,c=3} = (2\text{ m})(3\text{ m})(1\text{ m/sec}) + (1\text{ m})(3\text{ m})(1\text{ m/sec}) + (1\text{ m})(2\text{ m})(-3\text{ m/sec}) = 3\text{ m}^3/\text{sec}$
and the volume is increasing; $S = 2ab + 2ac + 2bc \Rightarrow \frac{dS}{dt} = \frac{\partial S}{\partial a}\frac{da}{dt} + \frac{\partial S}{\partial b}\frac{db}{dt} + \frac{\partial S}{\partial c}\frac{dc}{dt}$
$= 2(b+c)\frac{da}{dt} + 2(a+c)\frac{db}{dt} + 2(a+b)\frac{dc}{dt} \Rightarrow \left.\frac{dS}{dt}\right|_{a=1,b=2,c=3}$
$= 2(5\text{ m})(1\text{ m/sec}) + 2(4\text{ m})(1\text{ m/sec}) + 2(3\text{ m})(-3\text{ m/sec}) = 0\text{ m}^2/\text{sec}$ and the surface area is not changing;
$D = \sqrt{a^2+b^2+c^2} \Rightarrow \frac{dD}{dt} = \frac{\partial D}{\partial a}\frac{da}{dt} + \frac{\partial D}{\partial b}\frac{db}{dt} + \frac{\partial D}{\partial c}\frac{dc}{dt} = \frac{1}{\sqrt{a^2+b^2+c^2}}\left(a\frac{da}{dt} + b\frac{db}{dt} + c\frac{dc}{dt}\right) \Rightarrow \left.\frac{dD}{dt}\right|_{a=1,b=2,c=3}$
$= \left(\frac{1}{\sqrt{14}\text{ m}}\right)[(1\text{ m})(1\text{ m/sec}) + (2\text{ m})(1\text{ m/sec}) + (3\text{ m})(-3\text{ m/sec})] = -\frac{6}{\sqrt{14}}\text{ m/sec} < 0 \Rightarrow$ the diagonals are decreasing in length

43. $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x} = \frac{\partial f}{\partial u}(1) + \frac{\partial f}{\partial v}(0) + \frac{\partial f}{\partial w}(-1) = \frac{\partial f}{\partial u} - \frac{\partial f}{\partial w}$,
$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial y} = \frac{\partial f}{\partial u}(-1) + \frac{\partial f}{\partial v}(1) + \frac{\partial f}{\partial w}(0) = -\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}$, and
$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial z} = \frac{\partial f}{\partial u}(0) + \frac{\partial f}{\partial v}(-1) + \frac{\partial f}{\partial w}(1) = -\frac{\partial f}{\partial v} + \frac{\partial f}{\partial w} \Rightarrow \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0$

44. (a) $\frac{\partial w}{\partial r} = f_x\frac{\partial x}{\partial r} + f_y\frac{\partial y}{\partial r} = f_x\cos\theta + f_y\sin\theta$ and $\frac{\partial w}{\partial \theta} = f_x(-r\sin\theta) + f_y(r\cos\theta) \Rightarrow \frac{1}{r}\frac{\partial w}{\partial \theta} = -f_x\sin\theta + f_y\cos\theta$

(b) $\frac{\partial w}{\partial r}\sin\theta = f_x\sin\theta\cos\theta + f_y\sin^2\theta$ and $\left(\frac{\cos\theta}{r}\right)\frac{\partial w}{\partial \theta} = -f_x\sin\theta\cos\theta + f_y\cos^2\theta$
$\Rightarrow f_y = (\sin\theta)\frac{\partial w}{\partial r} + \left(\frac{\cos\theta}{r}\right)\frac{\partial w}{\partial \theta}$; then $\frac{\partial w}{\partial r} = f_x\cos\theta + \left[(\sin\theta)\frac{\partial w}{\partial r} + \left(\frac{\cos\theta}{r}\right)\frac{\partial w}{\partial \theta}\right](\sin\theta) \Rightarrow f_x\cos\theta$
$= \frac{\partial w}{\partial r} - (\sin^2\theta)\frac{\partial w}{\partial r} - \left(\frac{\sin\theta\cos\theta}{r}\right)\frac{\partial w}{\partial \theta} = (1-\sin^2\theta)\frac{\partial w}{\partial r} - \left(\frac{\sin\theta\cos\theta}{r}\right)\frac{\partial w}{\partial \theta} \Rightarrow f_x = (\cos\theta)\frac{\partial w}{\partial r} - \left(\frac{\sin\theta}{r}\right)\frac{\partial w}{\partial \theta}$

(c) $(f_x)^2 = (\cos^2\theta)\left(\frac{\partial w}{\partial r}\right)^2 - \left(\frac{2\sin\theta\cos\theta}{r}\right)\left(\frac{\partial w}{\partial r}\frac{\partial w}{\partial \theta}\right) + \left(\frac{\sin^2\theta}{r^2}\right)\left(\frac{\partial w}{\partial \theta}\right)^2$ and
$(f_y)^2 = (\sin^2\theta)\left(\frac{\partial w}{\partial r}\right)^2 + \left(\frac{2\sin\theta\cos\theta}{r}\right)\left(\frac{\partial w}{\partial r}\frac{\partial w}{\partial \theta}\right) + \left(\frac{\cos^2\theta}{r^2}\right)\left(\frac{\partial w}{\partial \theta}\right)^2 \Rightarrow (f_x)^2 + (f_y)^2 = \left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2$

45. $w_x = \frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x} = x\frac{\partial w}{\partial u} + y\frac{\partial w}{\partial v} \Rightarrow w_{xx} = \frac{\partial w}{\partial u} + x\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial u}\right) + y\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial v}\right)$
$= \frac{\partial w}{\partial u} + x\left(\frac{\partial^2 w}{\partial u^2}\frac{\partial u}{\partial x} + \frac{\partial^2 w}{\partial v \partial u}\frac{\partial v}{\partial x}\right) + y\left(\frac{\partial^2 w}{\partial u \partial v}\frac{\partial u}{\partial x} + \frac{\partial^2 w}{\partial v^2}\frac{\partial v}{\partial x}\right) = \frac{\partial w}{\partial u} + x\left(x\frac{\partial^2 w}{\partial u^2} + y\frac{\partial^2 w}{\partial v \partial u}\right) + y\left(x\frac{\partial^2 w}{\partial u \partial v} + y\frac{\partial^2 w}{\partial v^2}\right)$
$= \frac{\partial w}{\partial u} + x^2\frac{\partial^2 w}{\partial u^2} + 2xy\frac{\partial^2 w}{\partial v \partial u} + y^2\frac{\partial^2 w}{\partial v^2}$; $w_y = \frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y} = -y\frac{\partial w}{\partial u} + x\frac{\partial w}{\partial v}$
$\Rightarrow w_{yy} = -\frac{\partial w}{\partial u} - y\left(\frac{\partial^2 w}{\partial u^2}\frac{\partial u}{\partial y} + \frac{\partial^2 w}{\partial v \partial u}\frac{\partial v}{\partial y}\right) + x\left(\frac{\partial^2 w}{\partial u \partial v}\frac{\partial u}{\partial y} + \frac{\partial^2 w}{\partial v^2}\frac{\partial v}{\partial y}\right)$
$= -\frac{\partial w}{\partial u} - y\left(-y\frac{\partial^2 w}{\partial u^2} + x\frac{\partial^2 w}{\partial v \partial u}\right) + x\left(-y\frac{\partial^2 w}{\partial u \partial v} + x\frac{\partial^2 w}{\partial v^2}\right) = -\frac{\partial w}{\partial u} + y^2\frac{\partial^2 w}{\partial u^2} - 2xy\frac{\partial^2 w}{\partial v \partial u} + x^2\frac{\partial^2 w}{\partial v^2}$; thus
$w_{xx} + w_{yy} = (x^2+y^2)\frac{\partial^2 w}{\partial u^2} + (x^2+y^2)\frac{\partial^2 w}{\partial v^2} = (x^2+y^2)(w_{uu} + w_{vv}) = 0$, since $w_{uu} + w_{vv} = 0$

46. $\frac{\partial w}{\partial x} = f'(u)(1) + g'(v)(1) = f'(u) + g'(v) \Rightarrow w_{xx} = f''(u)(1) + g''(v)(1) = f''(u) + g''(v)$;
$\frac{\partial w}{\partial y} = f'(u)(i) + g'(v)(-i) \Rightarrow w_{yy} = f''(u)(i^2) + g''(v)(i^2) = -f''(u) - g''(v) \Rightarrow w_{xx} + w_{yy} = 0$

47. $f_x(x,y,z) = \cos t$, $f_y(x,y,z) = \sin t$, and $f_z(x,y,z) = t^2 + t - 2 \Rightarrow \frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$
$= (\cos t)(-\sin t) + (\sin t)(\cos t) + (t^2+t-2)(1) = t^2 + t - 2$; $\frac{df}{dt} = 0 \Rightarrow t^2 + t - 2 = 0 \Rightarrow t = -2$
or $t = 1$; $t = -2 \Rightarrow x = \cos(-2)$, $y = \sin(-2)$, $z = -2$ for the point $(\cos(-2), \sin(-2), -2)$; $t = 1 \Rightarrow x = \cos 1$, $y = \sin 1$, $z = 1$ for the point $(\cos 1, \sin 1, 1)$

48. $\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} = (2xe^{2y}\cos 3z)(-\sin t) + (2x^2e^{2y}\cos 3z)\left(\frac{1}{t+2}\right) + (-3x^2e^{2y}\sin 3z)(1)$
$= -2xe^{2y}\cos 3z \sin t + \frac{2x^2e^{2y}\cos 3z}{t+2} - 3x^2e^{2y}\sin 3z$; at the point on the curve $z = 0 \Rightarrow t = z = 0$
$\Rightarrow \left.\frac{dw}{dt}\right|_{(1,\ln 2,0)} = 0 + \frac{2(1)^2(4)(1)}{2} - 0 = 4$

49. (a) $\frac{\partial T}{\partial x} = 8x - 4y$ and $\frac{\partial T}{\partial y} = 8y - 4x \Rightarrow \frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} = (8x-4y)(-\sin t) + (8y-4x)(\cos t)$
$= (8\cos t - 4\sin t)(-\sin t) + (8\sin t - 4\cos t)(\cos t) = 4\sin^2 t - 4\cos^2 t \Rightarrow \frac{d^2T}{dt^2} = 16\sin t\cos t$;
$\frac{dT}{dt} = 0 \Rightarrow 4\sin^2 t - 4\cos^2 t = 0 \Rightarrow \sin^2 t = \cos^2 t \Rightarrow \sin t = \cos t$ or $\sin t = -\cos t \Rightarrow t = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}$ on the interval $0 \le t \le 2\pi$;
$\left.\frac{d^2T}{dt^2}\right|_{t=\frac{\pi}{4}} = 16\sin\frac{\pi}{4}\cos\frac{\pi}{4} > 0 \Rightarrow$ T has a minimum at $(x,y) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$;
$\left.\frac{d^2T}{dt^2}\right|_{t=\frac{3\pi}{4}} = 16\sin\frac{3\pi}{4}\cos\frac{3\pi}{4} < 0 \Rightarrow$ T has a maximum at $(x,y) = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$;
$\left.\frac{d^2T}{dt^2}\right|_{t=\frac{5\pi}{4}} = 16\sin\frac{5\pi}{4}\cos\frac{5\pi}{4} > 0 \Rightarrow$ T has a minimum at $(x,y) = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$;
$\left.\frac{d^2T}{dt^2}\right|_{t=\frac{7\pi}{4}} = 16\sin\frac{7\pi}{4}\cos\frac{7\pi}{4} < 0 \Rightarrow$ T has a maximum at $(x,y) = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

(b) $T = 4x^2 - 4xy + 4y^2 \Rightarrow \frac{\partial T}{\partial x} = 8x - 4y$, and $\frac{\partial T}{\partial y} = 8y - 4x$ so the extreme values occur at the four points found in part (a): $T\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = T\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = 4\left(\frac{1}{2}\right) - 4\left(-\frac{1}{2}\right) + 4\left(\frac{1}{2}\right) = 6$, the maximum and
$T\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = T\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = 4\left(\frac{1}{2}\right) - 4\left(\frac{1}{2}\right) + 4\left(\frac{1}{2}\right) = 2$, the minimum

50. (a) $\frac{\partial T}{\partial x} = y$ and $\frac{\partial T}{\partial y} = x \Rightarrow \frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} = y\left(-2\sqrt{2}\sin t\right) + x\left(\sqrt{2}\cos t\right)$
$= \left(\sqrt{2}\sin t\right)\left(-2\sqrt{2}\sin t\right) + \left(2\sqrt{2}\cos t\right)\left(\sqrt{2}\cos t\right) = -4\sin^2 t + 4\cos^2 t = -4\sin^2 t + 4(1 - \sin^2 t)$
$= 4 - 8\sin^2 t \Rightarrow \frac{d^2T}{dt^2} = -16\sin t\cos t$; $\frac{dT}{dt} = 0 \Rightarrow 4 - 8\sin^2 t = 0 \Rightarrow \sin^2 t = \frac{1}{2} \Rightarrow \sin t = \pm\frac{1}{\sqrt{2}} \Rightarrow t = \frac{\pi}{4}$, $\frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ on the interval $0 \le t \le 2\pi$;
$\left.\frac{d^2T}{dt^2}\right|_{t=\frac{\pi}{4}} = -8\sin 2\left(\frac{\pi}{4}\right) = -8 \Rightarrow$ T has a maximum at $(x,y) = (2,1)$;
$\left.\frac{d^2T}{dt^2}\right|_{t=\frac{3\pi}{4}} = -8\sin 2\left(\frac{3\pi}{4}\right) = 8 \Rightarrow$ T has a minimum at $(x,y) = (-2,1)$;

$\left.\frac{d^2T}{dt^2}\right|_{t=\frac{5\pi}{4}} = -8 \sin 2\left(\frac{5\pi}{4}\right) = -8 \Rightarrow$ T has a maximum at $(x, y) = (-2, -1)$;

$\left.\frac{d^2T}{dt^2}\right|_{t=\frac{7\pi}{4}} = -8 \sin 2\left(\frac{7\pi}{4}\right) = 8 \Rightarrow$ T has a minimum at $(x, y) = (2, -1)$

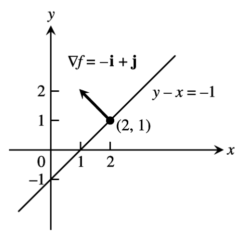
(b) $T = xy - 2 \Rightarrow \frac{\partial T}{\partial x} = y$ and $\frac{\partial T}{\partial y} = x$ so the extreme values occur at the four points found in part (a): $T(2, 1) = T(-2, -1) = 0$, the maximum and $T(-2, 1) = T(2, -1) = -4$, the minimum

51. $G(u, x) = \int_a^u g(t, x)\,dt$ where $u = f(x) \Rightarrow \frac{dG}{dx} = \frac{\partial G}{\partial u}\frac{du}{dx} + \frac{\partial G}{\partial x}\frac{dx}{dx} = g(u, x)f'(x) + \int_a^u g_x(t, x)\,dt$; thus $F(x) = \int_0^{x^2} \sqrt{t^4 + x^3}\,dt \Rightarrow F'(x) = \sqrt{(x^2)^4 + x^3}\,(2x) + \int_0^{x^2} \frac{\partial}{\partial x}\sqrt{t^4 + x^3}\,dt = 2x\sqrt{x^8 + x^3} + \int_0^{x^2} \frac{3x^2}{2\sqrt{t^4 + x^3}}\,dt$

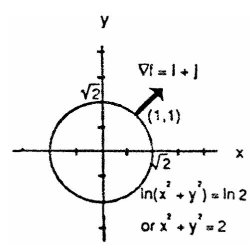
52. Using the result in Exercise 51, $F(x) = \int_{x^2}^1 \sqrt{t^3 + x^2}\,dt = -\int_1^{x^2} \sqrt{t^3 + x^2}\,dt \Rightarrow F'(x)$
$= \left[-\sqrt{(x^2)^3 + x^2}\,x^2 - \int_1^{x^2} \frac{\partial}{\partial x}\sqrt{t^3 + x^2}\,dt\right] = -x^2\sqrt{x^6 + x^2} + \int_{x^2}^1 \frac{x}{\sqrt{t^3 + x^2}}\,dt$

14.5 DIRECTIONAL DERIVATIVES AND GRADIENT VECTORS

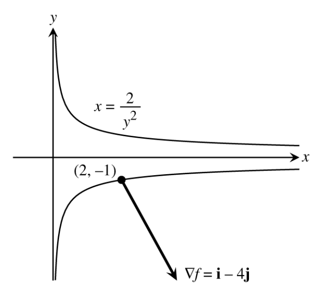
1. $\frac{\partial f}{\partial x} = -1,\ \frac{\partial f}{\partial y} = 1 \Rightarrow \nabla f = -\mathbf{i} + \mathbf{j}$; $f(2, 1) = -1$
$\Rightarrow -1 = y - x$ is the level curve



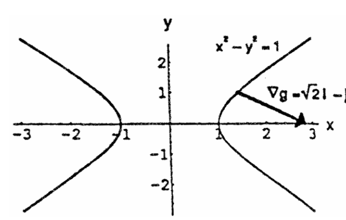
2. $\frac{\partial f}{\partial x} = \frac{2x}{x^2 + y^2} \Rightarrow \frac{\partial f}{\partial x}(1, 1) = 1$; $\frac{\partial f}{\partial y} = \frac{2y}{x^2 + y^2}$
$\Rightarrow \frac{\partial f}{\partial y}(1, 1) = 1 \Rightarrow \nabla f = \mathbf{i} + \mathbf{j}$; $f(1, 1) = \ln 2 \Rightarrow \ln 2$
$= \ln(x^2 + y^2) \Rightarrow 2 = x^2 + y^2$ is the level curve



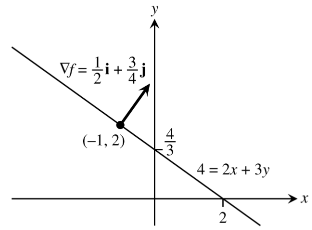
3. $\frac{\partial g}{\partial x} = y^2 \Rightarrow \frac{\partial g}{\partial x}(2, -1) = 1$; $\frac{\partial g}{\partial y} = 2xy \Rightarrow \frac{\partial g}{\partial x}(2, -1) = -4$;
$\Rightarrow \nabla g = \mathbf{i} - 4\mathbf{j}$; $g(2, -1) = 2 \Rightarrow x = \frac{2}{y^2}$ is the level curve



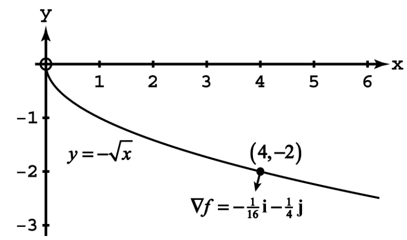
4. $\frac{\partial g}{\partial x} = x \Rightarrow \frac{\partial g}{\partial x}\left(\sqrt{2}, 1\right) = \sqrt{2}$; $\frac{\partial g}{\partial y} = -y$
$\Rightarrow \frac{\partial g}{\partial y}\left(\sqrt{2}, 1\right) = -1 \Rightarrow \nabla g = \sqrt{2}\,\mathbf{i} - \mathbf{j}$;
$g\left(\sqrt{2}, 1\right) = \frac{1}{2} \Rightarrow \frac{1}{2} = \frac{x^2}{2} - \frac{y^2}{2}$ or $1 = x^2 - y^2$ is the level curve



5. $\frac{\partial f}{\partial x} = \frac{1}{\sqrt{2x+3y}} \Rightarrow \frac{\partial f}{\partial x}(-1,2) = \frac{1}{2}$; $\frac{\partial f}{\partial y} = \frac{3}{2\sqrt{2x+3y}}$
$\Rightarrow \frac{\partial f}{\partial x}(-1,2) = \frac{3}{4}$; $\Rightarrow \nabla f = \frac{1}{2}\mathbf{i} + \frac{3}{4}\mathbf{j}$; $f(-1,2) = 2$
$\Rightarrow 4 = 2x + 3y$ is the level curve



6. $\frac{\partial f}{\partial x} = \frac{y}{2y^2\sqrt{x} + 2x^{3/2}} \Rightarrow \frac{\partial f}{\partial x}(4,-2) = -\frac{1}{16}$;
$\frac{\partial f}{\partial y} = -\frac{\sqrt{x}}{2y^2+x} \Rightarrow \frac{\partial f}{\partial y}(4,-2) = -\frac{1}{4} \Rightarrow \nabla f = -\frac{1}{16}\mathbf{i} - \frac{1}{4}\mathbf{j}$;
$f(4,-2) = -\frac{\pi}{4} \Rightarrow y = -\sqrt{x}$ is the level curve



7. $\frac{\partial f}{\partial x} = 2x + \frac{z}{x} \Rightarrow \frac{\partial f}{\partial x}(1,1,1) = 3$; $\frac{\partial f}{\partial y} = 2y \Rightarrow \frac{\partial f}{\partial y}(1,1,1) = 2$; $\frac{\partial f}{\partial z} = -4z + \ln x \Rightarrow \frac{\partial f}{\partial z}(1,1,1) = -4$;
thus $\nabla f = 3\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$

8. $\frac{\partial f}{\partial x} = -6xz + \frac{z}{x^2z^2+1} \Rightarrow \frac{\partial f}{\partial x}(1,1,1) = -\frac{11}{2}$; $\frac{\partial f}{\partial y} = -6yz \Rightarrow \frac{\partial f}{\partial y}(1,1,1) = -6$; $\frac{\partial f}{\partial z} = 6z^2 - 3(x^2+y^2) + \frac{x}{x^2z^2+1}$
$\Rightarrow \frac{\partial f}{\partial z}(1,1,1) = \frac{1}{2}$; thus $\nabla f = -\frac{11}{2}\mathbf{i} - 6\mathbf{j} + \frac{1}{2}\mathbf{k}$

9. $\frac{\partial f}{\partial x} = -\frac{x}{(x^2+y^2+z^2)^{3/2}} + \frac{1}{x} \Rightarrow \frac{\partial f}{\partial x}(-1,2,-2) = -\frac{26}{27}$; $\frac{\partial f}{\partial y} = -\frac{y}{(x^2+y^2+z^2)^{3/2}} + \frac{1}{y} \Rightarrow \frac{\partial f}{\partial y}(-1,2,-2) = \frac{23}{54}$;
$\frac{\partial f}{\partial z} = -\frac{z}{(x^2+y^2+z^2)^{3/2}} + \frac{1}{z} \Rightarrow \frac{\partial f}{\partial z}(-1,2,-2) = -\frac{23}{54}$; thus $\nabla f = -\frac{26}{27}\mathbf{i} + \frac{23}{54}\mathbf{j} - \frac{23}{54}\mathbf{k}$

10. $\frac{\partial f}{\partial x} = e^{x+y}\cos z + \frac{y+1}{\sqrt{1-x^2}} \Rightarrow \frac{\partial f}{\partial x}\left(0,0,\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} + 1$; $\frac{\partial f}{\partial y} = e^{x+y}\cos z + \sin^{-1} x \Rightarrow \frac{\partial f}{\partial y}\left(0,0,\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$;
$\frac{\partial f}{\partial z} = -e^{x+y}\sin z \Rightarrow \frac{\partial f}{\partial z}\left(0,0,\frac{\pi}{6}\right) = -\frac{1}{2}$; thus $\nabla f = \left(\frac{\sqrt{3}+2}{2}\right)\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j} - \frac{1}{2}\mathbf{k}$

11. $\mathbf{u} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{4\mathbf{i}+3\mathbf{j}}{\sqrt{4^2+3^2}} = \frac{4}{5}\mathbf{i} + \frac{3}{5}\mathbf{j}$; $f_x(x,y) = 2y \Rightarrow f_x(5,5) = 10$; $f_y(x,y) = 2x - 6y \Rightarrow f_y(5,5) = -20$
$\Rightarrow \nabla f = 10\mathbf{i} - 20\mathbf{j} \Rightarrow (D_{\mathbf{u}}f)_{P_0} = \nabla f \cdot \mathbf{u} = 10\left(\frac{4}{5}\right) - 20\left(\frac{3}{5}\right) = -4$

12. $\mathbf{u} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{3\mathbf{i}-4\mathbf{j}}{\sqrt{3^2+(-4)^2}} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$; $f_x(x,y) = 4x \Rightarrow f_x(-1,1) = -4$; $f_y(x,y) = 2y \Rightarrow f_y(-1,1) = 2$
$\Rightarrow \nabla f = -4\mathbf{i} + 2\mathbf{j} \Rightarrow (D_{\mathbf{u}}f)_{P_0} = \nabla f \cdot \mathbf{u} = -\frac{12}{5} - \frac{8}{5} = -4$

13. $\mathbf{u} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{12\mathbf{i}+5\mathbf{j}}{\sqrt{12^2+5^2}} = \frac{12}{13}\mathbf{i} + \frac{5}{13}\mathbf{j}$; $g_x(x,y) = \frac{y^2+2}{(xy+2)^2} \Rightarrow g_x(1,-1) = 3$; $g_y(x,y) = -\frac{x^2+2}{(xy+2)^2} \Rightarrow g_y(1,-1) = -3$
$\Rightarrow \nabla g = 3\mathbf{i} - 3\mathbf{j} \Rightarrow (D_{\mathbf{u}}g)_{P_0} = \nabla g \cdot \mathbf{u} = \frac{36}{13} - \frac{15}{13} = \frac{21}{13}$

14. $\mathbf{u} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{3\mathbf{i}-2\mathbf{j}}{\sqrt{3^2+(-2)^2}} = \frac{3}{\sqrt{13}}\mathbf{i} - \frac{2}{\sqrt{13}}\mathbf{j}$; $h_x(x,y) = \frac{\left(\frac{-y}{x^2}\right)}{\left(\frac{y}{x}\right)^2+1} + \frac{\left(\frac{y}{2}\right)\sqrt{3}}{\sqrt{1-\left(\frac{x^2y^2}{4}\right)}} \Rightarrow h_x(1,1) = \frac{1}{2}$;
$h_y(x,y) = \frac{\left(\frac{1}{x}\right)}{\left(\frac{y}{x}\right)^2+1} + \frac{\left(\frac{x}{2}\right)\sqrt{3}}{\sqrt{1-\left(\frac{x^2y^2}{4}\right)}} \Rightarrow h_y(1,1) = \frac{3}{2} \Rightarrow \nabla h = \frac{1}{2}\mathbf{i} + \frac{3}{2}\mathbf{j} \Rightarrow (D_{\mathbf{u}}h)_{P_0} = \nabla h \cdot \mathbf{u} = \frac{3}{2\sqrt{13}} - \frac{6}{2\sqrt{13}}$
$= -\frac{3}{2\sqrt{13}}$

15. $\mathbf{u} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{3\mathbf{i}+6\mathbf{j}-2\mathbf{k}}{\sqrt{3^2+6^2+(-2)^2}} = \frac{3}{7}\mathbf{i} + \frac{6}{7}\mathbf{j} - \frac{2}{7}\mathbf{k}$; $f_x(x,y,z) = y+z \Rightarrow f_x(1,-1,2) = 1$; $f_y(x,y,z) = x+z$ $\Rightarrow f_y(1,-1,2) = 3$; $f_z(x,y,z) = y+x \Rightarrow f_z(1,-1,2) = 0 \Rightarrow \nabla f = \mathbf{i} + 3\mathbf{j} \Rightarrow (D_\mathbf{u}f)_{P_0} = \nabla f \cdot \mathbf{u} = \frac{3}{7} + \frac{18}{7} = 3$

16. $\mathbf{u} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{1^2+1^2+1^2}} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$; $f_x(x,y,z) = 2x \Rightarrow f_x(1,1,1) = 2$; $f_y(x,y,z) = 4y$ $\Rightarrow f_y(1,1,1) = 4$; $f_z(x,y,z) = -6z \Rightarrow f_z(1,1,1) = -6 \Rightarrow \nabla f = 2\mathbf{i} + 4\mathbf{j} - 6\mathbf{k} \Rightarrow (D_\mathbf{u}f)_{P_0} = \nabla f \cdot \mathbf{u}$ $= 2\left(\frac{1}{\sqrt{3}}\right) + 4\left(\frac{1}{\sqrt{3}}\right) - 6\left(\frac{1}{\sqrt{3}}\right) = 0$

17. $\mathbf{u} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{2\mathbf{i}+\mathbf{j}-2\mathbf{k}}{\sqrt{2^2+1^2+(-2)^2}} = \frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$; $g_x(x,y,z) = 3e^x \cos yz \Rightarrow g_x(0,0,0) = 3$; $g_y(x,y,z) = -3ze^x \sin yz$ $\Rightarrow g_y(0,0,0) = 0$; $g_z(x,y,z) = -3ye^x \sin yz \Rightarrow g_z(0,0,0) = 0 \Rightarrow \nabla g = 3\mathbf{i} \Rightarrow (D_\mathbf{u}g)_{P_0} = \nabla g \cdot \mathbf{u} = 2$

18. $\mathbf{u} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{\mathbf{i}+2\mathbf{j}+2\mathbf{k}}{\sqrt{1^2+2^2+2^2}} = \frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$; $h_x(x,y,z) = -y \sin xy + \frac{1}{x} \Rightarrow h_x\left(1,0,\frac{1}{2}\right) = 1$; $h_y(x,y,z) = -x \sin xy + ze^{yz} \Rightarrow h_y\left(1,0,\frac{1}{2}\right) = \frac{1}{2}$; $h_z(x,y,z) = ye^{yz} + \frac{1}{z} \Rightarrow h_z\left(1,0,\frac{1}{2}\right) = 2 \Rightarrow \nabla h = \mathbf{i} + \frac{1}{2}\mathbf{j} + 2\mathbf{k}$ $\Rightarrow (D_\mathbf{u}h)_{P_0} = \nabla h \cdot \mathbf{u} = \frac{1}{3} + \frac{1}{3} + \frac{4}{3} = 2$

19. $\nabla f = (2x+y)\mathbf{i} + (x+2y)\mathbf{j} \Rightarrow \nabla f(-1,1) = -\mathbf{i} + \mathbf{j} \Rightarrow \mathbf{u} = \frac{\nabla f}{|\nabla f|} = \frac{-\mathbf{i}+\mathbf{j}}{\sqrt{(-1)^2+1^2}} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$; f increases most rapidly in the direction $\mathbf{u} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$ and decreases most rapidly in the direction $-\mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$; $(D_\mathbf{u}f)_{P_0} = \nabla f \cdot \mathbf{u} = |\nabla f| = \sqrt{2}$ and $(D_{-\mathbf{u}}f)_{P_0} = -\sqrt{2}$

20. $\nabla f = (2xy + ye^{xy} \sin y)\mathbf{i} + (x^2 + xe^{xy} \sin y + e^{xy} \cos y)\mathbf{j} \Rightarrow \nabla f(1,0) = 2\mathbf{j} \Rightarrow \mathbf{u} = \frac{\nabla f}{|\nabla f|} = \mathbf{j}$; f increases most rapidly in the direction $\mathbf{u} = \mathbf{j}$ and decreases most rapidly in the direction $-\mathbf{u} = -\mathbf{j}$; $(D_\mathbf{u}f)_{P_0} = \nabla f \cdot \mathbf{u} = |\nabla f|$ $= 2$ and $(D_{-\mathbf{u}}f)_{P_0} = -2$

21. $\nabla f = \frac{1}{y}\mathbf{i} - \left(\frac{x}{y^2} + z\right)\mathbf{j} - y\mathbf{k} \Rightarrow \nabla f(4,1,1) = \mathbf{i} - 5\mathbf{j} - \mathbf{k} \Rightarrow \mathbf{u} = \frac{\nabla f}{|\nabla f|} = \frac{\mathbf{i}-5\mathbf{j}-\mathbf{k}}{\sqrt{1^2+(-5)^2+(-1)^2}} = \frac{1}{3\sqrt{3}}\mathbf{i} - \frac{5}{3\sqrt{3}}\mathbf{j} - \frac{1}{3\sqrt{3}}\mathbf{k}$; f increases most rapidly in the direction of $\mathbf{u} = \frac{1}{3\sqrt{3}}\mathbf{i} - \frac{5}{3\sqrt{3}}\mathbf{j} - \frac{1}{3\sqrt{3}}\mathbf{k}$ and decreases most rapidly in the direction $-\mathbf{u} = -\frac{1}{3\sqrt{3}}\mathbf{i} + \frac{5}{3\sqrt{3}}\mathbf{j} + \frac{1}{3\sqrt{3}}\mathbf{k}$; $(D_\mathbf{u}f)_{P_0} = \nabla f \cdot \mathbf{u} = |\nabla f| = 3\sqrt{3}$ and $(D_{-\mathbf{u}}f)_{P_0} = -3\sqrt{3}$

22. $\nabla g = e^y\mathbf{i} + xe^y\mathbf{j} + 2z\mathbf{k} \Rightarrow \nabla g\left(1, \ln 2, \frac{1}{2}\right) = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{u} = \frac{\nabla g}{|\nabla g|} = \frac{2\mathbf{i}+2\mathbf{j}+\mathbf{k}}{\sqrt{2^2+2^2+1^2}} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}$; g increases most rapidly in the direction $\mathbf{u} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}$ and decreases most rapidly in the direction $-\mathbf{u} = -\frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}$; $(D_\mathbf{u}g)_{P_0} = \nabla g \cdot \mathbf{u} = |\nabla g| = 3$ and $(D_{-\mathbf{u}}g)_{P_0} = -3$

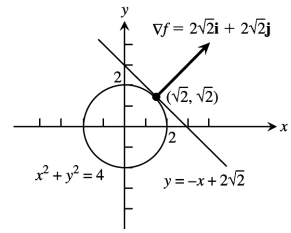
23. $\nabla f = \left(\frac{1}{x} + \frac{1}{x}\right)\mathbf{i} + \left(\frac{1}{y} + \frac{1}{y}\right)\mathbf{j} + \left(\frac{1}{z} + \frac{1}{z}\right)\mathbf{k} \Rightarrow \nabla f(1,1,1) = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \Rightarrow \mathbf{u} = \frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$; f increases most rapidly in the direction $\mathbf{u} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$ and decreases most rapidly in the direction $-\mathbf{u} = -\frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}$; $(D_\mathbf{u}f)_{P_0} = \nabla f \cdot \mathbf{u} = |\nabla f| = 2\sqrt{3}$ and $(D_{-\mathbf{u}}f)_{P_0} = -2\sqrt{3}$

24. $\nabla h = \left(\frac{2x}{x^2+y^2-1}\right)\mathbf{i} + \left(\frac{2y}{x^2+y^2-1} + 1\right)\mathbf{j} + 6\mathbf{k} \Rightarrow \nabla h(1,1,0) = 2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k} \Rightarrow \mathbf{u} = \frac{\nabla h}{|\nabla h|} = \frac{2\mathbf{i}+3\mathbf{j}+6\mathbf{k}}{\sqrt{2^2+3^2+6^2}}$ $= \frac{2}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}$; h increases most rapidly in the direction $\mathbf{u} = \frac{2}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}$ and decreases most rapidly in the direction $-\mathbf{u} = -\frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} - \frac{6}{7}\mathbf{k}$; $(D_\mathbf{u}h)_{P_0} = \nabla h \cdot \mathbf{u} = |\nabla h| = 7$ and $(D_{-\mathbf{u}}h)_{P_0} = -7$

25. $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} \Rightarrow \nabla f\left(\sqrt{2}, \sqrt{2}\right) = 2\sqrt{2}\,\mathbf{i} + 2\sqrt{2}\,\mathbf{j}$

$\Rightarrow$ Tangent line: $2\sqrt{2}\left(x - \sqrt{2}\right) + 2\sqrt{2}\left(y - \sqrt{2}\right) = 0$

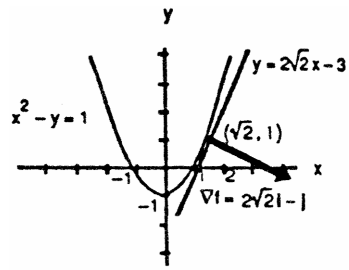
$\Rightarrow \sqrt{2}x + \sqrt{2}y = 4$



26. $\nabla f = 2x\mathbf{i} - \mathbf{j} \Rightarrow \nabla f\left(\sqrt{2}, 1\right) = 2\sqrt{2}\,\mathbf{i} - \mathbf{j}$

$\Rightarrow$ Tangent line: $2\sqrt{2}\left(x - \sqrt{2}\right) - (y - 1) = 0$

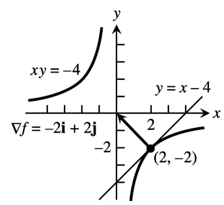
$\Rightarrow y = 2\sqrt{2}x - 3$



27. $\nabla f = y\mathbf{i} + x\mathbf{j} \Rightarrow \nabla f(2, -2) = -2\mathbf{i} + 2\mathbf{j}$

$\Rightarrow$ Tangent line: $-2(x - 2) + 2(y + 2) = 0$

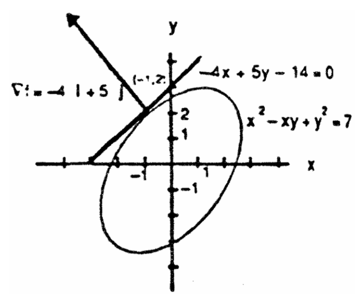
$\Rightarrow y = x - 4$



28. $\nabla f = (2x - y)\mathbf{i} + (2y - x)\mathbf{j} \Rightarrow \nabla f(-1, 2) = -4\mathbf{i} + 5\mathbf{j}$

$\Rightarrow$ Tangent line: $-4(x + 1) + 5(y - 2) = 0$

$\Rightarrow -4x + 5y - 14 = 0$



29. $\nabla f = (2x - y)\mathbf{i} + (-x + 2y - 1)\mathbf{j}$

(a) $\nabla f(1, -1) = 3\mathbf{i} - 4\mathbf{j} \Rightarrow |\nabla f(1, -1)| = 5 \Rightarrow D_{\mathbf{u}}f(1, -1) = 5$ in the direction of $\mathbf{u} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$

(b) $-\nabla f(1, -1) = -3\mathbf{i} + 4\mathbf{j} \Rightarrow |\nabla f(1, -1)| = 5 \Rightarrow D_{\mathbf{u}}f(1, -1) = -5$ in the direction of $\mathbf{u} = -\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$

(c) $D_{\mathbf{u}}f(1, -1) = 0$ in the direction of $\mathbf{u} = \frac{4}{5}\mathbf{i} + \frac{3}{5}\mathbf{j}$ or $\mathbf{u} = -\frac{4}{5}\mathbf{i} - \frac{3}{5}\mathbf{j}$

(d) Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} \Rightarrow |\mathbf{u}| = \sqrt{u_1^2 + u_2^2} = 1 \Rightarrow u_1^2 + u_2^2 = 1$; $D_{\mathbf{u}}f(1, -1) = \nabla f(1, -1) \cdot \mathbf{u} = (3\mathbf{i} - 4\mathbf{j}) \cdot (u_1\mathbf{i} + u_2\mathbf{j})$
$= 3u_1 - 4u_2 = 4 \Rightarrow u_2 = \frac{3}{4}u_1 - 1 \Rightarrow u_1^2 + \left(\frac{3}{4}u_1 - 1\right)^2 = 1 \Rightarrow \frac{25}{16}u_1^2 - \frac{3}{2}u_1 = 0 \Rightarrow u_1 = 0$ or $u_1 = \frac{24}{25}$;
$u_1 = 0 \Rightarrow u_2 = -1 \Rightarrow \mathbf{u} = -\mathbf{j}$, or $u_1 = \frac{24}{25} \Rightarrow u_2 = -\frac{7}{25} \Rightarrow \mathbf{u} = \frac{24}{25}\mathbf{i} - \frac{7}{25}\mathbf{j}$

(e) Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} \Rightarrow |\mathbf{u}| = \sqrt{u_1^2 + u_2^2} = 1 \Rightarrow u_1^2 + u_2^2 = 1$; $D_{\mathbf{u}}f(1, -1) = \nabla f(1, -1) \cdot \mathbf{u} = (3\mathbf{i} - 4\mathbf{j}) \cdot (u_1\mathbf{i} + u_2\mathbf{j})$
$= 3u_1 - 4u_2 = -3 \Rightarrow u_1 = \frac{4}{3}u_2 - 1 \Rightarrow \left(\frac{4}{3}u_2 - 1\right)^2 + u_2^2 = 1 \Rightarrow \frac{25}{9}u_2^2 - \frac{8}{3}u_2 = 0 \Rightarrow u_2 = 0$ or $u_2 = \frac{24}{25}$;
$u_2 = 0 \Rightarrow u_1 = -1 \Rightarrow \mathbf{u} = -\mathbf{i}$, or $u_2 = \frac{24}{25} \Rightarrow u_2 = \frac{7}{25} \Rightarrow \mathbf{u} = \frac{7}{25}\mathbf{i} + \frac{24}{25}\mathbf{j}$

.

30. $\nabla f = \frac{2y}{(x+y)^2}\mathbf{i} - \frac{2x}{(x+y)^2}\mathbf{j}$

(a) $\nabla f\left(-\frac{1}{2}, \frac{3}{2}\right) = 3\mathbf{i} + \mathbf{j} \Rightarrow \left|\nabla f\left(-\frac{1}{2}, \frac{3}{2}\right)\right| = \sqrt{10} \Rightarrow D_{\mathbf{u}}f\left(-\frac{1}{2}, \frac{3}{2}\right) = \sqrt{10}$ in the direction of $\mathbf{u} = \frac{3}{\sqrt{10}}\mathbf{i} + \frac{1}{\sqrt{10}}\mathbf{j}$

(b) $-\nabla f\left(-\frac{1}{2}, \frac{3}{2}\right) = -3\mathbf{i} - \mathbf{j} \Rightarrow \left|\nabla f\left(-\frac{1}{2}, \frac{3}{2}\right)\right| = \sqrt{10} \Rightarrow D_{\mathbf{u}}f(1, -1) = -\sqrt{10}$ in the direction of $\mathbf{u} = -\frac{3}{\sqrt{10}}\mathbf{i} - \frac{1}{\sqrt{10}}\mathbf{j}$

(c) $D_{\mathbf{u}}f\left(-\frac{1}{2}, \frac{3}{2}\right) = 0$ in the direction of $\mathbf{u} = \frac{1}{\sqrt{10}}\mathbf{i} - \frac{3}{\sqrt{10}}\mathbf{j}$ or $\mathbf{u} = -\frac{1}{\sqrt{10}}\mathbf{i} + \frac{3}{\sqrt{10}}\mathbf{j}$

(d) Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} \Rightarrow |\mathbf{u}| = \sqrt{u_1^2 + u_2^2} = 1 \Rightarrow u_1^2 + u_2^2 = 1$; $D_{\mathbf{u}}f\left(-\frac{1}{2}, \frac{3}{2}\right) = \nabla f\left(-\frac{1}{2}, \frac{3}{2}\right) \cdot \mathbf{u} = (3\mathbf{i} + \mathbf{j}) \cdot (u_1\mathbf{i} + u_2\mathbf{j})$
$= 3u_1 + u_2 = -2 \Rightarrow u_2 = -3u_1 - 2 \Rightarrow u_1^2 + (-3u_1 - 2)^2 = 1 \Rightarrow 10u_1^2 + 12u_1 + 3 = 0 \Rightarrow u_1 = \frac{-6 \pm \sqrt{6}}{10}$
$u_1 = \frac{-6+\sqrt{6}}{10} \Rightarrow u_2 = \frac{-2-3\sqrt{6}}{10} \Rightarrow \mathbf{u} = \frac{-6+\sqrt{6}}{10}\mathbf{i} + \frac{-2-3\sqrt{6}}{10}\mathbf{j}$, or $u_1 = \frac{-6-\sqrt{6}}{10} \Rightarrow u_2 = \frac{-2+3\sqrt{6}}{10}$
$\Rightarrow \mathbf{u} = \frac{-6-\sqrt{6}}{10}\mathbf{i} + \frac{-2+3\sqrt{6}}{10}\mathbf{j}$

(e) Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} \Rightarrow |\mathbf{u}| = \sqrt{u_1^2 + u_2^2} = 1 \Rightarrow u_1^2 + u_2^2 = 1$; $D_{\mathbf{u}}f\left(-\frac{1}{2}, \frac{3}{2}\right) = \nabla f\left(-\frac{1}{2}, \frac{3}{2}\right) \cdot \mathbf{u} = (3\mathbf{i} + \mathbf{j}) \cdot (u_1\mathbf{i} + u_2\mathbf{j})$
$= 3u_1 + u_2 = 1 \Rightarrow u_2 = 1 - 3u_1 \Rightarrow u_1^2 + (1 - 3u_1)^2 = 1 \Rightarrow 10u_1^2 - 6u_1 = 0 \Rightarrow u_1 = 0$ or $u_1 = \frac{3}{5}$;
$u_1 = 0 \Rightarrow u_2 = 1 \Rightarrow \mathbf{u} = \mathbf{j}$, or $u_1 = \frac{3}{5} \Rightarrow u_2 = -\frac{4}{5} \Rightarrow \mathbf{u} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$

31. $\nabla f = y\mathbf{i} + (x + 2y)\mathbf{j} \Rightarrow \nabla f(3, 2) = 2\mathbf{i} + 7\mathbf{j}$; a vector orthogonal to ∇f is $\mathbf{v} = 7\mathbf{i} - 2\mathbf{j} \Rightarrow \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{7\mathbf{i} - 2\mathbf{j}}{\sqrt{7^2 + (-2)^2}}$
$= \frac{7}{\sqrt{53}}\mathbf{i} - \frac{2}{\sqrt{53}}\mathbf{j}$ and $-\mathbf{u} = -\frac{7}{\sqrt{53}}\mathbf{i} + \frac{2}{\sqrt{53}}\mathbf{j}$ are the directions where the derivative is zero

32. $\nabla f = \frac{4xy^2}{(x^2 + y^2)^2}\mathbf{i} - \frac{4x^2y}{(x^2 + y^2)^2}\mathbf{j} \Rightarrow \nabla f(1, 1) = \mathbf{i} - \mathbf{j}$; a vector orthogonal to ∇f is $\mathbf{v} = \mathbf{i} + \mathbf{j}$
$\Rightarrow \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{i} + \mathbf{j}}{\sqrt{1^2 + 1^2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$ and $-\mathbf{u} = -\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$ are the directions where the derivative is zero

33. $\nabla f = (2x - 3y)\mathbf{i} + (-3x + 8y)\mathbf{j} \Rightarrow \nabla f(1, 2) = -4\mathbf{i} + 13\mathbf{j} \Rightarrow |\nabla f(1, 2)| = \sqrt{(-4)^2 + (13)^2} = \sqrt{185}$; no, the maximum rate of change is $\sqrt{185} < 14$

34. $\nabla T = 2y\mathbf{i} + (2x - z)\mathbf{j} - y\mathbf{k} \Rightarrow \nabla T(1, -1, 1) = -2\mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow |\nabla T(1, -1, 1)| = \sqrt{(-2)^2 + 1^2 + 1^2} = \sqrt{6}$; no, the minimum rate of change is $-\sqrt{6} > -3$

35. $\nabla f = f_x(1, 2)\mathbf{i} + f_y(1, 2)\mathbf{j}$ and $\mathbf{u}_1 = \frac{\mathbf{i} + \mathbf{j}}{\sqrt{1^2 + 1^2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} \Rightarrow (D_{\mathbf{u}_1}f)(1, 2) = f_x(1, 2)\left(\frac{1}{\sqrt{2}}\right) + f_y(1, 2)\left(\frac{1}{\sqrt{2}}\right)$
$= 2\sqrt{2} \Rightarrow f_x(1, 2) + f_y(1, 2) = 4$; $\mathbf{u}_2 = -\mathbf{j} \Rightarrow (D_{\mathbf{u}_2}f)(1, 2) = f_x(1, 2)(0) + f_y(1, 2)(-1) = -3 \Rightarrow -f_y(1, 2) = -3$
$\Rightarrow f_y(1, 2) = 3$; then $f_x(1, 2) + 3 = 4 \Rightarrow f_x(1, 2) = 1$; thus $\nabla f(1, 2) = \mathbf{i} + 3\mathbf{j}$ and $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{-\mathbf{i} - 2\mathbf{j}}{\sqrt{(-1)^2 + (-2)^2}}$
$= -\frac{1}{\sqrt{5}}\mathbf{i} - \frac{2}{\sqrt{5}}\mathbf{j} \Rightarrow (D_{\mathbf{u}}f)_{P_0} = \nabla f \cdot \mathbf{u} = -\frac{1}{\sqrt{5}} - \frac{6}{\sqrt{5}} = -\frac{7}{\sqrt{5}}$

36. (a) $(D_{\mathbf{u}}f)_P = 2\sqrt{3} \Rightarrow |\nabla f| = 2\sqrt{3}$; $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{i} + \mathbf{j} - \mathbf{k}}{\sqrt{1^2 + 1^2 + (-1)^2}} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}$; thus $\mathbf{u} = \frac{\nabla f}{|\nabla f|}$
$\Rightarrow \nabla f = |\nabla f|\mathbf{u} \Rightarrow \nabla f = 2\sqrt{3}\left(\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}\right) = 2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$

(b) $\mathbf{v} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{i} + \mathbf{j}}{\sqrt{1^2 + 1^2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} \Rightarrow (D_{\mathbf{u}}f)_{P_0} = \nabla f \cdot \mathbf{u} = 2\left(\frac{1}{\sqrt{2}}\right) + 2\left(\frac{1}{\sqrt{2}}\right) - 2(0) = 2\sqrt{2}$

37. The directional derivative is the scalar component. With ∇f evaluated at P_0, the scalar component of ∇f in the direction of $\mathbf{u}$ is $\nabla f \cdot \mathbf{u} = (D_{\mathbf{u}}f)_{P_0}$.

38. $D_{\mathbf{i}}f = \nabla f \cdot \mathbf{i} = (f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}) \cdot \mathbf{i} = f_x$; similarly, $D_{\mathbf{j}}f = \nabla f \cdot \mathbf{j} = f_y$ and $D_{\mathbf{k}}f = \nabla f \cdot \mathbf{k} = f_z$

39. If (x, y) is a point on the line, then $\mathbf{T}(x, y) = (x - x_0)\mathbf{i} + (y - y_0)\mathbf{j}$ is a vector parallel to the line $\Rightarrow \mathbf{T} \cdot \mathbf{N} = 0$
$\Rightarrow A(x - x_0) + B(y - y_0) = 0$, as claimed.

40. (a) $\nabla(kf) = \frac{\partial(kf)}{\partial x}\mathbf{i} + \frac{\partial(kf)}{\partial y}\mathbf{j} + \frac{\partial(kf)}{\partial z}\mathbf{k} = k\left(\frac{\partial f}{\partial x}\right)\mathbf{i} + k\left(\frac{\partial f}{\partial y}\right)\mathbf{j} + k\left(\frac{\partial f}{\partial z}\right)\mathbf{k} = k\left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right) = k\nabla f$

(b) $\nabla(f+g) = \frac{\partial(f+g)}{\partial x}\mathbf{i} + \frac{\partial(f+g)}{\partial y}\mathbf{j} + \frac{\partial(f+g)}{\partial z}\mathbf{k} = \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial x}\right)\mathbf{i} + \left(\frac{\partial f}{\partial y} + \frac{\partial g}{\partial y}\right)\mathbf{j} + \left(\frac{\partial f}{\partial z} + \frac{\partial g}{\partial z}\right)\mathbf{k}$
$= \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial g}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial g}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} + \frac{\partial g}{\partial z}\mathbf{k} = \left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right) + \left(\frac{\partial g}{\partial x}\mathbf{i} + \frac{\partial g}{\partial y}\mathbf{j} + \frac{\partial g}{\partial z}\mathbf{k}\right) = \nabla f + \nabla g$

(c) $\nabla(f-g) = \nabla f - \nabla g$ (Substitute $-g$ for g in part (b) above)

(d) $\nabla(fg) = \frac{\partial(fg)}{\partial x}\mathbf{i} + \frac{\partial(fg)}{\partial y}\mathbf{j} + \frac{\partial(fg)}{\partial z}\mathbf{k} = \left(\frac{\partial f}{\partial x}g + \frac{\partial g}{\partial x}f\right)\mathbf{i} + \left(\frac{\partial f}{\partial y}g + \frac{\partial g}{\partial y}f\right)\mathbf{j} + \left(\frac{\partial f}{\partial z}g + \frac{\partial g}{\partial z}f\right)\mathbf{k}$
$= \left(\frac{\partial f}{\partial x}g\right)\mathbf{i} + \left(\frac{\partial g}{\partial x}f\right)\mathbf{i} + \left(\frac{\partial f}{\partial y}g\right)\mathbf{j} + \left(\frac{\partial g}{\partial y}f\right)\mathbf{j} + \left(\frac{\partial f}{\partial z}g\right)\mathbf{k} + \left(\frac{\partial g}{\partial z}f\right)\mathbf{k}$
$= f\left(\frac{\partial g}{\partial x}\mathbf{i} + \frac{\partial g}{\partial y}\mathbf{j} + \frac{\partial g}{\partial z}\mathbf{k}\right) + g\left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right) = f\nabla g + g\nabla f$

(e) $\nabla\left(\frac{f}{g}\right) = \frac{\partial\left(\frac{f}{g}\right)}{\partial x}\mathbf{i} + \frac{\partial\left(\frac{f}{g}\right)}{\partial y}\mathbf{j} + \frac{\partial\left(\frac{f}{g}\right)}{\partial z}\mathbf{k} = \left(\frac{g\frac{\partial f}{\partial x} - f\frac{\partial g}{\partial x}}{g^2}\right)\mathbf{i} + \left(\frac{g\frac{\partial f}{\partial y} - f\frac{\partial g}{\partial y}}{g^2}\right)\mathbf{j} + \left(\frac{g\frac{\partial f}{\partial z} - f\frac{\partial g}{\partial z}}{g^2}\right)\mathbf{k}$
$= \left(\frac{g\frac{\partial f}{\partial x}\mathbf{i} + g\frac{\partial f}{\partial y}\mathbf{j} + g\frac{\partial f}{\partial z}\mathbf{k}}{g^2}\right) - \left(\frac{f\frac{\partial g}{\partial x}\mathbf{i} + f\frac{\partial g}{\partial y}\mathbf{j} + f\frac{\partial g}{\partial z}\mathbf{k}}{g^2}\right) = \frac{g\left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right)}{g^2} - \frac{f\left(\frac{\partial g}{\partial x}\mathbf{i} + \frac{\partial g}{\partial y}\mathbf{j} + \frac{\partial g}{\partial z}\mathbf{k}\right)}{g^2}$
$= \frac{g\nabla f}{g^2} - \frac{f\nabla g}{g^2} = \frac{g\nabla f - f\nabla g}{g^2}$

14.6 TANGENT PLANES AND DIFFERENTIALS

1. (a) $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow \nabla f(1,1,1) = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \Rightarrow$ Tangent plane: $2(x-1) + 2(y-1) + 2(z-1) = 0$
$\Rightarrow x + y + z = 3$;
(b) Normal line: $x = 1 + 2t,\ y = 1 + 2t,\ z = 1 + 2t$

2. (a) $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} - 2z\mathbf{k} \Rightarrow \nabla f(3,5,-4) = 6\mathbf{i} + 10\mathbf{j} + 8\mathbf{k} \Rightarrow$ Tangent plane: $6(x-3) + 10(y-5) + 8(z+4) = 0$
$\Rightarrow 3x + 5y + 4z = 18$;
(b) Normal line: $x = 3 + 6t,\ y = 5 + 10t,\ z = -4 + 8t$

3. (a) $\nabla f = -2x\mathbf{i} + 2\mathbf{k} \Rightarrow \nabla f(2,0,2) = -4\mathbf{i} + 2\mathbf{k} \Rightarrow$ Tangent plane: $-4(x-2) + 2(z-2) = 0$
$\Rightarrow -4x + 2z + 4 = 0 \Rightarrow -2x + z + 2 = 0$;
(b) Normal line: $x = 2 - 4t,\ y = 0,\ z = 2 + 2t$

4. (a) $\nabla f = (2x+2y)\mathbf{i} + (2x-2y)\mathbf{j} + 2z\mathbf{k} \Rightarrow \nabla f(1,-1,3) = 4\mathbf{j} + 6\mathbf{k} \Rightarrow$ Tangent plane: $4(y+1) + 6(z-3) = 0$
$\Rightarrow 2y + 3z = 7$;
(b) Normal line: $x = 1,\ y = -1 + 4t,\ z = 3 + 6t$

5. (a) $\nabla f = (-\pi\sin\pi x - 2xy + ze^{xz})\,\mathbf{i} + (-x^2 + z)\,\mathbf{j} + (xe^{xz} + y)\,\mathbf{k} \Rightarrow \nabla f(0,1,2) = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k} \Rightarrow$ Tangent plane:
$2(x-0) + 2(y-1) + 1(z-2) = 0 \Rightarrow 2x + 2y + z - 4 = 0$;
(b) Normal line: $x = 2t,\ y = 1 + 2t,\ z = 2 + t$

6. (a) $\nabla f = (2x-y)\mathbf{i} - (x+2y)\mathbf{j} - \mathbf{k} \Rightarrow \nabla f(1,1,-1) = \mathbf{i} - 3\mathbf{j} - \mathbf{k} \Rightarrow$ Tangent plane:
$1(x-1) - 3(y-1) - 1(z+1) = 0 \Rightarrow x - 3y - z = -1$;
(b) Normal line: $x = 1 + t,\ y = 1 - 3t,\ z = -1 - t$

7. (a) $\nabla f = \mathbf{i} + \mathbf{j} + \mathbf{k}$ for all points $\Rightarrow \nabla f(0,1,0) = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow$ Tangent plane: $1(x-0) + 1(y-1) + 1(z-0) = 0$
$\Rightarrow x + y + z - 1 = 0$;
(b) Normal line: $x = t,\ y = 1 + t,\ z = t$

8. (a) $\nabla f = (2x - 2y - 1)\mathbf{i} + (2y - 2x + 3)\mathbf{j} - \mathbf{k} \Rightarrow \nabla f(2,-3,18) = 9\mathbf{i} - 7\mathbf{j} - \mathbf{k} \Rightarrow$ Tangent plane:
$9(x-2) - 7(y+3) - 1(z-18) = 0 \Rightarrow 9x - 7y - z = 21$;
(b) Normal line: $x = 2 + 9t,\ y = -3 - 7t,\ z = 18 - t$

9. $z = f(x, y) = \ln(x^2 + y^2) \Rightarrow f_x(x, y) = \frac{2x}{x^2+y^2}$ and $f_y(x, y) = \frac{2y}{x^2+y^2} \Rightarrow f_x(1, 0) = 2$ and $f_y(1, 0) = 0 \Rightarrow$ from Eq. (4) the tangent plane at $(1, 0, 0)$ is $2(x - 1) - z = 0$ or $2x - z - 2 = 0$

10. $z = f(x, y) = e^{-(x^2+y^2)} \Rightarrow f_x(x, y) = -2xe^{-(x^2+y^2)}$ and $f_y(x, y) = -2ye^{-(x^2+y^2)} \Rightarrow f_x(0, 0) = 0$ and $f_y(0, 0) = 0$ $\Rightarrow$ from Eq. (4) the tangent plane at $(0, 0, 1)$ is $z - 1 = 0$ or $z = 1$

11. $z = f(x, y) = \sqrt{y - x} \Rightarrow f_x(x, y) = -\frac{1}{2}(y - x)^{-1/2}$ and $f_y(x, y) = \frac{1}{2}(y - x)^{-1/2} \Rightarrow f_x(1, 2) = -\frac{1}{2}$ and $f_y(1, 2) = \frac{1}{2}$ $\Rightarrow$ from Eq. (4) the tangent plane at $(1, 2, 1)$ is $-\frac{1}{2}(x - 1) + \frac{1}{2}(y - 2) - (z - 1) = 0 \Rightarrow x - y + 2z - 1 = 0$

12. $z = f(x, y) = 4x^2 + y^2 \Rightarrow f_x(x, y) = 8x$ and $f_y(x, y) = 2y \Rightarrow f_x(1, 1) = 8$ and $f_y(1, 1) = 2 \Rightarrow$ from Eq. (4) the tangent plane at $(1, 1, 5)$ is $8(x - 1) + 2(y - 1) - (z - 5) = 0$ or $8x + 2y - z - 5 = 0$

13. $\nabla f = \mathbf{i} + 2y\mathbf{j} + 2\mathbf{k} \Rightarrow \nabla f(1, 1, 1) = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\nabla g = \mathbf{i}$ for all points; $\mathbf{v} = \nabla f \times \nabla g$
$\Rightarrow \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 2 \\ 1 & 0 & 0 \end{vmatrix} = 2\mathbf{j} - 2\mathbf{k} \Rightarrow$ Tangent line: $x = 1, y = 1 + 2t, z = 1 - 2t$

14. $\nabla f = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} \Rightarrow \nabla f(1, 1, 1) = \mathbf{i} + \mathbf{j} + \mathbf{k}$; $\nabla g = 2x\mathbf{i} + 4y\mathbf{j} + 6z\mathbf{k} \Rightarrow \nabla g(1, 1, 1) = 2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$;
$\Rightarrow \mathbf{v} = \nabla f \times \nabla g \Rightarrow \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 2 & 4 & 6 \end{vmatrix} = 2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k} \Rightarrow$ Tangent line: $x = 1 + 2t, y = 1 - 4t, z = 1 + 2t$

15. $\nabla f = 2x\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \Rightarrow \nabla f\left(1, 1, \frac{1}{2}\right) = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\nabla g = \mathbf{j}$ for all points; $\mathbf{v} = \nabla f \times \nabla g$
$\Rightarrow \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 2 \\ 0 & 1 & 0 \end{vmatrix} = -2\mathbf{i} + 2\mathbf{k} \Rightarrow$ Tangent line: $x = 1 - 2t, y = 1, z = \frac{1}{2} + 2t$

16. $\nabla f = \mathbf{i} + 2y\mathbf{j} + \mathbf{k} \Rightarrow \nabla f\left(\frac{1}{2}, 1, \frac{1}{2}\right) = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\nabla g = \mathbf{j}$ for all points; $\mathbf{v} = \nabla f \times \nabla g$
$\Rightarrow \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -\mathbf{i} + \mathbf{k} \Rightarrow$ Tangent line: $x = \frac{1}{2} - t, y = 1, z = \frac{1}{2} + t$

17. $\nabla f = (3x^2 + 6xy^2 + 4y)\mathbf{i} + (6x^2y + 3y^2 + 4x)\mathbf{j} - 2z\mathbf{k} \Rightarrow \nabla f(1, 1, 3) = 13\mathbf{i} + 13\mathbf{j} - 6\mathbf{k}$; $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$
$\Rightarrow \nabla g(1, 1, 3) = 2\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$; $\mathbf{v} = \nabla f \times \nabla g \Rightarrow \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 13 & 13 & -6 \\ 2 & 2 & 6 \end{vmatrix} = 90\mathbf{i} - 90\mathbf{j} \Rightarrow$ Tangent line:
$x = 1 + 90t, y = 1 - 90t, z = 3$

18. $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} \Rightarrow \nabla f\left(\sqrt{2}, \sqrt{2}, 4\right) = 2\sqrt{2}\,\mathbf{i} + 2\sqrt{2}\,\mathbf{j}$; $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k} \Rightarrow \nabla g\left(\sqrt{2}, \sqrt{2}, 4\right)$
$= 2\sqrt{2}\mathbf{i} + 2\sqrt{2}\mathbf{j} - \mathbf{k}$; $\mathbf{v} = \nabla f \times \nabla g \Rightarrow \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2\sqrt{2} & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 2\sqrt{2} & -1 \end{vmatrix} = -2\sqrt{2}\,\mathbf{i} + 2\sqrt{2}\,\mathbf{j} \Rightarrow$ Tangent line:
$x = \sqrt{2} - 2\sqrt{2}\,t, y = \sqrt{2} + 2\sqrt{2}\,t, z = 4$

19. $\nabla f = \left(\frac{x}{x^2+y^2+z^2}\right)\mathbf{i} + \left(\frac{y}{x^2+y^2+z^2}\right)\mathbf{j} + \left(\frac{z}{x^2+y^2+z^2}\right)\mathbf{k} \Rightarrow \nabla f(3, 4, 12) = \frac{3}{169}\mathbf{i} + \frac{4}{169}\mathbf{j} + \frac{12}{169}\mathbf{k}$;
$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}}{\sqrt{3^2 + 6^2 + (-2)^2}} = \frac{3}{7}\mathbf{i} + \frac{6}{7}\mathbf{j} - \frac{2}{7}\mathbf{k} \Rightarrow \nabla f \cdot \mathbf{u} = \frac{9}{1183}$ and $df = (\nabla f \cdot \mathbf{u})\,ds = \left(\frac{9}{1183}\right)(0.1) \approx 0.0008$

20. $\nabla f = (e^x \cos yz)\,\mathbf{i} - (ze^x \sin yz)\,\mathbf{j} - (ye^x \sin yz)\,\mathbf{k} \Rightarrow \nabla f(0,0,0) = \mathbf{i}\,;\ \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2\mathbf{i}+2\mathbf{j}-2\mathbf{k}}{\sqrt{2^2+2^2+(-2)^2}}$
$= \frac{1}{\sqrt{3}}\,\mathbf{i} + \frac{1}{\sqrt{3}}\,\mathbf{j} - \frac{1}{\sqrt{3}}\,\mathbf{k} \Rightarrow \nabla f \cdot \mathbf{u} = \frac{1}{\sqrt{3}}$ and $df = (\nabla f \cdot \mathbf{u})\, ds = \frac{1}{\sqrt{3}}(0.1) \approx 0.0577$

21. $\nabla g = (1+\cos z)\mathbf{i} + (1-\sin z)\mathbf{j} + (-x\sin z - y\cos z)\mathbf{k} \Rightarrow \nabla g(2,-1,0) = 2\mathbf{i} + \mathbf{j} + \mathbf{k};\ \mathbf{A} = \overrightarrow{P_0P_1} = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$
$\Rightarrow \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{-2\mathbf{i}+2\mathbf{j}+2\mathbf{k}}{\sqrt{(-2)^2+2^2+2^2}} = -\frac{1}{\sqrt{3}}\,\mathbf{i} + \frac{1}{\sqrt{3}}\,\mathbf{j} + \frac{1}{\sqrt{3}}\,\mathbf{k} \Rightarrow \nabla g \cdot \mathbf{u} = 0$ and $dg = (\nabla g \cdot \mathbf{u})\, ds = (0)(0.2) = 0$

22. $\nabla h = [-\pi y \sin(\pi xy) + z^2]\,\mathbf{i} - [\pi x \sin(\pi xy)]\,\mathbf{j} + 2xz\mathbf{k} \Rightarrow \nabla h(-1,-1,-1) = (\pi \sin \pi + 1)\mathbf{i} + (\pi \sin \pi)\mathbf{j} + 2\mathbf{k}$
$= \mathbf{i} + 2\mathbf{k}\,;\ \mathbf{v} = \overrightarrow{P_0P_1} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ where $P_1 = (0,0,0) \Rightarrow \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{1^2+1^2+1^2}} = \frac{1}{\sqrt{3}}\,\mathbf{i} + \frac{1}{\sqrt{3}}\,\mathbf{j} + \frac{1}{\sqrt{3}}\,\mathbf{k}$
$\Rightarrow \nabla h \cdot \mathbf{u} = \frac{3}{\sqrt{3}} = \sqrt{3}$ and $dh = (\nabla h \cdot \mathbf{u})\, ds = \sqrt{3}(0.1) \approx 0.1732$

23. (a) The unit tangent vector at $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ in the direction of motion is $\mathbf{u} = \frac{\sqrt{3}}{2}\,\mathbf{i} - \frac{1}{2}\,\mathbf{j}\,;$
$\nabla T = (\sin 2y)\mathbf{i} + (2x\cos 2y)\mathbf{j} \Rightarrow \nabla T\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = \left(\sin \sqrt{3}\right)\mathbf{i} + \left(\cos \sqrt{3}\right)\mathbf{j} \Rightarrow D_{\mathbf{u}}T\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = \nabla T \cdot \mathbf{u}$
$= \frac{\sqrt{3}}{2} \sin \sqrt{3} - \frac{1}{2}\cos\sqrt{3} \approx 0.935°$ C/ft

(b) $\mathbf{r}(t) = (\sin 2t)\mathbf{i} + (\cos 2t)\mathbf{j} \Rightarrow \mathbf{v}(t) = (2\cos 2t)\mathbf{i} - (2\sin 2t)\mathbf{j}$ and $|\mathbf{v}| = 2;\ \frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt}$
$= \nabla T \cdot \mathbf{v} = \left(\nabla T \cdot \frac{\mathbf{v}}{|\mathbf{v}|}\right)|\mathbf{v}| = (D_{\mathbf{u}}T)\,|\mathbf{v}|$, where $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}$; at $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ we have $\mathbf{u} = \frac{\sqrt{3}}{2}\,\mathbf{i} - \frac{1}{2}\,\mathbf{j}$ from part (a)
$\Rightarrow \frac{dT}{dt} = \left(\frac{\sqrt{3}}{2}\sin\sqrt{3} - \frac{1}{2}\cos\sqrt{3}\right) \cdot 2 = \sqrt{3}\sin\sqrt{3} - \cos\sqrt{3} \approx 1.87°$ C/sec

24. (a) $\nabla T = (4x - yz)\mathbf{i} - xz\mathbf{j} - xy\mathbf{k} \Rightarrow \nabla T(8,6,-4) = 56\mathbf{i} + 32\mathbf{j} - 48\mathbf{k}\,;\ \mathbf{r}(t) = 2t^2\mathbf{i} + 3t\mathbf{j} - t^2\mathbf{k} \Rightarrow$ the particle is at the point $P(8,6,-4)$ when $t = 2$; $\mathbf{v}(t) = 4t\mathbf{i} + 3\mathbf{j} - 2t\mathbf{k} \Rightarrow \mathbf{v}(2) = 8\mathbf{i} + 3\mathbf{j} - 4\mathbf{k} \Rightarrow \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}$
$= \frac{8}{\sqrt{89}}\,\mathbf{i} + \frac{3}{\sqrt{89}}\,\mathbf{j} - \frac{4}{\sqrt{89}}\,\mathbf{k} \Rightarrow D_{\mathbf{u}}T(8,6,-4) = \nabla T \cdot \mathbf{u} = \frac{1}{\sqrt{89}}[56 \cdot 8 + 32 \cdot 3 - 48 \cdot (-4)] = \frac{736}{\sqrt{89}}$ ° C/m

(b) $\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} = \nabla T \cdot \mathbf{v} = (\nabla T \cdot \mathbf{u})\,|\mathbf{v}| \Rightarrow$ at $t = 2$, $\frac{dT}{dt} = D_{\mathbf{u}}T\big|_{t=2}\,\mathbf{v}(2) = \left(\frac{736}{\sqrt{89}}\right)\sqrt{89} = 736°$ C/sec

25. (a) $f(0,0) = 1,\ f_x(x,y) = 2x \Rightarrow f_x(0,0) = 0,\ f_y(x,y) = 2y \Rightarrow f_y(0,0) = 0 \Rightarrow L(x,y) = 1 + 0(x-0) + 0(y-0) = 1$

(b) $f(1,1) = 3,\ f_x(1,1) = 2,\ f_y(1,1) = 2 \Rightarrow L(x,y) = 3 + 2(x-1) + 2(y-1) = 2x + 2y - 1$

26. (a) $f(0,0) = 4,\ f_x(x,y) = 2(x+y+2) \Rightarrow f_x(0,0) = 4,\ f_y(x,y) = 2(x+y+2) \Rightarrow f_y(0,0) = 4$
$\Rightarrow L(x,y) = 4 + 4(x-0) + 4(y-0) = 4x + 4y + 4$

(b) $f(1,2) = 25,\ f_x(1,2) = 10,\ f_y(1,2) = 10 \Rightarrow L(x,y) = 25 + 10(x-1) + 10(y-2) = 10x + 10y - 5$

27. (a) $f(0,0) = 5,\ f_x(x,y) = 3$ for all (x,y), $f_y(x,y) = -4$ for all $(x,y) \Rightarrow L(x,y) = 5 + 3(x-0) - 4(y-0) = 3x - 4y + 5$

(b) $f(1,1) = 4,\ f_x(1,1) = 3,\ f_y(1,1) = -4 \Rightarrow L(x,y) = 4 + 3(x-1) - 4(y-1) = 3x - 4y + 5$

28. (a) $f(1,1) = 1,\ f_x(x,y) = 3x^2y^4 \Rightarrow f_x(1,1) = 3,\ f_y(x,y) = 4x^3y^3 \Rightarrow f_y(1,1) = 4$
$\Rightarrow L(x,y) = 1 + 3(x-1) + 4(y-1) = 3x + 4y - 6$

(b) $f(0,0) = 0,\ f_x(0,0) = 0,\ f_y(0,0) = 0 \Rightarrow L(x,y) = 0$

29. (a) $f(0,0) = 1,\ f_x(x,y) = e^x \cos y \Rightarrow f_x(0,0) = 1,\ f_y(x,y) = -e^x \sin y \Rightarrow f_y(0,0) = 0$
$\Rightarrow L(x,y) = 1 + 1(x-0) + 0(y-0) = x + 1$

(b) $f\left(0, \frac{\pi}{2}\right) = 0,\ f_x\left(0, \frac{\pi}{2}\right) = 0,\ f_y\left(0, \frac{\pi}{2}\right) = -1 \Rightarrow L(x,y) = 0 + 0(x-0) - 1\left(y - \frac{\pi}{2}\right) = -y + \frac{\pi}{2}$

30. (a) $f(0,0) = 1$, $f_x(x,y) = -e^{2y-x} \Rightarrow f_x(0,0) = -1$, $f_y(x,y) = 2e^{2y-x} \Rightarrow f_y(0,0) = 2$
$\Rightarrow L(x,y) = 1 - 1(x-0) + 2(y-0) = -x + 2y + 1$
(b) $f(1,2) = e^3$, $f_x(1,2) = -e^3$, $f_y(1,2) = 2e^3 \Rightarrow L(x,y) = e^3 - e^3(x-1) + 2e^3(y-2) = -e^3x + 2e^3y - 2e^3$

31. (a) $W(20, 25) = 11°F$; $W(30, -10) = -39°F$; $W(15, 15) = 0°F$
(b) $W(10, -40) = -65.5°F$; $W(50, -40) = -88°F$; $W(60, 30) = 10.2°F$;
(c) $W(25, 5) = -17.4088°F$; $\frac{\partial W}{\partial V} = -\frac{5.72}{V^{0.84}} + \frac{0.0684t}{V^{0.84}} \Rightarrow \frac{\partial W}{\partial V}(25, 5) = -0.36$; $\frac{\partial W}{\partial T} = 0.6215 + 0.4275V^{0.16}$
$\Rightarrow \frac{\partial W}{\partial T}(25, 5) = 1.3370 \Rightarrow L(V, T) = -17.4088 - 0.36(V - 25) + 1.337(T - 5) = 1.337T - 0.36V - 15.0938$
(d) i) $W(24, 6) \approx L(24, 6) = -15.7118 \approx -15.7°F$
ii) $W(27, 2) \approx L(27, 2) = -22.1398 \approx -22.1°F$
ii) $W(5, -10) \approx L(5, -10) = -30.2638 \approx -30.2°F$ This value is very different because the point $(5, -10)$ is not close to the point $(25, 5)$.

32. $W(50, -20) = -59.5298°F$; $\frac{\partial W}{\partial V} = -\frac{5.72}{V^{0.84}} + \frac{0.0684t}{V^{0.84}} \Rightarrow \frac{\partial W}{\partial V}(50, -20) = -0.2651$; $\frac{\partial W}{\partial T} = 0.6215 + 0.4275V^{0.16}$
$\Rightarrow \frac{\partial W}{\partial T}(50, -20) = 1.4209 \Rightarrow L(V, T) = -59.5298 - 0.2651(V - 50) + 1.4209(T + 20)$
$= 1.4209T - 0.2651V - 17.8568$
(a) $W(49, -22) \approx L(49, -22) = -62.1065 \approx -62.1°F$
(b) $W(53, -19) \approx L(53, -19) = -58.9042 \approx -58.9°F$
(c) $W(60, -30) \approx L(60, -30) = -76.3898 \approx -76.4°F$

33. $f(2,1) = 3$, $f_x(x,y) = 2x - 3y \Rightarrow f_x(2,1) = 1$, $f_y(x,y) = -3x \Rightarrow f_y(2,1) = -6 \Rightarrow L(x,y) = 3 + 1(x-2) - 6(y-1)$
$= 7 + x - 6y$; $f_{xx}(x,y) = 2$, $f_{yy}(x,y) = 0$, $f_{xy}(x,y) = -3 \Rightarrow M = 3$; thus $|E(x,y)| \le \left(\frac{1}{2}\right)(3)\left(|x-2| + |y-1|\right)^2$
$\le \left(\frac{3}{2}\right)(0.1 + 0.1)^2 = 0.06$

34. $f(2,2) = 11$, $f_x(x,y) = x + y + 3 \Rightarrow f_x(2,2) = 7$, $f_y(x,y) = x + \frac{y}{2} - 3 \Rightarrow f_y(2,2) = 0$
$\Rightarrow L(x,y) = 11 + 7(x-2) + 0(y-2) = 7x - 3$; $f_{xx}(x,y) = 1$, $f_{yy}(x,y) = \frac{1}{2}$, $f_{xy}(x,y) = 1$
$\Rightarrow M = 1$; thus $|E(x,y)| \le \left(\frac{1}{2}\right)(1)\left(|x-2| + |y-2|\right)^2 \le \left(\frac{1}{2}\right)(0.1 + 0.1)^2 = 0.02$

35. $f(0,0) = 1$, $f_x(x,y) = \cos y \Rightarrow f_x(0,0) = 1$, $f_y(x,y) = 1 - x\sin y \Rightarrow f_y(0,0) = 1$
$\Rightarrow L(x,y) = 1 + 1(x-0) + 1(y-0) = x + y + 1$; $f_{xx}(x,y) = 0$, $f_{yy}(x,y) = -x\cos y$, $f_{xy}(x,y) = -\sin y \Rightarrow M = 1$;
thus $|E(x,y)| \le \left(\frac{1}{2}\right)(1)\left(|x| + |y|\right)^2 \le \left(\frac{1}{2}\right)(0.2 + 0.2)^2 = 0.08$

36. $f(1,2) = 6$, $f_x(x,y) = y^2 - y\sin(x-1) \Rightarrow f_x(1,2) = 4$, $f_y(x,y) = 2xy + \cos(x-1) \Rightarrow f_y(1,2) = 5$
$\Rightarrow L(x,y) = 6 + 4(x-1) + 5(y-2) = 4x + 5y - 8$; $f_{xx}(x,y) = -y\cos(x-1)$, $f_{yy}(x,y) = 2x$,
$f_{xy}(x,y) = 2y - \sin(x-1)$; $|x-1| \le 0.1 \Rightarrow 0.9 \le x \le 1.1$ and $|y-2| \le 0.1 \Rightarrow 1.9 \le y \le 2.1$; thus the max of $|f_{xx}(x,y)|$ on R is 2.1, the max of $|f_{yy}(x,y)|$ on R is 2.2, and the max of $|f_{xy}(x,y)|$ on R is $2(2.1) - \sin(0.9 - 1)$
$\le 4.3 \Rightarrow M = 4.3$; thus $|E(x,y)| \le \left(\frac{1}{2}\right)(4.3)\left(|x-1| + |y-2|\right)^2 \le (2.15)(0.1 + 0.1)^2 = 0.086$

37. $f(0,0) = 1$, $f_x(x,y) = e^x\cos y \Rightarrow f_x(0,0) = 1$, $f_y(x,y) = -e^x\sin y \Rightarrow f_y(0,0) = 0$
$\Rightarrow L(x,y) = 1 + 1(x-0) + 0(y-0) = 1 + x$; $f_{xx}(x,y) = e^x\cos y$, $f_{yy}(x,y) = -e^x\cos y$, $f_{xy}(x,y) = -e^x\sin y$;
$|x| \le 0.1 \Rightarrow -0.1 \le x \le 0.1$ and $|y| \le 0.1 \Rightarrow -0.1 \le y \le 0.1$; thus the max of $|f_{xx}(x,y)|$ on R is $e^{0.1}\cos(0.1)$
≤ 1.11, the max of $|f_{yy}(x,y)|$ on R is $e^{0.1}\cos(0.1) \le 1.11$, and the max of $|f_{xy}(x,y)|$ on R is $e^{0.1}\sin(0.1)$
$\le 0.12 \Rightarrow M = 1.11$; thus $|E(x,y)| \le \left(\frac{1}{2}\right)(1.11)\left(|x| + |y|\right)^2 \le (0.555)(0.1 + 0.1)^2 = 0.0222$

38. $f(1,1) = 0$, $f_x(x,y) = \frac{1}{x} \Rightarrow f_x(1,1) = 1$, $f_y(x,y) = \frac{1}{y} \Rightarrow f_y(1,1) = 1 \Rightarrow L(x,y) = 0 + 1(x-1) + 1(y-1)$ $= x + y - 2$; $f_{xx}(x,y) = -\frac{1}{x^2}$, $f_{yy}(x,y) = -\frac{1}{y^2}$, $f_{xy}(x,y) = 0$; $|x-1| \le 0.2 \Rightarrow 0.98 \le x \le 1.2$ so the max of $|f_{xx}(x,y)|$ on R is $\frac{1}{(0.98)^2} \le 1.04$; $|y-1| \le 0.2 \Rightarrow 0.98 \le y \le 1.2$ so the max of $|f_{yy}(x,y)|$ on R is $\frac{1}{(0.98)^2} \le 1.04 \Rightarrow M = 1.04$; thus $|E(x,y)| \le \left(\frac{1}{2}\right)(1.04)\left(|x-1| + |y-1|\right)^2 \le (0.52)(0.2+0.2)^2 = 0.0832$

39. (a) $f(1,1,1) = 3$, $f_x(1,1,1) = y + z|_{(1,1,1)} = 2$, $f_y(1,1,1) = x + z|_{(1,1,1)} = 2$, $f_z(1,1,1) = y + x|_{(1,1,1)} = 2$ $\Rightarrow L(x,y,z) = 3 + 2(x-1) + 2(y-1) + 2(z-1) = 2x + 2y + 2z - 3$
(b) $f(1,0,0) = 0$, $f_x(1,0,0) = 0$, $f_y(1,0,0) = 1$, $f_z(1,0,0) = 1 \Rightarrow L(x,y,z) = 0 + 0(x-1) + (y-0) + (z-0) = y + z$
(c) $f(0,0,0) = 0$, $f_x(0,0,0) = 0$, $f_y(0,0,0) = 0$, $f_z(0,0,0) = 0 \Rightarrow L(x,y,z) = 0$

40. (a) $f(1,1,1) = 3$, $f_x(1,1,1) = 2x|_{(1,1,1)} = 2$, $f_y(1,1,1) = 2y|_{(1,1,1)} = 2$, $f_z(1,1,1) = 2z|_{(1,1,1)} = 2$ $\Rightarrow L(x,y,z) = 3 + 2(x-1) + 2(y-1) + 2(z-1) = 2x + 2y + 2z - 3$
(b) $f(0,1,0) = 1$, $f_x(0,1,0) = 0$, $f_y(0,1,0) = 2$, $f_z(0,1,0) = 0 \Rightarrow L(x,y,z) = 1 + 0(x-0) + 2(y-1) + 0(z-0)$ $= 2y - 1$
(c) $f(1,0,0) = 1$, $f_x(1,0,0) = 2$, $f_y(1,0,0) = 0$, $f_z(1,0,0) = 0 \Rightarrow L(x,y,z) = 1 + 2(x-1) + 0(y-0) + 0(z-0)$ $= 2x - 1$

41. (a) $f(1,0,0) = 1$, $f_x(1,0,0) = \left.\frac{x}{\sqrt{x^2+y^2+z^2}}\right|_{(1,0,0)} = 1$, $f_y(1,0,0) = \left.\frac{y}{\sqrt{x^2+y^2+z^2}}\right|_{(1,0,0)} = 0$, $f_z(1,0,0) = \left.\frac{z}{\sqrt{x^2+y^2+z^2}}\right|_{(1,0,0)} = 0 \Rightarrow L(x,y,z) = 1 + 1(x-1) + 0(y-0) + 0(z-0) = x$
(b) $f(1,1,0) = \sqrt{2}$, $f_x(1,1,0) = \frac{1}{\sqrt{2}}$, $f_y(1,1,0) = \frac{1}{\sqrt{2}}$, $f_z(1,1,0) = 0$ $\Rightarrow L(x,y,z) = \sqrt{2} + \frac{1}{\sqrt{2}}(x-1) + \frac{1}{\sqrt{2}}(y-1) + 0(z-0) = \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y$
(c) $f(1,2,2) = 3$, $f_x(1,2,2) = \frac{1}{3}$, $f_y(1,2,2) = \frac{2}{3}$, $f_z(1,2,2) = \frac{2}{3} \Rightarrow L(x,y,z) = 3 + \frac{1}{3}(x-1) + \frac{2}{3}(y-2) + \frac{2}{3}(z-2)$ $= \frac{1}{3}x + \frac{2}{3}y + \frac{2}{3}z$

42. (a) $f\left(\frac{\pi}{2},1,1\right) = 1$, $f_x\left(\frac{\pi}{2},1,1\right) = \left.\frac{y\cos xy}{z}\right|_{\left(\frac{\pi}{2},1,1\right)} = 0$, $f_y\left(\frac{\pi}{2},1,1\right) = \left.\frac{x\cos xy}{z}\right|_{\left(\frac{\pi}{2},1,1\right)} = 0$, $f_z\left(\frac{\pi}{2},1,1\right) = \left.\frac{-\sin xy}{z^2}\right|_{\left(\frac{\pi}{2},1,1\right)} = -1 \Rightarrow L(x,y,z) = 1 + 0\left(x - \frac{\pi}{2}\right) + 0(y-1) - 1(z-1) = 2 - z$
(b) $f(2,0,1) = 0$, $f_x(2,0,1) = 0$, $f_y(2,0,1) = 2$, $f_z(2,0,1) = 0 \Rightarrow L(x,y,z) = 0 + 0(x-2) + 2(y-0) + 0(z-1) = 2y$

43. (a) $f(0,0,0) = 2$, $f_x(0,0,0) = e^x|_{(0,0,0)} = 1$, $f_y(0,0,0) = -\sin(y+z)|_{(0,0,0)} = 0$, $f_z(0,0,0) = -\sin(y+z)|_{(0,0,0)} = 0 \Rightarrow L(x,y,z) = 2 + 1(x-0) + 0(y-0) + 0(z-0) = 2 + x$
(b) $f\left(0,\frac{\pi}{2},0\right) = 1$, $f_x\left(0,\frac{\pi}{2},0\right) = 1$, $f_y\left(0,\frac{\pi}{2},0\right) = -1$, $f_z\left(0,\frac{\pi}{2},0\right) = -1 \Rightarrow L(x,y,z)$ $= 1 + 1(x-0) - 1\left(y - \frac{\pi}{2}\right) - 1(z-0) = x - y - z + \frac{\pi}{2} + 1$
(c) $f\left(0,\frac{\pi}{4},\frac{\pi}{4}\right) = 1$, $f_x\left(0,\frac{\pi}{4},\frac{\pi}{4}\right) = 1$, $f_y\left(0,\frac{\pi}{4},\frac{\pi}{4}\right) = -1$, $f_z\left(0,\frac{\pi}{4},\frac{\pi}{4}\right) = -1 \Rightarrow L(x,y,z)$ $= 1 + 1(x-0) - 1\left(y - \frac{\pi}{4}\right) - 1\left(z - \frac{\pi}{4}\right) = x - y - z + \frac{\pi}{2} + 1$

44. (a) $f(1,0,0) = 0$, $f_x(1,0,0) = \left.\frac{yz}{(xyz)^2+1}\right|_{(1,0,0)} = 0$, $f_y(1,0,0) = \left.\frac{xz}{(xyz)^2+1}\right|_{(1,0,0)} = 0$, $f_z(1,0,0) = \left.\frac{xy}{(xyz)^2+1}\right|_{(1,0,0)} = 0 \Rightarrow L(x,y,z) = 0$
(b) $f(1,1,0) = 0$, $f_x(1,1,0) = 0$, $f_y(1,1,0) = 0$, $f_z(1,1,0) = 1 \Rightarrow L(x,y,z) = 0 + 0(x-1) + 0(y-1) + 1(z-0) = z$
(c) $f(1,1,1) = \frac{\pi}{4}$, $f_x(1,1,1) = \frac{1}{2}$, $f_y(1,1,1) = \frac{1}{2}$, $f_z(1,1,1) = \frac{1}{2} \Rightarrow L(x,y,z) = \frac{\pi}{4} + \frac{1}{2}(x-1) + \frac{1}{2}(y-1) + \frac{1}{2}(z-1)$ $= \frac{1}{2}x + \frac{1}{2}y + \frac{1}{2}z + \frac{\pi}{4} - \frac{3}{2}$

45. $f(x, y, z) = xz - 3yz + 2$ at $P_0(1, 1, 2) \Rightarrow f(1, 1, 2) = -2$; $f_x = z$, $f_y = -3z$, $f_z = x - 3y \Rightarrow L(x, y, z) = -2 + 2(x - 1) - 6(y - 1) - 2(z - 2) = 2x - 6y - 2z + 6$; $f_{xx} = 0$, $f_{yy} = 0$, $f_{zz} = 0$, $f_{xy} = 0$, $f_{yz} = -3 \Rightarrow M = 3$; thus, $|E(x, y, z)| \le \left(\frac{1}{2}\right)(3)(0.01 + 0.01 + 0.02)^2 = 0.0024$

46. $f(x, y, z) = x^2 + xy + yz + \frac{1}{4}z^2$ at $P_0(1, 1, 2) \Rightarrow f(1, 1, 2) = 5$; $f_x = 2x + y$, $f_y = x + z$, $f_z = y + \frac{1}{2}z \Rightarrow L(x, y, z) = 5 + 3(x - 1) + 3(y - 1) + 2(z - 2) = 3x + 3y + 2z - 5$; $f_{xx} = 2$, $f_{yy} = 0$, $f_{zz} = \frac{1}{2}$, $f_{xy} = 1$, $f_{xz} = 0$, $f_{yz} = 1 \Rightarrow M = 2$; thus $|E(x, y, z)| \le \left(\frac{1}{2}\right)(2)(0.01 + 0.01 + 0.08)^2 = 0.01$

47. $f(x, y, z) = xy + 2yz - 3xz$ at $P_0(1, 1, 0) \Rightarrow f(1, 1, 0) = 1$; $f_x = y - 3z$, $f_y = x + 2z$, $f_z = 2y - 3x \Rightarrow L(x, y, z) = 1 + (x - 1) + (y - 1) - (z - 0) = x + y - z - 1$; $f_{xx} = 0$, $f_{yy} = 0$, $f_{zz} = 0$, $f_{xy} = 1$, $f_{xz} = -3$, $f_{yz} = 2 \Rightarrow M = 3$; thus $|E(x, y, z)| \le \left(\frac{1}{2}\right)(3)(0.01 + 0.01 + 0.01)^2 = 0.00135$

48. $f(x, y, z) = \sqrt{2}\cos x \sin(y + z)$ at $P_0\left(0, 0, \frac{\pi}{4}\right) \Rightarrow f\left(0, 0, \frac{\pi}{4}\right) = 1$; $f_x = -\sqrt{2}\sin x \sin(y + z)$, $f_y = \sqrt{2}\cos x \cos(y + z)$, $f_z = \sqrt{2}\cos x \cos(y + z) \Rightarrow L(x, y, z) = 1 - 0(x - 0) + (y - 0) + \left(z - \frac{\pi}{4}\right) = y + z - \frac{\pi}{4} + 1$; $f_{xx} = -\sqrt{2}\cos x \sin(y + z)$, $f_{yy} = -\sqrt{2}\cos x \sin(y + z)$, $f_{zz} = -\sqrt{2}\cos x \sin(y + z)$, $f_{xy} = -\sqrt{2}\sin x \cos(y + z)$, $f_{xz} = -\sqrt{2}\sin x \cos(y + z)$, $f_{yz} = -\sqrt{2}\cos x \sin(y + z)$. The absolute value of each of these second partial derivatives is bounded above by $\sqrt{2} \Rightarrow M = \sqrt{2}$; thus $|E(x, y, z)| \le \left(\frac{1}{2}\right)\left(\sqrt{2}\right)(0.01 + 0.01 + 0.01)^2 = 0.000636$.

49. $T_x(x, y) = e^y + e^{-y}$ and $T_y(x, y) = x\left(e^y - e^{-y}\right) \Rightarrow dT = T_x(x, y)\,dx + T_y(x, y)\,dy = \left(e^y + e^{-y}\right)dx + x\left(e^y - e^{-y}\right)dy \Rightarrow dT|_{(2, \ln 2)} = 2.5\,dx + 3.0\,dy$. If $|dx| \le 0.1$ and $|dy| \le 0.02$, then the maximum possible error in the computed value of T is $(2.5)(0.1) + (3.0)(0.02) = 0.31$ in magnitude.

50. $V_r = 2\pi rh$ and $V_h = \pi r^2 \Rightarrow dV = V_r\,dr + V_h\,dh \Rightarrow \frac{dV}{V} = \frac{2\pi rh\,dr + \pi r^2\,dh}{\pi r^2 h} = \frac{2}{r}\,dr + \frac{1}{h}\,dh$; now $\left|\frac{dr}{r} \cdot 100\right| \le 1$ and $\left|\frac{dh}{h} \cdot 100\right| \le 1 \Rightarrow \left|\frac{dV}{V} \cdot 100\right| \le \left|\left(2\frac{dr}{r}\right)(100) + \left(\frac{dh}{h}\right)(100)\right| \le 2\left|\frac{dr}{r} \cdot 100\right| + \left|\frac{dh}{h} \cdot 100\right| \le 2(1) + 1 = 3 \Rightarrow 3\%$

51. $\frac{dx}{x} \le 0.02$, $\frac{dy}{y} \le 0.03$

(a) $S = 2x^2 + 4xy \Rightarrow dS = (4x + 4y)dx + 4x\,dy = (4x^2 + 4xy)\frac{dx}{x} + 4xy\frac{dy}{y} \le (4x^2 + 4xy)(0.02) + (4xy)(0.03) = 0.04(2x^2) + 0.05(4xy) \le 0.05(2x^2) + 0.05(4xy) = (0.05)(2x^2 + 4xy) = 0.05S$

(b) $V = x^2 y \Rightarrow dV = 2xy\,dx + x^2 dy = 2x^2 y\frac{dx}{x} + x^2 y\frac{dy}{y} \le (2x^2 y)(0.02) + (x^2 y)(0.03) = 0.07(x^2 y) = 0.07V$

52. $V = \frac{4\pi}{3}r^3 + \pi r^2 h \Rightarrow dV = (4\pi r^2 + 2\pi rh)dr + \pi r^2 dh$; $r = 10$, $h = 15$, $dr = \frac{1}{2}$ and $dh = 0 \Rightarrow dV = \left(4\pi(10)^2 + 2\pi(10)(15)\right)\left(\frac{1}{2}\right) + \pi(10)^2(0) = 350\pi\ \text{cm}^3$

53. $V_r = 2\pi rh$ and $V_h = \pi r^2 \Rightarrow dV = V_r\,dr + V_h\,dh \Rightarrow dV = 2\pi rh\,dr + \pi r^2\,dh \Rightarrow dV|_{(5,12)} = 120\pi\,dr + 25\pi\,dh$; $|dr| \le 0.1$ cm and $|dh| \le 0.1$ cm $\Rightarrow dV \le (120\pi)(0.1) + (25\pi)(0.1) = 14.5\pi\ \text{cm}^3$; $V(5, 12) = 300\pi\ \text{cm}^3 \Rightarrow$ maximum percentage error is $\pm\frac{14.5\pi}{300\pi} \times 100 = \pm 4.83\%$

54. (a) $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \Rightarrow -\frac{1}{R^2}\,dR = -\frac{1}{R_1^2}\,dR_1 - \frac{1}{R_2^2}\,dR_2 \Rightarrow dR = \left(\frac{R}{R_1}\right)^2 dR_1 + \left(\frac{R}{R_2}\right)^2 dR_2$

(b) $dR = R^2\left[\left(\frac{1}{R_1^2}\right)dR_1 + \left(\frac{1}{R_2^2}\right)dR_2\right] \Rightarrow dR|_{(100,400)} = R^2\left[\frac{1}{(100)^2}\,dR_1 + \frac{1}{(400)^2}\,dR_2\right] \Rightarrow$ R will be more sensitive to a variation in R_1 since $\frac{1}{(100)^2} > \frac{1}{(400)^2}$

(c) From part (a), $dR = \left(\frac{R}{R_1}\right)^2 dR_1 + \left(\frac{R}{R_2}\right)^2 dR_2$ so that R_1 changing from 20 to 20.1 ohms $\Rightarrow dR_1 = 0.1$ ohm and R_2 changing from 25 to 24.9 ohms $\Rightarrow dR_2 = -0.1$ ohms; $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \Rightarrow R = \frac{100}{9}$ ohms $\Rightarrow dR|_{(20,25)} = \frac{\left(\frac{100}{9}\right)^2}{(20)^2}(0.1) + \frac{\left(\frac{100}{9}\right)^2}{(25)^2}(-0.1) \approx 0.011$ ohms $\Rightarrow$ percentage change is $\frac{dR}{R}\Big|_{(20,25)} \times 100$ $= \frac{0.011}{\left(\frac{100}{9}\right)} \times 100 \approx 0.1\%$

55. $A = xy \Rightarrow dA = x\,dy + y\,dx$; if $x > y$ then a 1-unit change in y gives a greater change in dA than a 1-unit change in x. Thus, pay more attention to y which is the smaller of the two dimensions.

56. (a) $f_x(x, y) = 2x(y + 1) \Rightarrow f_x(1, 0) = 2$ and $f_y(x, y) = x^2 \Rightarrow f_y(1, 0) = 1 \Rightarrow df = 2\,dx + 1\,dy \Rightarrow$ df is more sensitive to changes in x

(b) $df = 0 \Rightarrow 2\,dx + dy = 0 \Rightarrow 2\,\frac{dx}{dy} + 1 = 0 \Rightarrow \frac{dx}{dy} = -\frac{1}{2}$

57. (a) $r^2 = x^2 + y^2 \Rightarrow 2r\,dr = 2x\,dx + 2y\,dy \Rightarrow dr = \frac{x}{r}\,dx + \frac{y}{r}\,dy \Rightarrow dr|_{(3,4)} = \left(\frac{3}{5}\right)(\pm 0.01) + \left(\frac{4}{5}\right)(\pm 0.01)$ $= \pm\frac{0.07}{5} = \pm 0.014 \Rightarrow \left|\frac{dr}{r} \times 100\right| = \left|\pm\frac{0.014}{5} \times 100\right| = 0.28\%$; $d\theta = \frac{\left(-\frac{y}{x^2}\right)}{\left(\frac{y}{x}\right)^2 + 1}\,dx + \frac{\left(\frac{1}{x}\right)}{\left(\frac{y}{x}\right)^2 + 1}\,dy$ $= \frac{-y}{y^2 + x^2}\,dx + \frac{x}{y^2 + x^2}\,dy \Rightarrow d\theta|_{(3,4)} = \left(\frac{-4}{25}\right)(\pm 0.01) + \left(\frac{3}{25}\right)(\pm 0.01) = \frac{\mp 0.04}{25} + \frac{\pm 0.03}{25}$ $\Rightarrow$ maximum change in $d\theta$ occurs when dx and dy have opposite signs ($dx = 0.01$ and $dy = -0.01$ or vice versa) $\Rightarrow d\theta = \frac{\pm 0.07}{25} \approx \pm 0.0028$; $\theta = \tan^{-1}\left(\frac{4}{3}\right) \approx 0.927255218 \Rightarrow \left|\frac{d\theta}{\theta} \times 100\right| = \left|\frac{\pm 0.0028}{0.927255218} \times 100\right|$ $\approx 0.30\%$

(b) the radius r is more sensitive to changes in y, and the angle θ is more sensitive to changes in x

58. (a) $V = \pi r^2 h \Rightarrow dV = 2\pi rh\,dr + \pi r^2\,dh \Rightarrow$ at $r = 1$ and $h = 5$ we have $dV = 10\pi\,dr + \pi\,dh \Rightarrow$ the volume is about 10 times more sensitive to a change in r

(b) $dV = 0 \Rightarrow 0 = 2\pi rh\,dr + \pi r^2\,dh = 2h\,dr + r\,dh = 10\,dr + dh \Rightarrow dr = -\frac{1}{10}\,dh$; choose $dh = 1.5$ $\Rightarrow dr = -0.15 \Rightarrow h = 6.5$ in. and $r = 0.85$ in. is one solution for $\Delta V \approx dV = 0$

59. $f(a, b, c, d) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \Rightarrow f_a = d,\ f_b = -c,\ f_c = -b,\ f_d = a \Rightarrow df = d\,da - c\,db - b\,dc + a\,dd$; since $|a|$ is much greater than $|b|$, $|c|$, and $|d|$, the function f is most sensitive to a change in d.

60. $u_x = e^y,\ u_y = xe^y + \sin z,\ u_z = y\cos z \Rightarrow du = e^y\,dx + (xe^y + \sin z)\,dy + (y\cos z)\,dz$ $\Rightarrow du|_{(2,\ln 3,\frac{\pi}{2})} = 3\,dx + 7\,dy + 0\,dz = 3\,dx + 7\,dy \Rightarrow$ magnitude of the maximum possible error $\le 3(0.2) + 7(0.6) = 4.8$

61. $Q_K = \frac{1}{2}\left(\frac{2KM}{h}\right)^{-1/2}\left(\frac{2M}{h}\right)$, $Q_M = \frac{1}{2}\left(\frac{2KM}{h}\right)^{-1/2}\left(\frac{2K}{h}\right)$, and $Q_h = \frac{1}{2}\left(\frac{2KM}{h}\right)^{-1/2}\left(\frac{-2KM}{h^2}\right)$ $\Rightarrow dQ = \frac{1}{2}\left(\frac{2KM}{h}\right)^{-1/2}\left(\frac{2M}{h}\right)dK + \frac{1}{2}\left(\frac{2KM}{h}\right)^{-1/2}\left(\frac{2K}{h}\right)dM + \frac{1}{2}\left(\frac{2KM}{h}\right)^{-1/2}\left(\frac{-2KM}{h^2}\right)dh$ $= \frac{1}{2}\left(\frac{2KM}{h}\right)^{-1/2}\left[\frac{2M}{h}\,dK + \frac{2K}{h}\,dM - \frac{2KM}{h^2}\,dh\right] \Rightarrow dQ|_{(2,20,0.0.05)}$ $= \frac{1}{2}\left[\frac{(2)(2)(20)}{0.05}\right]^{-1/2}\left[\frac{(2)(20)}{0.05}\,dK + \frac{(2)(2)}{0.05}\,dM - \frac{(2)(2)(20)}{(0.05)^2}\,dh\right] = (0.0125)(800\,dK + 80\,dM - 32{,}000\,dh)$ $\Rightarrow$ Q is most sensitive to changes in h

62. $A = \frac{1}{2}ab\sin C \Rightarrow A_a = \frac{1}{2}b\sin C,\ A_b = \frac{1}{2}a\sin C,\ A_c = \frac{1}{2}ab\cos C$ $\Rightarrow dA = \left(\frac{1}{2}b\sin C\right)da + \left(\frac{1}{2}a\sin C\right)db + \left(\frac{1}{2}ab\cos C\right)dC$; $dC = |2°| = |0.0349|$ radians, $da = |0.5|$ ft, $db = |0.5|$ ft; at $a = 150$ ft, $b = 200$ ft, and $C = 60°$, we see that the change is approximately $dA = \frac{1}{2}(200)(\sin 60°)\,|0.5| + \frac{1}{2}(150)(\sin 60°)\,|0.5| + \frac{1}{2}(200)(150)(\cos 60°)\,|0.0349| = \pm 338\ \text{ft}^2$

63. $z = f(x, y) \Rightarrow g(x, y, z) = f(x, y) - z = 0 \Rightarrow g_x(x, y, z) = f_x(x, y), g_y(x, y, z) = f_y(x, y)$ and $g_z(x, y, z) = -1$ $\Rightarrow g_x(x_0, y_0, f(x_0, y_0)) = f_x(x_0, y_0), g_y(x_0, y_0, f(x_0, y_0)) = f_y(x_0, y_0)$ and $g_z(x_0, y_0, f(x_0, y_0)) = -1 \Rightarrow$ the tangent plane at the point P_0 is $f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - [z - f(x_0, y_0)] = 0$ or $z = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + f(x_0, y_0)$

64. $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} = 2(\cos t + t \sin t)\mathbf{i} + 2(\sin t - t \cos t)\mathbf{j}$ and $\mathbf{v} = (t \cos t)\mathbf{i} + (t \sin t)\mathbf{j} \Rightarrow \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}$ $= \frac{(t \cos t)\mathbf{i} + (t \sin t)\mathbf{j}}{\sqrt{(t \cos t)^2 + (t \sin t)^2}} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$ since $t > 0 \Rightarrow (D_\mathbf{u}f)_{P_0} = \nabla f \cdot \mathbf{u}$ $= 2(\cos t + t \sin t)(\cos t) + 2(\sin t - t \cos t)(\sin t) = 2$

65. $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = (2 \cos t)\mathbf{i} + (2 \sin t)\mathbf{j} + 2t\mathbf{k}$ and $\mathbf{v} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}$ $= \frac{(-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \mathbf{k}}{\sqrt{(\sin t)^2 + (\cos t)^2 + 1^2}} = \left(\frac{-\sin t}{\sqrt{2}}\right)\mathbf{i} + \left(\frac{\cos t}{\sqrt{2}}\right)\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k} \Rightarrow (D_\mathbf{u}f)_{P_0} = \nabla f \cdot \mathbf{u}$ $= (2 \cos t)\left(\frac{-\sin t}{\sqrt{2}}\right) + (2 \sin t)\left(\frac{\cos t}{\sqrt{2}}\right) + (2t)\left(\frac{1}{\sqrt{2}}\right) = \frac{2t}{\sqrt{2}} \Rightarrow (D_\mathbf{u}f)\left(\frac{-\pi}{4}\right) = \frac{-\pi}{2\sqrt{2}}$, $(D_\mathbf{u}f)(0) = 0$ and $(D_\mathbf{u}f)\left(\frac{\pi}{4}\right) = \frac{\pi}{2\sqrt{2}}$

66. $r = \sqrt{t}\mathbf{i} + \sqrt{t}\mathbf{j} - \frac{1}{4}(t + 3)\mathbf{k} \Rightarrow \mathbf{v} = \frac{1}{2}t^{-1/2}\mathbf{i} + \frac{1}{2}t^{-1/2}\mathbf{j} - \frac{1}{4}\mathbf{k}$; $t = 1 \Rightarrow x = 1, y = 1, z = -1 \Rightarrow P_0 = (1, 1, -1)$ and $\mathbf{v}(1) = \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} - \frac{1}{4}\mathbf{k}$; $f(x, y, z) = x^2 + y^2 - z - 3 = 0 \Rightarrow \nabla f = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k}$ $\Rightarrow \nabla f(1, 1, -1) = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$; therefore $\mathbf{v} = \frac{1}{4}(\nabla f) \Rightarrow$ the curve is normal to the surface

67. $r = \sqrt{t}\mathbf{i} + \sqrt{t}\mathbf{j} + (2t - 1)\mathbf{k} \Rightarrow \mathbf{v} = \frac{1}{2}t^{-1/2}\mathbf{i} + \frac{1}{2}t^{-1/2}\mathbf{j} + 2\mathbf{k}$; $t = 1 \Rightarrow x = 1, y = 1, z = 1 \Rightarrow P_0 = (1, 1, 1)$ and $\mathbf{v}(1) = \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + 2\mathbf{k}$; $f(x, y, z) = x^2 + y^2 - z - 1 = 0 \Rightarrow \nabla f = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k} \Rightarrow \nabla f(1, 1, 1) = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$; now $\mathbf{v}(1) \cdot \nabla f(1, 1, 1) = 0$, thus the curve is tangent to the surface when $t = 1$

14.7 EXTREME VALUES AND SADDLE POINTS

1. $f_x(x, y) = 2x + y + 3 = 0$ and $f_y(x, y) = x + 2y - 3 = 0 \Rightarrow x = -3$ and $y = 3 \Rightarrow$ critical point is $(-3, 3)$; $f_{xx}(-3, 3) = 2, f_{yy}(-3, 3) = 2, f_{xy}(-3, 3) = 1 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 3 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum of $f(-3, 3) = -5$

2. $f_x(x, y) = 2y - 10x + 4 = 0$ and $f_y(x, y) = 2x - 4y + 4 = 0 \Rightarrow x = \frac{2}{3}$ and $y = \frac{4}{3} \Rightarrow$ critical point is $\left(\frac{2}{3}, \frac{4}{3}\right)$; $f_{xx}\left(\frac{2}{3}, \frac{4}{3}\right) = -10, f_{yy}\left(\frac{2}{3}, \frac{4}{3}\right) = -4, f_{xy}\left(\frac{2}{3}, \frac{4}{3}\right) = 2 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 36 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f\left(\frac{2}{3}, \frac{4}{3}\right) = 0$

3. $f_x(x, y) = 2x + y + 3 = 0$ and $f_y(x, y) = x + 2 = 0 \Rightarrow x = -2$ and $y = 1 \Rightarrow$ critical point is $(-2, 1)$; $f_{xx}(-2, 1) = 2, f_{yy}(-2, 1) = 0, f_{xy}(-2, 1) = 1 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -1 < 0 \Rightarrow$ saddle point

4. $f_x(x, y) = 5y - 14x + 3 = 0$ and $f_y(x, y) = 5x - 6 = 0 \Rightarrow x = \frac{6}{5}$ and $y = \frac{69}{25} \Rightarrow$ critical point is $\left(\frac{6}{5}, \frac{69}{25}\right)$; $f_{xx}\left(\frac{6}{5}, \frac{69}{25}\right) = -14, f_{yy}\left(\frac{6}{5}, \frac{69}{25}\right) = 0, f_{xy}\left(\frac{6}{5}, \frac{69}{25}\right) = 5 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -25 < 0 \Rightarrow$ saddle point

5. $f_x(x, y) = 2y - 2x + 3 = 0$ and $f_y(x, y) = 2x - 4y = 0 \Rightarrow x = 3$ and $y = \frac{3}{2} \Rightarrow$ critical point is $\left(3, \frac{3}{2}\right)$; $f_{xx}\left(3, \frac{3}{2}\right) = -2, f_{yy}\left(3, \frac{3}{2}\right) = -4, f_{xy}\left(3, \frac{3}{2}\right) = 2 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 4 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f\left(3, \frac{3}{2}\right) = \frac{17}{2}$

6. $f_x(x, y) = 2x - 4y = 0$ and $f_y(x, y) = -4x + 2y + 6 = 0 \Rightarrow x = 2$ and $y = 1 \Rightarrow$ critical point is $(2, 1)$; $f_{xx}(2, 1) = 2, f_{yy}(2, 1) = 2, f_{xy}(2, 1) = -4 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -12 < 0 \Rightarrow$ saddle point

7. $f_x(x,y) = 4x + 3y - 5 = 0$ and $f_y(x,y) = 3x + 8y + 2 = 0 \Rightarrow x = 2$ and $y = -1 \Rightarrow$ critical point is $(2,-1)$; $f_{xx}(2,-1) = 4$, $f_{yy}(2,-1) = 8$, $f_{xy}(2,-1) = 3 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 23 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum of $f(2,-1) = -6$

8. $f_x(x,y) = 2x - 2y - 2 = 0$ and $f_y(x,y) = -2x + 4y + 2 = 0 \Rightarrow x = 1$ and $y = 0 \Rightarrow$ critical point is $(1,0)$; $f_{xx}(1,0) = 2$, $f_{yy}(1,0) = 4$, $f_{xy}(1,0) = -2 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 4 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum of $f(1,0) = 0$

9. $f_x(x,y) = 2x - 2 = 0$ and $f_y(x,y) = -2y + 4 = 0 \Rightarrow x = 1$ and $y = 2 \Rightarrow$ critical point is $(1,2)$; $f_{xx}(1,2) = 2$, $f_{yy}(1,2) = -2$, $f_{xy}(1,2) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -4 < 0 \Rightarrow$ saddle point

10. $f_x(x,y) = 2x + 2y = 0$ and $f_y(x,y) = 2x = 0 \Rightarrow x = 0$ and $y = 0 \Rightarrow$ critical point is $(0,0)$; $f_{xx}(0,0) = 2$, $f_{yy}(0,0) = 0$, $f_{xy}(0,0) = 2 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -4 < 0 \Rightarrow$ saddle point

11. $f_x(x,y) = \frac{112x - 8x}{\sqrt{56x^2 - 8y^2 - 16x - 31}} - 8 = 0$ and $f_y(x,y) = \frac{-8y}{\sqrt{56x^2 - 8y^2 - 16x - 31}} = 0 \Rightarrow$ critical point is $\left(\frac{16}{7}, 0\right)$; $f_{xx}\left(\frac{16}{7}, 0\right) = -\frac{8}{15}$, $f_{yy}\left(\frac{16}{7}, 0\right) = -\frac{8}{15}$, $f_{xy}\left(\frac{16}{7}, 0\right) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = \frac{64}{225} > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f\left(\frac{16}{7}, 0\right) = -\frac{16}{7}$

12. $f_x(x,y) = \frac{-2x}{3(x^2+y^2)^{2/3}} = 0$ and $f_y(x,y) = \frac{-2y}{3(x^2+y^2)^{2/3}} = 0 \Rightarrow$ there are no solutions to the system $f_x(x,y) = 0$ and $f_y(x,y) = 0$, however, we must also consider where the partials are undefined, and this occurs when $x = 0$ and $y = 0$ $\Rightarrow$ critical point is $(0,0)$. Note that the partial derivatives are defined at every other point other than $(0,0)$. We cannot use the second derivative test, but this is the only possible local maximum, local minimum, or saddle point. $f(x,y)$ has a local maximum of $f(0,0) = 1$ at $(0,0)$ since $f(x,y) = 1 - \sqrt[3]{x^2 + y^2} \le 1$ for all (x,y) other than $(0,0)$.

13. $f_x(x,y) = 3x^2 - 2y = 0$ and $f_y(x,y) = -3y^2 - 2x = 0 \Rightarrow x = 0$ and $y = 0$, or $x = -\frac{2}{3}$ and $y = \frac{2}{3} \Rightarrow$ critical points are $(0,0)$ and $\left(-\frac{2}{3}, \frac{2}{3}\right)$; for $(0,0)$: $f_{xx}(0,0) = 6x|_{(0,0)} = 0$, $f_{yy}(0,0) = -6y|_{(0,0)} = 0$, $f_{xy}(0,0) = -2$ $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -4 < 0 \Rightarrow$ saddle point; for $\left(-\frac{2}{3}, \frac{2}{3}\right)$: $f_{xx}\left(-\frac{2}{3}, \frac{2}{3}\right) = -4$, $f_{yy}\left(-\frac{2}{3}, \frac{2}{3}\right) = -4$, $f_{xy}\left(-\frac{2}{3}, \frac{2}{3}\right) = -2$ $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 12 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f\left(-\frac{2}{3}, \frac{2}{3}\right) = \frac{170}{27}$

14. $f_x(x,y) = 3x^2 + 3y = 0$ and $f_y(x,y) = 3x + 3y^2 = 0 \Rightarrow x = 0$ and $y = 0$, or $x = -1$ and $y = -1 \Rightarrow$ critical points are $(0,0)$ and $(-1,-1)$; for $(0,0)$: $f_{xx}(0,0) = 6x|_{(0,0)} = 0$, $f_{yy}(0,0) = 6y|_{(0,0)} = 0$, $f_{xy}(0,0) = 3 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -9 < 0 \Rightarrow$ saddle point; for $(-1,-1)$: $f_{xx}(-1,-1) = -6$, $f_{yy}(-1,-1) = -6$, $f_{xy}(-1,-1) = 3 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 27 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f(-1,-1) = 1$

15. $f_x(x,y) = 12x - 6x^2 + 6y = 0$ and $f_y(x,y) = 6y + 6x = 0 \Rightarrow x = 0$ and $y = 0$, or $x = 1$ and $y = -1 \Rightarrow$ critical points are $(0,0)$ and $(1,-1)$; for $(0,0)$: $f_{xx}(0,0) = 12 - 12x|_{(0,0)} = 12$, $f_{yy}(0,0) = 6$, $f_{xy}(0,0) = 6 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 36 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum of $f(0,0) = 0$; for $(1,-1)$: $f_{xx}(1,-1) = 0$, $f_{yy}(1,-1) = 6$, $f_{xy}(1,-1) = 6 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -36 < 0 \Rightarrow$ saddle point

16. $f_x(x,y) = 3x^2 + 6x = 0 \Rightarrow x = 0$ or $x = -2$; $f_y(x,y) = 3y^2 - 6y = 0 \Rightarrow y = 0$ or $y = 2 \Rightarrow$ the critical points are $(0,0)$, $(0,2)$, $(-2,0)$, and $(-2,2)$; for $(0,0)$: $f_{xx}(0,0) = 6x + 6|_{(0,0)} = 6$, $f_{yy}(0,0) = 6y - 6|_{(0,0)} = -6$, $f_{xy}(0,0) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -36 < 0 \Rightarrow$ saddle point; for $(0,2)$: $f_{xx}(0,2) = 6$, $f_{yy}(0,2) = 6$, $f_{xy}(0,2) = 0$ $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 36 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum of $f(0,2) = -12$; for $(-2,0)$: $f_{xx}(-2,0) = -6$, $f_{yy}(-2,0) = -6$, $f_{xy}(-2,0) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 36 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f(-2,0) = -4$; for $(-2,2)$: $f_{xx}(-2,2) = -6$, $f_{yy}(-2,2) = 6$, $f_{xy}(-2,2) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -36 < 0 \Rightarrow$ saddle point

17. $f_x(x,y) = 3x^2 + 3y^2 - 15 = 0$ and $f_y(x,y) = 6xy + 3y^2 - 15 = 0 \Rightarrow$ critical points are $(2,1)$, $(-2,-1)$, $\left(0,\sqrt{5}\right)$, and $\left(0,-\sqrt{5}\right)$; for $(2,1)$: $f_{xx}(2,1) = 6x|_{(2,1)} = 12$, $f_{yy}(2,1) = (6x+6y)|_{(2,1)} = 18$, $f_{xy}(2,1) = 6y|_{(2,1)} = 6$ $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 180 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum of $f(2,1) = -30$; for $(-2,-1)$: $f_{xx}(-2,-1) = 6x|_{(-2,-1)}$ $= -12$, $f_{yy}(-2,-1) = (6x+6y)|_{(-2,-1)} = -18$, $f_{xy}(-2,-1) = 6y|_{(-2,-1)} = -6 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 180 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f(-2,-1) = 30$; for $\left(0,\sqrt{5}\right)$: $f_{xx}\left(0,\sqrt{5}\right) = 6x\Big|_{\left(0,\sqrt{5}\right)} = 0$, $f_{yy}\left(0,\sqrt{5}\right)$ $= (6x+6y)|_{\left(0,\sqrt{5}\right)} = 6\sqrt{5}$, $f_{xy}\left(0,\sqrt{5}\right) = 6y\Big|_{\left(0,\sqrt{5}\right)} = 6\sqrt{5} \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -180 < 0 \Rightarrow$ saddle point; for $\left(0,-\sqrt{5}\right)$: $f_{xx}\left(0,-\sqrt{5}\right) = 6x\Big|_{\left(0,-\sqrt{5}\right)} = 0$, $f_{yy}\left(0,-\sqrt{5}\right) = (6x+6y)|_{\left(0,-\sqrt{5}\right)} = -6\sqrt{5}$, $f_{xy}\left(0,-\sqrt{5}\right) = 6y\Big|_{\left(0,-\sqrt{5}\right)} = -6\sqrt{5} \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -180 < 0 \Rightarrow$ saddle point.

18. $f_x(x,y) = 6x^2 - 18x = 0 \Rightarrow 6x(x-3) = 0 \Rightarrow x = 0$ or $x = 3$; $f_y(x,y) = 6y^2 + 6y - 12 = 0 \Rightarrow 6(y+2)(y-1) = 0$ $\Rightarrow y = -2$ or $y = 1 \Rightarrow$ the critical points are $(0,-2)$, $(0,1)$, $(3,-2)$, and $(3,1)$; $f_{xx}(x,y) = 12x - 18$, $f_{yy}(x,y) = 12y + 6$, and $f_{xy}(x,y) = 0$; for $(0,-2)$: $f_{xx}(0,-2) = -18$, $f_{yy}(0,-2) = -18$, $f_{xy}(0,-2) = 0$ $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 324 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f(0,-2) = 20$; for $(0,1)$: $f_{xx}(0,1) = -18$, $f_{yy}(0,1) = 18$, $f_{xy}(0,1) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -324 < 0 \Rightarrow$ saddle point; for $(3,-2)$: $f_{xx}(3,-2) = 18$, $f_{yy}(3,-2) = -18$, $f_{xy}(3,-2) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -324 < 0 \Rightarrow$ saddle point; for $(3,1)$: $f_{xx}(3,1) = 18$, $f_{yy}(3,1) = 18$, $f_{xy}(3,1) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 324 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum of $f(3,1) = -34$

19. $f_x(x,y) = 4y - 4x^3 = 0$ and $f_y(x,y) = 4x - 4y^3 = 0 \Rightarrow x = y \Rightarrow x(1-x^2) = 0 \Rightarrow x = 0, 1, -1 \Rightarrow$ the critical points are $(0,0)$, $(1,1)$, and $(-1,-1)$; for $(0,0)$: $f_{xx}(0,0) = -12x^2|_{(0,0)} = 0$, $f_{yy}(0,0) = -12y^2|_{(0,0)} = 0$, $f_{xy}(0,0) = 4 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -16 < 0 \Rightarrow$ saddle point; for $(1,1)$: $f_{xx}(1,1) = -12$, $f_{yy}(1,1) = -12$, $f_{xy}(1,1) = 4$ $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 128 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f(1,1) = 2$; for $(-1,-1)$: $f_{xx}(-1,-1) = -12$, $f_{yy}(-1,-1) = -12$, $f_{xy}(-1,-1) = 4 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 128 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f(-1,-1) = 2$

20. $f_x(x,y) = 4x^3 + 4y = 0$ and $f_y(x,y) = 4y^3 + 4x = 0 \Rightarrow x = -y \Rightarrow -x^3 + x = 0 \Rightarrow x(1-x^2) = 0 \Rightarrow x = 0, 1, -1$ $\Rightarrow$ the critical points are $(0,0)$, $(1,-1)$, and $(-1,1)$; $f_{xx}(x,y) = 12x^2$, $f_{yy}(x,y) = 12y^2$, and $f_{xy}(x,y) = 4$; for $(0,0)$: $f_{xx}(0,0) = 0$, $f_{yy}(0,0) = 0$, $f_{xy}(0,0) = 4 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -16 < 0 \Rightarrow$ saddle point; for $(1,-1)$: $f_{xx}(1,-1) = 12$, $f_{yy}(1,-1) = 12$, $f_{xy}(1,-1) = 4 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 128 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum of $f(1,-1) = -2$; for $(-1,1)$: $f_{xx}(-1,1) = 12$, $f_{yy}(-1,1) = 12$, $f_{xy}(-1,1) = 4 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 128 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum of $f(-1,1) = -2$

21. $f_x(x,y) = \frac{-2x}{(x^2+y^2-1)^2} = 0$ and $f_y(x,y) = \frac{-2y}{(x^2+y^2-1)^2} = 0 \Rightarrow x = 0$ and $y = 0 \Rightarrow$ the critical point is $(0,0)$; $f_{xx} = \frac{4x^2-2y^2+2}{(x^2+y^2-1)^3}$, $f_{yy} = \frac{-2x^2+4y^2+2}{(x^2+y^2-1)^3}$, $f_{xy} = \frac{8xy}{(x^2+y^2-1)^3}$; $f_{xx}(0,0) = -2$, $f_{yy}(0,0) = -2$, $f_{xy}(0,0) = 0$ $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 4 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f(0,0) = -1$

22. $f_x(x,y) = -\frac{1}{x^2} + y = 0$ and $f_y(x,y) = x - \frac{1}{y^2} = 0 \Rightarrow x = 1$ and $y = 1 \Rightarrow$ the critical point is $(1,1)$; $f_{xx} = \frac{2}{x^3}$, $f_{yy} = \frac{2}{y^3}$, $f_{xy} = 1$; $f_{xx}(1,1) = 2$, $f_{yy}(1,1) = 2$, $f_{xy}(1,1) = 1 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 3 > 0$ and $f_{xx} > 2 \Rightarrow$ local minimum of $f(1,1) = 3$

23. $f_x(x,y) = y\cos x = 0$ and $f_y(x,y) = \sin x = 0 \Rightarrow x = n\pi$, n an integer, and $y = 0 \Rightarrow$ the critical points are $(n\pi, 0)$, n an integer (Note: cos x and sin x cannot both be 0 for the same x, so sin x must be 0 and $y = 0$); $f_{xx} = -y\sin x$, $f_{yy} = 0$, $f_{xy} = \cos x$; $f_{xx}(n\pi,0) = 0$, $f_{yy}(n\pi,0) = 0$, $f_{xy}(n\pi,0) = 1$ if n is even and $f_{xy}(n\pi,0) = -1$ if n is odd $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -1 < 0 \Rightarrow$ saddle point.

24. $f_x(x, y) = 2e^{2x} \cos y = 0$ and $f_y(x, y) = -e^{2x} \sin y = 0 \Rightarrow$ no solution since $e^{2x} \neq 0$ for any x and the functions cos y and sin y cannot equal 0 for the same y $\Rightarrow$ no critical points $\Rightarrow$ no extrema and no saddle points

25. $f_x(x, y) = (2x - 4)e^{x^2+y^2-4x} = 0$ and $f_y(x, y) = 2ye^{x^2+y^2-4x} = 0 \Rightarrow$ critical point is $(2, 0)$; $f_{xx}(2, 0) = \frac{2}{e^4}$, $f_{xy}(2, 0) = 0$, $f_{yy}(2, 0) = \frac{2}{e^4} \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = \frac{4}{e^8} > 0$ and $f_{xx} > 0 \Rightarrow$ local mimimum of $f(2, 0) = \frac{1}{e^4}$

26. $f_x(x, y) = -ye^x = 0$ and $f_y(x, y) = e^y - e^x = 0 \Rightarrow$ critical point is $(0, 0)$; $f_{xx}(2, 0) = 0$, $f_{xy}(2, 0) = -1$, $f_{yy}(2, 0) = 1$ $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -1 < 0 \Rightarrow$ saddle point

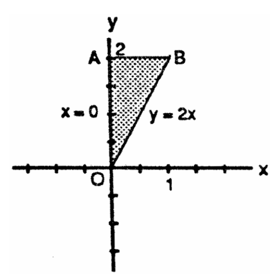
27. $f_x(x, y) = 2xe^{-y} = 0$ and $f_y(x, y) = 2ye^{-y} - e^{-y}(x^2 + y^2) = 0 \Rightarrow$ critical points are $(0, 0)$ and $(0, 2)$; for $(0, 0)$: $f_{xx}(0, 0) = 2e^{-y}|_{(0,0)} = 2$, $f_{yy}(0, 0) = (2e^{-y} - 4ye^{-y} + e^{-y}(x^2 + y^2))|_{(0,0)} = 2$, $f_{xy}(0, 0) = -2xe^{-y}|_{(0,0)} = 0$ $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 4 > 0$ and $f_{xx} > 0 \Rightarrow$ local mimimum of $f(0, 0) = 0$; for $(0, 2)$: $f_{xx}(0, 2) = 2e^{-y}|_{(0,2)} = \frac{2}{e^2}$, $f_{yy}(0, 2) = (2e^{-y} - 4ye^{-y} + e^{-y}(x^2 + y^2))|_{(0,2)} = -\frac{2}{e^2}$, $f_{xy}(0, 2) = -2xe^{-y}|_{(0,2)} = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -\frac{4}{e^4} < 0$ $\Rightarrow$ saddle point

28. $f_x(x, y) = e^x(x^2 - 2x + y^2) = 0$ and $f_y(x, y) = -2ye^x = 0 \Rightarrow$ critical points are $(0, 0)$ and $(-2, 0)$; for $(0, 0)$: $f_{xx}(0, 0) = e^x(x^2 + 4x + 2 - y^2)|_{(0,0)} = 2$, $f_{yy}(0, 0) = -2e^x|_{(0,0)} = -2$, $f_{xy}(0, 0) = -2ye^x|_{(0,0)} = 0$ $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -4 < 0$ and $f_{xx} > 0 \Rightarrow$ saddle point; for $(-2, 0)$: $f_{xx}(-2, 0) = e^x(x^2 + 4x + 2 - y^2)|_{(-2,0)} = -\frac{2}{e^2}$, $f_{yy}(-2, 0) = -2e^x|_{(-2,0)} = -\frac{2}{e^2}$, $f_{xy}(-2, 0) = -2ye^x|_{(-2,0)} = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = \frac{4}{e^4} > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f(-2, 0) = \frac{4}{e^2}$

29. $f_x(x, y) = -4 + \frac{2}{x} = 0$ and $f_y(x, y) = -1 + \frac{1}{y} = 0 \Rightarrow$ critical point is $\left(\frac{1}{2}, 1\right)$; $f_{xx}\left(\frac{1}{2}, 1\right) = -8$, $f_{yy}\left(\frac{1}{2}, 1\right) = -1$, $f_{xy}\left(\frac{1}{2}, 1\right) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 8 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f\left(\frac{1}{2}, 1\right) = -3 - 2\ln 2$

30. $f_x(x, y) = 2x + \frac{1}{x+y} = 0$ and $f_y(x, y) = -1 + \frac{1}{x+y} = 0 \Rightarrow$ critical point is $\left(-\frac{1}{2}, \frac{3}{2}\right)$; $f_{xx}\left(-\frac{1}{2}, \frac{3}{2}\right) = 1$, $f_{yy}\left(-\frac{1}{2}, \frac{3}{2}\right) = -1$, $f_{xy}\left(-\frac{1}{2}, \frac{3}{2}\right) = -1 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -2 < 0 \Rightarrow$ saddle point

31. (i) On OA, $f(x, y) = f(0, y) = y^2 - 4y + 1$ on $0 \le y \le 2$;
$f'(0, y) = 2y - 4 = 0 \Rightarrow y = 2$;
$f(0, 0) = 1$ and $f(0, 2) = -3$

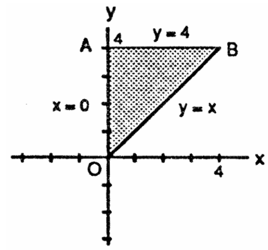


(ii) On AB, $f(x, y) = f(x, 2) = 2x^2 - 4x - 3$ on $0 \le x \le 1$;
$f'(x, 2) = 4x - 4 = 0 \Rightarrow x = 1$;
$f(0, 2) = -3$ and $f(1, 2) = -5$

(iii) On OB, $f(x, y) = f(x, 2x) = 6x^2 - 12x + 1$ on $0 \le x \le 1$; endpoint values have been found above;
$f'(x, 2x) = 12x - 12 = 0 \Rightarrow x = 1$ and $y = 2$, but $(1, 2)$ is not an interior point of OB

(iv) For interior points of the triangular region, $f_x(x, y) = 4x - 4 = 0$ and $f_y(x, y) = 2y - 4 = 0$ $\Rightarrow x = 1$ and $y = 2$, but $(1, 2)$ is not an interior point of the region. Therefore, the absolute maximum is 1 at $(0, 0)$ and the absolute minimum is -5 at $(1, 2)$.

32. (i) On OA, $D(x, y) = D(0, y) = y^2 + 1$ on $0 \le y \le 4$; $D'(0, y) = 2y = 0 \Rightarrow y = 0$; $D(0, 0) = 1$ and $D(0, 4) = 17$

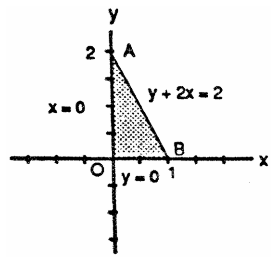


(ii) On AB, $D(x, y) = D(x, 4) = x^2 - 4x + 17$ on $0 \le x \le 4$; $D'(x, 4) = 2x - 4 = 0 \Rightarrow x = 2$ and $(2, 4)$ is an interior point of AB; $D(2, 4) = 13$ and $D(4, 4) = D(0, 4) = 17$

(iii) On OB, $D(x, y) = D(x, x) = x^2 + 1$ on $0 \le x \le 4$; $D'(x, x) = 2x = 0 \Rightarrow x = 0$ and $y = 0$, which is not an interior point of OB; endpoint values have been found above

(iv) For interior points of the triangular region, $f_x(x, y) = 2x - y = 0$ and $f_y(x, y) = -x + 2y = 0 \Rightarrow x = 0$ and $y = 0$, which is not an interior point of the region. Therefore, the absolute maximum is 17 at $(0, 4)$ and $(4, 4)$, and the absolute minimum is 1 at $(0, 0)$.

33. (i) On OA, $f(x, y) = f(0, y) = y^2$ on $0 \le y \le 2$; $f'(0, y) = 2y = 0 \Rightarrow y = 0$ and $x = 0$; $f(0, 0) = 0$ and $f(0, 2) = 4$

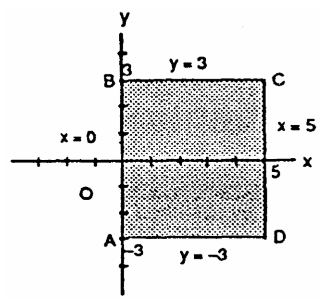


(ii) On OB, $f(x, y) = f(x, 0) = x^2$ on $0 \le x \le 1$; $f'(x, 0) = 2x = 0 \Rightarrow x = 0$ and $y = 0$; $f(0, 0) = 0$ and $f(1, 0) = 1$

(iii) On AB, $f(x, y) = f(x, -2x + 2) = 5x^2 - 8x + 4$ on $0 \le x \le 1$; $f'(x, -2x + 2) = 10x - 8 = 0 \Rightarrow x = \frac{4}{5}$ and $y = \frac{2}{5}$; $f\left(\frac{4}{5}, \frac{2}{5}\right) = \frac{4}{5}$; endpoint values have been found above.

(iv) For interior points of the triangular region, $f_x(x, y) = 2x = 0$ and $f_y(x, y) = 2y = 0 \Rightarrow x = 0$ and $y = 0$, but $(0, 0)$ is not an interior point of the region. Therefore the absolute maximum is 4 at $(0, 2)$ and the absolute minimum is 0 at $(0, 0)$.

34. (i) On AB, $T(x, y) = T(0, y) = y^2$ on $-3 \le y \le 3$; $T'(0, y) = 2y = 0 \Rightarrow y = 0$ and $x = 0$; $T(0, 0) = 0$, $T(0, -3) = 9$, and $T(0, 3) = 9$



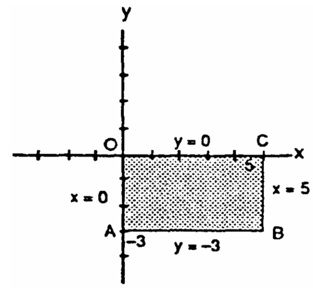
(ii) On BC, $T(x, y) = T(x, 3) = x^2 - 3x + 9$ on $0 \le x \le 5$; $T'(x, 3) = 2x - 3 = 0 \Rightarrow x = \frac{3}{2}$ and $y = 3$; $T\left(\frac{3}{2}, 3\right) = \frac{27}{4}$ and $T(5, 3) = 19$

(iii) On CD, $T(x, y) = T(5, y) = y^2 + 5y - 5$ on $-3 \le y \le 3$; $T'(5, y) = 2y + 5 = 0 \Rightarrow y = -\frac{5}{2}$ and $x = 5$; $T\left(5, -\frac{5}{2}\right) = -\frac{45}{4}$, $T(5, -3) = -11$ and $T(5, 3) = 19$

(iv) On AD, $T(x, y) = T(x, -3) = x^2 - 9x + 9$ on $0 \le x \le 5$; $T'(x, -3) = 2x - 9 = 0 \Rightarrow x = \frac{9}{2}$ and $y = -3$; $T\left(\frac{9}{2}, -3\right) = -\frac{45}{4}$, $T(0, -3) = 9$ and $T(5, -3) = -11$

(v) For interior points of the rectangular region, $T_x(x, y) = 2x + y - 6 = 0$ and $T_y(x, y) = x + 2y = 0 \Rightarrow x = 4$ and $y = -2 \Rightarrow (4, -2)$ is an interior critical point with $T(4, -2) = -12$. Therefore the absolute maximum is 19 at $(5, 3)$ and the absolute minimum is -12 at $(4, -2)$.

35. (i) On OC, $T(x, y) = T(x, 0) = x^2 - 6x + 2$ on $0 \le x \le 5$; $T'(x, 0) = 2x - 6 = 0 \Rightarrow x = 3$ and $y = 0$; $T(3, 0) = -7$, $T(0, 0) = 2$, and $T(5, 0) = -3$



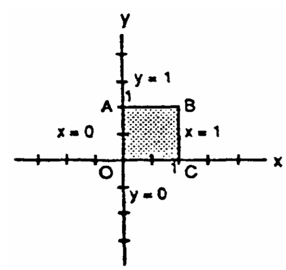
(ii) On CB, $T(x, y) = T(5, y) = y^2 + 5y - 3$ on $-3 \le y \le 0$; $T'(5, y) = 2y + 5 = 0 \Rightarrow y = -\frac{5}{2}$ and $x = 5$; $T\left(5, -\frac{5}{2}\right) = -\frac{37}{4}$ and $T(5, -3) = -9$

(iii) On AB, $T(x, y) = T(x, -3) = x^2 - 9x + 11$ on $0 \le x \le 5$; $T'(x, -3) = 2x - 9 = 0 \Rightarrow x = \frac{9}{2}$ and $y = -3$; $T\left(\frac{9}{2}, -3\right) = -\frac{37}{4}$ and $T(0, -3) = 11$

(iv) On AO, $T(x, y) = T(0, y) = y^2 + 2$ on $-3 \le y \le 0$; $T'(0, y) = 2y = 0 \Rightarrow y = 0$ and $x = 0$, but $(0, 0)$ is not an interior point of AO

(v) For interior points of the rectangular region, $T_x(x, y) = 2x + y - 6 = 0$ and $T_y(x, y) = x + 2y = 0 \Rightarrow x = 4$ and $y = -2$, an interior critical point with $T(4, -2) = -10$. Therefore the absolute maximum is 11 at $(0, -3)$ and the absolute minimum is -10 at $(4, -2)$.

36. (i) On OA, $f(x, y) = f(0, y) = -24y^2$ on $0 \le y \le 1$; $f'(0, y) = -48y = 0 \Rightarrow y = 0$ and $x = 0$, but $(0, 0)$ is not an interior point of OA; $f(0, 0) = 0$ and $f(0, 1) = -24$



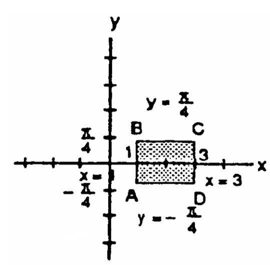
(ii) On AB, $f(x, y) = f(x, 1) = 48x - 32x^3 - 24$ on $0 \le x \le 1$; $f'(x, 1) = 48 - 96x^2 = 0 \Rightarrow x = \frac{1}{\sqrt{2}}$ and $y = 1$, or $x = -\frac{1}{\sqrt{2}}$ and $y = 1$, but $\left(-\frac{1}{\sqrt{2}}, 1\right)$ is not in the interior of AB; $f\left(\frac{1}{\sqrt{2}}, 1\right) = 16\sqrt{2} - 24$ and $f(1, 1) = -8$

(iii) On BC, $f(x, y) = f(1, y) = 48y - 32 - 24y^2$ on $0 \le y \le 1$; $f'(1, y) = 48 - 48y = 0 \Rightarrow y = 1$ and $x = 1$, but $(1, 1)$ is not an interior point of BC; $f(1, 0) = -32$ and $f(1, 1) = -8$

(iv) On OC, $f(x, y) = f(x, 0) = -32x^3$ on $0 \le x \le 1$; $f'(x, 0) = -96x^2 = 0 \Rightarrow x = 0$ and $y = 0$, but $(0, 0)$ is not an interior point of OC; $f(0, 0) = 0$ and $f(1, 0) = -32$

(v) For interior points of the rectangular region, $f_x(x, y) = 48y - 96x^2 = 0$ and $f_y(x, y) = 48x - 48y = 0$ $\Rightarrow x = 0$ and $y = 0$, or $x = \frac{1}{2}$ and $y = \frac{1}{2}$, but $(0, 0)$ is not an interior point of the region; $f\left(\frac{1}{2}, \frac{1}{2}\right) = 2$. Therefore the absolute maximum is 2 at $\left(\frac{1}{2}, \frac{1}{2}\right)$ and the absolute minimum is -32 at $(1, 0)$.

37. (i) On AB, $f(x, y) = f(1, y) = 3\cos y$ on $-\frac{\pi}{4} \le y \le \frac{\pi}{4}$; $f'(1, y) = -3\sin y = 0 \Rightarrow y = 0$ and $x = 1$; $f(1, 0) = 3$, $f\left(1, -\frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2}$, and $f\left(1, \frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2}$



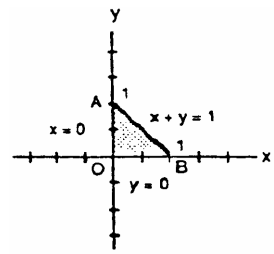
(ii) On CD, $f(x, y) = f(3, y) = 3\cos y$ on $-\frac{\pi}{4} \le y \le \frac{\pi}{4}$; $f'(3, y) = -3\sin y = 0 \Rightarrow y = 0$ and $x = 3$; $f(3, 0) = 3$, $f\left(3, -\frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2}$ and $f\left(3, \frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2}$

(iii) On BC, $f(x, y) = f\left(x, \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}(4x - x^2)$ on $1 \le x \le 3$; $f'\left(x, \frac{\pi}{4}\right) = \sqrt{2}(2 - x) = 0 \Rightarrow x = 2$ and $y = \frac{\pi}{4}$; $f\left(2, \frac{\pi}{4}\right) = 2\sqrt{2}$, $f\left(1, \frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2}$, and $f\left(3, \frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2}$

(iv) On AD, $f(x, y) = f\left(x, -\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}(4x - x^2)$ on $1 \le x \le 3$; $f'\left(x, -\frac{\pi}{4}\right) = \sqrt{2}(2 - x) = 0 \Rightarrow x = 2$ and $y = -\frac{\pi}{4}$; $f\left(2, -\frac{\pi}{4}\right) = 2\sqrt{2}$, $f\left(1, -\frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2}$, and $f\left(3, -\frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2}$

(v) For interior points of the region, $f_x(x, y) = (4 - 2x)\cos y = 0$ and $f_y(x, y) = -(4x - x^2)\sin y = 0 \Rightarrow x = 2$ and $y = 0$, which is an interior critical point with $f(2, 0) = 4$. Therefore the absolute maximum is 4 at $(2, 0)$ and the absolute minimum is $\frac{3\sqrt{2}}{2}$ at $\left(3, -\frac{\pi}{4}\right)$, $\left(3, \frac{\pi}{4}\right)$, $\left(1, -\frac{\pi}{4}\right)$, and $\left(1, \frac{\pi}{4}\right)$.

38. (i) On OA, $f(x, y) = f(0, y) = 2y + 1$ on $0 \le y \le 1$; $f'(0, y) = 2 \Rightarrow$ no interior critical points; $f(0, 0) = 1$ and $f(0, 1) = 3$



(ii) On OB, $f(x, y) = f(x, 0) = 4x + 1$ on $0 \le x \le 1$; $f'(x, 0) = 4 \Rightarrow$ no interior critical points; $f(1, 0) = 5$

(iii) On AB, $f(x, y) = f(x, -x + 1) = 8x^2 - 6x + 3$ on $0 \le x \le 1$; $f'(x, -x + 1) = 16x - 6 = 0 \Rightarrow x = \frac{3}{8}$ and $y = \frac{5}{8}$; $f\left(\frac{3}{8}, \frac{5}{8}\right) = \frac{15}{8}$, $f(0, 1) = 3$, and $f(1, 0) = 5$

(iv) For interior points of the triangular region, $f_x(x, y) = 4 - 8y = 0$ and $f_y(x, y) = -8x + 2 = 0$ $\Rightarrow y = \frac{1}{2}$ and $x = \frac{1}{4}$ which is an interior critical point with $f\left(\frac{1}{4}, \frac{1}{2}\right) = 2$. Therefore the absolute maximum is 5 at $(1, 0)$ and the absolute minimum is 1 at $(0, 0)$.

39. Let $F(a, b) = \int_a^b (6 - x - x^2)\, dx$ where $a \le b$. The boundary of the domain of F is the line $a = b$ in the ab-plane, and $F(a, a) = 0$, so F is identically 0 on the boundary of its domain. For interior critical points we have: $\frac{\partial F}{\partial a} = -(6 - a - a^2) = 0 \Rightarrow a = -3, 2$ and $\frac{\partial F}{\partial b} = (6 - b - b^2) = 0 \Rightarrow b = -3, 2$. Since $a \le b$, there is only one interior critical point $(-3, 2)$ and $F(-3, 2) = \int_{-3}^{2} (6 - x - x^2)\, dx$ gives the area under the parabola $y = 6 - x - x^2$ that is above the x-axis. Therefore, $a = -3$ and $b = 2$.

40. Let $F(a, b) = \int_a^b (24 - 2x - x^2)^{1/3}\, dx$ where $a \le b$. The boundary of the domain of F is the line $a = b$ and on this line F is identically 0. For interior critical points we have: $\frac{\partial F}{\partial a} = -(24 - 2a - a^2)^{1/3} = 0 \Rightarrow a = 4, -6$ and $\frac{\partial F}{\partial b} = (24 - 2b - b^2)^{1/3} = 0 \Rightarrow b = 4, -6$. Since $a \le b$, there is only one critical point $(-6, 4)$ and $F(-6, 4) = \int_{-6}^{4} (24 - 2x - x^2)\, dx$ gives the area under the curve $y = (24 - 2x - x^2)^{1/3}$ that is above the x-axis. Therefore, $a = -6$ and $b = 4$.

41. $T_x(x, y) = 2x - 1 = 0$ and $T_y(x, y) = 4y = 0 \Rightarrow x = \frac{1}{2}$ and $y = 0$ with $T\left(\frac{1}{2}, 0\right) = -\frac{1}{4}$; on the boundary $x^2 + y^2 = 1$: $T(x, y) = -x^2 - x + 2$ for $-1 \le x \le 1 \Rightarrow T'(x, y) = -2x - 1 = 0 \Rightarrow x = -\frac{1}{2}$ and $y = \pm\frac{\sqrt{3}}{2}$; $T\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = \frac{9}{4}$, $T\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) = \frac{9}{4}$, $T(-1, 0) = 2$, and $T(1, 0) = 0 \Rightarrow$ the hottest is $2\frac{1}{4}\,°$ at $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$; the coldest is $-\frac{1}{4}\,°$ at $\left(\frac{1}{2}, 0\right)$.

42. $f_x(x, y) = y + 2 - \frac{2}{x} = 0$ and $f_y(x, y) = x - \frac{1}{y} = 0 \Rightarrow x = \frac{1}{2}$ and $y = 2$; $f_{xx}\left(\frac{1}{2}, 2\right) = \left.\frac{2}{x^2}\right|_{\left(\frac{1}{2}, 2\right)} = 8$, $f_{yy}\left(\frac{1}{2}, 2\right) = \left.\frac{1}{y^2}\right|_{\left(\frac{1}{2}, 2\right)} = \frac{1}{4}$, $f_{xy}\left(\frac{1}{2}, 2\right) = 1 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 1 > 0$ and $f_{xx} > 0 \Rightarrow$ a local minimum of $f\left(\frac{1}{2}, 2\right)$ $= 2 - \ln\frac{1}{2} = 2 + \ln 2$

43. (a) $f_x(x, y) = 2x - 4y = 0$ and $f_y(x, y) = 2y - 4x = 0 \Rightarrow x = 0$ and $y = 0$; $f_{xx}(0, 0) = 2$, $f_{yy}(0, 0) = 2$, $f_{xy}(0, 0) = -4 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -12 < 0 \Rightarrow$ saddle point at $(0, 0)$

(b) $f_x(x, y) = 2x - 2 = 0$ and $f_y(x, y) = 2y - 4 = 0 \Rightarrow x = 1$ and $y = 2$; $f_{xx}(1, 2) = 2$, $f_{yy}(1, 2) = 2$, $f_{xy}(1, 2) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 4 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum at $(1, 2)$

(c) $f_x(x, y) = 9x^2 - 9 = 0$ and $f_y(x, y) = 2y + 4 = 0 \Rightarrow x = \pm 1$ and $y = -2$; $f_{xx}(1, -2) = 18x\big|_{(1,-2)} = 18$, $f_{yy}(1, -2) = 2$, $f_{xy}(1, -2) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 36 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum at $(1, -2)$; $f_{xx}(-1, -2) = -18$, $f_{yy}(-1, -2) = 2$, $f_{xy}(-1, -2) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -36 < 0 \Rightarrow$ saddle point at $(-1, -2)$

44. (a) Minimum at $(0, 0)$ since $f(x, y) > 0$ for all other (x, y)
(b) Maximum of 1 at $(0, 0)$ since $f(x, y) < 1$ for all other (x, y)
(c) Neither since $f(x, y) < 0$ for $x < 0$ and $f(x, y) > 0$ for $x > 0$
(d) Neither since $f(x, y) < 0$ for $x < 0$ and $f(x, y) > 0$ for $x > 0$
(e) Neither since $f(x, y) < 0$ for $x < 0$ and $y > 0$, but $f(x, y) > 0$ for $x > 0$ and $y > 0$
(f) Minimum at $(0, 0)$ since $f(x, y) > 0$ for all other (x, y)

45. If $k = 0$, then $f(x, y) = x^2 + y^2 \Rightarrow f_x(x, y) = 2x = 0$ and $f_y(x, y) = 2y = 0 \Rightarrow x = 0$ and $y = 0 \Rightarrow (0, 0)$ is the only critical point. If $k \neq 0$, $f_x(x, y) = 2x + ky = 0 \Rightarrow y = -\frac{2}{k}x$; $f_y(x, y) = kx + 2y = 0 \Rightarrow kx + 2\left(-\frac{2}{k}x\right) = 0$ $\Rightarrow kx - \frac{4x}{k} = 0 \Rightarrow \left(k - \frac{4}{k}\right)x = 0 \Rightarrow x = 0$ or $k = \pm 2 \Rightarrow y = \left(-\frac{2}{k}\right)(0) = 0$ or $y = \pm x$; in any case $(0, 0)$ is a critical point.

46. (See Exercise 45 above): $f_{xx}(x, y) = 2$, $f_{yy}(x, y) = 2$, and $f_{xy}(x, y) = k \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 4 - k^2$; f will have a saddle point at $(0, 0)$ if $4 - k^2 < 0 \Rightarrow k > 2$ or $k < -2$; f will have a local minimum at $(0, 0)$ if $4 - k^2 > 0 \Rightarrow -2 < k < 2$; the test is inconclusive if $4 - k^2 = 0 \Rightarrow k = \pm 2$.

47. No; for example $f(x, y) = xy$ has a saddle point at $(a, b) = (0, 0)$ where $f_x = f_y = 0$.

48. If $f_{xx}(a, b)$ and $f_{yy}(a, b)$ differ in sign, then $f_{xx}(a, b)\, f_{yy}(a, b) < 0$ so $f_{xx}f_{yy} - f_{xy}^2 < 0$. The surface must therefore have a saddle point at (a, b) by the second derivative test.

49. We want the point on $z = 10 - x^2 - y^2$ where the tangent plane is parallel to the plane $x + 2y + 3z = 0$. To find a normal vector to $z = 10 - x^2 - y^2$ let $w = z + x^2 + y^2 - 10$. Then $\nabla w = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}$ is normal to $z = 10 - x^2 - y^2$ at (x, y). The vector ∇w is parallel to $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ which is normal to the plane $x + 2y + 3z = 0$ if $6x\mathbf{i} + 6y\mathbf{j} + 3\mathbf{k} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ or $x = \frac{1}{6}$ and $y = \frac{1}{3}$. Thus the point is $\left(\frac{1}{6}, \frac{1}{3}, 10 - \frac{1}{36} - \frac{1}{9}\right)$ or $\left(\frac{1}{6}, \frac{1}{3}, \frac{355}{36}\right)$.

50. We want the point on $z = x^2 + y^2 + 10$ where the tangent plane is parallel to the plane $x + 2y - z = 0$. Let $w = z - x^2 - y^2 - 10$, then $\nabla w = -2x\mathbf{i} - 2y\mathbf{j} + \mathbf{k}$ is normal to $z = x^2 + y^2 + 10$ at (x, y). The vector ∇w is parallel to $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ which is normal to the plane if $x = \frac{1}{2}$ and $y = 1$. Thus the point $\left(\frac{1}{2}, 1, \frac{1}{4} + 1 + 10\right)$ or $\left(\frac{1}{2}, 1, \frac{45}{4}\right)$ is the point on the surface $z = x^2 + y^2 + 10$ nearest the plane $x + 2y - z = 0$.

51. $d(x, y, z) = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2} \Rightarrow$ we can minimize $d(x, y, z)$ by minimizing $D(x, y, z) = x^2 + y^2 + z^2$; $3x + 2y + z = 6 \Rightarrow z = 6 - 3x - 2y \Rightarrow D(x, y) = x^2 + y^2 + (6 - 3x - 2y)^2 \Rightarrow D_x(x, y) = 2x - 6(6 - 3x - 2y) = 0$ and $D_y(x, y) = 2y - 4(6 - 3x - 2y) = 0 \Rightarrow$ critical point is $\left(\frac{9}{7}, \frac{6}{7}\right) \Rightarrow z = \frac{3}{7}$; $D_{xx}\left(\frac{9}{7}, \frac{6}{7}\right) = 20$, $D_{yy}\left(\frac{1}{2}, 1\right) = 10$, $D_{xy}\left(\frac{1}{2}, 1\right) = 12 \Rightarrow D_{xx}D_{yy} - D_{xy}^2 = 56 > 0$ and $D_{xx} > 0 \Rightarrow$ local minimum of $d\left(\frac{9}{7}, \frac{6}{7}, \frac{3}{7}\right) = \frac{3\sqrt{14}}{7}$

52. $d(x, y, z) = \sqrt{(x-2)^2 + (y+1)^2 + (z-1)^2} \Rightarrow$ we can minimize $d(x, y, z)$ by minimizing $D(x, y, z) = (x-2)^2 + (y+1)^2 + (z-1)^2$; $x + y - z = 2 \Rightarrow z = x + y - 2$ $\Rightarrow D(x, y) = (x-2)^2 + (y+1)^2 + (x+y-3)^2 \Rightarrow D_x(x, y) = 2(x-2) + 2(x+y-3) = 0$ and $D_y(x, y) = 2(y+1) + 2(x+y-3) = 0 \Rightarrow$ critical point is $\left(\frac{8}{3}, -\frac{1}{3}\right) \Rightarrow z = \frac{1}{3}$; $D_{xx}\left(\frac{8}{3}, -\frac{1}{3}\right) = 4$, $D_{yy}\left(\frac{8}{3}, -\frac{1}{3}\right) = 4$, $D_{xy}\left(\frac{8}{3}, -\frac{1}{3}\right) = 2 \Rightarrow D_{xx}D_{yy} - D_{xy}^2 = 12 > 0$ and $D_{xx} > 0 \Rightarrow$ local minimum of $d\left(\frac{8}{3}, -\frac{1}{3}, \frac{1}{3}\right) = \frac{2}{\sqrt{3}}$

53. $s(x, y, z) = x^2 + y^2 + z^2$; $x + y + z = 9 \Rightarrow z = 9 - x - y \Rightarrow s(x, y) = x^2 + y^2 + (9 - x - y)^2$
$\Rightarrow s_x(x, y) = 2x - 2(9 - x - y) = 0$ and $s_y(x, y) = 2y - 2(9 - x - y) = 0 \Rightarrow$ critical point is $(3, 3) \Rightarrow z = 3$;
$s_{xx}(3, 3) = 4$, $s_{yy}(3, 3) = 4$, $s_{xy}(3, 3) = 2 \Rightarrow s_{xx}s_{yy} - s_{xy}^2 = 12 > 0$ and $s_{xx} > 0 \Rightarrow$ local minimum of $s(3, 3, 3) = 27$

54. $p(x, y, z) = xyz$; $x + y + z = 3 \Rightarrow z = 3 - x - y \Rightarrow p(x, y) = xy(3 - x - y) = 3xy - x^2y - xy^2$
$\Rightarrow p_x(x, y) = 3y - 2xy - y^2 = 0$ and $p_y(x, y) = 3x - x^2 - 2xy = 0 \Rightarrow$ critical points are $(0, 0)$, $(0, 3)$, $(3, 0)$, and $(1, 1)$; for $(0, 0) \Rightarrow z = 3$; $p_{xx}(0, 0) = 0$, $p_{yy}(0, 0) = 0$, $p_{xy}(0, 0) = 3 \Rightarrow p_{xx}p_{yy} - p_{xy}^2 = -9 < 0 \Rightarrow$ saddle point;
for $(0, 3) \Rightarrow z = 0$; $p_{xx}(0, 3) = -6$, $p_{yy}(0, 3) = 0$, $p_{xy}(0, 3) = -3 \Rightarrow p_{xx}p_{yy} - p_{xy}^2 = -9 < 0 \Rightarrow$ saddle point;
for $(3, 0) \Rightarrow z = 0$; $p_{xx}(3, 0) = 0$, $p_{yy}(3, 0) = -6$, $p_{xy}(3, 0) = -3 \Rightarrow p_{xx}p_{yy} - p_{xy}^2 = -9 < 0 \Rightarrow$ saddle point;
for $(1, 1) \Rightarrow z = 1$; $p_{xx}(1, 1) = -2$, $p_{yy}(1, 1) = -2$, $p_{xy}(1, 1) = -1 \Rightarrow p_{xx}p_{yy} - p_{xy}^2 = 3 > 0$ and $p_{xx} < 0 \Rightarrow$ local maximum of $p(1, 1, 1) = 1$

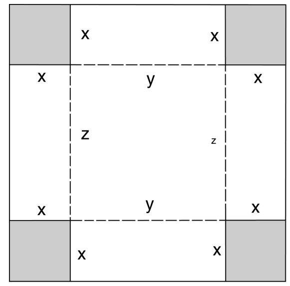
55. $s(x, y, z) = xy + yz + xz$; $x + y + z = 6 \Rightarrow z = 6 - x - y \Rightarrow s(x, y) = xy + y(6 - x - y) + x(6 - x - y)$
$= 6x + 6y - xy - x^2 - y^2 \Rightarrow s_x(x, y) = 6 - 2x - y = 0$ and $s_y(x, y) = 6 - x - 2y = 0 \Rightarrow$ critical point is $(2, 2)$
$\Rightarrow z = 2$; $s_{xx}(2, 2) = -2$, $s_{yy}(2, 2) = -2$, $s_{xy}(2, 2) = -1 \Rightarrow s_{xx}s_{yy} - s_{xy}^2 = 3 > 0$ and $s_{xx} < 0 \Rightarrow$ local maximum of $s(2, 2, 2) = 12$

56. $d(x, y, z) = \sqrt{(x + 6)^2 + (y - 4)^2 + (z - 0)^2} \Rightarrow$ we can minimize $d(x, y, z)$ by minimizing
$D(x, y, z) = (x + 6)^2 + (y - 4)^2 + z^2$; $z = \sqrt{x^2 + y^2} \Rightarrow D(x, y) = (x + 6)^2 + (y - 4)^2 + x^2 + y^2$
$= 2x^2 + 2y^2 + 12x - 8y + 52 \Rightarrow D_x(x, y) = 4x + 12 = 0$ and $D_y(x, y) = 4y - 8 = 0 \Rightarrow$ critical point is $(-3, 2)$
$\Rightarrow z = \sqrt{13}$; $D_{xx}(-3, 2) = 4$, $D_{yy}(-3, 2) = 4$, $D_{xy}(-3, 2) = 0 \Rightarrow D_{xx}D_{yy} - D_{xy}^2 = 16 > 0$ and $D_{xx} > 0 \Rightarrow$ local minimum of $d\left(-3, 2, \sqrt{13}\right) = \sqrt{26}$

57. $V(x, y, z) = (2x)(2y)(2z) = 8xyz$; $x^2 + y^2 + z^2 = 4 \Rightarrow z = \sqrt{4 - x^2 - y^2} \Rightarrow V(x, y) = 8xy\sqrt{4 - x^2 - y^2}$,
$x \geq 0$ and $y \geq 0 \Rightarrow V_x(x, y) = \frac{32y - 16x^2y - 8y^3}{\sqrt{4 - x^2 - y^2}} = 0$ and $V_y(x, y) = \frac{32x - 16xy^2 - 8x^3}{\sqrt{4 - x^2 - y^2}} = 0 \Rightarrow$ critical points are
$(0, 0)$, $\left(\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$, $\left(\frac{2}{\sqrt{3}}, -\frac{2}{\sqrt{3}}\right)$, $\left(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$, and $\left(-\frac{2}{\sqrt{3}}, -\frac{2}{\sqrt{3}}\right)$. Only $(0, 0)$ and $\left(\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$ satisfy $x \geq 0$ and $y \geq 0$
$V(0, 0) = 0$ and $V\left(\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right) = \frac{64}{3\sqrt{3}}$; On $x = 0$, $0 \leq y \leq 2 \Rightarrow V(0, y) = 8(0)y\sqrt{4 - 0^2 - y^2} = 0$, no critical points,
$V(0, 0) = 0$, $V(0, 2) = 0$; On $y = 0$, $0 \leq x \leq 2 \Rightarrow V(x, 0) = 8x(0)\sqrt{4 - x^2 - 0^2} = 0$, no critical points, $V(0, 0) = 0$,
$V(0, 2) = 0$; On $y = \sqrt{4 - x^2}$, $0 \leq x \leq 2 \Rightarrow V\left(x, \sqrt{4 - x^2}\right) = 8x\sqrt{4 - x^2}\sqrt{4 - x^2 - \left(\sqrt{4 - x^2}\right)^2} = 0$
no critical points, $V(0, 2) = 0$, $V(2, 0) = 0$. Thus, there is a maximum volume of $\frac{64}{3\sqrt{3}}$ if the box is $\frac{2}{\sqrt{3}} \times \frac{2}{\sqrt{3}} \times \frac{2}{\sqrt{3}}$.

58. $S(x, y, z) = 2xy + 2yz + 2xz$; $xyz = 27 \Rightarrow z = \frac{27}{xy} \Rightarrow S(x, y, z) = 2xy + 2y\left(\frac{27}{xy}\right) + 2x\left(\frac{27}{xy}\right) = 2xy + \frac{54}{x} + \frac{54}{y}$, $x > 0$,
$y > 0$; $S_x(x, y) = 2y - \frac{54}{x^2} = 0$ and $S_y(x, y) = 2x - \frac{54}{y^2} = 0 \Rightarrow$ Critical point is $(3, 3) \Rightarrow z = 3$; $S_{xx}(3, 3) = 4$,
$S_{yy}(3, 3) = 4$, $D_{xy}(3, 3) = 2 \Rightarrow D_{xx}D_{yy} - D_{xy}^2 = 12 > 0$ and $D_{xx} > 0 \Rightarrow$ local minimum of $S(3, 3, 3) = 54$

59. Let x = height of the box, y = width, and z = length, cut out squares of length x from corner of the material See diagram at right. Fold along the dashed lines to form the box. From the diagram we see that the length of the material is $2x + y$ and the width is $2x + z$. Thus $(2x + y)(2x + z) = 12$ $\Rightarrow z = \frac{2(6 - 2x^2 + xy)}{2x + y}$. Since $V(x, y, z) = x y z$ $\Rightarrow V(x, y) = \frac{2x y(6 - 2x^2 + xy)}{2x + y}$, where $x > 0$, $y > 0$.



$V_x(x, y) = \frac{4(3y^2 - 4x^3y - 4x^2y^2 - xy^3)}{(2x + y)^2} = 0$ and $V_y(x, y) = \frac{2(12x^2 - 4x^4 - 4x^3y - x^2y^2)}{(2x + y)^2} = 0 \Rightarrow$ critical points are $\left(\sqrt{3}, 0\right)$, $\left(-\sqrt{3}, 0\right)$, $\left(\frac{1}{\sqrt{3}}, \frac{4}{\sqrt{3}}\right)$, and $\left(-\frac{1}{\sqrt{3}}, -\frac{4}{\sqrt{3}}\right)$. Only $\left(\sqrt{3}, 0\right)$ and $\left(\frac{1}{\sqrt{3}}, \frac{4}{\sqrt{3}}\right)$ satisfy $x > 0$ and $y > 0$. For $\left(\sqrt{3}, 0\right)$: $z = 0$; $V_{xx}\left(\sqrt{3}, 0\right) = 0$, $V_{yy}\left(\sqrt{3}, 0\right) = -2\sqrt{3}$, $V_{xy}\left(\sqrt{3}, 0\right) = -4\sqrt{3} \Rightarrow V_{xx}V_{yy} - V_{xy}^2 = -48 < 0 \Rightarrow$ saddle point. For $\left(\frac{1}{\sqrt{3}}, \frac{4}{\sqrt{3}}\right)$: $z = \frac{4}{\sqrt{3}}$; $V_{xx}\left(\frac{1}{\sqrt{3}}, \frac{4}{\sqrt{3}}\right) = -\frac{80}{3\sqrt{3}}$, $V_{yy}\left(\frac{1}{\sqrt{3}}, \frac{4}{\sqrt{3}}\right) = -\frac{2}{3\sqrt{3}}$, $V_{xy}\left(\frac{1}{\sqrt{3}}, \frac{4}{\sqrt{3}}\right) = -\frac{4}{3\sqrt{3}} \Rightarrow V_{xx}V_{yy} - V_{xy}^2 = \frac{16}{3} > 0$ and $V_{xx} < 0 \Rightarrow$ local maximum of $V\left(\frac{1}{\sqrt{3}}, \frac{4}{\sqrt{3}}, \frac{4}{\sqrt{3}}\right) = \frac{16}{3\sqrt{3}}$

60. (a) (i) On $x = 0$, $f(x, y) = f(0, y) = y^2 - y + 1$ for $0 \le y \le 1$; $f'(0, y) = 2y - 1 = 0 \Rightarrow y = \frac{1}{2}$ and $x = 0$; $f\left(0, \frac{1}{2}\right) = \frac{3}{4}$, $f(0, 0) = 1$, and $f(0, 1) = 1$

(ii) On $y = 1$, $f(x, y) = f(x, 1) = x^2 + x + 1$ for $0 \le x \le 1$; $f'(x, 1) = 2x + 1 = 0 \Rightarrow x = -\frac{1}{2}$ and $y = 1$, but $\left(-\frac{1}{2}, 1\right)$ is outside the domain; $f(0, 1) = 1$ and $f(1, 1) = 3$

(iii) On $x = 1$, $f(x, y) = f(1, y) = y^2 + y + 1$ for $0 \le y \le 1$; $f'(1, y) = 2y + 1 = 0 \Rightarrow y = -\frac{1}{2}$ and $x = 1$, but $\left(1, -\frac{1}{2}\right)$ is outside the domain; $f(1, 0) = 1$ and $f(1, 1) = 3$

(iv) On $y = 0$, $f(x, y) = f(x, 0) = x^2 - x + 1$ for $0 \le x \le 1$; $f'(x, 0) = 2x - 1 = 0 \Rightarrow x = \frac{1}{2}$ and $y = 0$; $f\left(\frac{1}{2}, 0\right) = \frac{3}{4}$; $f(0, 0) = 1$, and $f(1, 0) = 1$

(v) On the interior of the square, $f_x(x, y) = 2x + 2y - 1 = 0$ and $f_y(x, y) = 2y + 2x - 1 = 0 \Rightarrow 2x + 2y = 1$ $\Rightarrow (x + y) = \frac{1}{2}$. Then $f(x, y) = x^2 + y^2 + 2xy - x - y + 1 = (x + y)^2 - (x + y) + 1 = \frac{3}{4}$ is the absolute minimum value when $2x + 2y = 1$.

(b) The absolute maximum is $f(1, 1) = 3$.

61. (a) $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \frac{dx}{dt} + \frac{dy}{dt} = -2\sin t + 2\cos t = 0 \Rightarrow \cos t = \sin t \Rightarrow x = y$

(i) On the semicircle $x^2 + y^2 = 4$, $y \ge 0$, we have $t = \frac{\pi}{4}$ and $x = y = \sqrt{2} \Rightarrow f\left(\sqrt{2}, \sqrt{2}\right) = 2\sqrt{2}$. At the endpoints, $f(-2, 0) = -2$ and $f(2, 0) = 2$. Therefore the absolute minimum is $f(-2, 0) = -2$ when $t = \pi$; the absolute maximum is $f\left(\sqrt{2}, \sqrt{2}\right) = 2\sqrt{2}$ when $t = \frac{\pi}{4}$.

(ii) On the quartercircle $x^2 + y^2 = 4$, $x \ge 0$ and $y \ge 0$, the endpoints give $f(0, 2) = 2$ and $f(2, 0) = 2$. Therefore the absolute minimum is $f(2, 0) = 2$ and $f(0, 2) = 2$ when $t = 0, \frac{\pi}{2}$ respectively; the absolute maximum is $f\left(\sqrt{2}, \sqrt{2}\right) = 2\sqrt{2}$ when $t = \frac{\pi}{4}$.

(b) $\frac{dg}{dt} = \frac{\partial g}{\partial x}\frac{dx}{dt} + \frac{\partial g}{\partial y}\frac{dy}{dt} = y\frac{dx}{dt} + x\frac{dy}{dt} = -4\sin^2 t + 4\cos^2 t = 0 \Rightarrow \cos t = \pm\sin t \Rightarrow x = \pm y$.

(i) On the semicircle $x^2 + y^2 = 4$, $y \ge 0$, we obtain $x = y = \sqrt{2}$ at $t = \frac{\pi}{4}$ and $x = -\sqrt{2}$, $y = \sqrt{2}$ at $t = \frac{3\pi}{4}$. Then $g\left(\sqrt{2}, \sqrt{2}\right) = 2$ and $g\left(-\sqrt{2}, \sqrt{2}\right) = -2$. At the endpoints, $g(-2, 0) = g(2, 0) = 0$. Therefore the absolute minimum is $g\left(-\sqrt{2}, \sqrt{2}\right) = -2$ when $t = \frac{3\pi}{4}$; the absolute maximum is $g\left(\sqrt{2}, \sqrt{2}\right) = 2$ when $t = \frac{\pi}{4}$.

(ii) On the quartercircle $x^2 + y^2 = 4$, $x \ge 0$ and $y \ge 0$, the endpoints give $g(0,2) = 0$ and $g(2,0) = 0$. Therefore the absolute minimum is $g(2,0) = 0$ and $g(0,2) = 0$ when $t = 0, \frac{\pi}{2}$ respectively; the absolute maximum is $g\left(\sqrt{2}, \sqrt{2}\right) = 2$ when $t = \frac{\pi}{4}$.

(c) $\frac{dh}{dt} = \frac{\partial h}{\partial x}\frac{dx}{dt} + \frac{\partial h}{\partial y}\frac{dy}{dt} = 4x\frac{dx}{dt} + 2y\frac{dy}{dt} = (8\cos t)(-2\sin t) + (4\sin t)(2\cos t) = -8\cos t\sin t = 0$
$\Rightarrow\ t = 0, \frac{\pi}{2}, \pi$ yielding the points $(2,0)$, $(0,2)$ for $0 \le t \le \pi$.

(i) On the semicircle $x^2 + y^2 = 4$, $y \ge 0$ we have $h(2,0) = 8$, $h(0,2) = 4$, and $h(-2,0) = 8$. Therefore, the absolute minimum is $h(0,2) = 4$ when $t = \frac{\pi}{2}$; the absolute maximum is $h(2,0) = 8$ and $h(-2,0) = 8$ when $t = 0, \pi$ respectively.

(ii) On the quartercircle $x^2 + y^2 = 4$, $x \ge 0$ and $y \ge 0$ the absolute minimum is $h(0,2) = 4$ when $t = \frac{\pi}{2}$; the absolute maximum is $h(2,0) = 8$ when $t = 0$.

62. (a) $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = 2\frac{dx}{dt} + 3\frac{dy}{dt} = -6\sin t + 6\cos t = 0\ \Rightarrow\ \sin t = \cos t\ \Rightarrow\ t = \frac{\pi}{4}$ for $0 \le t \le \pi$.

(i) On the semi-ellipse, $\frac{x^2}{9} + \frac{y^2}{4} = 1$, $y \ge 0$, $f(x,y) = 2x + 3y = 6\cos t + 6\sin t = 6\left(\frac{\sqrt{2}}{2}\right) + 6\left(\frac{\sqrt{2}}{2}\right) = 6\sqrt{2}$ at $t = \frac{\pi}{4}$. At the endpoints, $f(-3,0) = -6$ and $f(3,0) = 6$. The absolute minimum is $f(-3,0) = -6$ when $t = \pi$; the absolute maximum is $f\left(\frac{3\sqrt{2}}{2}, \sqrt{2}\right) = 6\sqrt{2}$ when $t = \frac{\pi}{4}$.

(ii) On the quarter ellipse, at the endpoints $f(0,2) = 6$ and $f(3,0) = 6$. The absolute minimum is $f(3,0) = 6$ and $f(0,2) = 6$ when $t = 0, \frac{\pi}{2}$ respectively; the absolute maximum is $f\left(\frac{3\sqrt{2}}{2}, \sqrt{2}\right) = 6\sqrt{2}$ when $t = \frac{\pi}{4}$.

(b) $\frac{dg}{dt} = \frac{\partial g}{\partial x}\frac{dx}{dt} + \frac{\partial g}{\partial y}\frac{dy}{dt} = y\frac{dx}{dt} + x\frac{dy}{dt} = (2\sin t)(-3\sin t) + (3\cos t)(2\cos t) = 6\left(\cos^2 t - \sin^2 t\right) = 6\cos 2t = 0$
$\Rightarrow\ t = \frac{\pi}{4}, \frac{3\pi}{4}$ for $0 \le t \le \pi$.

(i) On the semi-ellipse, $g(x,y) = xy = 6\sin t\cos t$. Then $g\left(\frac{3\sqrt{2}}{2}, \sqrt{2}\right) = 3$ when $t = \frac{\pi}{4}$, and $g\left(-\frac{3\sqrt{2}}{2}, \sqrt{2}\right) = -3$ when $t = \frac{3\pi}{4}$. At the endpoints, $g(-3,0) = g(3,0) = 0$. The absolute minimum is $g\left(-\frac{3\sqrt{2}}{2}, \sqrt{2}\right) = -3$ when $t = \frac{3\pi}{4}$; the absolute maximum is $g\left(\frac{3\sqrt{2}}{2}, \sqrt{2}\right) = 3$ when $t = \frac{\pi}{4}$.

(ii) On the quarter ellipse, at the endpoints $g(0,2) = 0$ and $g(3,0) = 0$. The absolute minimum is $g(3,0) = 0$ and $g(0,2) = 0$ at $t = 0, \frac{\pi}{2}$ respectively; the absolute maximum is $g\left(\frac{3\sqrt{2}}{2}, \sqrt{2}\right) = 3$ when $t = \frac{\pi}{4}$.

(c) $\frac{dh}{dt} = \frac{\partial h}{\partial x}\frac{dx}{dt} + \frac{\partial h}{\partial y}\frac{dy}{dt} = 2x\frac{dx}{dt} + 6y\frac{dy}{dt} = (6\cos t)(-3\sin t) + (12\sin t)(2\cos t) = 6\sin t\cos t = 0$
$\Rightarrow\ t = 0, \frac{\pi}{2}, \pi$ for $0 \le t \le \pi$, yielding the points $(3,0)$, $(0,2)$, and $(-3,0)$.

(i) On the semi-ellipse, $y \ge 0$ so that $h(3,0) = 9$, $h(0,2) = 12$, and $h(-3,0) = 9$. The absolute minimum is $h(3,0) = 9$ and $h(-3,0) = 9$ when $t = 0, \pi$ respectively; the absolute maximum is $h(0,2) = 12$ when $t = \frac{\pi}{2}$.

(ii) On the quarter ellipse, the absolute minimum is $h(3,0) = 9$ when $t = 0$; the absolute maximum is $h(0,2) = 12$ when $t = \frac{\pi}{2}$.

63. $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = y\frac{dx}{dt} + x\frac{dy}{dt}$

(i) $x = 2t$ and $y = t + 1\ \Rightarrow\ \frac{df}{dt} = (t+1)(2) + (2t)(1) = 4t + 2 = 0\ \Rightarrow\ t = -\frac{1}{2}\ \Rightarrow\ x = -1$ and $y = \frac{1}{2}$ with $f\left(-1, \frac{1}{2}\right) = -\frac{1}{2}$. The absolute minimum is $f\left(-1, \frac{1}{2}\right) = -\frac{1}{2}$ when $t = -\frac{1}{2}$; there is no absolute maximum.

(ii) For the endpoints: $t = -1\ \Rightarrow\ x = -2$ and $y = 0$ with $f(-2,0) = 0$; $t = 0\ \Rightarrow\ x = 0$ and $y = 1$ with $f(0,1) = 0$. The absolute minimum is $f\left(-1, \frac{1}{2}\right) = -\frac{1}{2}$ when $t = -\frac{1}{2}$; the absolute maximum is $f(0,1) = 0$ and $f(-2,0) = 0$ when $t = -1, 0$ respectively.

(iii) There are no interior critical points. For the endpoints: $t = 0\ \Rightarrow\ x = 0$ and $y = 1$ with $f(0,1) = 0$; $t = 1\ \Rightarrow\ x = 2$ and $y = 2$ with $f(2,2) = 4$. The absolute minimum is $f(0,1) = 0$ when $t = 0$; the absolute maximum is $f(2,2) = 4$ when $t = 1$.

64. (a) $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$

(i) $x = t$ and $y = 2 - 2t \Rightarrow \frac{df}{dt} = (2t)(1) + 2(2-2t)(-2) = 10t - 8 = 0 \Rightarrow t = \frac{4}{5} \Rightarrow x = \frac{4}{5}$ and $y = \frac{2}{5}$ with $f\left(\frac{4}{5}, \frac{2}{5}\right) = \frac{16}{25} + \frac{4}{25} = \frac{4}{5}$. The absolute minimum is $f\left(\frac{4}{5}, \frac{2}{5}\right) = \frac{4}{5}$ when $t = \frac{4}{5}$; there is no absolute maximum along the line.

(ii) For the endpoints: $t = 0 \Rightarrow x = 0$ and $y = 2$ with $f(0,2) = 4$; $t = 1 \Rightarrow x = 1$ and $y = 0$ with $f(1,0) = 1$. The absolute minimum is $f\left(\frac{4}{5}, \frac{2}{5}\right) = \frac{4}{5}$ at the interior critical point when $t = \frac{4}{5}$; the absolute maximum is $f(0,2) = 4$ at the endpoint when $t = 0$.

(b) $\frac{dg}{dt} = \frac{\partial g}{\partial x}\frac{dx}{dt} + \frac{\partial g}{\partial y}\frac{dy}{dt} = \left[\frac{-2x}{(x^2+y^2)^2}\right]\frac{dx}{dt} + \left[\frac{-2y}{(x^2+y^2)^2}\right]\frac{dy}{dt}$

(i) $x = t$ and $y = 2 - 2t \Rightarrow x^2 + y^2 = 5t^2 - 8t + 4 \Rightarrow \frac{dg}{dt} = -\left(5t^2 - 8t + 4\right)^{-2}[(-2t)(1) + (-2)(2-2t)(-2)]$ $= -\left(5t^2 - 8t + 4\right)^{-2}(-10t + 8) = 0 \Rightarrow t = \frac{4}{5} \Rightarrow x = \frac{4}{5}$ and $y = \frac{2}{5}$ with $g\left(\frac{4}{5}, \frac{2}{5}\right) = \frac{1}{\left(\frac{4}{5}\right)} = \frac{5}{4}$. The absolute maximum is $g\left(\frac{4}{5}, \frac{2}{5}\right) = \frac{5}{4}$ when $t = \frac{4}{5}$; there is no absolute minimum along the line since x and y can be as large as we please.

(ii) For the endpoints: $t = 0 \Rightarrow x = 0$ and $y = 2$ with $g(0,2) = \frac{1}{4}$; $t = 1 \Rightarrow x = 1$ and $y = 0$ with $g(1,0) = 1$. The absolute minimum is $g(0,2) = \frac{1}{4}$ when $t = 0$; the absolute maximum is $g\left(\frac{4}{5}, \frac{2}{5}\right) = \frac{5}{4}$ when $t = \frac{4}{5}$.

65. $w = (mx_1 + b - y_1)^2 + (mx_2 + b - y_2)^2 + \cdots + (mx_n + b - y_n)^2$

$\Rightarrow \frac{\partial w}{\partial m} = 2(mx_1 + b - y_1)(x_1) + 2(mx_2 + b - y_2)(x_2) + \cdots + 2(mx_n + b - y_n)(x_n)$

$\Rightarrow \frac{\partial w}{\partial b} = 2(mx_1 + b - y_1)(1) + 2(mx_2 + b - y_2)(1) + \cdots + 2(mx_n + b - y_n)(1)$

$\frac{\partial w}{\partial m} = 0 \Rightarrow 2\left[(mx_1 + b - y_1)(x_1) + (mx_2 + b - y_2)(x_2) + \cdots + (mx_n + b - y_n)(x_n)\right] = 0$

$\Rightarrow mx_1^2 + bx_1 - x_1y_1 + mx_2^2 + bx_2 - x_2y_2 + \cdots + mx_n^2 + bx_n - x_ny_n = 0$

$\Rightarrow m(x_1^2 + x_2^2 + \cdots + x_n^2) + b(x_1 + x_2 + \cdots + x_n) - (x_1y_1 + x_2y_2 + \cdots + x_ny_n) = 0$

$\Rightarrow m\sum_{k=1}^{n}(x_k^2) + b\sum_{k=1}^{n}x_k - \sum_{k=1}^{n}(x_ky_k) = 0$

$\frac{\partial w}{\partial b} = 0 \Rightarrow 2\left[(mx_1 + b - y_1) + (mx_2 + b - y_2) + \cdots + (mx_n + b - y_n)\right] = 0$

$\Rightarrow mx_1 + b - y_1 + mx_2 + b - y_2 + \cdots + mx_n + b - y_n = 0$

$\Rightarrow m(x_1 + x_2 + \cdots + x_n) + (b + b + \cdots + b) - (y_1 + y_2 + \cdots + y_n) = 0$

$\Rightarrow m\sum_{k=1}^{n}x_k + b\sum_{k=1}^{n}1 - \sum_{k=1}^{n}y_k = 0 \Rightarrow m\sum_{k=1}^{n}x_k + bn - \sum_{k=1}^{n}y_k = 0 \Rightarrow b = \frac{1}{n}\left(\sum_{k=1}^{n}y_k - m\sum_{k=1}^{n}x_k\right).$

Substituting for b in the equation obtained for $\frac{\partial w}{\partial m}$ we get $m\sum_{k=1}^{n}(x_k^2) + \frac{1}{n}\left(\sum_{k=1}^{n}y_k - m\sum_{k=1}^{n}x_k\right)\sum_{k=1}^{n}x_k - \sum_{k=1}^{n}(x_ky_k) = 0.$

Multiply both sides by n to obtain $mn\sum_{k=1}^{n}(x_k^2) + \left(\sum_{k=1}^{n}y_k - m\sum_{k=1}^{n}x_k\right)\sum_{k=1}^{n}x_k - n\sum_{k=1}^{n}(x_ky_k) = 0$

$\Rightarrow mn\sum_{k=1}^{n}(x_k^2) + \left(\sum_{k=1}^{n}x_k\right)\left(\sum_{k=1}^{n}y_k\right) - m\left(\sum_{k=1}^{n}x_k\right)^2 - n\sum_{k=1}^{n}(x_ky_k) = 0$

$\Rightarrow mn\sum_{k=1}^{n}(x_k^2) - m\left(\sum_{k=1}^{n}x_k\right)^2 = n\sum_{k=1}^{n}(x_ky_k) - \left(\sum_{k=1}^{n}x_k\right)\left(\sum_{k=1}^{n}y_k\right)$

$\Rightarrow m\left[n\sum_{k=1}^{n}(x_k^2) - \left(\sum_{k=1}^{n}x_k\right)^2\right] = n\sum_{k=1}^{n}(x_ky_k) - \left(\sum_{k=1}^{n}x_k\right)\left(\sum_{k=1}^{n}y_k\right)$

$\Rightarrow m = \frac{n\sum_{k=1}^{n}(x_ky_k) - \left(\sum_{k=1}^{n}x_k\right)\left(\sum_{k=1}^{n}y_k\right)}{n\sum_{k=1}^{n}(x_k^2) - \left(\sum_{k=1}^{n}x_k\right)^2} = \frac{\left(\sum_{k=1}^{n}x_k\right)\left(\sum_{k=1}^{n}y_k\right) - n\sum_{k=1}^{n}(x_ky_k)}{\left(\sum_{k=1}^{n}x_k\right)^2 - n\sum_{k=1}^{n}(x_k^2)}$

To show that these values for m and b minimize the sum of the squares of the distances, use second derivative test.

$\frac{\partial^2 w}{\partial m^2} = 2x_1^2 + 2x_2^2 + \cdots + 2x_n^2 = 2\sum_{k=1}^{n}(x_k^2)$; $\frac{\partial^2 w}{\partial m\,\partial b} = 2x_1 + 2x_2 + \cdots + 2x_n = 2\sum_{k=1}^{n}x_k$; $\frac{\partial^2 w}{\partial b^2} = 2 + 2 + \cdots + 2 = 2n$

The discriminant is: $\left(\frac{\partial^2 w}{\partial m^2}\right)\left(\frac{\partial^2 w}{\partial b^2}\right) - \left(\frac{\partial^2 w}{\partial m\,\partial b}\right)^2 = \left[2\sum_{k=1}^{n}(x_k^2)\right](2n) - \left[2\sum_{k=1}^{n}x_k\right]^2 = 4\left[n\sum_{k=1}^{n}(x_k^2) - \left(\sum_{k=1}^{n}x_k\right)^2\right].$

Now, $n\sum_{k=1}^{n}(x_k^2) - \left(\sum_{k=1}^{n}x_k\right)^2 = n(x_1^2 + x_2^2 + \cdots + x_n^2) - (x_1 + x_2 + \cdots + x_n)(x_1 + x_2 + \cdots + x_n)$

$= n\,x_1^2 + n\,x_2^2 + \cdots + n\,x_n^2 - x_1^2 - x_1x_2 - \cdots - x_1x_n - x_2x_1 - x_2^2 - \cdots - x_2x_n - x_nx_1 - x_nx_2 - \cdots - x_n^2$

$= (n-1)\,x_1^2 + (n-1)\,x_2^2 + \cdots + (n-1)\,x_n^2 - 2\,x_1x_2 - 2\,x_1x_3 - \cdots - 2\,x_1x_n - 2\,x_2x_3 - \cdots - 2\,x_2x_n - \cdots - 2\,x_{n-1}x_n$

$= (x_1^2 - 2\,x_1x_2 + x_2^2) + (x_1^2 - 2\,x_1x_3 + x_3^2) + \cdots + (x_1^2 - 2\,x_1x_n + x_n^2) + (x_2^2 - 2\,x_2x_3 + x_3^2) + \cdots + (x_2^2 - 2\,x_2x_n + x_n^2)$
$+ \cdots + (x_{n-1}^2 - 2\,x_{n-1}x_n + x_n^2)$

$= (x_1 - x_2)^2 + (x_1 - x_3)^2 + \cdots + (x_1 - x_n)^2 + (x_2 - x_3)^2 + \cdots + (x_2 - x_n)^2 + \cdots + (x_{n-1} - x_n)^2 \geq 0.$

Thus we have : $\left(\frac{\partial^2 w}{\partial m^2}\right)\left(\frac{\partial^2 w}{\partial b^2}\right) - \left(\frac{\partial^2 w}{\partial m\,\partial b}\right)^2 = 4\left[n\sum_{k=1}^{n}(x_k^2) - \left(\sum_{k=1}^{n}x_k\right)^2\right] \geq 4(0) = 0$. If $x_1 = x_2 = \cdots = x_n$ then $\left(\frac{\partial^2 w}{\partial m^2}\right)\left(\frac{\partial^2 w}{\partial b^2}\right) - \left(\frac{\partial^2 w}{\partial m\,\partial b}\right)^2 = 0$. Also, $\frac{\partial^2 w}{\partial m^2} = 2\sum_{k=1}^{n}(x_k^2) \geq 0$. If $x_1 = x_2 = \cdots = x_n = 0$, then $\frac{\partial^2 w}{\partial m^2} = 0$.

Provided that at least one x_i is nonzero and different from the rest of x_j, $j \neq i$, then $\left(\frac{\partial^2 w}{\partial m^2}\right)\left(\frac{\partial^2 w}{\partial b^2}\right) - \left(\frac{\partial^2 w}{\partial m\,\partial b}\right)^2 > 0$ and $\frac{\partial^2 w}{\partial m^2} > 0 \Rightarrow$ the values given above for m and b minimize w.

66. $m = \frac{(0)(5) - 3(6)}{(0)^2 - 3(8)} = \frac{3}{4}$ and
$b = \frac{1}{3}\left[5 - \frac{3}{4}(0)\right] = \frac{5}{3}$
$\Rightarrow y = \frac{3}{4}x + \frac{5}{3}$; $y\big|_{x=4} = \frac{14}{3}$

k	x_k	y_k	x_k^2	x_ky_k
1	−2	0	4	0
2	0	2	0	0
3	2	3	4	6
Σ	0	5	8	6

67. $m = \frac{(2)(-1) - 3(-14)}{(2)^2 - 3(10)} = -\frac{20}{13}$ and
$b = \frac{1}{3}\left[-1 - \left(-\frac{20}{13}\right)(2)\right] = \frac{9}{13}$
$\Rightarrow y = -\frac{20}{13}x + \frac{9}{13}$; $y\big|_{x=4} = -\frac{71}{13}$

k	x_k	y_k	x_k^2	x_ky_k
1	−1	2	1	−2
2	0	1	0	0
3	3	−4	9	−12
Σ	2	−1	10	−14

68. $m = \frac{(3)(5) - 3(8)}{(3)^2 - 3(5)} = \frac{3}{2}$ and
$b = \frac{1}{3}\left[5 - \frac{3}{2}(3)\right] = \frac{1}{6}$
$\Rightarrow y = \frac{3}{2}x + \frac{1}{6}$; $y\big|_{x=4} = \frac{37}{6}$

k	x_k	y_k	x_k^2	x_ky_k
1	0	0	0	0
2	1	2	1	2
3	2	3	4	6
Σ	3	5	5	8

69-74. Example CAS commands:

Maple:

```
f := (x,y) -> x^2+y^3-3*x*y;
x0,x1 := -5,5;
y0,y1 := -5,5;
plot3d( f(x,y), x=x0..x1, y=y0..y1, axes=boxed, shading=zhue, title="#69(a) (Section 14.7)" );
plot3d( f(x,y), x=x0..x1, y=y0..y1, grid=[40,40], axes=boxed, shading=zhue, style=patchcontour, title="#69(b)
    (Section 14.7)" );
fx := D[1](f);                                          # (c)
fy := D[2](f);
crit_pts := solve( {fx(x,y)=0,fy(x,y)=0}, {x,y} );
fxx := D[1](fx);                                        # (d)
fxy := D[2](fx);
```

```
fyy := D[2](fy);
discr := unapply( fxx(x,y)*fyy(x,y)-fxy(x,y)^2, (x,y) );
for CP in {crit_pts} do                                  # (e)
  eval( [x,y,fxx(x,y),discr(x,y)], CP );
end do;
# (0,0) is a saddle point
# ( 9/4,  3/2) is a local minimum
```

Mathematica: (assigned functions and bounds will vary)

```
Clear[x,y,f]
f[x_,y_]:= x^2 + y^3 − 3x y
xmin= −5;  xmax= 5;  ymin= −5;  ymax= 5;
Plot3D[f[x,y], {x, xmin, xmax}, {y, ymin, ymax}, AxesLabel → {x, y, z}]
ContourPlot[f[x,y], {x, xmin, xmax}, {y, ymin, ymax}, ContourShading → False, Contours → 40]
fx= D[f[x,y], x];
fy= D[f[x,y], y];
critical=Solve[{fx==0, fy==0},{x, y}]
fxx= D[fx, x];
fxy= D[fx, y];
fyy= D[fy, y];
discriminant= fxx fyy − fxy^2
{{x, y}, f[x, y], discriminant, fxx} /.critical
```

14.8 LAGRANGE MULTIPLIERS

1. $\nabla f = y\mathbf{i} + x\mathbf{j}$ and $\nabla g = 2x\mathbf{i} + 4y\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow y\mathbf{i} + x\mathbf{j} = \lambda(2x\mathbf{i} + 4y\mathbf{j}) \Rightarrow y = 2x\lambda$ and $x = 4y\lambda$ $\Rightarrow x = 8x\lambda^2 \Rightarrow \lambda = \pm \frac{\sqrt{2}}{4}$ or $x = 0$.

   CASE 1: If $x = 0$, then $y = 0$. But $(0, 0)$ is not on the ellipse so $x \neq 0$.

   CASE 2: $x \neq 0 \Rightarrow \lambda = \pm \frac{\sqrt{2}}{4} \Rightarrow x = \pm \sqrt{2}y \Rightarrow \left(\pm \sqrt{2}y\right)^2 + 2y^2 = 1 \Rightarrow y = \pm \frac{1}{2}$.

   Therefore f takes on its extreme values at $\left(\pm \frac{\sqrt{2}}{2}, \frac{1}{2}\right)$ and $\left(\pm \frac{\sqrt{2}}{2}, -\frac{1}{2}\right)$. The extreme values of f on the ellipse are $\pm \frac{\sqrt{2}}{2}$.

2. $\nabla f = y\mathbf{i} + x\mathbf{j}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow y\mathbf{i} + x\mathbf{j} = \lambda(2x\mathbf{i} + 2y\mathbf{j}) \Rightarrow y = 2x\lambda$ and $x = 2y\lambda$ $\Rightarrow x = 4x\lambda^2 \Rightarrow x = 0$ or $\lambda = \pm \frac{1}{2}$.

   CASE 1: If $x = 0$, then $y = 0$. But $(0, 0)$ is not on the circle $x^2 + y^2 - 10 = 0$ so $x \neq 0$.

   CASE 2: $x \neq 0 \Rightarrow \lambda = \pm \frac{1}{2} \Rightarrow y = 2x\left(\pm \frac{1}{2}\right) = \pm x \Rightarrow x^2 + (\pm x)^2 - 10 = 0 \Rightarrow x = \pm \sqrt{5} \Rightarrow y = \pm \sqrt{5}$.

   Therefore f takes on its extreme values at $\left(\pm \sqrt{5}, \sqrt{5}\right)$ and $\left(\pm \sqrt{5}, -\sqrt{5}\right)$. The extreme values of f on the circle are 5 and -5.

3. $\nabla f = -2x\mathbf{i} - 2y\mathbf{j}$ and $\nabla g = \mathbf{i} + 3\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow -2x\mathbf{i} - 2y\mathbf{j} = \lambda(\mathbf{i} + 3\mathbf{j}) \Rightarrow x = -\frac{\lambda}{2}$ and $y = -\frac{3\lambda}{2}$ $\Rightarrow \left(-\frac{\lambda}{2}\right) + 3\left(-\frac{3\lambda}{2}\right) = 10 \Rightarrow \lambda = -2 \Rightarrow x = 1$ and $y = 3 \Rightarrow$ f takes on its extreme value at $(1, 3)$ on the line. The extreme value is $f(1, 3) = 49 - 1 - 9 = 39$.

4. $\nabla f = 2xy\mathbf{i} + x^2\mathbf{j}$ and $\nabla g = \mathbf{i} + \mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow 2xy\mathbf{i} + x^2\mathbf{j} = \lambda(\mathbf{i} + \mathbf{j}) \Rightarrow 2xy = \lambda$ and $x^2 = \lambda$ $\Rightarrow 2xy = x^2 \Rightarrow x = 0$ or $2y = x$.

   CASE 1: If $x = 0$, then $x + y = 3 \Rightarrow y = 3$.

CASE 2: If $x \neq 0$, then $2y = x$ so that $x + y = 3 \Rightarrow 2y + y = 3 \Rightarrow y = 1 \Rightarrow x = 2$.
Therefore f takes on its extreme values at $(0, 3)$ and $(2, 1)$. The extreme values of f are $f(0, 3) = 0$ and $f(2, 1) = 4$.

5. We optimize $f(x, y) = x^2 + y^2$, the square of the distance to the origin, subject to the constraint $g(x, y) = xy^2 - 54 = 0$. Thus $\nabla f = 2x\mathbf{i} + 2y\mathbf{j}$ and $\nabla g = y^2\mathbf{i} + 2xy\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow 2x\mathbf{i} + 2y\mathbf{j} = \lambda\,(y^2\mathbf{i} + 2xy\mathbf{j}) \Rightarrow 2x = \lambda y^2$ and $2y = 2\lambda xy$.
CASE 1: If $y = 0$, then $x = 0$. But $(0, 0)$ does not satisfy the constraint $xy^2 = 54$ so $y \neq 0$.
CASE 2: If $y \neq 0$, then $2 = 2\lambda x \Rightarrow x = \frac{1}{\lambda} \Rightarrow 2\left(\frac{1}{\lambda}\right) = \lambda y^2 \Rightarrow y^2 = \frac{2}{\lambda^2}$. Then $xy^2 = 54 \Rightarrow \left(\frac{1}{\lambda}\right)\left(\frac{2}{\lambda^2}\right) = 54$
$\Rightarrow \lambda^3 = \frac{1}{27} \Rightarrow \lambda = \frac{1}{3} \Rightarrow x = 3$ and $y^2 = 18 \Rightarrow x = 3$ and $y = \pm 3\sqrt{2}$.
Therefore $\left(3, \pm 3\sqrt{2}\right)$ are the points on the curve $xy^2 = 54$ nearest the origin (since $xy^2 = 54$ has points increasingly far away as y gets close to 0, no points are farthest away).

6. We optimize $f(x, y) = x^2 + y^2$, the square of the distance to the origin subject to the constraint $g(x, y) = x^2y - 2 = 0$. Thus $\nabla f = 2x\mathbf{i} + 2y\mathbf{j}$ and $\nabla g = 2xy\mathbf{i} + x^2\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow 2x = 2xy\lambda$ and $2y = x^2\lambda \Rightarrow \lambda = \frac{2y}{x^2}$, since $x = 0 \Rightarrow y = 0$ (but $g(0, 0) \neq 0$). Thus $x \neq 0$ and $2x = 2xy\left(\frac{2y}{x^2}\right) \Rightarrow x^2 = 2y^2 \Rightarrow (2y^2)\,y - 2 = 0 \Rightarrow y = 1$ (since $y > 0$) $\Rightarrow x = \pm\sqrt{2}$. Therefore $\left(\pm\sqrt{2}, 1\right)$ are the points on the curve $x^2y = 2$ nearest the origin (since $x^2y = 2$ has points increasingly far away as x gets close to 0, no points are farthest away).

7. (a) $\nabla f = \mathbf{i} + \mathbf{j}$ and $\nabla g = y\mathbf{i} + x\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow \mathbf{i} + \mathbf{j} = \lambda(y\mathbf{i} + x\mathbf{j}) \Rightarrow 1 = \lambda y$ and $1 = \lambda x \Rightarrow y = \frac{1}{\lambda}$ and $x = \frac{1}{\lambda} \Rightarrow \frac{1}{\lambda^2} = 16 \Rightarrow \lambda = \pm\frac{1}{4}$. Use $\lambda = \frac{1}{4}$ since $x > 0$ and $y > 0$. Then $x = 4$ and $y = 4 \Rightarrow$ the minimum value is 8 at the point $(4, 4)$. Now, $xy = 16$, $x > 0$, $y > 0$ is a branch of a hyperbola in the first quadrant with the x-and y-axes as asymptotes. The equations $x + y = c$ give a family of parallel lines with $m = -1$. As these lines move away from the origin, the number c increases. Thus the minimum value of c occurs where $x + y = c$ is tangent to the hyperbola's branch.
(b) $\nabla f = y\mathbf{i} + x\mathbf{j}$ and $\nabla g = \mathbf{i} + \mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow y\mathbf{i} + x\mathbf{j} = \lambda(\mathbf{i} + \mathbf{j}) \Rightarrow y = \lambda = x\ y + y = 16 \Rightarrow y = 8 \Rightarrow x = 8 \Rightarrow f(8, 8) = 64$ is the maximum value. The equations $xy = c$ ($x > 0$ and $y > 0$ or $x < 0$ and $y < 0$ to get a maximum value) give a family of hyperbolas in the first and third quadrants with the x- and y-axes as asymptotes. The maximum value of c occurs where the hyperbola $xy = c$ is tangent to the line $x + y = 16$.

8. Let $f(x, y) = x^2 + y^2$ be the square of the distance from the origin. Then $\nabla f = 2x\mathbf{i} + 2y\mathbf{j}$ and $\nabla g = (2x + y)\mathbf{i} + (2y + x)\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow 2x = \lambda(2x + y)$ and $2y = \lambda(2y + x) \Rightarrow \frac{2y}{2y+x} = \lambda$
$\Rightarrow 2x = \left(\frac{2y}{2y+x}\right)(2x + y) \Rightarrow x(2y + x) = y(2x + y) \Rightarrow x^2 = y^2 \Rightarrow y = \pm x$.
CASE 1: $y = x \Rightarrow x^2 + x(x) + x^2 - 1 = 0 \Rightarrow x = \pm\frac{1}{\sqrt{3}}$ and $y = x$.
CASE 2: $y = -x \Rightarrow x^2 + x(-x) + (-x)^2 - 1 = 0 \Rightarrow x = \pm 1$ and $y = -x$. Thus $f\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \frac{2}{3}$
$= f\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$ and $f(1, -1) = 2 = f(-1, 1)$.
Therefore the points $(1, -1)$ and $(-1, 1)$ are the farthest away; $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and $\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$ are the closest points to the origin.

9. $V = \pi r^2 h \Rightarrow 16\pi = \pi r^2 h \Rightarrow 16 = r^2 h \Rightarrow g(r, h) = r^2 h - 16$; $S = 2\pi rh + 2\pi r^2 \Rightarrow \nabla S = (2\pi h + 4\pi r)\mathbf{i} + 2\pi r\mathbf{j}$ and $\nabla g = 2rh\mathbf{i} + r^2\mathbf{j}$ so that $\nabla S = \lambda \nabla g \Rightarrow (2\pi rh + 4\pi r)\mathbf{i} + 2\pi r\mathbf{j} = \lambda\,(2rh\mathbf{i} + r^2\mathbf{j}) \Rightarrow 2\pi rh + 4\pi r = 2rh\lambda$ and $2\pi r = \lambda r^2 \Rightarrow r = 0$ or $\lambda = \frac{2\pi}{r}$. But $r = 0$ gives no physical can, so $r \neq 0 \Rightarrow \lambda = \frac{2\pi}{r} \Rightarrow 2\pi h + 4\pi r = 2rh\left(\frac{2\pi}{r}\right) \Rightarrow 2r = h \Rightarrow 16 = r^2(2r) \Rightarrow r = 2 \Rightarrow h = 4$; thus $r = 2$ cm and $h = 4$ cm give the only extreme surface area of $24\pi\ \text{cm}^2$. Since $r = 4$ cm and $h = 1$ cm $\Rightarrow V = 16\pi\ \text{cm}^3$ and $S = 40\pi\ \text{cm}^2$, which is a larger surface area, then $24\pi\ \text{cm}^2$ must be the minimum surface area.

10. For a cylinder of radius r and height h we want to maximize the surface area $S = 2\pi rh$ subject to the constraint $g(r,h) = r^2 + \left(\frac{h}{2}\right)^2 - a^2 = 0$. Thus $\nabla S = 2\pi h\mathbf{i} + 2\pi r\mathbf{j}$ and $\nabla g = 2r\mathbf{i} + \frac{h}{2}\mathbf{j}$ so that $\nabla S = \lambda \nabla g \Rightarrow 2\pi h = 2\lambda r$ and $2\pi r = \frac{\lambda h}{2} \Rightarrow \frac{\pi h}{r} = \lambda$ and $2\pi r = \left(\frac{\pi h}{r}\right)\left(\frac{h}{2}\right) \Rightarrow 4r^2 = h^2 \Rightarrow h = 2r \Rightarrow r^2 + \frac{4r^2}{4} = a^2 \Rightarrow 2r^2 = a^2 \Rightarrow r = \frac{a}{\sqrt{2}}$ $\Rightarrow h = a\sqrt{2} \Rightarrow S = 2\pi\left(\frac{a}{\sqrt{2}}\right)\left(a\sqrt{2}\right) = 2\pi a^2$.

11. $A = (2x)(2y) = 4xy$ subject to $g(x,y) = \frac{x^2}{16} + \frac{y^2}{9} - 1 = 0$; $\nabla A = 4y\mathbf{i} + 4x\mathbf{j}$ and $\nabla g = \frac{x}{8}\mathbf{i} + \frac{2y}{9}\mathbf{j}$ so that $\nabla A = \lambda \nabla g \Rightarrow 4y\mathbf{i} + 4x\mathbf{j} = \lambda\left(\frac{x}{8}\mathbf{i} + \frac{2y}{9}\mathbf{j}\right) \Rightarrow 4y = \left(\frac{x}{8}\right)\lambda$ and $4x = \left(\frac{2y}{9}\right)\lambda \Rightarrow \lambda = \frac{32y}{x}$ and $4x = \left(\frac{2y}{9}\right)\left(\frac{32y}{x}\right)$ $\Rightarrow y = \pm\frac{3}{4}x \Rightarrow \frac{x^2}{16} + \frac{\left(\frac{\pm 3}{4}x\right)^2}{9} = 1 \Rightarrow x^2 = 8 \Rightarrow x = \pm 2\sqrt{2}$. We use $x = 2\sqrt{2}$ since x represents distance. Then $y = \frac{3}{4}\left(2\sqrt{2}\right) = \frac{3\sqrt{2}}{2}$, so the length is $2x = 4\sqrt{2}$ and the width is $2y = 3\sqrt{2}$.

12. $P = 4x + 4y$ subject to $g(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$; $\nabla P = 4\mathbf{i} + 4\mathbf{j}$ and $\nabla g = \frac{2x}{a^2}\mathbf{i} + \frac{2y}{b^2}\mathbf{j}$ so that $\nabla P = \lambda \nabla g$ $\Rightarrow 4 = \left(\frac{2x}{a^2}\right)\lambda$ and $4 = \left(\frac{2y}{b^2}\right)\lambda \Rightarrow \lambda = \frac{2a^2}{x}$ and $4 = \left(\frac{2y}{b^2}\right)\left(\frac{2a^2}{x}\right) \Rightarrow y = \left(\frac{b^2}{a^2}\right)x \Rightarrow \frac{x^2}{a^2} + \frac{\left(\frac{b^2}{a^2}\right)^2 x^2}{b^2} = 1 \Rightarrow \frac{x^2}{a^2} + \frac{b^2x^2}{a^4}$ $= 1 \Rightarrow (a^2 + b^2)\,x^2 = a^4 \Rightarrow x = \frac{a^2}{\sqrt{a^2+b^2}}$, since $x > 0 \Rightarrow y = \left(\frac{b^2}{a^2}\right)x = \frac{b^2}{\sqrt{a^2+b^2}} \Rightarrow$ width $= 2x = \frac{2a^2}{\sqrt{a^2+b^2}}$ and height $= 2y = \frac{2b^2}{\sqrt{a^2+b^2}} \Rightarrow$ perimeter is $P = 4x + 4y = \frac{4a^2+4b^2}{\sqrt{a^2+b^2}} = 4\sqrt{a^2+b^2}$

13. $\nabla f = 2x\mathbf{i} + 2y\mathbf{j}$ and $\nabla g = (2x-2)\mathbf{i} + (2y-4)\mathbf{j}$ so that $\nabla f = \lambda \nabla g = 2x\mathbf{i} + 2y\mathbf{j} = \lambda[(2x-2)\mathbf{i} + (2y-4)\mathbf{j}]$ $\Rightarrow 2x = \lambda(2x-2)$ and $2y = \lambda(2y-4) \Rightarrow x = \frac{\lambda}{\lambda-1}$ and $y = \frac{2\lambda}{\lambda-1}$, $\lambda \neq 1 \Rightarrow y = 2x \Rightarrow x^2 - 2x + (2x)^2 - 4(2x) = 0$ $\Rightarrow x = 0$ and $y = 0$, or $x = 2$ and $y = 4$. Therefore $f(0,0) = 0$ is the minimum value and $f(2,4) = 20$ is the maximum value. (Note that $\lambda = 1$ gives $2x = 2x - 2$ or $0 = -2$, which is impossible.)

14. $\nabla f = 3\mathbf{i} - \mathbf{j}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow 3 = 2\lambda x$ and $-1 = 2\lambda y \Rightarrow \lambda = \frac{3}{2x}$ and $-1 = 2\left(\frac{3}{2x}\right)y$ $\Rightarrow y = -\frac{x}{3} \Rightarrow x^2 + \left(-\frac{x}{3}\right)^2 = 4 \Rightarrow 10x^2 = 36 \Rightarrow x = \pm\frac{6}{\sqrt{10}} \Rightarrow x = \frac{6}{\sqrt{10}}$ and $y = -\frac{2}{\sqrt{10}}$, or $x = -\frac{6}{\sqrt{10}}$ and $y = \frac{2}{\sqrt{10}}$. Therefore $f\left(\frac{6}{\sqrt{10}}, -\frac{2}{\sqrt{10}}\right) = \frac{20}{\sqrt{10}} + 6 = 2\sqrt{10} + 6 \approx 12.325$ is the maximum value, and $f\left(-\frac{6}{\sqrt{10}}, \frac{2}{\sqrt{10}}\right)$ $= -2\sqrt{10} + 6 \approx -0.325$ is the minimum value.

15. $\nabla T = (8x-4y)\mathbf{i} + (-4x+2y)\mathbf{j}$ and $g(x,y) = x^2 + y^2 - 25 = 0 \Rightarrow \nabla g = 2x\mathbf{i} + 2y\mathbf{j}$ so that $\nabla T = \lambda \nabla g$ $\Rightarrow (8x-4y)\mathbf{i} + (-4x+2y)\mathbf{j} = \lambda(2x\mathbf{i} + 2y\mathbf{j}) \Rightarrow 8x - 4y = 2\lambda x$ and $-4x + 2y = 2\lambda y \Rightarrow y = \frac{-2x}{\lambda-1}$, $\lambda \neq 1$ $\Rightarrow 8x - 4\left(\frac{-2x}{\lambda-1}\right) = 2\lambda x \Rightarrow x = 0$, or $\lambda = 0$, or $\lambda = 5$.

CASE 1: $x = 0 \Rightarrow y = 0$; but $(0,0)$ is not on $x^2 + y^2 = 25$ so $x \neq 0$.

CASE 2: $\lambda = 0 \Rightarrow y = 2x \Rightarrow x^2 + (2x)^2 = 25 \Rightarrow x = \pm\sqrt{5}$ and $y = 2x$.

CASE 3: $\lambda = 5 \Rightarrow y = \frac{-2x}{4} = -\frac{x}{2} \Rightarrow x^2 + \left(-\frac{x}{2}\right)^2 = 25 \Rightarrow x = \pm 2\sqrt{5} \Rightarrow x = 2\sqrt{5}$ and $y = -\sqrt{5}$, or $x = -2\sqrt{5}$ and $y = \sqrt{5}$.

Therefore $T\left(\sqrt{5}, 2\sqrt{5}\right) = 0° = T\left(-\sqrt{5}, -2\sqrt{5}\right)$ is the minimum value and $T\left(2\sqrt{5}, -\sqrt{5}\right) = 125°$ $= T\left(-2\sqrt{5}, \sqrt{5}\right)$ is the maximum value. (Note: $\lambda = 1 \Rightarrow x = 0$ from the equation $-4x + 2y = 2\lambda y$; but we found $x \neq 0$ in CASE 1.)

16. The surface area is given by $S = 4\pi r^2 + 2\pi rh$ subject to the constraint $V(r,h) = \frac{4}{3}\pi r^3 + \pi r^2 h = 8000$. Thus $\nabla S = (8\pi r + 2\pi h)\mathbf{i} + 2\pi r\mathbf{j}$ and $\nabla V = (4\pi r^2 + 2\pi rh)\,\mathbf{i} + \pi r^2\mathbf{j}$ so that $\nabla S = \lambda \nabla V = (8\pi r + 2\pi h)\mathbf{i} + 2\pi r\mathbf{j}$ $= \lambda\left[(4\pi r^2 + 2\pi rh)\,\mathbf{i} + \pi r^2\mathbf{j}\right] \Rightarrow 8\pi r + 2\pi h = \lambda\,(4\pi r^2 + 2\pi rh)$ and $2\pi r = \lambda\pi r^2 \Rightarrow r = 0$ or $2 = r\lambda$. But $r \neq 0$

so $2 = r\lambda \Rightarrow \lambda = \frac{2}{r} \Rightarrow 4r + h = \frac{2}{r}(2r^2 + rh) \Rightarrow h = 0 \Rightarrow$ the tank is a sphere (there is no cylindrical part) and $\frac{4}{3}\pi r^3 = 8000 \Rightarrow r = 10\left(\frac{6}{\pi}\right)^{1/3}$.

17. Let $f(x,y,z) = (x-1)^2 + (y-1)^2 + (z-1)^2$ be the square of the distance from $(1,1,1)$. Then $\nabla f = 2(x-1)\mathbf{i} + 2(y-1)\mathbf{j} + 2(z-1)\mathbf{k}$ and $\nabla g = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ so that $\nabla f = \lambda \nabla g$ $\Rightarrow 2(x-1)\mathbf{i} + 2(y-1)\mathbf{j} + 2(z-1)\mathbf{k} = \lambda(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \Rightarrow 2(x-1) = \lambda, 2(y-1) = 2\lambda, 2(z-1) = 3\lambda$ $\Rightarrow 2(y-1) = 2[2(x-1)]$ and $2(z-1) = 3[2(x-1)] \Rightarrow x = \frac{y+1}{2} \Rightarrow z + 2 = 3\left(\frac{y+1}{2}\right)$ or $z = \frac{3y-1}{2}$; thus $\frac{y+1}{2} + 2y + 3\left(\frac{3y-1}{2}\right) - 13 = 0 \Rightarrow y = 2 \Rightarrow x = \frac{3}{2}$ and $z = \frac{5}{2}$. Therefore the point $\left(\frac{3}{2}, 2, \frac{5}{2}\right)$ is closest (since no point on the plane is farthest from the point $(1,1,1)$).

18. Let $f(x,y,z) = (x-1)^2 + (y+1)^2 + (z-1)^2$ be the square of the distance from $(1,-1,1)$. Then $\nabla f = 2(x-1)\mathbf{i} + 2(y+1)\mathbf{j} + 2(z-1)\mathbf{k}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow x - 1 = \lambda x, y + 1 = \lambda y$ and $z - 1 = \lambda z \Rightarrow x = \frac{1}{1-\lambda}, y = -\frac{1}{1-\lambda}$, and $z = \frac{1}{1-\lambda}$ for $\lambda \neq 1 \Rightarrow \left(\frac{1}{1-\lambda}\right)^2 + \left(\frac{-1}{1-\lambda}\right)^2 + \left(\frac{1}{1-\lambda}\right)^2 = 4$ $\Rightarrow \frac{1}{1-\lambda} = \pm\frac{2}{\sqrt{3}} \Rightarrow x = \frac{2}{\sqrt{3}}, y = -\frac{2}{\sqrt{3}}, z = \frac{2}{\sqrt{3}}$ or $x = -\frac{2}{\sqrt{3}}, y = \frac{2}{\sqrt{3}}, z = -\frac{2}{\sqrt{3}}$. The largest value of f occurs where $x < 0$, $y > 0$, and $z < 0$ or at the point $\left(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}, -\frac{2}{\sqrt{3}}\right)$ on the sphere.

19. Let $f(x,y,z) = x^2 + y^2 + z^2$ be the square of the distance from the origin. Then $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ and $\nabla g = 2x\mathbf{i} - 2y\mathbf{j} - 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \lambda(2x\mathbf{i} - 2y\mathbf{j} - 2z\mathbf{k}) \Rightarrow 2x = 2x\lambda, 2y = 2y\lambda$, and $2z = -2z\lambda \Rightarrow x = 0$ or $\lambda = 1$.
CASE 1: $\lambda = 1 \Rightarrow 2y = -2y \Rightarrow y = 0; 2z = -2z \Rightarrow z = 0 \Rightarrow x^2 - 1 = 0 \Rightarrow x^2 - 1 = 0 \Rightarrow x = \pm 1$ and $y = z = 0$.
CASE 2: $x = 0 \Rightarrow y^2 - z^2 = 1$, which has no solution.
Therefore the points on the unit circle $x^2 + y^2 = 1$, are the points on the surface $x^2 + y^2 - z^2 = 1$ closest to the origin. The minimum distance is 1.

20. Let $f(x,y,z) = x^2 + y^2 + z^2$ be the square of the distance to the origin. Then $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ and $\nabla g = y\mathbf{i} + x\mathbf{j} - \mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \lambda(y\mathbf{i} + x\mathbf{j} - \mathbf{k}) \Rightarrow 2x = \lambda y, 2y = \lambda x$, and $2z = -\lambda$ $\Rightarrow x = \frac{\lambda y}{2} \Rightarrow 2y = \lambda\left(\frac{\lambda y}{2}\right) \Rightarrow y = 0$ or $\lambda = \pm 2$.
CASE 1: $y = 0 \Rightarrow x = 0 \Rightarrow -z + 1 = 0 \Rightarrow z = 1$.
CASE 2: $\lambda = 2 \Rightarrow x = y$ and $z = -1 \Rightarrow x^2 - (-1) + 1 = 0 \Rightarrow x^2 + 2 = 0$, so no solution.
CASE 3: $\lambda = -2 \Rightarrow x = -y$ and $z = 1 \Rightarrow (-y)y - 1 + 1 = 0 \Rightarrow y = 0$, again.
Therefore $(0,0,1)$ is the point on the surface closest to the origin since this point gives the only extreme value and there is no maximum distance from the surface to the origin.

21. Let $f(x,y,z) = x^2 + y^2 + z^2$ be the square of the distance to the origin. Then $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ and $\nabla g = -y\mathbf{i} - x\mathbf{j} + 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \lambda(-y\mathbf{i} - x\mathbf{j} + 2z\mathbf{k}) \Rightarrow 2x = -y\lambda, 2y = -x\lambda$, and $2z = 2z\lambda \Rightarrow \lambda = 1$ or $z = 0$.
CASE 1: $\lambda = 1 \Rightarrow 2x = -y$ and $2y = -x \Rightarrow y = 0$ and $x = 0 \Rightarrow z^2 - 4 = 0 \Rightarrow z = \pm 2$ and $x = y = 0$.
CASE 2: $z = 0 \Rightarrow -xy - 4 = 0 \Rightarrow y = -\frac{4}{x}$. Then $2x = \frac{4}{x}\lambda \Rightarrow \lambda = \frac{x^2}{2}$, and $-\frac{8}{x} = -x\lambda \Rightarrow -\frac{8}{x} = -x\left(\frac{x^2}{2}\right)$
$\Rightarrow x^4 = 16 \Rightarrow x = \pm 2$. Thus, $x = 2$ and $y = -2$, or $x = -2$ and $y = 2$.
Therefore we get four points: $(2,-2,0)$, $(-2,2,0)$, $(0,0,2)$ and $(0,0,-2)$. But the points $(0,0,2)$ and $(0,0,-2)$ are closest to the origin since they are 2 units away and the others are $2\sqrt{2}$ units away.

22. Let $f(x,y,z) = x^2 + y^2 + z^2$ be the square of the distance to the origin. Then $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ and $\nabla g = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow 2x = \lambda yz, 2y = \lambda xz$, and $2z = \lambda xy \Rightarrow 2x^2 = \lambda xyz$ and $2y^2 = \lambda yxz$ $\Rightarrow x^2 = y^2 \Rightarrow y = \pm x \Rightarrow z = \pm x \Rightarrow x(\pm x)(\pm x) = 1 \Rightarrow x = \pm 1 \Rightarrow$ the points are $(1,1,1)$, $(1,-1,-1)$, $(-1,-1,1)$, and $(-1,1,-1)$.

23. $\nabla f = \mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow \mathbf{i} - 2\mathbf{j} + 5\mathbf{k} = \lambda(2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}) \Rightarrow 1 = 2x\lambda$, $-2 = 2y\lambda$, and $5 = 2z\lambda \Rightarrow x = \frac{1}{2\lambda}, y = -\frac{1}{\lambda} = -2x$, and $z = \frac{5}{2\lambda} = 5x \Rightarrow x^2 + (-2x)^2 + (5x)^2 = 30 \Rightarrow x = \pm 1$. Thus, $x = 1, y = -2, z = 5$ or $x = -1, y = 2, z = -5$. Therefore $f(1, -2, 5) = 30$ is the maximum value and $f(-1, 2, -5) = -30$ is the minimum value.

24. $\nabla f = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} = \lambda(2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}) \Rightarrow 1 = 2x\lambda$, $2 = 2y\lambda$, and $3 = 2z\lambda \Rightarrow x = \frac{1}{2\lambda}, y = \frac{1}{\lambda} = 2x$, and $z = \frac{3}{2\lambda} = 3x \Rightarrow x^2 + (2x)^2 + (3x)^2 = 25 \Rightarrow x = \pm \frac{5}{\sqrt{14}}$. Thus, $x = \frac{5}{\sqrt{14}}, y = \frac{10}{\sqrt{14}}, z = \frac{15}{\sqrt{14}}$ or $x = -\frac{5}{\sqrt{14}}, y = -\frac{10}{\sqrt{14}}, z = -\frac{15}{\sqrt{14}}$. Therefore $f\left(\frac{5}{\sqrt{14}}, \frac{10}{\sqrt{14}}, \frac{15}{\sqrt{14}}\right) = 5\sqrt{14}$ is the maximum value and $f\left(-\frac{5}{\sqrt{14}}, -\frac{10}{\sqrt{14}}, -\frac{15}{\sqrt{14}}\right) = -5\sqrt{14}$ is the minimum value.

25. $f(x, y, z) = x^2 + y^2 + z^2$ and $g(x, y, z) = x + y + z - 9 = 0 \Rightarrow \nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ and $\nabla g = \mathbf{i} + \mathbf{j} + \mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k}) \Rightarrow 2x = \lambda, 2y = \lambda$, and $2z = \lambda \Rightarrow x = y = z \Rightarrow x + x + x - 9 = 0 \Rightarrow x = 3, y = 3$, and $z = 3$.

26. $f(x, y, z) = xyz$ and $g(x, y, z) = x + y + z^2 - 16 = 0 \Rightarrow \nabla f = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ and $\nabla g = \mathbf{i} + \mathbf{j} + 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} = \lambda(\mathbf{i} + \mathbf{j} + 2z\mathbf{k}) \Rightarrow yz = \lambda, xz = \lambda$, and $xy = 2z\lambda \Rightarrow yz = xz \Rightarrow z = 0$ or $y = x$. But $z > 0$ so that $y = x \Rightarrow x^2 = 2z\lambda$ and $xz = \lambda$. Then $x^2 = 2z(xz) \Rightarrow x = 0$ or $x = 2z^2$. But $x > 0$ so that $x = 2z^2 \Rightarrow y = 2z^2 \Rightarrow 2z^2 + 2z^2 + z^2 = 16 \Rightarrow z = \pm \frac{4}{\sqrt{5}}$. We use $z = \frac{4}{\sqrt{5}}$ since $z > 0$. Then $x = \frac{32}{5}$ and $y = \frac{32}{5}$ which yields $f\left(\frac{32}{5}, \frac{32}{5}, \frac{4}{\sqrt{5}}\right) = \frac{4096}{25\sqrt{5}}$.

27. $V = xyz$ and $g(x, y, z) = x^2 + y^2 + z^2 - 1 = 0 \Rightarrow \nabla V = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ so that $\nabla V = \lambda \nabla g \Rightarrow yz = \lambda x, xz = \lambda y$, and $xy = \lambda z \Rightarrow xyz = \lambda x^2$ and $xyz = \lambda y^2 \Rightarrow y = \pm x \Rightarrow z = \pm x \Rightarrow x^2 + x^2 + x^2 = 1 \Rightarrow x = \frac{1}{\sqrt{3}}$ since $x > 0 \Rightarrow$ the dimensions of the box are $\frac{1}{\sqrt{3}}$ by $\frac{1}{\sqrt{3}}$ by $\frac{1}{\sqrt{3}}$ for maximum volume. (Note that there is no minimum volume since the box could be made arbitrarily thin.)

28. $V = xyz$ with x, y, z all positive and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$; thus $V = xyz$ and $g(x, y, z) = bcx + acy + abz - abc = 0 \Rightarrow \nabla V = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ and $\nabla g = bc\mathbf{i} + ac\mathbf{j} + ab\mathbf{k}$ so that $\nabla V = \lambda \nabla g \Rightarrow yz = \lambda bc, xz = \lambda ac$, and $xy = \lambda ab \Rightarrow xyz = \lambda bcx, xyz = \lambda acy$, and $xyz = \lambda abz \Rightarrow \lambda \neq 0$. Also, $\lambda bcx = \lambda acy = \lambda abz \Rightarrow bx = ay, cy = bz$, and $cx = az \Rightarrow y = \frac{b}{a}x$ and $z = \frac{c}{a}x$. Then $\frac{x}{a} + \frac{y}{b} + \frac{c}{z} = 1 \Rightarrow \frac{x}{a} + \frac{1}{b}\left(\frac{b}{a}x\right) + \frac{1}{c}\left(\frac{c}{a}x\right) = 1 \Rightarrow \frac{3x}{a} = 1 \Rightarrow x = \frac{a}{3} \Rightarrow y = \left(\frac{b}{a}\right)\left(\frac{a}{3}\right) = \frac{b}{3}$ and $z = \left(\frac{c}{a}\right)\left(\frac{a}{3}\right) = \frac{c}{3} \Rightarrow V = xyz = \left(\frac{a}{3}\right)\left(\frac{b}{3}\right)\left(\frac{c}{3}\right) = \frac{abc}{27}$ is the maximum volume. (Note that there is no minimum volume since the box could be made arbitrarily thin.)

29. $\nabla T = 16x\mathbf{i} + 4z\mathbf{j} + (4y - 16)\mathbf{k}$ and $\nabla g = 8x\mathbf{i} + 2y\mathbf{j} + 8z\mathbf{k}$ so that $\nabla T = \lambda \nabla g \Rightarrow 16x\mathbf{i} + 4z\mathbf{j} + (4y - 16)\mathbf{k} = \lambda(8x\mathbf{i} + 2y\mathbf{j} + 8z\mathbf{k}) \Rightarrow 16x = 8x\lambda, 4z = 2y\lambda$, and $4y - 16 = 8z\lambda \Rightarrow \lambda = 2$ or $x = 0$.

CASE 1: $\lambda = 2 \Rightarrow 4z = 2y(2) \Rightarrow z = y$. Then $4z - 16 = 16z \Rightarrow z = -\frac{4}{3} \Rightarrow y = -\frac{4}{3}$. Then $4x^2 + \left(-\frac{4}{3}\right)^2 + 4\left(-\frac{4}{3}\right)^2 = 16 \Rightarrow x = \pm \frac{4}{3}$.

CASE 2: $x = 0 \Rightarrow \lambda = \frac{2z}{y} \Rightarrow 4y - 16 = 8z\left(\frac{2z}{y}\right) \Rightarrow y^2 - 4y = 4z^2 \Rightarrow 4(0)^2 + y^2 + (y^2 - 4y) - 16 = 0 \Rightarrow y^2 - 2y - 8 = 0 \Rightarrow (y - 4)(y + 2) = 0 \Rightarrow y = 4$ or $y = -2$. Now $y = 4 \Rightarrow 4z^2 = 4^2 - 4(4) \Rightarrow z = 0$ and $y = -2 \Rightarrow 4z^2 = (-2)^2 - 4(-2) \Rightarrow z = \pm\sqrt{3}$.

The temperatures are $T\left(\pm\frac{4}{3}, -\frac{4}{3}, -\frac{4}{3}\right) = 642\frac{2}{3}^{\circ}$, $T(0, 4, 0) = 600^{\circ}$, $T\left(0, -2, \sqrt{3}\right) = \left(600 - 24\sqrt{3}\right)^{\circ}$, and $T\left(0, -2, -\sqrt{3}\right) = \left(600 + 24\sqrt{3}\right)^{\circ} \approx 641.6^{\circ}$. Therefore $\left(\pm\frac{4}{3}, -\frac{4}{3}, -\frac{4}{3}\right)$ are the hottest points on the space probe.

30. $\nabla T = 400yz^2\mathbf{i} + 400xz^2\mathbf{j} + 800xyz\mathbf{k}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ so that $\nabla T = \lambda \nabla g$ $\Rightarrow 400yz^2\mathbf{i} + 400xz^2\mathbf{j} + 800xyz\mathbf{k} = \lambda(2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}) \Rightarrow 400yz^2 = 2x\lambda$, $400xz^2 = 2y\lambda$, and $800xyz = 2z\lambda$. Solving this system yields the points $(0, \pm 1, 0)$, $(\pm 1, 0, 0)$, and $\left(\pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{\sqrt{2}}{2}\right)$. The corresponding temperatures are $T(0, \pm 1, 0) = 0$, $T(\pm 1, 0, 0) = 0$, and $T\left(\pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{\sqrt{2}}{2}\right) = \pm 50$. Therefore 50 is the maximum temperature at $\left(\frac{1}{2}, \frac{1}{2}, \pm \frac{\sqrt{2}}{2}\right)$ and $\left(-\frac{1}{2}, -\frac{1}{2}, \pm \frac{\sqrt{2}}{2}\right)$; -50 is the minimum temperature at $\left(\frac{1}{2}, -\frac{1}{2}, \pm \frac{\sqrt{2}}{2}\right)$ and $\left(-\frac{1}{2}, \frac{1}{2}, \pm \frac{\sqrt{2}}{2}\right)$.

31. $\nabla U = (y + 2)\mathbf{i} + x\mathbf{j}$ and $\nabla g = 2\mathbf{i} + \mathbf{j}$ so that $\nabla U = \lambda \nabla g \Rightarrow (y + 2)\mathbf{i} + x\mathbf{j} = \lambda(2\mathbf{i} + \mathbf{j}) \Rightarrow y + 2 = 2\lambda$ and $x = \lambda \Rightarrow y + 2 = 2x \Rightarrow y = 2x - 2 \Rightarrow 2x + (2x - 2) = 30 \Rightarrow x = 8$ and $y = 14$. Therefore $U(8, 14) = \$128$ is the maximum value of U under the constraint.

32. $\nabla M = (6 + z)\mathbf{i} - 2y\mathbf{j} + x\mathbf{k}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ so that $\nabla M = \lambda \nabla g \Rightarrow (6 + z)\mathbf{i} - 2y\mathbf{j} + x\mathbf{k}$ $= \lambda(2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}) \Rightarrow 6 + z = 2x\lambda$, $-2y = 2y\lambda$, $x = 2z\lambda \Rightarrow \lambda = -1$ or $y = 0$.

CASE 1: $\lambda = -1 \Rightarrow 6 + z = -2x$ and $x = -2z \Rightarrow 6 + z = -2(-2z) \Rightarrow z = 2$ and $x = -4$. Then $(-4)^2 + y^2 + 2^2 - 36 = 0 \Rightarrow y = \pm 4$.

CASE 2: $y = 0$, $6 + z = 2x\lambda$, and $x = 2z\lambda \Rightarrow \lambda = \frac{x}{2z} \Rightarrow 6 + z = 2x\left(\frac{x}{2z}\right) \Rightarrow 6z + z^2 = x^2$ $\Rightarrow (6z + z^2) + 0^2 + z^2 = 36 \Rightarrow z = -6$ or $z = 3$. Now $z = -6 \Rightarrow x^2 = 0 \Rightarrow x = 0$; $z = 3$ $\Rightarrow x^2 = 27 \Rightarrow x = \pm 3\sqrt{3}$.

Therefore we have the points $\left(\pm 3\sqrt{3}, 0, 3\right)$, $(0, 0, -6)$, and $(-4, \pm 4, 2)$. Then $M\left(3\sqrt{3}, 0, 3\right) = 27\sqrt{3} + 60$ ≈ 106.8, $M\left(-3\sqrt{3}, 0, 3\right) = 60 - 27\sqrt{3} \approx 13.2$, $M(0, 0, -6) = 60$, and $M(-4, 4, 2) = 12 = M(-4, -4, 2)$. Therefore, the weakest field is at $(-4, \pm 4, 2)$.

33. Let $g_1(x, y, z) = 2x - y = 0$ and $g_2(x, y, z) = y + z = 0 \Rightarrow \nabla g_1 = 2\mathbf{i} - \mathbf{j}$, $\nabla g_2 = \mathbf{j} + \mathbf{k}$, and $\nabla f = 2x\mathbf{i} + 2\mathbf{j} - 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \Rightarrow 2x\mathbf{i} + 2\mathbf{j} - 2z\mathbf{k} = \lambda(2\mathbf{i} - \mathbf{j}) + \mu(\mathbf{j} + \mathbf{k}) \Rightarrow 2x\mathbf{i} + 2\mathbf{j} - 2z\mathbf{k} = 2\lambda\mathbf{i} + (\mu - \lambda)\mathbf{j} + \mu\mathbf{k}$ $\Rightarrow 2x = 2\lambda$, $2 = \mu - \lambda$, and $-2z = \mu \Rightarrow x = \lambda$. Then $2 = -2z - x \Rightarrow x = -2z - 2$ so that $2x - y = 0$ $\Rightarrow 2(-2z - 2) - y = 0 \Rightarrow -4z - 4 - y = 0$. This equation coupled with $y + z = 0$ implies $z = -\frac{4}{3}$ and $y = \frac{4}{3}$. Then $x = \frac{2}{3}$ so that $\left(\frac{2}{3}, \frac{4}{3}, -\frac{4}{3}\right)$ is the point that gives the maximum value $f\left(\frac{2}{3}, \frac{4}{3}, -\frac{4}{3}\right) = \left(\frac{2}{3}\right)^2 + 2\left(\frac{4}{3}\right) - \left(-\frac{4}{3}\right)^2 = \frac{4}{3}$.

34. Let $g_1(x, y, z) = x + 2y + 3z - 6 = 0$ and $g_2(x, y, z) = x + 3y + 9z - 9 = 0 \Rightarrow \nabla g_1 = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\nabla g_2 = \mathbf{i} + 3\mathbf{j} + 9\mathbf{k}$, and $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \Rightarrow 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ $= \lambda(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) + \mu(\mathbf{i} + 3\mathbf{j} + 9\mathbf{k}) \Rightarrow 2x = \lambda + \mu$, $2y = 2\lambda + 3\mu$, and $2z = 3\lambda + 9\mu$. Then $0 = x + 2y + 3z - 6$ $= \frac{1}{2}(\lambda + \mu) + (2\lambda + 3\mu) + \left(\frac{9}{2}\lambda + \frac{27}{2}\mu\right) - 6 \Rightarrow 7\lambda + 17\mu = 6$; $0 = x + 3y + 9z - 9$ $\Rightarrow \frac{1}{2}(\lambda + \mu) + \left(3\lambda + \frac{9}{2}\mu\right) + \left(\frac{27}{2}\lambda + \frac{81}{2}\mu\right) - 9 \Rightarrow 34\lambda + 91\mu = 18$. Solving these two equations for λ and μ gives $\lambda = \frac{240}{59}$ and $\mu = -\frac{78}{59} \Rightarrow x = \frac{\lambda + \mu}{2} = \frac{81}{59}$, $y = \frac{2\lambda + 3\mu}{2} = \frac{123}{59}$, and $z = \frac{3\lambda + 9\mu}{2} = \frac{9}{59}$. The minimum value is $f\left(\frac{81}{59}, \frac{123}{59}, \frac{9}{59}\right) = \frac{21{,}771}{59^2} = \frac{369}{59}$. (Note that there is no <u>maximum</u> value of f subject to the constraints because at least one of the variables x, y, or z can be made arbitrary and assume a value as large as we please.)

35. Let $f(x, y, z) = x^2 + y^2 + z^2$ be the square of the distance from the origin. We want to minimize $f(x, y, z)$ subject to the constraints $g_1(x, y, z) = y + 2z - 12 = 0$ and $g_2(x, y, z) = x + y - 6 = 0$. Thus $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$, $\nabla g_1 = \mathbf{j} + 2\mathbf{k}$, and $\nabla g_2 = \mathbf{i} + \mathbf{j}$ so that $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \Rightarrow 2x = \mu$, $2y = \lambda + \mu$, and $2z = 2\lambda$. Then $0 = y + 2z - 12$ $= \left(\frac{\lambda}{2} + \frac{\mu}{2}\right) + 2\lambda - 12 \Rightarrow \frac{5}{2}\lambda + \frac{1}{2}\mu = 12 \Rightarrow 5\lambda + \mu = 24$; $0 = x + y - 6 = \frac{\mu}{2} + \left(\frac{\lambda}{2} + \frac{\mu}{2}\right) - 6 \Rightarrow \frac{1}{2}\lambda + \mu = 6$ $\Rightarrow \lambda + 2\mu = 12$. Solving these two equations for λ and μ gives $\lambda = 4$ and $\mu = 4 \Rightarrow x = \frac{\mu}{2} = 2$, $y = \frac{\lambda + \mu}{2} = 4$, and $z = \lambda = 4$. The point $(2, 4, 4)$ on the line of intersection is closest to the origin. (There is no maximum distance from the origin since points on the line can be arbitrarily far away.)

36. The maximum value is $f\left(\frac{2}{3}, \frac{4}{3}, -\frac{4}{3}\right) = \frac{4}{3}$ from Exercise 33 above.

37. Let $g_1(x,y,z) = z - 1 = 0$ and $g_2(x,y,z) = x^2 + y^2 + z^2 - 10 = 0 \Rightarrow \nabla g_1 = \mathbf{k}$, $\nabla g_2 = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$, and $\nabla f = 2xyz\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k}$ so that $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \Rightarrow 2xyz\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k} = \lambda(\mathbf{k}) + \mu(2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k})$ $\Rightarrow 2xyz = 2x\mu$, $x^2z = 2y\mu$, and $x^2y = 2z\mu + \lambda \Rightarrow xyz = x\mu \Rightarrow x = 0$ or $yz = \mu \Rightarrow \mu = y$ since $z = 1$.
CASE 1: $x = 0$ and $z = 1 \Rightarrow y^2 - 9 = 0$ (from g_2) $\Rightarrow y = \pm 3$ yielding the points $(0, \pm 3, 1)$.
CASE 2: $\mu = y \Rightarrow x^2z = 2y^2 \Rightarrow x^2 = 2y^2$ (since $z = 1$) $\Rightarrow 2y^2 + y^2 + 1 - 10 = 0$ (from g_2) $\Rightarrow 3y^2 - 9 = 0$ $\Rightarrow y = \pm\sqrt{3} \Rightarrow x^2 = 2\left(\pm\sqrt{3}\right)^2 \Rightarrow x = \pm\sqrt{6}$ yielding the points $\left(\pm\sqrt{6}, \pm\sqrt{3}, 1\right)$.
Now $f(0, \pm 3, 1) = 1$ and $f\left(\pm\sqrt{6}, \pm\sqrt{3}, 1\right) = 6\left(\pm\sqrt{3}\right) + 1 = 1 \pm 6\sqrt{3}$. Therefore the maximum of f is $1 + 6\sqrt{3}$ at $\left(\pm\sqrt{6}, \sqrt{3}, 1\right)$, and the minimum of f is $1 - 6\sqrt{3}$ at $\left(\pm\sqrt{6}, -\sqrt{3}, 1\right)$.

38. (a) Let $g_1(x,y,z) = x + y + z - 40 = 0$ and $g_2(x,y,z) = x + y - z = 0 \Rightarrow \nabla g_1 = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\nabla g_2 = \mathbf{i} + \mathbf{j} - \mathbf{k}$, and $\nabla w = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ so that $\nabla w = \lambda \nabla g_1 + \mu \nabla g_2 \Rightarrow yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} = \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k}) + \mu(\mathbf{i} + \mathbf{j} - \mathbf{k})$ $\Rightarrow yz = \lambda + \mu$, $xz = \lambda + \mu$, and $xy = \lambda - \mu \Rightarrow yz = xz \Rightarrow z = 0$ or $y = x$.
CASE 1: $z = 0 \Rightarrow x + y = 40$ and $x + y = 0 \Rightarrow$ no solution.
CASE 2: $x = y \Rightarrow 2x + z - 40 = 0$ and $2x - z = 0 \Rightarrow z = 20 \Rightarrow x = 10$ and $y = 10 \Rightarrow w = (10)(10)(20) = 2000$

(b) $\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = -2\mathbf{i} + 2\mathbf{j}$ is parallel to the line of intersection $\Rightarrow$ the line is $x = -2t + 10$, $y = 2t + 10$, $z = 20$. Since $z = 20$, we see that $w = xyz = (-2t + 10)(2t + 10)(20) = (-4t^2 + 100)(20)$ which has its maximum when $t = 0 \Rightarrow x = 10$, $y = 10$, and $z = 20$.

39. Let $g_1(x,y,z) = y - x = 0$ and $g_2(x,y,z) = x^2 + y^2 + z^2 - 4 = 0$. Then $\nabla f = y\mathbf{i} + x\mathbf{j} + 2z\mathbf{k}$, $\nabla g_1 = -\mathbf{i} + \mathbf{j}$, and $\nabla g_2 = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \Rightarrow y\mathbf{i} + x\mathbf{j} + 2z\mathbf{k} = \lambda(-\mathbf{i} + \mathbf{j}) + \mu(2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k})$ $\Rightarrow y = -\lambda + 2x\mu$, $x = \lambda + 2y\mu$, and $2z = 2z\mu \Rightarrow z = 0$ or $\mu = 1$.
CASE 1: $z = 0 \Rightarrow x^2 + y^2 - 4 = 0 \Rightarrow 2x^2 - 4 = 0$ (since $x = y$) $\Rightarrow x = \pm\sqrt{2}$ and $y = \pm\sqrt{2}$ yielding the points $\left(\pm\sqrt{2}, \pm\sqrt{2}, 0\right)$.
CASE 2: $\mu = 1 \Rightarrow y = -\lambda + 2x$ and $x = \lambda + 2y \Rightarrow x + y = 2(x + y) \Rightarrow 2x = 2(2x)$ since $x = y \Rightarrow x = 0 \Rightarrow y = 0$ $\Rightarrow z^2 - 4 = 0 \Rightarrow z = \pm 2$ yielding the points $(0, 0, \pm 2)$.
Now, $f(0, 0, \pm 2) = 4$ and $f\left(\pm\sqrt{2}, \pm\sqrt{2}, 0\right) = 2$. Therefore the maximum value of f is 4 at $(0, 0, \pm 2)$ and the minimum value of f is 2 at $\left(\pm\sqrt{2}, \pm\sqrt{2}, 0\right)$.

40. Let $f(x,y,z) = x^2 + y^2 + z^2$ be the square of the distance from the origin. We want to minimize $f(x,y,z)$ subject to the constraints $g_1(x,y,z) = 2y + 4z - 5 = 0$ and $g_2(x,y,z) = 4x^2 + 4y^2 - z^2 = 0$. Thus $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$, $\nabla g_1 = 2\mathbf{j} + 4\mathbf{k}$, and $\nabla g_2 = 8x\mathbf{i} + 8y\mathbf{j} - 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \Rightarrow 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ $= \lambda(2\mathbf{j} + 4\mathbf{k}) + \mu(8x\mathbf{i} + 8y\mathbf{j} - 2z\mathbf{k}) \Rightarrow 2x = 8x\mu$, $2y = 2\lambda + 8y\mu$, and $2z = 4\lambda - 2z\mu \Rightarrow x = 0$ or $\mu = \frac{1}{4}$.
CASE 1: $x = 0 \Rightarrow 4(0)^2 + 4y^2 - z^2 = 0 \Rightarrow z = \pm 2y \Rightarrow 2y + 4(2y) - 5 = 0 \Rightarrow y = \frac{1}{2}$, or $2y + 4(-2y) - 5 = 0$ $\Rightarrow y = -\frac{5}{6}$ yielding the points $\left(0, \frac{1}{2}, 1\right)$ and $\left(0, -\frac{5}{6}, \frac{5}{3}\right)$.
CASE 2: $\mu = \frac{1}{4} \Rightarrow y = \lambda + y \Rightarrow \lambda = 0 \Rightarrow 2z = 4(0) - 2z\left(\frac{1}{4}\right) \Rightarrow z = 0 \Rightarrow 2y + 4(0) = 5 \Rightarrow y = \frac{5}{2}$ and $(0)^2 = 4x^2 + 4\left(\frac{5}{2}\right)^2 \Rightarrow$ no solution.
Then $f\left(0, \frac{1}{2}, 1\right) = \frac{5}{4}$ and $f\left(0, -\frac{5}{6}, \frac{5}{3}\right) = 25\left(\frac{1}{36} + \frac{1}{9}\right) = \frac{125}{36} \Rightarrow$ the point $\left(0, \frac{1}{2}, 1\right)$ is closest to the origin.

41. $\nabla f = \mathbf{i} + \mathbf{j}$ and $\nabla g = y\mathbf{i} + x\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow \mathbf{i} + \mathbf{j} = \lambda(y\mathbf{i} + x\mathbf{j}) \Rightarrow 1 = y\lambda$ and $1 = x\lambda \Rightarrow y = x$ $\Rightarrow y^2 = 16 \Rightarrow y = \pm 4 \Rightarrow (4, 4)$ and $(-4, -4)$ are candidates for the location of extreme values. But as $x \to \infty$, $y \to \infty$ and $f(x, y) \to \infty$; as $x \to -\infty$, $y \to 0$ and $f(x, y) \to -\infty$. Therefore no maximum or minimum value exists subject to the constraint.

42. Let $f(A, B, C) = \sum_{k=1}^{4} (Ax_k + By_k + C - z_k)^2 = C^2 + (B + C - 1)^2 + (A + B + C - 1)^2 + (A + C + 1)^2$. We want to minimize f. Then $f_A(A, B, C) = 4A + 2B + 4C$, $f_B(A, B, C) = 2A + 4B + 4C - 4$, and $f_C(A, B, C) = 4A + 4B + 8C - 2$. Set each partial derivative equal to 0 and solve the system to get $A = -\frac{1}{2}$, $B = \frac{3}{2}$, and $C = -\frac{1}{4}$ or the critical point of f is $\left(-\frac{1}{2}, \frac{3}{2}, -\frac{1}{4}\right)$.

43. (a) Maximize $f(a, b, c) = a^2b^2c^2$ subject to $a^2 + b^2 + c^2 = r^2$. Thus $\nabla f = 2ab^2c^2\mathbf{i} + 2a^2bc^2\mathbf{j} + 2a^2b^2c\mathbf{k}$ and $\nabla g = 2a\mathbf{i} + 2b\mathbf{j} + 2c\mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow 2ab^2c^2 = 2a\lambda$, $2a^2bc^2 = 2b\lambda$, and $2a^2b^2c = 2c\lambda$ $\Rightarrow 2a^2b^2c^2 = 2a^2\lambda = 2b^2\lambda = 2c^2\lambda \Rightarrow \lambda = 0$ or $a^2 = b^2 = c^2$.

CASE 1: $\lambda = 0 \Rightarrow a^2b^2c^2 = 0$.

CASE 2: $a^2 = b^2 = c^2 \Rightarrow f(a, b, c) = a^2a^2a^2$ and $3a^2 = r^2 \Rightarrow f(a, b, c) = \left(\frac{r^2}{3}\right)^3$ is the maximum value.

(b) The point $\left(\sqrt{a}, \sqrt{b}, \sqrt{c}\right)$ is on the sphere if $a + b + c = r^2$. Moreover, by part (a), $abc = f\left(\sqrt{a}, \sqrt{b}, \sqrt{c}\right)$ $\le \left(\frac{r^2}{3}\right)^3 \Rightarrow (abc)^{1/3} \le \frac{r^2}{3} = \frac{a+b+c}{3}$, as claimed.

44. Let $f(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} a_ix_i = a_1x_1 + a_2x_2 + \ldots + a_nx_n$ and $g(x_1, x_2, \ldots, x_n) = x_1^2 + x_2^2 + \ldots + x_n^2 - 1$. Then we want $\nabla f = \lambda \nabla g \Rightarrow a_1 = \lambda(2x_1), a_2 = \lambda(2x_2), \ldots, a_n = \lambda(2x_n), \lambda \ne 0 \Rightarrow x_i = \frac{a_i}{2\lambda} \Rightarrow \frac{a_1^2}{4\lambda^2} + \frac{a_2^2}{4\lambda^2} + \ldots + \frac{a_n^2}{4\lambda^2} = 1$ $\Rightarrow 4\lambda^2 = \sum_{i=1}^{n} a_i^2 \Rightarrow 2\lambda = \left(\sum_{i=1}^{n} a_i^2\right)^{1/2} \Rightarrow f(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} a_ix_i = \sum_{i=1}^{n} a_i \left(\frac{a_i}{2\lambda}\right) = \frac{1}{2\lambda}\sum_{i=1}^{n} a_i^2 = \left(\sum_{i=1}^{n} a_i^2\right)^{1/2}$ is the maximum value.

45-50. Example CAS commands:

Maple:

```
f := (x,y,z) -> x*y+y*z;
g1 := (x,y,z) -> x^2+y^2-2;
g2 := (x,y,z) -> x^2+z^2-2;
h := unapply( f(x,y,z)-lambda[1]*g1(x,y,z)-lambda[2]*g2(x,y,z), (x,y,z,lambda[1],lambda[2]) );    # (a)
hx := diff( h(x,y,z,lambda[1],lambda[2]), x );                                                    #(b)
hy := diff( h(x,y,z,lambda[1],lambda[2]), y );
hz := diff( h(x,y,z,lambda[1],lambda[2]), z );
hl1 := diff( h(x,y,z,lambda[1],lambda[2]), lambda[1] );
hl2 := diff( h(x,y,z,lambda[1],lambda[2]), lambda[2] );
sys := { hx=0, hy=0, hz=0, hl1=0, hl2=0 };
q1 := solve( sys, {x,y,z,lambda[1],lambda[2]} );                                                  # (c)
q2 := map(allvalues,{q1});
for p in q2 do                                                                                    # (d)
  eval( [x,y,z,f(x,y,z)], p );
  ``=evalf(eval( [x,y,z,f(x,y,z)], p ));
end do;
```

Mathematica: (assigned functions will vary)

```
Clear[x, y, z, lambda1, lambda2]
f[x_,y_,z_]:= x y + y z
g1[x_,y_,z_]:= x^2 + y^2 − 2
g2[x_,y_,z_]:= x^2 + z^2 − 2
h = f[x, y, z] − lambda1 g1[x, y, z] − lambda2 g2[x, y, z];
hx= D[h, x];  hy= D[h, y];  hz= D[h,z]; hL1=D[h, lambda1];  hL2= D[h, lambda2];
critical=Solve[{hx==0, hy==0, hz==0, hL1==0, hL2==0, g1[x,y,z]==0, g2[x,y,z]==0},
{x, y, z, lambda1, lambda2}]//N
{{x, y, z}, f[x, y, z]}/.critical
```

14.9 TAYLOR'S FORMULA FOR TWO VARIABLES

1. $f(x,y) = xe^y \Rightarrow f_x = e^y, f_y = xe^y, f_{xx} = 0, f_{xy} = e^y, f_{yy} = xe^y$
$\Rightarrow f(x,y) \approx f(0,0) + xf_x(0,0) + yf_y(0,0) + \frac{1}{2}\left[x^2 f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$
$= 0 + x \cdot 1 + y \cdot 0 + \frac{1}{2}\left(x^2 \cdot 0 + 2xy \cdot 1 + y^2 \cdot 0\right) = x + xy$ quadratic approximation;
$f_{xxx} = 0, f_{xxy} = 0, f_{xyy} = e^y, f_{yyy} = xe^y$
$\Rightarrow f(x,y) \approx \text{quadratic} + \frac{1}{6}\left[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right]$
$= x + xy + \frac{1}{6}\left(x^3 \cdot 0 + 3x^2 y \cdot 0 + 3xy^2 \cdot 1 + y^3 \cdot 0\right) = x + xy + \frac{1}{2} xy^2$, cubic approximation

2. $f(x,y) = e^x \cos y \Rightarrow f_x = e^x \cos y, f_y = -e^x \sin y, f_{xx} = e^x \cos y, f_{xy} = -e^x \sin y, f_{yy} = -e^x \cos y$
$\Rightarrow f(x,y) \approx f(0,0) + xf_x(0,0) + yf_y(0,0) + \frac{1}{2}\left[x^2 f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$
$= 1 + x \cdot 1 + y \cdot 0 + \frac{1}{2}\left[x^2 \cdot 1 + 2xy \cdot 0 + y^2 \cdot (-1)\right] = 1 + x + \frac{1}{2}\left(x^2 - y^2\right)$, quadratic approximation;
$f_{xxx} = e^x \cos y, f_{xxy} = -e^x \sin y, f_{xyy} = -e^x \cos y, f_{yyy} = e^x \sin y$
$\Rightarrow f(x,y) \approx \text{quadratic} + \frac{1}{6}\left[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right]$
$= 1 + x + \frac{1}{2}\left(x^2 - y^2\right) + \frac{1}{6}\left[x^3 \cdot 1 + 3x^2 y \cdot 0 + 3xy^2 \cdot (-1) + y^3 \cdot 0\right]$
$= 1 + x + \frac{1}{2}\left(x^2 - y^2\right) + \frac{1}{6}\left(x^3 - 3xy^2\right)$, cubic approximation

3. $f(x,y) = y \sin x \Rightarrow f_x = y \cos x, f_y = \sin x, f_{xx} = -y \sin x, f_{xy} = \cos x, f_{yy} = 0$
$\Rightarrow f(x,y) \approx f(0,0) + xf_x(0,0) + yf_y(0,0) + \frac{1}{2}\left[x^2 f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$
$= 0 + x \cdot 0 + y \cdot 0 + \frac{1}{2}\left(x^2 \cdot 0 + 2xy \cdot 1 + y^2 \cdot 0\right) = xy$, quadratic approximation;
$f_{xxx} = -y \cos x, f_{xxy} = -\sin x, f_{xyy} = 0, f_{yyy} = 0$
$\Rightarrow f(x,y) \approx \text{quadratic} + \frac{1}{6}\left[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right]$
$= xy + \frac{1}{6}\left(x^3 \cdot 0 + 3x^2 y \cdot 0 + 3xy^2 \cdot 0 + y^3 \cdot 0\right) = xy$, cubic approximation

4. $f(x,y) = \sin x \cos y \Rightarrow f_x = \cos x \cos y, f_y = -\sin x \sin y, f_{xx} = -\sin x \cos y, f_{xy} = -\cos x \sin y,$
$f_{yy} = -\sin x \cos y \Rightarrow f(x,y) \approx f(0,0) + xf_x(0,0) + yf_y(0,0) + \frac{1}{2}\left[x^2 f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$
$= 0 + x \cdot 1 + y \cdot 0 + \frac{1}{2}\left(x^2 \cdot 0 + 2xy \cdot 0 + y^2 \cdot 0\right) = x$, quadratic approximation;
$f_{xxx} = -\cos x \cos y, f_{xxy} = \sin x \sin y, f_{xyy} = -\cos x \cos y, f_{yyy} = \sin x \sin y$
$\Rightarrow f(x,y) \approx \text{quadratic} + \frac{1}{6}\left[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right]$
$= x + \frac{1}{6}\left[x^3 \cdot (-1) + 3x^2 y \cdot 0 + 3xy^2 \cdot (-1) + y^3 \cdot 0\right] = x - \frac{1}{6}\left(x^3 + 3xy^2\right)$, cubic approximation

5. $f(x,y) = e^x \ln(1+y) \Rightarrow f_x = e^x \ln(1+y), f_y = \frac{e^x}{1+y}, f_{xx} = e^x \ln(1+y), f_{xy} = \frac{e^x}{1+y}, f_{yy} = -\frac{e^x}{(1+y)^2}$
$\Rightarrow f(x,y) \approx f(0,0) + xf_x(0,0) + yf_y(0,0) + \frac{1}{2}\left[x^2 f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$
$= 0 + x \cdot 0 + y \cdot 1 + \frac{1}{2}\left[x^2 \cdot 0 + 2xy \cdot 1 + y^2 \cdot (-1)\right] = y + \frac{1}{2}\left(2xy - y^2\right)$, quadratic approximation;
$f_{xxx} = e^x \ln(1+y), f_{xxy} = \frac{e^x}{1+y}, f_{xyy} = -\frac{e^x}{(1+y)^2}, f_{yyy} = \frac{2e^x}{(1+y)^3}$

$\Rightarrow f(x,y) \approx \text{quadratic} + \frac{1}{6}\left[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right]$
$= y + \frac{1}{2}(2xy - y^2) + \frac{1}{6}\left[x^3 \cdot 0 + 3x^2 y \cdot 1 + 3xy^2 \cdot (-1) + y^3 \cdot 2\right]$
$= y + \frac{1}{2}(2xy - y^2) + \frac{1}{6}(3x^2 y - 3xy^2 + 2y^3)$, cubic approximation

6. $f(x,y) = \ln(2x + y + 1) \Rightarrow f_x = \frac{2}{2x+y+1}$, $f_y = \frac{1}{2x+y+1}$, $f_{xx} = \frac{-4}{(2x+y+1)^2}$, $f_{xy} = \frac{-2}{(2x+y+1)^2}$,
$f_{yy} = \frac{-1}{(2x+y+1)^2} \Rightarrow f(x,y) \approx f(0,0) + xf_x(0,0) + yf_y(0,0) + \frac{1}{2}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$
$= 0 + x \cdot 2 + y \cdot 1 + \frac{1}{2}\left[x^2 \cdot (-4) + 2xy \cdot (-2) + y^2 \cdot (-1)\right] = 2x + y + \frac{1}{2}(-4x^2 - 4xy - y^2)$
$= (2x + y) - \frac{1}{2}(2x + y)^2$, quadratic approximation;
$f_{xxx} = \frac{16}{(2x+y+1)^3}$, $f_{xxy} = \frac{8}{(2x+y+1)^3}$, $f_{xyy} = \frac{4}{(2x+y+1)^3}$, $f_{yyy} = \frac{2}{(2x+y+1)^3}$
$\Rightarrow f(x,y) \approx \text{quadratic} + \frac{1}{6}\left[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right]$
$= (2x + y) - \frac{1}{2}(2x + y)^2 + \frac{1}{6}(x^3 \cdot 16 + 3x^2 y \cdot 8 + 3xy^2 \cdot 4 + y^3 \cdot 2)$
$= (2x + y) - \frac{1}{2}(2x + y)^2 + \frac{1}{3}(8x^3 + 12x^2 y + 6xy^2 + y^2)$
$= (2x + y) - \frac{1}{2}(2x + y)^2 + \frac{1}{3}(2x + y)^3$, cubic approximation

7. $f(x,y) = \sin(x^2 + y^2) \Rightarrow f_x = 2x\cos(x^2 + y^2)$, $f_y = 2y\cos(x^2 + y^2)$, $f_{xx} = 2\cos(x^2 + y^2) - 4x^2\sin(x^2 + y^2)$,
$f_{xy} = -4xy\sin(x^2 + y^2)$, $f_{yy} = 2\cos(x^2 + y^2) - 4y^2\sin(x^2 + y^2)$
$\Rightarrow f(x,y) \approx f(0,0) + xf_x(0,0) + yf_y(0,0) + \frac{1}{2}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$
$= 0 + x \cdot 0 + y \cdot 0 + \frac{1}{2}(x^2 \cdot 2 + 2xy \cdot 0 + y^2 \cdot 2) = x^2 + y^2$, quadratic approximation;
$f_{xxx} = -12x\sin(x^2 + y^2) - 8x^3\cos(x^2 + y^2)$, $f_{xxy} = -4y\sin(x^2 + y^2) - 8x^2 y\cos(x^2 + y^2)$,
$f_{xyy} = -4x\sin(x^2 + y^2) - 8xy^2\cos(x^2 + y^2)$, $f_{yyy} = -12y\sin(x^2 + y^2) - 8y^3\cos(x^2 + y^2)$
$\Rightarrow f(x,y) \approx \text{quadratic} + \frac{1}{6}\left[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right]$
$= x^2 + y^2 + \frac{1}{6}(x^3 \cdot 0 + 3x^2 y \cdot 0 + 3xy^2 \cdot 0 + y^3 \cdot 0) = x^2 + y^2$, cubic approximation

8. $f(x,y) = \cos(x^2 + y^2) \Rightarrow f_x = -2x\sin(x^2 + y^2)$, $f_y = -2y\sin(x^2 + y^2)$,
$f_{xx} = -2\sin(x^2 + y^2) - 4x^2\cos(x^2 + y^2)$, $f_{xy} = -4xy\cos(x^2 + y^2)$, $f_{yy} = -2\sin(x^2 + y^2) - 4y^2\cos(x^2 + y^2)$
$\Rightarrow f(x,y) \approx f(0,0) + xf_x(0,0) + yf_y(0,0) + \frac{1}{2}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$
$= 1 + x \cdot 0 + y \cdot 0 + \frac{1}{2}\left[x^2 \cdot 0 + 2xy \cdot 0 + y^2 \cdot 0\right] = 1$, quadratic approximation;
$f_{xxx} = -12x\cos(x^2 + y^2) + 8x^3\sin(x^2 + y^2)$, $f_{xxy} = -4y\cos(x^2 + y^2) + 8x^2 y\sin(x^2 + y^2)$,
$f_{xyy} = -4x\cos(x^2 + y^2) + 8xy^2\sin(x^2 + y^2)$, $f_{yyy} = -12y\cos(x^2 + y^2) + 8y^3\sin(x^2 + y^2)$
$\Rightarrow f(x,y) \approx \text{quadratic} + \frac{1}{6}\left[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right]$
$= 1 + \frac{1}{6}(x^3 \cdot 0 + 3x^2 y \cdot 0 + 3xy^2 \cdot 0 + y^3 \cdot 0) = 1$, cubic approximation

9. $f(x,y) = \frac{1}{1-x-y} \Rightarrow f_x = \frac{1}{(1-x-y)^2} = f_y$, $f_{xx} = \frac{2}{(1-x-y)^3} = f_{xy} = f_{yy}$
$\Rightarrow f(x,y) \approx f(0,0) + xf_x(0,0) + yf_y(0,0) + \frac{1}{2}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$
$= 1 + x \cdot 1 + y \cdot 1 + \frac{1}{2}(x^2 \cdot 2 + 2xy \cdot 2 + y^2 \cdot 2) = 1 + (x + y) + (x^2 + 2xy + y^2)$
$= 1 + (x + y) + (x + y)^2$, quadratic approximation; $f_{xxx} = \frac{6}{(1-x-y)^4} = f_{xxy} = f_{xyy} = f_{yyy}$
$\Rightarrow f(x,y) \approx \text{quadratic} + \frac{1}{6}\left[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right]$
$= 1 + (x + y) + (x + y)^2 + \frac{1}{6}(x^3 \cdot 6 + 3x^2 y \cdot 6 + 3xy^2 \cdot 6 + y^3 \cdot 6)$
$= 1 + (x + y) + (x + y)^2 + (x^3 + 3x^2 y + 3xy^2 + y^3) = 1 + (x + y) + (x + y)^2 + (x + y)^3$, cubic approximation

10. $f(x,y) = \frac{1}{1-x-y+xy} \Rightarrow f_x = \frac{1-y}{(1-x-y+xy)^2}$, $f_y = \frac{1-x}{(1-x-y+xy)^2}$, $f_{xx} = \frac{2(1-y)^2}{(1-x-y+xy)^3}$,
$f_{xy} = \frac{1}{(1-x-y+xy)^2}$, $f_{yy} = \frac{2(1-x)^2}{(1-x-y+xy)^3}$
$\Rightarrow f(x,y) \approx f(0,0) + xf_x(0,0) + yf_y(0,0) + \frac{1}{2}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$
$= 1 + x \cdot 1 + y \cdot 1 + \frac{1}{2}(x^2 \cdot 2 + 2xy \cdot 1 + y^2 \cdot 2) = 1 + x + y + x^2 + xy + y^2$, quadratic approximation;

$f_{xxx} = \frac{6(1-y)^3}{(1-x-y+xy)^4}$, $f_{xxy} = \frac{[-4(1-x-y+xy)+6(1-y)(1-x)](1-y)}{(1-x-y+xy)^4}$,
$f_{xyy} = \frac{[-4(1-x-y+xy)+6(1-x)(1-y)](1-x)}{(1-x-y+xy)^4}$, $f_{yyy} = \frac{6(1-x)^3}{(1-x-y+xy)^4}$
$\Rightarrow f(x,y) \approx \text{quadratic} + \frac{1}{6}[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)]$
$= 1 + x + y + x^2 + xy + y^2 + \frac{1}{6}(x^3 \cdot 6 + 3x^2 y \cdot 2 + 3xy^2 \cdot 2 + y^3 \cdot 6)$
$= 1 + x + y + x^2 + xy + y^2 + x^3 + x^2 y + xy^2 + y^3$, cubic approximation

11. $f(x,y) = \cos x \cos y \Rightarrow f_x = -\sin x \cos y$, $f_y = -\cos x \sin y$, $f_{xx} = -\cos x \cos y$, $f_{xy} = \sin x \sin y$, $f_{yy} = -\cos x \cos y \Rightarrow f(x,y) \approx f(0,0) + x f_x(0,0) + y f_y(0,0) + \frac{1}{2}[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)]$
$= 1 + x \cdot 0 + y \cdot 0 + \frac{1}{2}[x^2 \cdot (-1) + 2xy \cdot 0 + y^2 \cdot (-1)] = 1 - \frac{x^2}{2} - \frac{y^2}{2}$, quadratic approximation. Since all partial derivatives of f are products of sines and cosines, the absolute value of these derivatives is less than or equal to 1 $\Rightarrow E(x,y) \le \frac{1}{6}[(0.1)^3 + 3(0.1)^3 + 3(0.1)^3 + 0.1)^3] \le 0.00134$.

12. $f(x,y) = e^x \sin y \Rightarrow f_x = e^x \sin y$, $f_y = e^x \cos y$, $f_{xx} = e^x \sin y$, $f_{xy} = e^x \cos y$, $f_{yy} = -e^x \sin y$
$\Rightarrow f(x,y) \approx f(0,0) + x f_x(0,0) + y f_y(0,0) + \frac{1}{2}[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)]$
$= 0 + x \cdot 0 + y \cdot 1 + \frac{1}{2}(x^2 \cdot 0 + 2xy \cdot 1 + y^2 \cdot 0) = y + xy$, quadratic approximation. Now, $f_{xxx} = e^x \sin y$, $f_{xxy} = e^x \cos y$, $f_{xyy} = -e^x \sin y$, and $f_{yyy} = -e^x \cos y$. Since $|x| \le 0.1$, $|e^x \sin y| \le |e^{0.1} \sin 0.1| \approx 0.11$ and $|e^x \cos y| \le |e^{0.1} \cos 0.1| \approx 1.11$. Therefore,
$E(x,y) \le \frac{1}{6}[(0.11)(0.1)^3 + 3(1.11)(0.1)^3 + 3(0.11)(0.1)^3 + (1.11)(0.1)^3] \le 0.000814$.

14.10 PARTIAL DERIVATIVES WITH CONSTRAINED VARIABLES

1. $w = x^2 + y^2 + z^2$ and $z = x^2 + y^2$:

(a) $\begin{pmatrix} y \\ z \end{pmatrix} \to \begin{pmatrix} x = x(y,z) \\ y = y \\ z = z \end{pmatrix} \to w \Rightarrow \left(\frac{\partial w}{\partial y}\right)_z = \frac{\partial w}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial y}$; $\frac{\partial z}{\partial y} = 0$ and $\frac{\partial z}{\partial y} = 2x\frac{\partial x}{\partial y} + 2y\frac{\partial y}{\partial y}$
$= 2x\frac{\partial x}{\partial y} + 2y \Rightarrow 0 = 2x\frac{\partial x}{\partial y} + 2y \Rightarrow \frac{\partial x}{\partial y} = -\frac{y}{x} \Rightarrow \left(\frac{\partial w}{\partial y}\right)_z = (2x)\left(-\frac{y}{x}\right) + (2y)(1) + (2z)(0) = -2y + 2y = 0$

(b) $\begin{pmatrix} x \\ z \end{pmatrix} \to \begin{pmatrix} x = x \\ y = y(x,z) \\ z = z \end{pmatrix} \to w \Rightarrow \left(\frac{\partial w}{\partial z}\right)_x = \frac{\partial w}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial z} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial z}$; $\frac{\partial x}{\partial z} = 0$ and $\frac{\partial z}{\partial z} = 2x\frac{\partial x}{\partial z} + 2y\frac{\partial y}{\partial z}$
$\Rightarrow 1 = 2y\frac{\partial y}{\partial z} \Rightarrow \frac{\partial y}{\partial z} = \frac{1}{2y} \Rightarrow \left(\frac{\partial w}{\partial z}\right)_x = (2x)(0) + (2y)\left(\frac{1}{2y}\right) + (2z)(1) = 1 + 2z$

(c) $\begin{pmatrix} y \\ z \end{pmatrix} \to \begin{pmatrix} x = x(y,z) \\ y = y \\ z = z \end{pmatrix} \to w \Rightarrow \left(\frac{\partial w}{\partial z}\right)_y = \frac{\partial w}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial z} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial z}$; $\frac{\partial y}{\partial z} = 0$ and $\frac{\partial z}{\partial z} = 2x\frac{\partial x}{\partial z} + 2y\frac{\partial y}{\partial z}$
$\Rightarrow 1 = 2x\frac{\partial x}{\partial z} \Rightarrow \frac{\partial x}{\partial z} = \frac{1}{2x} \Rightarrow \left(\frac{\partial w}{\partial z}\right)_y = (2x)\left(\frac{1}{2x}\right) + (2y)(0) + (2z)(1) = 1 + 2z$

2. $w = x^2 + y - z + \sin t$ and $x + y = t$:

(a) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} x = x \\ y = y \\ z = z \\ t = x + y \end{pmatrix} \to w \Rightarrow \left(\frac{\partial w}{\partial y}\right)_{x,z} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial y}$; $\frac{\partial x}{\partial y} = 0$, $\frac{\partial z}{\partial y} = 0$, and
$\frac{\partial t}{\partial y} = 1 \Rightarrow \left(\frac{\partial w}{\partial y}\right)_{x,t} = (2x)(0) + (1)(1) + (-1)(0) + (\cos t)(1) = 1 + \cos t = 1 + \cos(x+y)$

(b) $\begin{pmatrix} y \\ z \\ t \end{pmatrix} \to \begin{pmatrix} x = t - y \\ y = y \\ z = z \\ t = t \end{pmatrix} \to w \Rightarrow \left(\frac{\partial w}{\partial y}\right)_{z,t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial y}$; $\frac{\partial z}{\partial y} = 0$ and $\frac{\partial t}{\partial y} = 0$
$\Rightarrow \frac{\partial x}{\partial y} = \frac{\partial t}{\partial y} - \frac{\partial y}{\partial y} = -1 \Rightarrow \left(\frac{\partial w}{\partial y}\right)_{z,t} = (2x)(-1) + (1)(1) + (-1)(0) + (\cos t)(0) = 1 - 2(t-y) = 1 + 2y - 2t$

(c) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} x = x \\ y = y \\ z = z \\ t = x + y \end{pmatrix} \to w \Rightarrow \left(\frac{\partial w}{\partial z}\right)_{x,y} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial z} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial z} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial z}$; $\frac{\partial x}{\partial z} = 0$ and $\frac{\partial y}{\partial z} = 0$

$\Rightarrow \left(\frac{\partial w}{\partial z}\right)_{x,y} = (2x)(0) + (1)(0) + (-1)(1) + (\cos t)(0) = -1$

(d) $\begin{pmatrix} y \\ z \\ t \end{pmatrix} \to \begin{pmatrix} x = t - y \\ y = y \\ z = z \\ t = t \end{pmatrix} \to w \Rightarrow \left(\frac{\partial w}{\partial z}\right)_{y,t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial z} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial z} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial z}$; $\frac{\partial y}{\partial z} = 0$ and $\frac{\partial t}{\partial z} = 0$

$\Rightarrow \left(\frac{\partial w}{\partial z}\right)_{y,t} = (2x)(0) + (1)(0) + (-1)(1) + (\cos t)(0) = -1$

(e) $\begin{pmatrix} x \\ z \\ t \end{pmatrix} \to \begin{pmatrix} x = x \\ y = t - x \\ z = z \\ t = t \end{pmatrix} \to w \Rightarrow \left(\frac{\partial w}{\partial t}\right)_{x,z} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial t}$; $\frac{\partial x}{\partial t} = 0$ and $\frac{\partial z}{\partial t} = 0$

$\Rightarrow \left(\frac{\partial w}{\partial t}\right)_{x,z} = (2x)(0) + (1)(1) + (-1)(0) + (\cos t)(1) = 1 + \cos t$

(f) $\begin{pmatrix} y \\ z \\ t \end{pmatrix} \to \begin{pmatrix} x = t - y \\ y = y \\ z = z \\ t = t \end{pmatrix} \to w \Rightarrow \left(\frac{\partial w}{\partial t}\right)_{y,z} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial t}$; $\frac{\partial y}{\partial t} = 0$ and $\frac{\partial z}{\partial t} = 0$

$\Rightarrow \left(\frac{\partial w}{\partial t}\right)_{y,z} = (2x)(1) + (1)(0) + (-1)(0) + (\cos t)(1) = \cos t + 2x = \cos t + 2(t - y)$

3. $U = f(P, V, T)$ and $PV = nRT$

(a) $\begin{pmatrix} P \\ V \end{pmatrix} \to \begin{pmatrix} P = P \\ V = V \\ T = \frac{PV}{nR} \end{pmatrix} \to U \Rightarrow \left(\frac{\partial U}{\partial P}\right)_V = \frac{\partial U}{\partial P}\frac{\partial P}{\partial P} + \frac{\partial U}{\partial V}\frac{\partial V}{\partial P} + \frac{\partial U}{\partial T}\frac{\partial T}{\partial P} = \frac{\partial U}{\partial P} + \left(\frac{\partial U}{\partial V}\right)(0) + \left(\frac{\partial U}{\partial T}\right)\left(\frac{V}{nR}\right)$

$= \frac{\partial U}{\partial P} + \left(\frac{\partial U}{\partial T}\right)\left(\frac{V}{nR}\right)$

(b) $\begin{pmatrix} V \\ T \end{pmatrix} \to \begin{pmatrix} P = \frac{nRT}{V} \\ V = V \\ T = T \end{pmatrix} \to U \Rightarrow \left(\frac{\partial U}{\partial T}\right)_V = \frac{\partial U}{\partial P}\frac{\partial P}{\partial T} + \frac{\partial U}{\partial V}\frac{\partial V}{\partial T} + \frac{\partial U}{\partial T}\frac{\partial T}{\partial T} = \left(\frac{\partial U}{\partial P}\right)\left(\frac{nR}{V}\right) + \left(\frac{\partial U}{\partial V}\right)(0) + \frac{\partial U}{\partial T}$

$= \left(\frac{\partial U}{\partial P}\right)\left(\frac{nR}{V}\right) + \frac{\partial U}{\partial T}$

4. $w = x^2 + y^2 + z^2$ and $y \sin z + z \sin x = 0$

(a) $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x = x \\ y = y \\ z = z(x, y) \end{pmatrix} \to w \Rightarrow \left(\frac{\partial w}{\partial x}\right)_y = \frac{\partial w}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial x}$; $\frac{\partial y}{\partial x} = 0$ and

$(y \cos z)\frac{\partial z}{\partial x} + (\sin x)\frac{\partial z}{\partial x} + z \cos x = 0 \Rightarrow \frac{\partial z}{\partial x} = \frac{-z \cos x}{y \cos z + \sin x}$. At $(0, 1, \pi)$, $\frac{\partial z}{\partial x} = \frac{-\pi}{-1} = \pi$

$\Rightarrow \left(\frac{\partial w}{\partial x}\right)_{y|_{(0,1,\pi)}} = (2x)(1) + (2y)(0) + (2z)(\pi)\big|_{(0,1,\pi)} = 2\pi^2$

(b) $\begin{pmatrix} y \\ z \end{pmatrix} \to \begin{pmatrix} x = x(y, z) \\ y = y \\ z = z \end{pmatrix} \to w \Rightarrow \left(\frac{\partial w}{\partial z}\right)_y = \frac{\partial w}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial z} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial z} = (2x)\frac{\partial x}{\partial z} + (2y)(0) + (2z)(1)$

$= (2x)\frac{\partial x}{\partial z} + 2z$. Now $(\sin z)\frac{\partial y}{\partial z} + y \cos z + \sin x + (z \cos x)\frac{\partial x}{\partial z} = 0$ and $\frac{\partial y}{\partial z} = 0$

$\Rightarrow y \cos z + \sin x + (z \cos x)\frac{\partial x}{\partial z} = 0 \Rightarrow \frac{\partial x}{\partial z} = \frac{-y \cos z - \sin x}{z \cos x}$. At $(0, 1, \pi)$, $\frac{\partial x}{\partial z} = \frac{1 - 0}{(\pi)(1)} = \frac{1}{\pi}$

$\Rightarrow \left(\frac{\partial w}{\partial z}\right)_{y|_{(0,1,\pi)}} = 2(0)\left(\frac{1}{\pi}\right) + 2\pi = 2\pi$

5. $w = x^2y^2 + yz - z^3$ and $x^2 + y^2 + z^2 = 6$

(a) $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x = x \\ y = y \\ z = z(x, y) \end{pmatrix} \to w \Rightarrow \left(\frac{\partial w}{\partial y}\right)_x = \frac{\partial w}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial y}$

$= (2xy^2)(0) + (2x^2y + z)(1) + (y - 3z^2)\frac{\partial z}{\partial y} = 2x^2y + z + (y - 3z^2)\frac{\partial z}{\partial y}$. Now $(2x)\frac{\partial x}{\partial y} + 2y + (2z)\frac{\partial z}{\partial y} = 0$ and $\frac{\partial x}{\partial y} = 0 \Rightarrow 2y + (2z)\frac{\partial z}{\partial y} = 0 \Rightarrow \frac{\partial z}{\partial y} = -\frac{y}{z}$. At $(w, x, y, z) = (4, 2, 1, -1)$, $\frac{\partial z}{\partial y} = -\frac{1}{-1} = 1 \Rightarrow \left(\frac{\partial w}{\partial y}\right)_x\Big|_{(4,2,1,-1)}$
$= [(2)(2)^2(1) + (-1)] + [1 - 3(-1)^2](1) = 5$

(b) $\begin{pmatrix} y \\ z \end{pmatrix} \to \begin{pmatrix} x = x(y, z) \\ y = y \\ z = z \end{pmatrix} \to w \Rightarrow \left(\frac{\partial w}{\partial y}\right)_z = \frac{\partial w}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial y}$
$= (2xy^2)\frac{\partial x}{\partial y} + (2x^2y + z)(1) + (y - 3z^2)(0) = (2x^2y)\frac{\partial x}{\partial y} + 2x^2y + z$. Now $(2x)\frac{\partial x}{\partial y} + 2y + (2z)\frac{\partial z}{\partial y} = 0$ and $\frac{\partial z}{\partial y} = 0 \Rightarrow (2x)\frac{\partial x}{\partial y} + 2y = 0 \Rightarrow \frac{\partial x}{\partial y} = -\frac{y}{x}$. At $(w, x, y, z) = (4, 2, 1, -1)$, $\frac{\partial x}{\partial y} = -\frac{1}{2} \Rightarrow \left(\frac{\partial w}{\partial y}\right)_z\Big|_{(4,2,1,-1)}$
$= (2)(2)(1)^2\left(-\frac{1}{2}\right) + (2)(2)^2(1) + (-1) = 5$

6. $y = uv \Rightarrow 1 = v\frac{\partial u}{\partial y} + u\frac{\partial v}{\partial y}$; $x = u^2 + v^2$ and $\frac{\partial x}{\partial y} = 0 \Rightarrow 0 = 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} \Rightarrow \frac{\partial v}{\partial y} = \left(-\frac{u}{v}\right)\frac{\partial u}{\partial y} \Rightarrow 1$
$= v\frac{\partial u}{\partial y} + u\left(-\frac{u}{v}\frac{\partial u}{\partial y}\right) = \left(\frac{v^2 - u^2}{v}\right)\frac{\partial u}{\partial y} \Rightarrow \frac{\partial u}{\partial y} = \frac{v}{v^2 - u^2}$. At $(u, v) = \left(\sqrt{2}, 1\right)$, $\frac{\partial u}{\partial y} = \frac{1}{1^2 - \left(\sqrt{2}\right)^2} = -1$
$\Rightarrow \left(\frac{\partial u}{\partial y}\right)_x = -1$

7. $\begin{pmatrix} r \\ \theta \end{pmatrix} \to \begin{pmatrix} x = r\cos\theta \\ y = r\sin\theta \end{pmatrix} \Rightarrow \left(\frac{\partial x}{\partial r}\right)_\theta = \cos\theta$; $x^2 + y^2 = r^2 \Rightarrow 2x + 2y\frac{\partial y}{\partial x} = 2r\frac{\partial r}{\partial x}$ and $\frac{\partial y}{\partial x} = 0 \Rightarrow 2x = 2r\frac{\partial r}{\partial x}$
$\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} \Rightarrow \left(\frac{\partial r}{\partial x}\right)_y = \frac{x}{\sqrt{x^2 + y^2}}$

8. If x, y, and z are independent, then $\left(\frac{\partial w}{\partial x}\right)_{y,z} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial x}$
$= (2x)(1) + (-2y)(0) + (4)(0) + (1)\left(\frac{\partial t}{\partial x}\right) = 2x + \frac{\partial t}{\partial x}$. Thus $x + 2z + t = 25 \Rightarrow 1 + 0 + \frac{\partial t}{\partial x} = 0 \Rightarrow \frac{\partial t}{\partial x} = -1$
$\Rightarrow \left(\frac{\partial w}{\partial x}\right)_{y,z} = 2x - 1$. On the other hand, if x, y, and t are independent, then $\left(\frac{\partial w}{\partial x}\right)_{y,t}$
$= \frac{\partial w}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial x} = (2x)(1) + (-2y)(0) + 4\frac{\partial z}{\partial x} + (1)(0) = 2x + 4\frac{\partial z}{\partial x}$. Thus, $x + 2z + t = 25$
$\Rightarrow 1 + 2\frac{\partial z}{\partial x} + 0 = 0 \Rightarrow \frac{\partial z}{\partial x} = -\frac{1}{2} \Rightarrow \left(\frac{\partial w}{\partial x}\right)_{y,t} = 2x + 4\left(-\frac{1}{2}\right) = 2x - 2$.

9. If x is a differentiable function of y and z, then $f(x, y, z) = 0 \Rightarrow \frac{\partial f}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x} = 0$
$\Rightarrow \left(\frac{\partial x}{\partial y}\right)_z = -\frac{\partial f/\partial y}{\partial f/\partial z}$. Similarly, if y is a differentiable function of x and z, $\left(\frac{\partial y}{\partial z}\right)_x = -\frac{\partial f/\partial z}{\partial f/\partial x}$ and if z is a differentiable function of x and y, $\left(\frac{\partial z}{\partial x}\right)_y = -\frac{\partial f/\partial x}{\partial f/\partial y}$. Then $\left(\frac{\partial x}{\partial y}\right)_z\left(\frac{\partial y}{\partial z}\right)_x\left(\frac{\partial z}{\partial x}\right)_y$
$= \left(-\frac{\partial f/\partial y}{\partial f/\partial z}\right)\left(-\frac{\partial f/\partial z}{\partial f/\partial x}\right)\left(-\frac{\partial f/\partial x}{\partial f/\partial y}\right) = -1$.

10. $z = z + f(u)$ and $u = xy \Rightarrow \frac{\partial z}{\partial x} = 1 + \frac{df}{du}\frac{\partial u}{\partial x} = 1 + y\frac{df}{du}$; also $\frac{\partial z}{\partial y} = 0 + \frac{df}{du}\frac{\partial u}{\partial y} = x\frac{df}{du}$ so that $x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$
$= x\left(1 + y\frac{df}{du}\right) - y\left(x\frac{df}{du}\right) = x$

11. If x and y are independent, then $g(x, y, z) = 0 \Rightarrow \frac{\partial g}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial g}{\partial z}\frac{\partial z}{\partial y} = 0$ and $\frac{\partial x}{\partial y} = 0 \Rightarrow \frac{\partial g}{\partial y} + \frac{\partial g}{\partial z}\frac{\partial z}{\partial y} = 0$
$\Rightarrow \left(\frac{\partial z}{\partial y}\right)_x = -\frac{\partial g/\partial y}{\partial g/\partial z}$, as claimed.

12. Let x and y be independent. Then $f(x, y, z, w) = 0$, $g(x, y, z, w) = 0$ and $\frac{\partial y}{\partial x} = 0$
$\Rightarrow \frac{\partial f}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x} = 0$ and
$\frac{\partial g}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial g}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial g}{\partial w}\frac{\partial w}{\partial x} = \frac{\partial g}{\partial x} + \frac{\partial g}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial g}{\partial w}\frac{\partial w}{\partial x} = 0$ imply

$$\begin{cases} \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x} = -\frac{\partial f}{\partial x} \\ \frac{\partial g}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial g}{\partial w}\frac{\partial w}{\partial x} = -\frac{\partial g}{\partial x} \end{cases} \Rightarrow \left(\frac{\partial z}{\partial x}\right)_y = \frac{\begin{vmatrix} -\frac{\partial f}{\partial x} & \frac{\partial f}{\partial w} \\ -\frac{\partial g}{\partial x} & \frac{\partial g}{\partial w} \end{vmatrix}}{\begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial w} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial w} \end{vmatrix}} = \frac{-\frac{\partial f}{\partial x}\frac{\partial g}{\partial w} + \frac{\partial g}{\partial x}\frac{\partial f}{\partial w}}{\frac{\partial f}{\partial z}\frac{\partial g}{\partial w} - \frac{\partial g}{\partial z}\frac{\partial f}{\partial w}} = -\frac{\frac{\partial f}{\partial x}\frac{\partial g}{\partial w} - \frac{\partial f}{\partial w}\frac{\partial g}{\partial x}}{\frac{\partial f}{\partial z}\frac{\partial g}{\partial w} - \frac{\partial f}{\partial w}\frac{\partial g}{\partial z}}$$, as claimed.

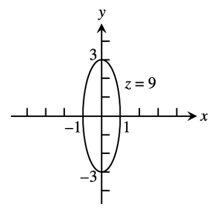
Likewise, $f(x, y, z, w) = 0$, $g(x, y, z, w) = 0$ and $\frac{\partial x}{\partial y} = 0 \Rightarrow \frac{\partial f}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial y}$ $= \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial y} = 0$ and (similarly) $\frac{\partial g}{\partial y} + \frac{\partial g}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial g}{\partial w}\frac{\partial w}{\partial y} = 0$ imply

$$\begin{cases} \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial y} = -\frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial g}{\partial w}\frac{\partial w}{\partial y} = -\frac{\partial g}{\partial y} \end{cases} \Rightarrow \left(\frac{\partial w}{\partial y}\right)_x = \frac{\begin{vmatrix} \frac{\partial f}{\partial z} & -\frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial z} & -\frac{\partial g}{\partial y} \end{vmatrix}}{\begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial w} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial w} \end{vmatrix}} = \frac{-\frac{\partial f}{\partial z}\frac{\partial g}{\partial y} + \frac{\partial g}{\partial z}\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial z}\frac{\partial g}{\partial w} - \frac{\partial g}{\partial z}\frac{\partial f}{\partial w}} = -\frac{\frac{\partial f}{\partial z}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial z}}{\frac{\partial f}{\partial z}\frac{\partial g}{\partial w} - \frac{\partial f}{\partial w}\frac{\partial g}{\partial z}}$$, as claimed.

CHAPTER 14 PRACTICE EXERCISES

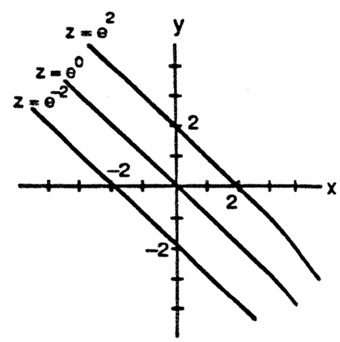
1. Domain: All points in the xy-plane
   Range: $z \geq 0$

   Level curves are ellipses with major axis along the y-axis and minor axis along the x-axis.



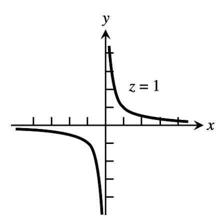
2. Domain: All points in the xy-plane
   Range: $0 < z < \infty$

   Level curves are the straight lines $x + y = \ln z$ with slope -1, and $z > 0$.



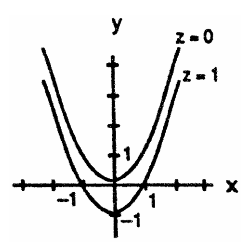
3. Domain: All (x, y) such that $x \neq 0$ and $y \neq 0$
   Range: $z \neq 0$

   Level curves are hyperbolas with the x- and y-axes as asymptotes.



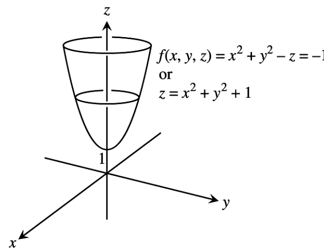
4. Domain: All (x, y) so that $x^2 - y \geq 0$
   Range: $z \geq 0$

   Level curves are the parabolas $y = x^2 - c$, $c \geq 0$.



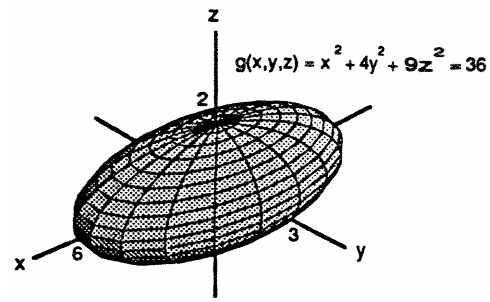
5. Domain: All points (x, y, z) in space
Range: All real numbers

Level surfaces are paraboloids of revolution with the z-axis as axis.



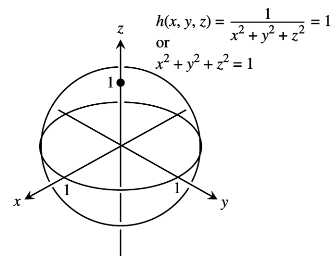
6. Domain: All points (x, y, z) in space
Range: Nonnegative real numbers

Level surfaces are ellipsoids with center (0, 0, 0).



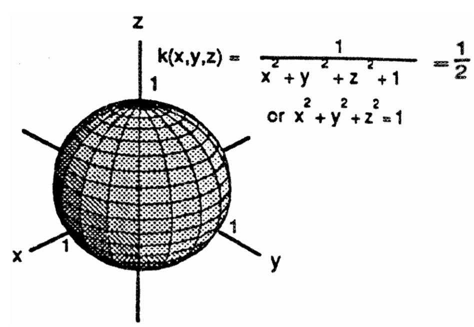
7. Domain: All (x, y, z) such that $(x, y, z) \neq (0, 0, 0)$
Range: Positive real numbers

Level surfaces are spheres with center (0, 0, 0) and radius $r > 0$.



8. Domain: All points (x, y, z) in space
Range: (0, 1]

Level surfaces are spheres with center (0, 0, 0) and radius $r > 0$.



9. $\lim\limits_{(x,y) \to (\pi, \ln 2)} e^y \cos x = e^{\ln 2} \cos \pi = (2)(-1) = -2$

10. $\lim\limits_{(x,y) \to (0,0)} \frac{2+y}{x+\cos y} = \frac{2+0}{0+\cos 0} = 2$

11. $\lim\limits_{\substack{(x,y) \to (1,1) \\ x \neq \pm y}} \frac{x-y}{x^2-y^2} = \lim\limits_{\substack{(x,y) \to (1,1) \\ x \neq \pm y}} \frac{x-y}{(x-y)(x+y)} = \lim\limits_{(x,y) \to (1,1)} \frac{1}{x+y} = \frac{1}{1+1} = \frac{1}{2}$

12. $\lim\limits_{(x,y) \to (1,1)} \frac{x^3y^3-1}{xy-1} = \lim\limits_{(x,y) \to (1,1)} \frac{(xy-1)(x^2y^2+xy+1)}{xy-1} = \lim\limits_{(x,y) \to (1,1)} (x^2y^2+xy+1) = 1^2 \cdot 1^2 + 1 \cdot 1 + 1 = 3$

13. $\lim\limits_{P \to (1,-1,e)} \ln|x+y+z| = \ln|1+(-1)+e| = \ln e = 1$

14. $\lim\limits_{P \to (1,-1,-1)} \tan^{-1}(x+y+z) = \tan^{-1}(1+(-1)+(-1)) = \tan^{-1}(-1) = -\frac{\pi}{4}$

15. Let $y = kx^2, k \neq 1$. Then $\lim\limits_{\substack{(x,y)\to(0,0) \\ y \neq x^2}} \frac{y}{x^2 - y} = \lim\limits_{(x,kx^2)\to(0,0)} \frac{kx^2}{x^2 - kx^2} = \frac{k}{1-k^2}$ which gives different limits for different values of k $\Rightarrow$ the limit does not exist.

16. Let $y = kx, k \neq 0$. Then $\lim\limits_{\substack{(x,y)\to(0,0) \\ xy \neq 0}} \frac{x^2+y^2}{xy} = \lim\limits_{(x,kx)\to(0,0)} \frac{x^2+(kx)^2}{x(kx)} = \frac{1+k^2}{k}$ which gives different limits for different values of k $\Rightarrow$ the limit does not exist.

17. Let $y = kx$. Then $\lim\limits_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \frac{x^2-k^2x^2}{x^2+k^2x^2} = \frac{1-k^2}{1+k^2}$ which gives different limits for different values of k $\Rightarrow$ the limit does not exist so f(0, 0) cannot be defined in a way that makes f continuous at the origin.

18. Along the x-axis, $y = 0$ and $\lim\limits_{(x,y)\to(0,0)} \frac{\sin(x-y)}{|x|+|y|} = \lim\limits_{x\to 0} \frac{\sin x}{|x|} = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$, so the limit fails to exist $\Rightarrow$ f is not continuous at (0, 0).

19. $\frac{\partial g}{\partial r} = \cos\theta + \sin\theta$, $\frac{\partial g}{\partial \theta} = -r\sin\theta + r\cos\theta$

20. $\frac{\partial f}{\partial x} = \frac{1}{2}\left(\frac{2x}{x^2+y^2}\right) + \frac{\left(-\frac{y}{x^2}\right)}{1+\left(\frac{y}{x}\right)^2} = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2} = \frac{x-y}{x^2+y^2}$, $\frac{\partial f}{\partial y} = \frac{1}{2}\left(\frac{2y}{x^2+y^2}\right) + \frac{\left(\frac{1}{x}\right)}{1+\left(\frac{y}{x}\right)^2} = \frac{y}{x^2+y^2} + \frac{x}{x^2+y^2} = \frac{x+y}{x^2+y^2}$

21. $\frac{\partial f}{\partial R_1} = -\frac{1}{R_1^2}$, $\frac{\partial f}{\partial R_2} = -\frac{1}{R_2^2}$, $\frac{\partial f}{\partial R_3} = -\frac{1}{R_3^2}$

22. $h_x(x,y,z) = 2\pi\cos(2\pi x + y - 3z)$, $h_y(x,y,z) = \cos(2\pi x + y - 3z)$, $h_z(x,y,z) = -3\cos(2\pi x + y - 3z)$

23. $\frac{\partial P}{\partial n} = \frac{RT}{V}$, $\frac{\partial P}{\partial R} = \frac{nT}{V}$, $\frac{\partial P}{\partial T} = \frac{nR}{V}$, $\frac{\partial P}{\partial V} = -\frac{nRT}{V^2}$

24. $f_r(r,\ell,T,w) = -\frac{1}{2r^2\ell}\sqrt{\frac{T}{\pi w}}$, $f_\ell(r,\ell,T,w) = -\frac{1}{2r\ell^2}\sqrt{\frac{T}{\pi w}}$, $f_T(r,\ell,T,w) = \left(\frac{1}{2r\ell}\right)\left(\frac{1}{\sqrt{\pi w}}\right)\left(\frac{1}{2\sqrt{T}}\right)$
$= \frac{1}{4r\ell}\sqrt{\frac{1}{T\pi w}} = \frac{1}{4r\ell T}\sqrt{\frac{T}{\pi w}}$, $f_w(r,\ell,T,w) = \left(\frac{1}{2r\ell}\right)\sqrt{\frac{T}{\pi}}\left(-\frac{1}{2}w^{-3/2}\right) = -\frac{1}{4r\ell w}\sqrt{\frac{T}{\pi w}}$

25. $\frac{\partial g}{\partial x} = \frac{1}{y}$, $\frac{\partial g}{\partial y} = 1 - \frac{x}{y^2} \Rightarrow \frac{\partial^2 g}{\partial x^2} = 0$, $\frac{\partial^2 g}{\partial y^2} = \frac{2x}{y^3}$, $\frac{\partial^2 g}{\partial y \partial x} = \frac{\partial^2 g}{\partial x \partial y} = -\frac{1}{y^2}$

26. $g_x(x,y) = e^x + y\cos x$, $g_y(x,y) = \sin x \Rightarrow g_{xx}(x,y) = e^x - y\sin x$, $g_{yy}(x,y) = 0$, $g_{xy}(x,y) = g_{yx}(x,y) = \cos x$

27. $\frac{\partial f}{\partial x} = 1 + y - 15x^2 + \frac{2x}{x^2+1}$, $\frac{\partial f}{\partial y} = x \Rightarrow \frac{\partial^2 f}{\partial x^2} = -30x + \frac{2-2x^2}{(x^2+1)^2}$, $\frac{\partial^2 f}{\partial y^2} = 0$, $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = 1$

28. $f_x(x,y) = -3y$, $f_y(x,y) = 2y - 3x - \sin y + 7e^y \Rightarrow f_{xx}(x,y) = 0$, $f_{yy}(x,y) = 2 - \cos y + 7e^y$, $f_{xy}(x,y) = f_{yx}(x,y) = -3$

29. $\frac{\partial w}{\partial x} = y\cos(xy+\pi)$, $\frac{\partial w}{\partial y} = x\cos(xy+\pi)$, $\frac{dx}{dt} = e^t$, $\frac{dy}{dt} = \frac{1}{t+1}$
$\Rightarrow \frac{dw}{dt} = [y\cos(xy+\pi)]e^t + [x\cos(xy+\pi)]\left(\frac{1}{t+1}\right)$; $t = 0 \Rightarrow x = 1$ and $y = 0$
$\Rightarrow \left.\frac{dw}{dt}\right|_{t=0} = 0 \cdot 1 + [1 \cdot (-1)]\left(\frac{1}{0+1}\right) = -1$

30. $\frac{\partial w}{\partial x} = e^y$, $\frac{\partial w}{\partial y} = xe^y + \sin z$, $\frac{\partial w}{\partial z} = y\cos z + \sin z$, $\frac{dx}{dt} = t^{-1/2}$, $\frac{dy}{dt} = 1 + \frac{1}{t}$, $\frac{dz}{dt} = \pi$
$\Rightarrow \frac{dw}{dt} = e^y t^{-1/2} + (xe^y + \sin z)\left(1 + \frac{1}{t}\right) + (y\cos z + \sin z)\pi$; $t = 1 \Rightarrow x = 2, y = 0$, and $z = \pi$
$\Rightarrow \left.\frac{dw}{dt}\right|_{t=1} = 1 \cdot 1 + (2 \cdot 1 - 0)(2) + (0 + 0)\pi = 5$

31. $\frac{\partial w}{\partial x} = 2\cos(2x - y)$, $\frac{\partial w}{\partial y} = -\cos(2x - y)$, $\frac{\partial x}{\partial r} = 1$, $\frac{\partial x}{\partial s} = \cos s$, $\frac{\partial y}{\partial r} = s$, $\frac{\partial y}{\partial s} = r$
$\Rightarrow \frac{\partial w}{\partial r} = [2\cos(2x - y)](1) + [-\cos(2x - y)](s)$; $r = \pi$ and $s = 0 \Rightarrow x = \pi$ and $y = 0$
$\Rightarrow \left.\frac{\partial w}{\partial r}\right|_{(\pi,0)} = (2\cos 2\pi) - (\cos 2\pi)(0) = 2$; $\frac{\partial w}{\partial s} = [2\cos(2x - y)](\cos s) + [-\cos(2x - y)](r)$
$\Rightarrow \left.\frac{\partial w}{\partial s}\right|_{(\pi,0)} = (2\cos 2\pi)(\cos 0) - (\cos 2\pi)(\pi) = 2 - \pi$

32. $\frac{\partial w}{\partial u} = \frac{dw}{dx}\frac{\partial x}{\partial u} = \left(\frac{x}{1+x^2} - \frac{1}{x^2+1}\right)(2e^u \cos v)$; $u = v = 0 \Rightarrow x = 2 \Rightarrow \left.\frac{\partial w}{\partial u}\right|_{(0,0)} = \left(\frac{2}{5} - \frac{1}{5}\right)(2) = \frac{2}{5}$;
$\frac{\partial w}{\partial v} = \frac{dw}{dx}\frac{\partial x}{\partial v} = \left(\frac{x}{1+x^2} - \frac{1}{x^2+1}\right)(-2e^u \sin v) \Rightarrow \left.\frac{\partial w}{\partial v}\right|_{(0,0)} = \left(\frac{2}{5} - \frac{1}{5}\right)(0) = 0$

33. $\frac{\partial f}{\partial x} = y + z$, $\frac{\partial f}{\partial y} = x + z$, $\frac{\partial f}{\partial z} = y + x$, $\frac{dx}{dt} = -\sin t$, $\frac{dy}{dt} = \cos t$, $\frac{dz}{dt} = -2\sin 2t$
$\Rightarrow \frac{df}{dt} = -(y + z)(\sin t) + (x + z)(\cos t) - 2(y + x)(\sin 2t)$; $t = 1 \Rightarrow x = \cos 1$, $y = \sin 1$, and $z = \cos 2$
$\Rightarrow \left.\frac{df}{dt}\right|_{t=1} = -(\sin 1 + \cos 2)(\sin 1) + (\cos 1 + \cos 2)(\cos 1) - 2(\sin 1 + \cos 1)(\sin 2)$

34. $\frac{\partial w}{\partial x} = \frac{dw}{ds}\frac{\partial s}{\partial x} = (5)\frac{dw}{ds}$ and $\frac{\partial w}{\partial y} = \frac{dw}{ds}\frac{\partial s}{\partial y} = (1)\frac{dw}{ds} = \frac{dw}{ds} \Rightarrow \frac{\partial w}{\partial x} - 5\frac{\partial w}{\partial y} = 5\frac{dw}{ds} - 5\frac{dw}{ds} = 0$

35. $F(x, y) = 1 - x - y^2 - \sin xy \Rightarrow F_x = -1 - y\cos xy$ and $F_y = -2y - x\cos xy \Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-1 - y\cos xy}{-2y - x\cos xy}$
$= \frac{1 + y\cos xy}{-2y - x\cos xy} \Rightarrow$ at $(x, y) = (0, 1)$ we have $\left.\frac{dy}{dx}\right|_{(0,1)} = \frac{1+1}{-2} = -1$

36. $F(x, y) = 2xy + e^{x+y} - 2 \Rightarrow F_x = 2y + e^{x+y}$ and $F_y = 2x + e^{x+y} \Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2y + e^{x+y}}{2x + e^{x+y}}$
$\Rightarrow$ at $(x, y) = (0, \ln 2)$ we have $\left.\frac{dy}{dx}\right|_{(0,\ln 2)} = -\frac{2\ln 2 + 2}{0 + 2} = -(\ln 2 + 1)$

37. $\nabla f = (-\sin x \cos y)\mathbf{i} - (\cos x \sin y)\mathbf{j} \Rightarrow \nabla f|_{\left(\frac{\pi}{4}, \frac{\pi}{4}\right)} = -\frac{1}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} \Rightarrow |\nabla f| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$;
$\mathbf{u} = \frac{\nabla f}{|\nabla f|} = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j} \Rightarrow$ f increases most rapidly in the direction $\mathbf{u} = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$ and decreases most rapidly in the direction $-\mathbf{u} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$; $(D_{\mathbf{u}}f)_{P_0} = |\nabla f| = \frac{\sqrt{2}}{2}$ and $(D_{-\mathbf{u}}f)_{P_0} = -\frac{\sqrt{2}}{2}$;
$\mathbf{u}_1 = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{3\mathbf{i} + 4\mathbf{j}}{\sqrt{3^2 + 4^2}} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j} \Rightarrow (D_{\mathbf{u}_1}f)_{P_0} = \nabla f \cdot \mathbf{u}_1 = \left(-\frac{1}{2}\right)\left(\frac{3}{5}\right) + \left(-\frac{1}{2}\right)\left(\frac{4}{5}\right) = -\frac{7}{10}$

38. $\nabla f = 2xe^{-2y}\mathbf{i} - 2x^2e^{-2y}\mathbf{j} \Rightarrow \nabla f|_{(1,0)} = 2\mathbf{i} - 2\mathbf{j} \Rightarrow |\nabla f| = \sqrt{2^2 + (-2)^2} = 2\sqrt{2}$; $\mathbf{u} = \frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$
$\Rightarrow$ f increases most rapidly in the direction $\mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$ and decreases most rapidly in the direction $-\mathbf{u} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$; $(D_{\mathbf{u}}f)_{P_0} = |\nabla f| = 2\sqrt{2}$ and $(D_{-\mathbf{u}}f)_{P_0} = -2\sqrt{2}$; $\mathbf{u}_1 = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{i} + \mathbf{j}}{\sqrt{1^2 + 1^2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$
$\Rightarrow (D_{\mathbf{u}_1}f)_{P_0} = \nabla f \cdot \mathbf{u}_1 = (2)\left(\frac{1}{\sqrt{2}}\right) + (-2)\left(\frac{1}{\sqrt{2}}\right) = 0$

39. $\nabla f = \left(\frac{2}{2x + 3y + 6z}\right)\mathbf{i} + \left(\frac{3}{2x + 3y + 6z}\right)\mathbf{j} + \left(\frac{6}{2x + 3y + 6z}\right)\mathbf{k} \Rightarrow \nabla f|_{(-1,-1,1)} = 2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$;
$\mathbf{u} = \frac{\nabla f}{|\nabla f|} = \frac{2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}}{\sqrt{2^2 + 3^2 + 6^2}} = \frac{2}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k} \Rightarrow$ f increases most rapidly in the direction $\mathbf{u} = \frac{2}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}$ and decreases most rapidly in the direction $-\mathbf{u} = -\frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} - \frac{6}{7}\mathbf{k}$; $(D_{\mathbf{u}}f)_{P_0} = |\nabla f| = 7$, $(D_{-\mathbf{u}}f)_{P_0} = -7$;
$\mathbf{u}_1 = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k} \Rightarrow (D_{\mathbf{u}_1}f)_{P_0} = (D_{\mathbf{u}}f)_{P_0} = 7$

40. $\nabla f = (2x + 3y)\mathbf{i} + (3x + 2)\mathbf{j} + (1 - 2z)\mathbf{k} \Rightarrow \nabla f|_{(0,0,0)} = 2\mathbf{j} + \mathbf{k}$; $\mathbf{u} = \frac{\nabla f}{|\nabla f|} = \frac{2}{\sqrt{5}}\mathbf{j} + \frac{1}{\sqrt{5}}\mathbf{k} \Rightarrow$ f increases most rapidly in the direction $\mathbf{u} = \frac{2}{\sqrt{5}}\mathbf{j} + \frac{1}{\sqrt{5}}\mathbf{k}$ and decreases most rapidly in the direction $-\mathbf{u} = -\frac{2}{\sqrt{5}}\mathbf{j} - \frac{1}{\sqrt{5}}\mathbf{k}$;
$(D_{\mathbf{u}}f)_{P_0} = |\nabla f| = \sqrt{5}$ and $(D_{-\mathbf{u}}f)_{P_0} = -\sqrt{5}$; $\mathbf{u}_1 = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{i} + \mathbf{j} + \mathbf{k}}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$
$\Rightarrow (D_{\mathbf{u}_1}f)_{P_0} = \nabla f \cdot \mathbf{u}_1 = (0)\left(\frac{1}{\sqrt{3}}\right) + (2)\left(\frac{1}{\sqrt{3}}\right) + (1)\left(\frac{1}{\sqrt{3}}\right) = \frac{3}{\sqrt{3}} = \sqrt{3}$

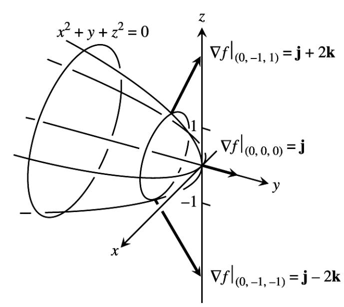
41. $\mathbf{r} = (\cos 3t)\mathbf{i} + (\sin 3t)\mathbf{j} + 3t\mathbf{k} \Rightarrow \mathbf{v}(t) = (-3\sin 3t)\mathbf{i} + (3\cos 3t)\mathbf{j} + 3\mathbf{k} \Rightarrow \mathbf{v}\left(\frac{\pi}{3}\right) = -3\mathbf{j} + 3\mathbf{k}$
$\Rightarrow \mathbf{u} = -\frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}$; $f(x, y, z) = xyz \Rightarrow \nabla f = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$; $t = \frac{\pi}{3}$ yields the point on the helix $(-1, 0, \pi)$
$\Rightarrow \nabla f|_{(-1,0,\pi)} = -\pi\mathbf{j} \Rightarrow \nabla f \cdot \mathbf{u} = (-\pi\mathbf{j}) \cdot \left(-\frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}\right) = \frac{\pi}{\sqrt{2}}$

42. $f(x, y, z) = xyz \Rightarrow \nabla f = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$; at $(1, 1, 1)$ we get $\nabla f = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow$ the maximum value of $D_{\mathbf{u}}f|_{(1,1,1)} = |\nabla f| = \sqrt{3}$

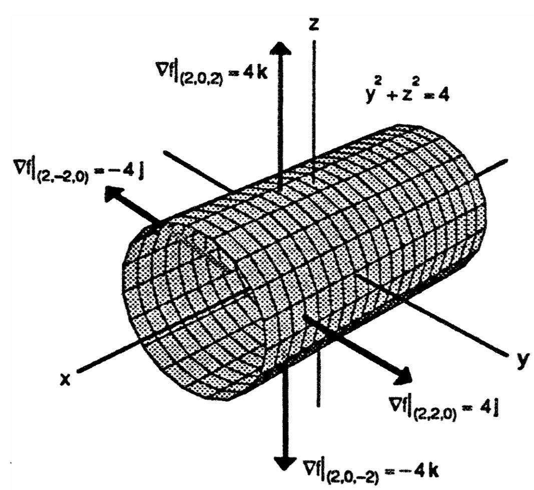
43. (a) Let $\nabla f = a\mathbf{i} + b\mathbf{j}$ at $(1, 2)$. The direction toward $(2, 2)$ is determined by $\mathbf{v}_1 = (2-1)\mathbf{i} + (2-2)\mathbf{j} = \mathbf{i} = \mathbf{u}$ so that $\nabla f \cdot \mathbf{u} = 2 \Rightarrow a = 2$. The direction toward $(1, 1)$ is determined by $\mathbf{v}_2 = (1-1)\mathbf{i} + (1-2)\mathbf{j} = -\mathbf{j} = \mathbf{u}$ so that $\nabla f \cdot \mathbf{u} = -2 \Rightarrow -b = -2 \Rightarrow b = 2$. Therefore $\nabla f = 2\mathbf{i} + 2\mathbf{j}$; $f_x(1, 2) = f_y(1, 2) = 2$.
(b) The direction toward $(4, 6)$ is determined by $\mathbf{v}_3 = (4-1)\mathbf{i} + (6-2)\mathbf{j} = 3\mathbf{i} + 4\mathbf{j} \Rightarrow \mathbf{u} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$
$\Rightarrow \nabla f \cdot \mathbf{u} = \frac{14}{5}$.

44. (a) True (b) False (c) True (d) True

45. $\nabla f = 2x\mathbf{i} + \mathbf{j} + 2z\mathbf{k} \Rightarrow$
$\nabla f|_{(0,-1,-1)} = \mathbf{j} - 2\mathbf{k}$,
$\nabla f|_{(0,0,0)} = \mathbf{j}$,
$\nabla f|_{(0,-1,1)} = \mathbf{j} + 2\mathbf{k}$



46. $\nabla f = 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow$
$\nabla f|_{(2,2,0)} = 4\mathbf{j}$,
$\nabla f|_{(2,-2,0)} = -4\mathbf{j}$,
$\nabla f|_{(2,0,2)} = 4\mathbf{k}$,
$\nabla f|_{(2,0,-2)} = -4\mathbf{k}$



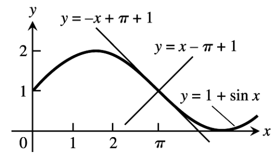
47. $\nabla f = 2x\mathbf{i} - \mathbf{j} - 5\mathbf{k} \Rightarrow \nabla f|_{(2,-1,1)} = 4\mathbf{i} - \mathbf{j} - 5\mathbf{k} \Rightarrow$ Tangent Plane: $4(x-2) - (y+1) - 5(z-1) = 0$
$\Rightarrow 4x - y - 5z = 4$; Normal Line: $x = 2 + 4t, y = -1 - t, z = 1 - 5t$

48. $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k} \Rightarrow \nabla f|_{(1,1,2)} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k} \Rightarrow$ Tangent Plane: $2(x-1) + 2(y-1) + (z-2) = 0$
$\Rightarrow 2x + 2y + z - 6 = 0$; Normal Line: $x = 1 + 2t, y = 1 + 2t, z = 2 + t$

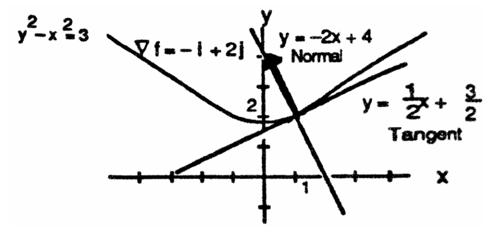
49. $\frac{\partial z}{\partial x} = \frac{2x}{x^2+y^2} \Rightarrow \frac{\partial z}{\partial x}\Big|_{(0,1,0)} = 0$ and $\frac{\partial z}{\partial y} = \frac{2y}{x^2+y^2} \Rightarrow \frac{\partial z}{\partial y}\Big|_{(0,1,0)} = 2$; thus the tangent plane is
$2(y-1) - (z-0) = 0$ or $2y - z - 2 = 0$

50. $\frac{\partial z}{\partial x} = -2x\left(x^2+y^2\right)^{-2} \Rightarrow \left.\frac{\partial z}{\partial x}\right|_{(1,1,\frac{1}{2})} = -\frac{1}{2}$ and $\frac{\partial z}{\partial y} = -2y\left(x^2+y^2\right)^{-2} \Rightarrow \left.\frac{\partial z}{\partial y}\right|_{(1,1,\frac{1}{2})} = -\frac{1}{2}$; thus the tangent plane is $-\frac{1}{2}(x-1) - \frac{1}{2}(y-1) - \left(z-\frac{1}{2}\right) = 0$ or $x+y+2z-3=0$

51. $\nabla f = (-\cos x)\mathbf{i} + \mathbf{j} \Rightarrow \left.\nabla f\right|_{(\pi,1)} = \mathbf{i}+\mathbf{j} \Rightarrow$ the tangent line is $(x-\pi)+(y-1)=0 \Rightarrow x+y=\pi+1$; the normal line is $y-1 = 1(x-\pi) \Rightarrow y = x-\pi+1$



52. $\nabla f = -x\mathbf{i} + y\mathbf{j} \Rightarrow \left.\nabla f\right|_{(1,2)} = -\mathbf{i}+2\mathbf{j} \Rightarrow$ the tangent line is $-(x-1)+2(y-2)=0 \Rightarrow y = \frac{1}{2}x + \frac{3}{2}$; the normal line is $y-2 = -2(x-1) \Rightarrow y = -2x+4$



53. Let $f(x,y,z) = x^2+2y+2z-4$ and $g(x,y,z) = y-1$. Then $\nabla f = 2x\mathbf{i}+2\mathbf{j}+2\mathbf{k}\big|_{(1,1,\frac{1}{2})} = 2\mathbf{i}+2\mathbf{j}+2\mathbf{k}$ and $\nabla g = \mathbf{j} \Rightarrow \nabla f \times \nabla g = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 2 \\ 0 & 1 & 0 \end{vmatrix} = -2\mathbf{i}+2\mathbf{k} \Rightarrow$ the line is $x = 1-2t$, $y=1$, $z = \frac{1}{2}+2t$

54. Let $f(x,y,z) = x+y^2+z-2$ and $g(x,y,z) = y-1$. Then $\nabla f = \mathbf{i}+2y\mathbf{j}+\mathbf{k}\big|_{(\frac{1}{2},1,\frac{1}{2})} = \mathbf{i}+2\mathbf{j}+\mathbf{k}$ and $\nabla g = \mathbf{j} \Rightarrow \nabla f \times \nabla g = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -\mathbf{i}+\mathbf{k} \Rightarrow$ the line is $x = \frac{1}{2}-t$, $y=1$, $z=\frac{1}{2}+t$

55. $f\left(\frac{\pi}{4},\frac{\pi}{4}\right) = \frac{1}{2}$, $f_x\left(\frac{\pi}{4},\frac{\pi}{4}\right) = \cos x\cos y\big|_{(\pi/4,\pi/4)} = \frac{1}{2}$, $f_y\left(\frac{\pi}{4},\frac{\pi}{4}\right) = -\sin x\sin y\big|_{(\pi/4,\pi/4)} = -\frac{1}{2}$
$\Rightarrow L(x,y) = \frac{1}{2} + \frac{1}{2}\left(x-\frac{\pi}{4}\right) - \frac{1}{2}\left(y-\frac{\pi}{4}\right) = \frac{1}{2}+\frac{1}{2}x-\frac{1}{2}y$; $f_{xx}(x,y) = -\sin x\cos y$, $f_{yy}(x,y) = -\sin x\cos y$, and $f_{xy}(x,y) = -\cos x\sin y$. Thus an upper bound for E depends on the bound M used for $|f_{xx}|$, $|f_{xy}|$, and $|f_{yy}|$.
With $M = \frac{\sqrt{2}}{2}$ we have $|E(x,y)| \le \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right)\left(\left|x-\frac{\pi}{4}\right| + \left|y-\frac{\pi}{4}\right|\right)^2 \le \frac{\sqrt{2}}{4}(0.2)^2 \le 0.0142$;
with $M=1$, $|E(x,y)| \le \frac{1}{2}(1)\left(\left|x-\frac{\pi}{4}\right| + \left|y-\frac{\pi}{4}\right|\right)^2 = \frac{1}{2}(0.2)^2 = 0.02$.

56. $f(1,1) = 0$, $f_x(1,1) = y\big|_{(1,1)} = 1$, $f_y(1,1) = x-6y\big|_{(1,1)} = -5 \Rightarrow L(x,y) = (x-1)-5(y-1) = x-5y+4$;
$f_{xx}(x,y) = 0$, $f_{yy}(x,y) = -6$, and $f_{xy}(x,y) = 1 \Rightarrow$ maximum of $|f_{xx}|$, $|f_{yy}|$, and $|f_{xy}|$ is $6 \Rightarrow M = 6$
$\Rightarrow |E(x,y)| \le \frac{1}{2}(6)\left(|x-1|+|y-1|\right)^2 = \frac{1}{2}(6)(0.1+0.2)^2 = 0.27$

57. $f(1,0,0) = 0$, $f_x(1,0,0) = y-3z\big|_{(1,0,0)} = 0$, $f_y(1,0,0) = x+2z\big|_{(1,0,0)} = 1$, $f_z(1,0,0) = 2y-3x\big|_{(1,0,0)} = -3$
$\Rightarrow L(x,y,z) = 0(x-1)+(y-0)-3(z-0) = y-3z$; $f(1,1,0) = 1$, $f_x(1,1,0) = 1$, $f_y(1,1,0) = 1$, $f_z(1,1,0) = -1$
$\Rightarrow L(x,y,z) = 1+(x-1)+(y-1)-1(z-0) = x+y-z-1$

58. $f\left(0,0,\frac{\pi}{4}\right) = 1$, $f_x\left(0,0,\frac{\pi}{4}\right) = -\sqrt{2}\sin x\sin(y+z)\Big|_{(0,0,\frac{\pi}{4})} = 0$, $f_y\left(0,0,\frac{\pi}{4}\right) = \sqrt{2}\cos x\cos(y+z)\Big|_{(0,0,\frac{\pi}{4})} = 1$,
$f_z\left(0,0,\frac{\pi}{4}\right) = \sqrt{2}\cos x\cos(y+z)\Big|_{(0,0,\frac{\pi}{4})} = 1 \Rightarrow L(x,y,z) = 1+1(y-0)+1\left(z-\frac{\pi}{4}\right) = 1+y+z-\frac{\pi}{4}$;
$f\left(\frac{\pi}{4},\frac{\pi}{4},0\right) = \frac{\sqrt{2}}{2}$, $f_x\left(\frac{\pi}{4},\frac{\pi}{4},0\right) = -\frac{\sqrt{2}}{2}$, $f_y\left(\frac{\pi}{4},\frac{\pi}{4},0\right) = \frac{\sqrt{2}}{2}$, $f_z\left(\frac{\pi}{4},\frac{\pi}{4},0\right) = \frac{\sqrt{2}}{2}$
$\Rightarrow L(x,y,z) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right) + \frac{\sqrt{2}}{2}\left(y-\frac{\pi}{4}\right) + \frac{\sqrt{2}}{2}(z-0) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y + \frac{\sqrt{2}}{2}z$

59. $V = \pi r^2 h \Rightarrow dV = 2\pi rh\, dr + \pi r^2\, dh \Rightarrow dV|_{(1.5,5280)} = 2\pi(1.5)(5280)\, dr + \pi(1.5)^2\, dh = 15{,}840\pi\, dr + 2.25\pi\, dh$. You should be more careful with the diameter since it has a greater effect on dV.

60. $df = (2x - y)\, dx + (-x + 2y)\, dy \Rightarrow df|_{(1,2)} = 3\, dy \Rightarrow$ f is more sensitive to changes in y; in fact, near the point $(1, 2)$ a change in x does not change f.

61. $dI = \frac{1}{R}\, dV - \frac{V}{R^2}\, dR \Rightarrow dI\big|_{(24,100)} = \frac{1}{100}\, dV - \frac{24}{100^2}\, dR \Rightarrow dI\big|_{dV=-1,dR=-20} = -0.01 + (480)(.0001) = 0.038$, or increases by 0.038 amps; % change in $V = (100)\left(-\frac{1}{24}\right) \approx -4.17\%$; % change in $R = \left(-\frac{20}{100}\right)(100) = -20\%$; $I = \frac{24}{100} = 0.24 \Rightarrow$ estimated % change in $I = \frac{dI}{I} \times 100 = \frac{0.038}{0.24} \times 100 \approx 15.83\% \Rightarrow$ more sensitive to voltage change.

62. $A = \pi ab \Rightarrow dA = \pi b\, da + \pi a\, db \Rightarrow dA|_{(10,16)} = 16\pi\, da + 10\pi\, db$; $da = \pm 0.1$ and $db = \pm 0.1$ $\Rightarrow dA = \pm 26\pi(0.1) = \pm 2.6\pi$ and $A = \pi(10)(16) = 160\pi \Rightarrow \left|\frac{dA}{A} \times 100\right| = \left|\frac{2.6\pi}{160\pi} \times 100\right| \approx 1.625\%$

63. (a) $y = uv \Rightarrow dy = v\, du + u\, dv$; percentage change in $u \le 2\% \Rightarrow |du| \le 0.02$, and percentage change in $v \le 3\%$ $\Rightarrow |dv| \le 0.03$; $\frac{dy}{y} = \frac{v\,du + u\,dv}{uv} = \frac{du}{u} + \frac{dv}{v} \Rightarrow \left|\frac{dy}{y} \times 100\right| = \left|\frac{du}{u} \times 100 + \frac{dv}{v} \times 100\right| \le \left|\frac{du}{u} \times 100\right| + \left|\frac{dv}{v} \times 100\right|$ $\le 2\% + 3\% = 5\%$

(b) $z = u + v \Rightarrow \frac{dz}{z} = \frac{du + dv}{u + v} = \frac{du}{u + v} + \frac{dv}{u + v} \le \frac{du}{u} + \frac{dv}{v}$ (since $u > 0$, $v > 0$) $\Rightarrow \left|\frac{dz}{z} \times 100\right| \le \left|\frac{du}{u} \times 100 + \frac{dv}{v} \times 100\right| = \left|\frac{dy}{y} \times 100\right|$

64. $C = \frac{7}{71.84 w^{0.425} h^{0.725}} \Rightarrow C_w = \frac{(-0.425)(7)}{71.84 w^{1.425} h^{0.725}}$ and $C_h = \frac{(-0.725)(7)}{71.84 w^{0.425} h^{1.725}}$ $\Rightarrow dC = \frac{-2.975}{71.84 w^{1.425} h^{0.725}}\, dw + \frac{-5.075}{71.84 w^{0.425} h^{1.725}}\, dh$; thus when $w = 70$ and $h = 180$ we have $dC|_{(70,180)} \approx -(0.00000225)\, dw - (0.00000149)\, dh \Rightarrow$ 1 kg error in weight has more effect

65. $f_x(x, y) = 2x - y + 2 = 0$ and $f_y(x, y) = -x + 2y + 2 = 0 \Rightarrow x = -2$ and $y = -2 \Rightarrow (-2, -2)$ is the critical point; $f_{xx}(-2, -2) = 2$, $f_{yy}(-2, -2) = 2$, $f_{xy}(-2, -2) = -1 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 3 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum value of $f(-2, -2) = -8$

66. $f_x(x, y) = 10x + 4y + 4 = 0$ and $f_y(x, y) = 4x - 4y - 4 = 0 \Rightarrow x = 0$ and $y = -1 \Rightarrow (0, -1)$ is the critical point; $f_{xx}(0, -1) = 10$, $f_{yy}(0, -1) = -4$, $f_{xy}(0, -1) = 4 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -56 < 0 \Rightarrow$ saddle point with $f(0, -1) = 2$

67. $f_x(x, y) = 6x^2 + 3y = 0$ and $f_y(x, y) = 3x + 6y^2 = 0 \Rightarrow y = -2x^2$ and $3x + 6\left(4x^4\right) = 0 \Rightarrow x\left(1 + 8x^3\right) = 0$ $\Rightarrow x = 0$ and $y = 0$, or $x = -\frac{1}{2}$ and $y = -\frac{1}{2} \Rightarrow$ the critical points are $(0, 0)$ and $\left(-\frac{1}{2}, -\frac{1}{2}\right)$. For $(0, 0)$: $f_{xx}(0, 0) = 12x|_{(0,0)} = 0$, $f_{yy}(0, 0) = 12y|_{(0,0)} = 0$, $f_{xy}(0, 0) = 3 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -9 < 0 \Rightarrow$ saddle point with $f(0, 0) = 0$. For $\left(-\frac{1}{2}, -\frac{1}{2}\right)$: $f_{xx} = -6$, $f_{yy} = -6$, $f_{xy} = 3 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 27 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum value of $f\left(-\frac{1}{2}, -\frac{1}{2}\right) = \frac{1}{4}$

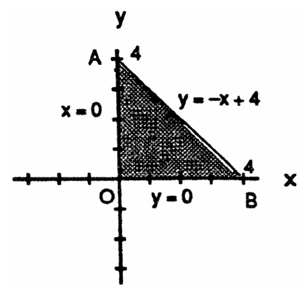
68. $f_x(x, y) = 3x^2 - 3y = 0$ and $f_y(x, y) = 3y^2 - 3x = 0 \Rightarrow y = x^2$ and $x^4 - x = 0 \Rightarrow x\left(x^3 - 1\right) = 0 \Rightarrow$ the critical points are $(0, 0)$ and $(1, 1)$. For $(0, 0)$: $f_{xx}(0, 0) = 6x|_{(0,0)} = 0$, $f_{yy}(0, 0) = 6y|_{(0,0)} = 0$, $f_{xy}(0, 0) = -3$ $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -9 < 0 \Rightarrow$ saddle point with $f(0, 0) = 15$. For $(1, 1)$: $f_{xx}(1, 1) = 6$, $f_{yy}(1, 1) = 6$, $f_{xy}(1, 1) = -3$ $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 27 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum value of $f(1, 1) = 14$

69. $f_x(x, y) = 3x^2 + 6x = 0$ and $f_y(x, y) = 3y^2 - 6y = 0 \Rightarrow x(x + 2) = 0$ and $y(y - 2) = 0 \Rightarrow x = 0$ or $x = -2$ and $y = 0$ or $y = 2 \Rightarrow$ the critical points are $(0, 0)$, $(0, 2)$, $(-2, 0)$, and $(-2, 2)$. For $(0, 0)$: $f_{xx}(0, 0) = 6x + 6|_{(0,0)}$ $= 6$, $f_{yy}(0, 0) = 6y - 6|_{(0,0)} = -6$, $f_{xy}(0, 0) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -36 < 0 \Rightarrow$ saddle point with $f(0, 0) = 0$. For $(0, 2)$: $f_{xx}(0, 2) = 6$, $f_{yy}(0, 2) = 6$, $f_{xy}(0, 2) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 36 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum value of

$f(0,2) = -4$. For $(-2,0)$: $f_{xx}(-2,0) = -6$, $f_{yy}(-2,0) = -6$, $f_{xy}(-2,0) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 36 > 0$ and $f_{xx} < 0$ $\Rightarrow$ local maximum value of $f(-2,0) = 4$. For $(-2,2)$: $f_{xx}(-2,2) = -6$, $f_{yy}(-2,2) = 6$, $f_{xy}(-2,2) = 0$ $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -36 < 0 \Rightarrow$ saddle point with $f(-2,2) = 0$.

70. $f_x(x,y) = 4x^3 - 16x = 0 \Rightarrow 4x(x^2 - 4) = 0 \Rightarrow x = 0, 2, -2$; $f_y(x,y) = 6y - 6 = 0 \Rightarrow y = 1$. Therefore the critical points are $(0,1)$, $(2,1)$, and $(-2,1)$. For $(0,1)$: $f_{xx}(0,1) = 12x^2 - 16|_{(0,1)} = -16$, $f_{yy}(0,1) = 6$, $f_{xy}(0,1) = 0$ $\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -96 < 0 \Rightarrow$ saddle point with $f(0,1) = -3$. For $(2,1)$: $f_{xx}(2,1) = 32$, $f_{yy}(2,1) = 6$, $f_{xy}(2,1) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 192 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum value of $f(2,1) = -19$. For $(-2,1)$: $f_{xx}(-2,1) = 32$, $f_{yy}(-2,1) = 6$, $f_{xy}(-2,1) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 192 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum value of $f(-2,1) = -19$.

71. (i) On OA, $f(x,y) = f(0,y) = y^2 + 3y$ for $0 \le y \le 4$ $\Rightarrow f'(0,y) = 2y + 3 = 0 \Rightarrow y = -\frac{3}{2}$. But $\left(0, -\frac{3}{2}\right)$ is not in the region.
Endpoints: $f(0,0) = 0$ and $f(0,4) = 28$.

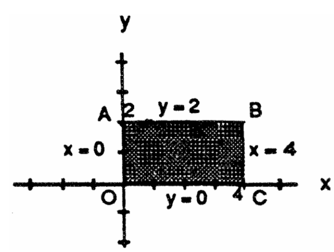


(ii) On AB, $f(x,y) = f(x, -x+4) = x^2 - 10x + 28$ for $0 \le x \le 4 \Rightarrow f'(x, -x+4) = 2x - 10 = 0$ $\Rightarrow x = 5, y = -1$. But $(5,-1)$ is not in the region.
Endpoints: $f(4,0) = 4$ and $f(0,4) = 28$.

(iii) On OB, $f(x,y) = f(x,0) = x^2 - 3x$ for $0 \le x \le 4 \Rightarrow f'(x,0) = 2x - 3 \Rightarrow x = \frac{3}{2}$ and $y = 0 \Rightarrow \left(\frac{3}{2}, 0\right)$ is a critical point with $f\left(\frac{3}{2}, 0\right) = -\frac{9}{4}$.
Endpoints: $f(0,0) = 0$ and $f(4,0) = 4$.

(iv) For the interior of the triangular region, $f_x(x,y) = 2x + y - 3 = 0$ and $f_y(x,y) = x + 2y + 3 = 0 \Rightarrow x = 3$ and $y = -3$. But $(3,-3)$ is not in the region. Therefore the absolute maximum is 28 at $(0,4)$ and the absolute minimum is $-\frac{9}{4}$ at $\left(\frac{3}{2}, 0\right)$.

72. (i) On OA, $f(x,y) = f(0,y) = -y^2 + 4y + 1$ for $0 \le y \le 2 \Rightarrow f'(0,y) = -2y + 4 = 0 \Rightarrow y = 2$ and $x = 0$. But $(0,2)$ is not in the interior of OA.
Endpoints: $f(0,0) = 1$ and $f(0,2) = 5$.



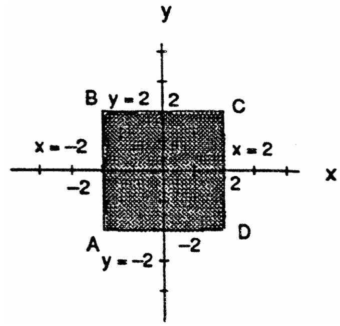
(ii) On AB, $f(x,y) = f(x,2) = x^2 - 2x + 5$ for $0 \le x \le 4$ $\Rightarrow f'(x,2) = 2x - 2 = 0 \Rightarrow x = 1$ and $y = 2$ $\Rightarrow (1,2)$ is an interior critical point of AB with $f(1,2) = 4$. Endpoints: $f(4,2) = 13$ and $f(0,2) = 5$.

(iii) On BC, $f(x,y) = f(4,y) = -y^2 + 4y + 9$ for $0 \le y \le 2 \Rightarrow f'(4,y) = -2y + 4 = 0 \Rightarrow y = 2$ and $x = 4$. But $(4,2)$ is not in the interior of BC. Endpoints: $f(4,0) = 9$ and $f(4,2) = 13$.

(iv) On OC, $f(x,y) = f(x,0) = x^2 - 2x + 1$ for $0 \le x \le 4 \Rightarrow f'(x,0) = 2x - 2 = 0 \Rightarrow x = 1$ and $y = 0 \Rightarrow (1,0)$ is an interior critical point of OC with $f(1,0) = 0$. Endpoints: $f(0,0) = 1$ and $f(4,0) = 9$.

(v) For the interior of the rectangular region, $f_x(x,y) = 2x - 2 = 0$ and $f_y(x,y) = -2y + 4 = 0 \Rightarrow x = 1$ and $y = 2$. But $(1,2)$ is not in the interior of the region. Therefore the absolute maximum is 13 at $(4,2)$ and the absolute minimum is 0 at $(1,0)$.

73. (i) On AB, $f(x,y) = f(-2,y) = y^2 - y - 4$ for $-2 \le y \le 2 \Rightarrow f'(-2,y) = 2y - 1 \Rightarrow y = \frac{1}{2}$ and $x = -2 \Rightarrow \left(-2, \frac{1}{2}\right)$ is an interior critical point in AB with $f\left(-2, \frac{1}{2}\right) = -\frac{17}{4}$. Endpoints: $f(-2,-2) = 2$ and $f(2,2) = -2$.



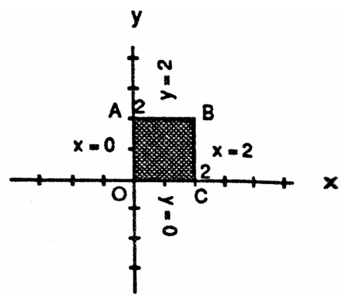
(ii) On BC, $f(x,y) = f(x,2) = -2$ for $-2 \le x \le 2$ $\Rightarrow f'(x,2) = 0 \Rightarrow$ no critical points in the interior of BC. Endpoints: $f(-2,2) = -2$ and $f(2,2) = -2$.

(iii) On CD, $f(x,y) = f(2,y) = y^2 - 5y + 4$ for $-2 \le y \le 2 \Rightarrow f'(2,y) = 2y - 5 = 0 \Rightarrow y = \frac{5}{2}$ and $x = 2$. But $\left(2, \frac{5}{2}\right)$ is not in the region. Endpoints: $f(2,-2) = 18$ and $f(2,2) = -2$.

(iv) On AD, $f(x,y) = f(x,-2) = 4x + 10$ for $-2 \le x \le 2 \Rightarrow f'(x,-2) = 4 \Rightarrow$ no critical points in the interior of AD. Endpoints: $f(-2,-2) = 2$ and $f(2,-2) = 18$.

(v) For the interior of the square, $f_x(x,y) = -y + 2 = 0$ and $f_y(x,y) = 2y - x - 3 = 0 \Rightarrow y = 2$ and $x = 1$ $\Rightarrow (1,2)$ is an interior critical point of the square with $f(1,2) = -2$. Therefore the absolute maximum is 18 at $(2,-2)$ and the absolute minimum is $-\frac{17}{4}$ at $\left(-2, \frac{1}{2}\right)$.

74. (i) On OA, $f(x,y) = f(0,y) = 2y - y^2$ for $0 \le y \le 2$ $\Rightarrow f'(0,y) = 2 - 2y = 0 \Rightarrow y = 1$ and $x = 0 \Rightarrow$ $(0,1)$ is an interior critical point of OA with $f(0,1) = 1$. Endpoints: $f(0,0) = 0$ and $f(0,2) = 0$.



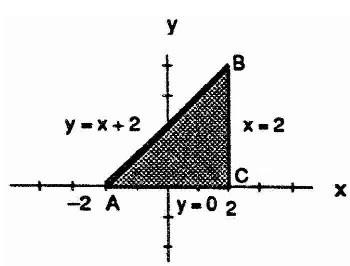
(ii) On AB, $f(x,y) = f(x,2) = 2x - x^2$ for $0 \le x \le 2$ $\Rightarrow f'(x,2) = 2 - 2x = 0 \Rightarrow x = 1$ and $y = 2$ $\Rightarrow (1,2)$ is an interior critical point of AB with $f(1,2) = 1$. Endpoints: $f(0,2) = 0$ and $f(2,2) = 0$.

(iii) On BC, $f(x,y) = f(2,y) = 2y - y^2$ for $0 \le y \le 2$ $\Rightarrow f'(2,y) = 2 - 2y = 0 \Rightarrow y = 1$ and $x = 2$ $\Rightarrow (2,1)$ is an interior critical point of BC with $f(2,1) = 1$. Endpoints: $f(2,0) = 0$ and $f(2,2) = 0$.

(iv) On OC, $f(x,y) = f(x,0) = 2x - x^2$ for $0 \le x \le 2 \Rightarrow f'(x,0) = 2 - 2x = 0 \Rightarrow x = 1$ and $y = 0 \Rightarrow (1,0)$ is an interior critical point of OC with $f(1,0) = 1$. Endpoints: $f(0,0) = 0$ and $f(0,2) = 0$.

(v) For the interior of the rectangular region, $f_x(x,y) = 2 - 2x = 0$ and $f_y(x,y) = 2 - 2y = 0 \Rightarrow x = 1$ and $y = 1 \Rightarrow (1,1)$ is an interior critical point of the square with $f(1,1) = 2$. Therefore the absolute maximum is 2 at $(1,1)$ and the absolute minimum is 0 at the four corners $(0,0)$, $(0,2)$, $(2,2)$, and $(2,0)$.

75. (i) On AB, $f(x,y) = f(x, x+2) = -2x + 4$ for $-2 \le x \le 2 \Rightarrow f'(x, x+2) = -2 = 0 \Rightarrow$ no critical points in the interior of AB. Endpoints: $f(-2,0) = 8$ and $f(2,4) = 0$.

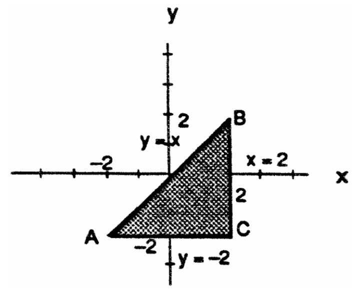


(ii) On BC, $f(x,y) = f(2,y) = -y^2 + 4y$ for $0 \le y \le 4$ $\Rightarrow f'(2,y) = -2y + 4 = 0 \Rightarrow y = 2$ and $x = 2$ $\Rightarrow (2,2)$ is an interior critical point of BC with $f(2,2) = 4$. Endpoints: $f(2,0) = 0$ and $f(2,4) = 0$.

(iii) On AC, $f(x,y) = f(x,0) = x^2 - 2x$ for $-2 \le x \le 2$ $\Rightarrow f'(x,0) = 2x - 2 \Rightarrow x = 1$ and $y = 0 \Rightarrow (1,0)$ is an interior critical point of AC with $f(1,0) = -1$. Endpoints: $f(-2,0) = 8$ and $f(2,0) = 0$.

(iv) For the interior of the triangular region, $f_x(x,y) = 2x - 2 = 0$ and $f_y(x,y) = -2y + 4 = 0 \Rightarrow x = 1$ and $y = 2 \Rightarrow (1,2)$ is an interior critical point of the region with $f(1,2) = 3$. Therefore the absolute maximum is 8 at $(-2,0)$ and the absolute minimum is -1 at $(1,0)$.

76. (i) On AB, $f(x,y) = f(x,x) = 4x^2 - 2x^4 + 16$ for $-2 \le x \le 2 \Rightarrow f'(x,x) = 8x - 8x^3 = 0 \Rightarrow x = 0$ and $y = 0$, or $x = 1$ and $y = 1$, or $x = -1$ and $y = -1$ $\Rightarrow (0,0), (1,1), (-1,-1)$ are all interior points of AB with $f(0,0) = 16$, $f(1,1) = 18$, and $f(-1,-1) = 18$. Endpoints: $f(-2,-2) = 0$ and $f(2,2) = 0$.

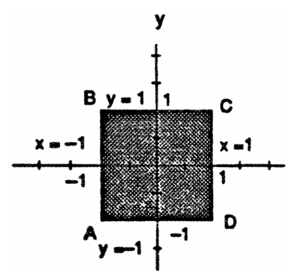


(ii) On BC, $f(x,y) = f(2,y) = 8y - y^4$ for $-2 \le y \le 2$ $\Rightarrow f'(2,y) = 8 - 4y^3 = 0 \Rightarrow y = \sqrt[3]{2}$ and $x = 2$ $\Rightarrow \left(2, \sqrt[3]{2}\right)$ is an interior critical point of BC with $f\left(2, \sqrt[3]{2}\right) = 6\sqrt[3]{2}$. Endpoints: $f(2,-2) = -32$ and $f(2,2) = 0$.

(iii) On AC, $f(x,y) = f(x,-2) = -8x - x^4$ for $-2 \le x \le 2 \Rightarrow f'(x,-2) = -8 - 4x^3 = 0 \Rightarrow x = \sqrt[3]{-2}$ and $y = -2$ $\Rightarrow \left(\sqrt[3]{-2}, -2\right)$ is an interior critical point of AC with $f\left(\sqrt[3]{-2}, -2\right) = 6\sqrt[3]{2}$. Endpoints: $f(-2,-2) = 0$ and $f(2,-2) = -32$.

(iv) For the interior of the triangular region, $f_x(x,y) = 4y - 4x^3 = 0$ and $f_y(x,y) = 4x - 4y^3 = 0 \Rightarrow x = 0$ and $y = 0$, or $x = 1$ and $y = 1$ or $x = -1$ and $y = -1$. But neither of the points $(0,0)$ and $(1,1)$, or $(-1,-1)$ are interior to the region. Therefore the absolute maximum is 18 at $(1,1)$ and $(-1,-1)$, and the absolute minimum is -32 at $(2,-2)$.

77. (i) On AB, $f(x,y) = f(-1,y) = y^3 - 3y^2 + 2$ for $-1 \le y \le 1 \Rightarrow f'(-1,y) = 3y^2 - 6y = 0 \Rightarrow y = 0$ and $x = -1$, or $y = 2$ and $x = -1 \Rightarrow (-1,0)$ is an interior critical point of AB with $f(-1,0) = 2$; $(-1,2)$ is outside the boundary. Endpoints: $f(-1,-1) = -2$ and $f(-1,1) = 0$.



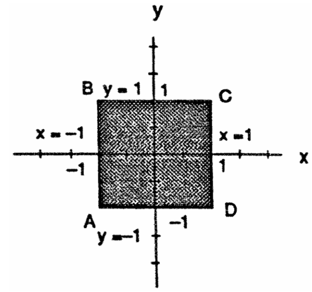
(ii) On BC, $f(x,y) = f(x,1) = x^3 + 3x^2 - 2$ for $-1 \le x \le 1 \Rightarrow f'(x,1) = 3x^2 + 6x = 0 \Rightarrow x = 0$ and $y = 1$, or $x = -2$ and $y = 1 \Rightarrow (0,1)$ is an interior critical point of BC with $f(0,1) = -2$; $(-2,1)$ is outside the boundary. Endpoints: $f(-1,1) = 0$ and $f(1,1) = 2$.

(iii) On CD, $f(x,y) = f(1,y) = y^3 - 3y^2 + 4$ for $-1 \le y \le 1 \Rightarrow f'(1,y) = 3y^2 - 6y = 0 \Rightarrow y = 0$ and $x = 1$, or $y = 2$ and $x = 1 \Rightarrow (1,0)$ is an interior critical point of CD with $f(1,0) = 4$; $(1,2)$ is outside the boundary. Endpoints: $f(1,1) = 2$ and $f(1,-1) = 0$.

(iv) On AD, $f(x,y) = f(x,-1) = x^3 + 3x^2 - 4$ for $-1 \le x \le 1 \Rightarrow f'(x,-1) = 3x^2 + 6x = 0 \Rightarrow x = 0$ and $y = -1$, or $x = -2$ and $y = -1 \Rightarrow (0,-1)$ is an interior point of AD with $f(0,-1) = -4$; $(-2,-1)$ is outside the boundary. Endpoints: $f(-1,-1) = -2$ and $f(1,-1) = 0$.

(v) For the interior of the square, $f_x(x,y) = 3x^2 + 6x = 0$ and $f_y(x,y) = 3y^2 - 6y = 0 \Rightarrow x = 0$ or $x = -2$, and $y = 0$ or $y = 2 \Rightarrow (0,0)$ is an interior critical point of the square region with $f(0,0) = 0$; the points $(0,2)$, $(-2,0)$, and $(-2,2)$ are outside the region. Therefore the absolute maximum is 4 at $(1,0)$ and the absolute minimum is -4 at $(0,-1)$.

78. (i) On AB, $f(x, y) = f(-1, y) = y^3 - 3y$ for $-1 \le y \le 1$ $\Rightarrow f'(-1, y) = 3y^2 - 3 = 0 \Rightarrow y = \pm 1$ and $x = -1$ yielding the corner points $(-1, -1)$ and $(-1, 1)$ with $f(-1, -1) = 2$ and $f(-1, 1) = -2$.



(ii) On BC, $f(x, y) = f(x, 1) = x^3 + 3x + 2$ for $-1 \le x \le 1 \Rightarrow f'(x, 1) = 3x^2 + 3 = 0 \Rightarrow$ no solution. Endpoints: $f(-1, 1) = -2$ and $f(1, 1) = 6$.

(iii) On CD, $f(x, y) = f(1, y) = y^3 + 3y + 2$ for $-1 \le y \le 1 \Rightarrow f'(1, y) = 3y^2 + 3 = 0 \Rightarrow$ no solution. Endpoints: $f(1, 1) = 6$ and $f(1, -1) = -2$.

(iv) On AD, $f(x, y) = f(x, -1) = x^3 - 3x$ for $-1 \le x \le 1 \Rightarrow f'(x, -1) = 3x^2 - 3 = 0 \Rightarrow x = \pm 1$ and $y = -1$ yielding the corner points $(-1, -1)$ and $(1, -1)$ with $f(-1, -1) = 2$ and $f(1, -1) = -2$

(v) For the interior of the square, $f_x(x, y) = 3x^2 + 3y = 0$ and $f_y(x, y) = 3y^2 + 3x = 0 \Rightarrow y = -x^2$ and $x^4 + x = 0 \Rightarrow x = 0$ or $x = -1 \Rightarrow y = 0$ or $y = -1 \Rightarrow (0, 0)$ is an interior critical point of the square region with $f(0, 0) = 1$; $(-1, -1)$ is on the boundary. Therefore the absolute maximum is 6 at $(1, 1)$ and the absolute minimum is -2 at $(1, -1)$ and $(-1, 1)$.

79. $\nabla f = 3x^2\mathbf{i} + 2y\mathbf{j}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow 3x^2\mathbf{i} + 2y\mathbf{j} = \lambda(2x\mathbf{i} + 2y\mathbf{j}) \Rightarrow 3x^2 = 2x\lambda$ and $2y = 2y\lambda \Rightarrow \lambda = 1$ or $y = 0$.

CASE 1: $\lambda = 1 \Rightarrow 3x^2 = 2x \Rightarrow x = 0$ or $x = \frac{2}{3}$; $x = 0 \Rightarrow y = \pm 1$ yielding the points $(0, 1)$ and $(0, -1)$; $x = \frac{2}{3}$ $\Rightarrow y = \pm \frac{\sqrt{5}}{3}$ yielding the points $\left(\frac{2}{3}, \frac{\sqrt{5}}{3}\right)$ and $\left(\frac{2}{3}, -\frac{\sqrt{5}}{3}\right)$.

CASE 2: $y = 0 \Rightarrow x^2 - 1 = 0 \Rightarrow x = \pm 1$ yielding the points $(1, 0)$ and $(-1, 0)$.

Evaluations give $f(0, \pm 1) = 1$, $f\left(\frac{2}{3}, \pm \frac{\sqrt{5}}{3}\right) = \frac{23}{27}$, $f(1, 0) = 1$, and $f(-1, 0) = -1$. Therefore the absolute maximum is 1 at $(0, \pm 1)$ and $(1, 0)$, and the absolute minimum is -1 at $(-1, 0)$.

80. $\nabla f = y\mathbf{i} + x\mathbf{j}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow y\mathbf{i} + x\mathbf{j} = \lambda(2x\mathbf{i} + 2y\mathbf{j}) \Rightarrow y = 2\lambda x$ and $xy = 2\lambda y \Rightarrow x = 2\lambda(2\lambda x) = 4\lambda^2 x \Rightarrow x = 0$ or $4\lambda^2 = 1$.

CASE 1: $x = 0 \Rightarrow y = 0$ but $(0, 0)$ does not lie on the circle, so no solution.

CASE 2: $4\lambda^2 = 1 \Rightarrow \lambda = \frac{1}{2}$ or $\lambda = -\frac{1}{2}$. For $\lambda = \frac{1}{2}$, $y = x \Rightarrow 1 = x^2 + y^2 = 2x^2 \Rightarrow x = y = \pm \frac{1}{\sqrt{2}}$ yielding the points $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$. For $\lambda = -\frac{1}{2}$, $y = -x \Rightarrow 1 = x^2 + y^2 = 2x^2 \Rightarrow x = \pm \frac{1}{\sqrt{2}}$ and $y = -x$ yielding the points $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.

Evaluations give the absolute maximum value $f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = \frac{1}{2}$ and the absolute minimum value $f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}$.

81. (i) $f(x, y) = x^2 + 3y^2 + 2y$ on $x^2 + y^2 = 1 \Rightarrow \nabla f = 2x\mathbf{i} + (6y + 2)\mathbf{j}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j}$ so that $\nabla f = \lambda \nabla g$ $\Rightarrow 2x\mathbf{i} + (6y + 2)\mathbf{j} = \lambda(2x\mathbf{i} + 2y\mathbf{j}) \Rightarrow 2x = 2x\lambda$ and $6y + 2 = 2y\lambda \Rightarrow \lambda = 1$ or $x = 0$.

CASE 1: $\lambda = 1 \Rightarrow 6y + 2 = 2y \Rightarrow y = -\frac{1}{2}$ and $x = \pm \frac{\sqrt{3}}{2}$ yielding the points $\left(\pm \frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$.

CASE 2: $x = 0 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$ yielding the points $(0, \pm 1)$.

Evaluations give $f\left(\pm \frac{\sqrt{3}}{2}, -\frac{1}{2}\right) = \frac{1}{2}$, $f(0, 1) = 5$, and $f(0, -1) = 1$. Therefore $\frac{1}{2}$ and 5 are the extreme values on the boundary of the disk.

(ii) For the interior of the disk, $f_x(x, y) = 2x = 0$ and $f_y(x, y) = 6y + 2 = 0 \Rightarrow x = 0$ and $y = -\frac{1}{3}$ $\Rightarrow \left(0, -\frac{1}{3}\right)$ is an interior critical point with $f\left(0, -\frac{1}{3}\right) = -\frac{1}{3}$. Therefore the absolute maximum of f on the disk is 5 at $(0, 1)$ and the absolute minimum of f on the disk is $-\frac{1}{3}$ at $\left(0, -\frac{1}{3}\right)$.

82. (i) $f(x, y) = x^2 + y^2 - 3x - xy$ on $x^2 + y^2 = 9 \Rightarrow \nabla f = (2x - 3 - y)\mathbf{i} + (2y - x)\mathbf{j}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j}$ so that $\nabla f = \lambda \nabla g \Rightarrow (2x - 3 - y)\mathbf{i} + (2y - x)\mathbf{j} = \lambda(2x\mathbf{i} + 2y\mathbf{j}) \Rightarrow 2x - 3 - y = 2x\lambda$ and $2y - x = 2y\lambda$ $\Rightarrow 2x(1 - \lambda) - y = 3$ and $-x + 2y(1 - \lambda) = 0 \Rightarrow 1 - \lambda = \frac{x}{2y}$ and $(2x)\left(\frac{x}{2y}\right) - y = 3 \Rightarrow x^2 - y^2 = 3y$ $\Rightarrow x^2 = y^2 + 3y$. Thus, $9 = x^2 + y^2 = y^2 + 3y + y^2 \Rightarrow 2y^2 + 3y - 9 = 0 \Rightarrow (2y - 3)(y + 3) = 0$ $\Rightarrow y = -3, \frac{3}{2}$. For $y = -3$, $x^2 + y^2 = 9 \Rightarrow x = 0$ yielding the point $(0, -3)$. For $y = \frac{3}{2}$, $x^2 + y^2 = 9$ $\Rightarrow x^2 + \frac{9}{4} = 9 \Rightarrow x^2 = \frac{27}{4} \Rightarrow x = \pm\frac{3\sqrt{3}}{2}$. Evaluations give $f(0, -3) = 9$, $f\left(-\frac{3\sqrt{3}}{2}, \frac{3}{2}\right) = 9 + \frac{27\sqrt{3}}{4}$ ≈ 20.691, and $f\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right) = 9 - \frac{27\sqrt{3}}{4} \approx -2.691$.

(ii) For the interior of the disk, $f_x(x, y) = 2x - 3 - y = 0$ and $f_y(x, y) = 2y - x = 0 \Rightarrow x = 2$ and $y = 1$ $\Rightarrow (2, 1)$ is an interior critical point of the disk with $f(2, 1) = -3$. Therefore, the absolute maximum of f on the disk is $9 + \frac{27\sqrt{3}}{4}$ at $\left(-\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$ and the absolute minimum of f on the disk is -3 at $(2, 1)$.

83. $\nabla f = \mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow \mathbf{i} - \mathbf{j} + \mathbf{k} = \lambda(2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}) \Rightarrow 1 = 2x\lambda$, $-1 = 2y\lambda$, $1 = 2z\lambda \Rightarrow x = -y = z = \frac{1}{\lambda}$. Thus $x^2 + y^2 + z^2 = 1 \Rightarrow 3x^2 = 1 \Rightarrow x = \pm\frac{1}{\sqrt{3}}$ yielding the points $\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$. Evaluations give the absolute maximum value of $f\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \frac{3}{\sqrt{3}} = \sqrt{3}$ and the absolute minimum value of $f\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) = -\sqrt{3}$.

84. Let $f(x, y, z) = x^2 + y^2 + z^2$ be the square of the distance to the origin and $g(x, y, z) = x^2 - zy - 4$. Then $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ and $\nabla g = 2x\mathbf{i} - z\mathbf{j} - y\mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow 2x = 2\lambda x$, $2y = -\lambda z$, and $2z = -\lambda y \Rightarrow x = 0$ or $\lambda = 1$.

CASE 1: $x = 0 \Rightarrow zy = -4 \Rightarrow z = -\frac{4}{y}$ and $y = -\frac{4}{z} \Rightarrow 2\left(-\frac{4}{y}\right) = -\lambda y$ and $2\left(-\frac{4}{z}\right) = -\lambda z \Rightarrow \frac{8}{\lambda} = y^2$ and $\frac{8}{\lambda} = z^2 \Rightarrow y^2 = z^2 \Rightarrow y = \pm z$. But $y = x \Rightarrow z^2 = -4$ leads to no solution, so $y = -z \Rightarrow z^2 = 4$ $\Rightarrow z = \pm 2$ yielding the points $(0, -2, 2)$ and $(0, 2, -2)$.

CASE 2: $\lambda = 1 \Rightarrow 2z = -y$ and $2y = -z \Rightarrow 2y = -\left(-\frac{y}{2}\right) \Rightarrow 4y = y \Rightarrow y = 0 \Rightarrow z = 0 \Rightarrow x^2 - 4 = 0 \Rightarrow$ $x = \pm 2$ yielding the points $(-2, 0, 0)$ and $(2, 0, 0)$.

Evaluations give $f(0, -2, 2) = f(0, 2, -2) = 8$ and $f(-2, 0, 0) = f(2, 0, 0) = 4$. Thus the points $(-2, 0, 0)$ and $(2, 0, 0)$ on the surface are closest to the origin.

85. The cost is $f(x, y, z) = 2axy + 2bxz + 2cyz$ subject to the constraint $xyz = V$. Then $\nabla f = \lambda \nabla g$ $\Rightarrow 2ay + 2bz = \lambda yz$, $2ax + 2cz = \lambda xz$, and $2bx + 2cy = \lambda xy \Rightarrow 2axy + 2bxz = \lambda xyz$, $2axy + 2cyz = \lambda xyz$, and $2bxz + 2cyz = \lambda xyz \Rightarrow 2axy + 2bxz = 2axy + 2cyz \Rightarrow y = \left(\frac{b}{c}\right)x$. Also $2axy + 2bxz = 2bxz + 2cyz \Rightarrow z = \left(\frac{a}{c}\right)x$. Then $x\left(\frac{b}{c}x\right)\left(\frac{a}{c}x\right) = V \Rightarrow x^3 = \frac{c^2V}{ab} \Rightarrow$ width $= x = \left(\frac{c^2V}{ab}\right)^{1/3}$, Depth $= y = \left(\frac{b}{c}\right)\left(\frac{c^2V}{ab}\right)^{1/3} = \left(\frac{b^2V}{ac}\right)^{1/3}$, and Height $= z = \left(\frac{a}{c}\right)\left(\frac{c^2V}{ab}\right)^{1/3} = \left(\frac{a^2V}{bc}\right)^{1/3}$.

86. The volume of the pyramid in the first octant formed by the plane is $V(a, b, c) = \frac{1}{3}\left(\frac{1}{2}ab\right)c = \frac{1}{6}abc$. The point $(2, 1, 2)$ on the plane $\Rightarrow \frac{2}{a} + \frac{1}{b} + \frac{2}{c} = 1$. We want to minimize V subject to the constraint $2bc + ac + 2ab = abc$. Thus, $\nabla V = \frac{bc}{6}\mathbf{i} + \frac{ac}{6}\mathbf{j} + \frac{ab}{6}\mathbf{k}$ and $\nabla g = (c + 2b - bc)\mathbf{i} + (2c + 2a - ac)\mathbf{j} + (2b + a - ab)\mathbf{k}$ so that $\nabla V = \lambda \nabla g$ $\Rightarrow \frac{bc}{6} = \lambda(c + 2b - bc)$, $\frac{ac}{6} = \lambda(2c + 2a - ac)$, and $\frac{ab}{6} = \lambda(2b + a - ab) \Rightarrow \frac{abc}{6} = \lambda(ac + 2ab - abc)$, $\frac{abc}{6} = \lambda(2bc + 2ab - abc)$, and $\frac{abc}{6} = \lambda(2bc + ac - abc) \Rightarrow \lambda ac = 2\lambda bc$ and $2\lambda ab = 2\lambda bc$. Now $\lambda \neq 0$ since $a \neq 0$, $b \neq 0$, and $c \neq 0 \Rightarrow ac = 2bc$ and $ab = bc \Rightarrow a = 2b = c$. Substituting into the constraint equation gives $\frac{2}{a} + \frac{2}{a} + \frac{2}{a} = 1 \Rightarrow a = 6 \Rightarrow b = 3$ and $c = 6$. Therefore the desired plane is $\frac{x}{6} + \frac{y}{3} + \frac{z}{6} = 1$ or $x + 2y + z = 6$.

87. $\nabla f = (y+z)\mathbf{i} + x\mathbf{j} + x\mathbf{k}$, $\nabla g = 2x\mathbf{i} + 2y\mathbf{j}$, and $\nabla h = z\mathbf{i} + x\mathbf{k}$ so that $\nabla f = \lambda \nabla g + \mu \nabla h$ $\Rightarrow (y+z)\mathbf{i} + x\mathbf{j} + x\mathbf{k} = \lambda(2x\mathbf{i} + 2y\mathbf{j}) + \mu(z\mathbf{i} + x\mathbf{k}) \Rightarrow y+z = 2\lambda x + \mu z$, $x = 2\lambda y$, $x = \mu x \Rightarrow x = 0$ or $\mu = 1$.

CASE 1: $x = 0$ which is impossible since $xz = 1$.

CASE 2: $\mu = 1 \Rightarrow y + z = 2\lambda x + z \Rightarrow y = 2\lambda x$ and $x = 2\lambda y \Rightarrow y = (2\lambda)(2\lambda y) \Rightarrow y = 0$ or $4\lambda^2 = 1$. If $y = 0$, then $x^2 = 1 \Rightarrow x = \pm 1$ so with $xz = 1$ we obtain the points $(1,0,1)$ and $(-1,0,-1)$. If $4\lambda^2 = 1$, then $\lambda = \pm \frac{1}{2}$. For $\lambda = -\frac{1}{2}$, $y = -x$ so $x^2 + y^2 = 1 \Rightarrow x^2 = \frac{1}{2}$ $\Rightarrow x = \pm \frac{1}{\sqrt{2}}$ with $xz = 1 \Rightarrow z = \pm\sqrt{2}$, and we obtain the points $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \sqrt{2}\right)$ and $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\sqrt{2}\right)$. For $\lambda = \frac{1}{2}$, $y = x \Rightarrow x^2 = \frac{1}{2} \Rightarrow x = \pm\frac{1}{\sqrt{2}}$ with $xz = 1 \Rightarrow z = \pm\sqrt{2}$, and we obtain the points $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \sqrt{2}\right)$ and $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\sqrt{2}\right)$.

Evaluations give $f(1,0,1) = 1$, $f(-1,0,-1) = 1$, $f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \sqrt{2}\right) = \frac{1}{2}$, $f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\sqrt{2}\right) = \frac{1}{2}$, $f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \sqrt{2}\right) = \frac{3}{2}$, and $f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\sqrt{2}\right) = \frac{3}{2}$. Therefore the absolute maximum is $\frac{3}{2}$ at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \sqrt{2}\right)$ and $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\sqrt{2}\right)$, and the absolute minimum is $\frac{1}{2}$ at $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\sqrt{2}\right)$ and $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \sqrt{2}\right)$.

88. Let $f(x,y,z) = x^2 + y^2 + z^2$ be the square of the distance to the origin. Then $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$, $\nabla g = \mathbf{i} + \mathbf{j} + \mathbf{k}$, and $\nabla h = 4x\mathbf{i} + 4y\mathbf{j} - 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g + \mu \nabla h \Rightarrow 2x = \lambda + 4x\mu$, $2y = \lambda + 4y\mu$, and $2z = \lambda - 2z\mu \Rightarrow \lambda = 2x(1-2\mu) = 2y(1-2\mu) = 2z(1+2\mu) \Rightarrow x = y$ or $\mu = \frac{1}{2}$.

CASE 1: $x = y \Rightarrow z^2 = 4x^2 \Rightarrow z = \pm 2x$ so that $x + y + z = 1 \Rightarrow x + x + 2x = 1$ or $x + x - 2x = 1$ (impossible) $\Rightarrow x = \frac{1}{4} \Rightarrow y = \frac{1}{4}$ and $z = \frac{1}{2}$ yielding the point $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)$.

CASE 2: $\mu = \frac{1}{2} \Rightarrow \lambda = 0 \Rightarrow 0 = 2z(1+1) \Rightarrow z = 0$ so that $2x^2 + 2y^2 = 0 \Rightarrow x = y = 0$. But the origin $(0,0,0)$ fails to satisfy the first constraint $x + y + z = 1$.

Therefore, the point $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)$ on the curve of intersection is closest to the origin.

89. (a) y, z are independent with $w = x^2 e^{yz}$ and $z = x^2 - y^2 \Rightarrow \frac{\partial w}{\partial y} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial y}$ $= (2xe^{yz})\frac{\partial x}{\partial y} + (zx^2e^{yz})(1) + (yx^2e^{yz})(0)$; $z = x^2 - y^2 \Rightarrow 0 = 2x\frac{\partial x}{\partial y} - 2y \Rightarrow \frac{\partial x}{\partial y} = \frac{y}{x}$; therefore, $\left(\frac{\partial w}{\partial y}\right)_z = (2xe^{yz})\left(\frac{y}{x}\right) + zx^2e^{yz} = (2y + zx^2)e^{yz}$

(b) z, x are independent with $w = x^2 e^{yz}$ and $z = x^2 - y^2 \Rightarrow \frac{\partial w}{\partial z} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial z} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial z}$ $= (2xe^{yz})(0) + (zx^2e^{yz})\frac{\partial y}{\partial z} + (yx^2e^{yz})(1)$; $z = x^2 - y^2 \Rightarrow 1 = 0 - 2y\frac{\partial y}{\partial z} \Rightarrow \frac{\partial y}{\partial z} = -\frac{1}{2y}$; therefore, $\left(\frac{\partial w}{\partial z}\right)_x = (zx^2e^{yz})\left(-\frac{1}{2y}\right) + yx^2e^{yz} = x^2e^{yz}\left(y - \frac{z}{2y}\right)$

(c) z, y are independent with $w = x^2 e^{yz}$ and $z = x^2 - y^2 \Rightarrow \frac{\partial w}{\partial z} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial z} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial z}$ $= (2xe^{yz})\frac{\partial x}{\partial z} + (zx^2e^{yz})(0) + (yx^2e^{yz})(1)$; $z = x^2 - y^2 \Rightarrow 1 = 2x\frac{\partial x}{\partial z} - 0 \Rightarrow \frac{\partial x}{\partial z} = \frac{1}{2x}$; therefore, $\left(\frac{\partial w}{\partial z}\right)_y = (2xe^{yz})\left(\frac{1}{2x}\right) + yx^2e^{yz} = (1 + x^2y)e^{yz}$

90. (a) T, P are independent with $U = f(P,V,T)$ and $PV = nRT \Rightarrow \frac{\partial U}{\partial T} = \frac{\partial U}{\partial P}\frac{\partial P}{\partial T} + \frac{\partial U}{\partial V}\frac{\partial V}{\partial T} + \frac{\partial U}{\partial T}\frac{\partial T}{\partial T}$ $= \left(\frac{\partial U}{\partial P}\right)(0) + \left(\frac{\partial U}{\partial V}\right)\left(\frac{\partial V}{\partial T}\right) + \left(\frac{\partial U}{\partial T}\right)(1)$; $PV = nRT \Rightarrow P\frac{\partial V}{\partial T} = nR \Rightarrow \frac{\partial V}{\partial T} = \frac{nR}{P}$; therefore, $\left(\frac{\partial U}{\partial T}\right)_P = \left(\frac{\partial U}{\partial V}\right)\left(\frac{nR}{P}\right) + \frac{\partial U}{\partial T}$

(b) V, T are independent with $U = f(P,V,T)$ and $PV = nRT \Rightarrow \frac{\partial U}{\partial V} = \frac{\partial U}{\partial P}\frac{\partial P}{\partial V} + \frac{\partial U}{\partial V}\frac{\partial V}{\partial V} + \frac{\partial U}{\partial T}\frac{\partial T}{\partial V}$ $= \left(\frac{\partial U}{\partial P}\right)\left(\frac{\partial P}{\partial V}\right) + \left(\frac{\partial U}{\partial V}\right)(1) + \left(\frac{\partial U}{\partial T}\right)(0)$; $PV = nRT \Rightarrow V\frac{\partial P}{\partial V} + P = (nR)\left(\frac{\partial T}{\partial V}\right) = 0 \Rightarrow \frac{\partial P}{\partial V} = -\frac{P}{V}$; therefore, $\left(\frac{\partial U}{\partial V}\right)_T = \left(\frac{\partial U}{\partial P}\right)\left(-\frac{P}{V}\right) + \frac{\partial U}{\partial V}$

91. Note that $x = r\cos\theta$ and $y = r\sin\theta \Rightarrow r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$. Thus,
$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial w}{\partial \theta}\frac{\partial \theta}{\partial x} = \left(\frac{\partial w}{\partial r}\right)\left(\frac{x}{\sqrt{x^2+y^2}}\right) + \left(\frac{\partial w}{\partial \theta}\right)\left(\frac{-y}{x^2+y^2}\right) = (\cos\theta)\frac{\partial w}{\partial r} - \left(\frac{\sin\theta}{r}\right)\frac{\partial w}{\partial \theta}$;
$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial w}{\partial \theta}\frac{\partial \theta}{\theta y} = \left(\frac{\partial w}{\partial r}\right)\left(\frac{y}{\sqrt{x^2+y^2}}\right) + \left(\frac{\partial w}{\partial \theta}\right)\left(\frac{x}{x^2+y^2}\right) = (\sin\theta)\frac{\partial w}{\partial r} + \left(\frac{\cos\theta}{r}\right)\frac{\partial w}{\partial \theta}$

92. $z_x = f_u\frac{\partial u}{\partial x} + f_v\frac{\partial v}{\partial x} = af_u + af_v$, and $z_y = f_u\frac{\partial u}{\partial y} + f_v\frac{\partial v}{\partial y} = bf_u - bf_v$

93. $\frac{\partial u}{\partial y} = b$ and $\frac{\partial u}{\partial x} = a \Rightarrow \frac{\partial w}{\partial x} = \frac{dw}{du}\frac{\partial u}{\partial x} = a\frac{dw}{du}$ and $\frac{\partial w}{\partial y} = \frac{dw}{du}\frac{\partial u}{\partial y} = b\frac{dw}{du} \Rightarrow \frac{1}{a}\frac{\partial w}{\partial x} = \frac{dw}{du}$ and $\frac{1}{b}\frac{\partial w}{\partial y} = \frac{dw}{du}$
$\Rightarrow \frac{1}{a}\frac{\partial w}{\partial x} = \frac{1}{b}\frac{\partial w}{\partial y} \Rightarrow b\frac{\partial w}{\partial x} = a\frac{\partial w}{\partial y}$

94. $\frac{\partial w}{\partial x} = \frac{2x}{x^2+y^2+2z} = \frac{2(r+s)}{(r+s)^2+(r-s)^2+4rs} = \frac{2(r+s)}{2(r^2+2rs+s^2)} = \frac{1}{r+s}$, $\frac{\partial w}{\partial y} = \frac{2y}{x^2+y^2+2z} = \frac{2(r-s)}{2(r+s)^2} = \frac{r-s}{(r+s)^2}$,
and $\frac{\partial w}{\partial z} = \frac{2}{x^2+y^2+2z} = \frac{1}{(r+s)^2} \Rightarrow \frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} = \frac{1}{r+s} + \frac{r-s}{(r+s)^2} + \left[\frac{1}{(r+s)^2}\right](2s) = \frac{2r+2s}{(r+s)^2}$
$= \frac{2}{r+s}$ and $\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} = \frac{1}{r+s} - \frac{r-s}{(r+s)^2} + \left[\frac{1}{(r+s)^2}\right](2r) = \frac{2}{r+s}$

95. $e^u\cos v - x = 0 \Rightarrow (e^u\cos v)\frac{\partial u}{\partial x} - (e^u\sin v)\frac{\partial v}{\partial x} = 1$; $e^u\sin v - y = 0 \Rightarrow (e^u\sin v)\frac{\partial u}{\partial x} + (e^u\cos v)\frac{\partial v}{\partial x} = 0$.
Solving this system yields $\frac{\partial u}{\partial x} = e^{-u}\cos v$ and $\frac{\partial v}{\partial x} = -e^{-u}\sin v$. Similarly, $e^u\cos v - x = 0$
$\Rightarrow (e^u\cos v)\frac{\partial u}{\partial y} - (e^u\sin v)\frac{\partial v}{\partial y} = 0$ and $e^u\sin v - y = 0 \Rightarrow (e^u\sin v)\frac{\partial u}{\partial y} + (e^u\cos v)\frac{\partial v}{\partial y} = 1$. Solving this second system yields $\frac{\partial u}{\partial y} = e^{-u}\sin v$ and $\frac{\partial v}{\partial y} = e^{-u}\cos v$. Therefore $\left(\frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j}\right)\cdot\left(\frac{\partial v}{\partial x}\mathbf{i} + \frac{\partial v}{\partial y}\mathbf{j}\right)$
$= [(e^{-u}\cos v)\mathbf{i} + (e^{-u}\sin v)\mathbf{j}]\cdot[(-e^{-u}\sin v)\mathbf{i} + (e^{-u}\cos v)\mathbf{j}] = 0 \Rightarrow$ the vectors are orthogonal $\Rightarrow$ the angle between the vectors is the constant $\frac{\pi}{2}$.

96. $\frac{\partial g}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = (-r\sin\theta)\frac{\partial f}{\partial x} + (r\cos\theta)\frac{\partial f}{\partial y}$
$\Rightarrow \frac{\partial^2 g}{\partial \theta^2} = (-r\sin\theta)\left(\frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial \theta} + \frac{\partial^2 f}{\partial y\partial x}\frac{\partial y}{\partial \theta}\right) - (r\cos\theta)\frac{\partial f}{\partial x} + (r\cos\theta)\left(\frac{\partial^2 f}{\partial x\partial y}\frac{\partial x}{\partial \theta} + \frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial \theta}\right) - (r\sin\theta)\frac{\partial f}{\partial y}$
$= (-r\sin\theta)\left(\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial \theta}\right) - (r\cos\theta) + (r\cos\theta)\left(\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial \theta}\right) - (r\sin\theta)$
$= (-r\sin\theta + r\cos\theta)(-r\sin\theta + r\cos\theta) - (r\cos\theta + r\sin\theta) = (-2)(-2) - (0+2) = 4 - 2 = 2$ at $(r,\theta) = \left(2, \frac{\pi}{2}\right)$.

97. $(y+z)^2 + (z-x)^2 = 16 \Rightarrow \nabla f = -2(z-x)\mathbf{i} + 2(y+z)\mathbf{j} + 2(y+2z-x)\mathbf{k}$; if the normal line is parallel to the yz-plane, then x is constant $\Rightarrow \frac{\partial f}{\partial x} = 0 \Rightarrow -2(z-x) = 0 \Rightarrow z = x \Rightarrow (y+z)^2 + (z-z)^2 = 16 \Rightarrow y + z = \pm 4$.
Let $x = t \Rightarrow z = t \Rightarrow y = -t \pm 4$. Therefore the points are $(t, -t\pm 4, t)$, t a real number.

98. Let $f(x,y,z) = xy + yz + zx - x - z^2 = 0$. If the tangent plane is to be parallel to the xy-plane, then ∇f is perpendicular to the xy-plane $\Rightarrow \nabla f\cdot\mathbf{i} = 0$ and $\nabla f\cdot\mathbf{j} = 0$. Now $\nabla f = (y+z-1)\mathbf{i} + (x+z)\mathbf{j} + (y+x-2z)\mathbf{k}$ so that $\nabla f\cdot\mathbf{i} = y + z - 1 = 0 \Rightarrow y + z = 1 \Rightarrow y = 1 - z$, and $\nabla f\cdot\mathbf{j} = x + z = 0 \Rightarrow x = -z$. Then
$-z(1-z) + (1-z)z + z(-z) - (-z) - z^2 = 0 \Rightarrow z - 2z^2 = 0 \Rightarrow z = \frac{1}{2}$ or $z = 0$. Now $z = \frac{1}{2} \Rightarrow x = -\frac{1}{2}$ and $y = \frac{1}{2}$
$\Rightarrow \left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ is one desired point; $z = 0 \Rightarrow x = 0$ and $y = 1 \Rightarrow (0,1,0)$ is a second desired point.

99. $\nabla f = \lambda(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \Rightarrow \frac{\partial f}{\partial x} = \lambda x \Rightarrow f(x,y,z) = \frac{1}{2}\lambda x^2 + g(y,z)$ for some function g $\Rightarrow \lambda y = \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y}$
$\Rightarrow g(y,z) = \frac{1}{2}\lambda y^2 + h(z)$ for some function h $\Rightarrow \lambda z = \frac{\partial f}{\partial z} = \frac{\partial g}{\partial z} = h'(z) \Rightarrow h(z) = \frac{1}{2}\lambda z^2 + C$ for some arbitrary constant C $\Rightarrow g(y,z) = \frac{1}{2}\lambda y^2 + \left(\frac{1}{2}\lambda z^2 + C\right) \Rightarrow f(x,y,z) = \frac{1}{2}\lambda x^2 + \frac{1}{2}\lambda y^2 + \frac{1}{2}\lambda z^2 + C \Rightarrow f(0,0,a) = \frac{1}{2}\lambda a^2 + C$ and $f(0,0,-a) = \frac{1}{2}\lambda(-a)^2 + C \Rightarrow f(0,0,a) = f(0,0,-a)$ for any constant a, as claimed.

100. $\left(\frac{df}{ds}\right)_{\mathbf{u},(0,0,0)} = \lim_{s \to 0} \frac{f(0+su_1, 0+su_2, 0+su_3) - f(0,0,0)}{s}$, $s > 0$

$= \lim_{s \to 0} \frac{\sqrt{s^2u_1^2 + s^2u_2^2 + s^2u_3^2} - 0}{s}$, $s > 0$

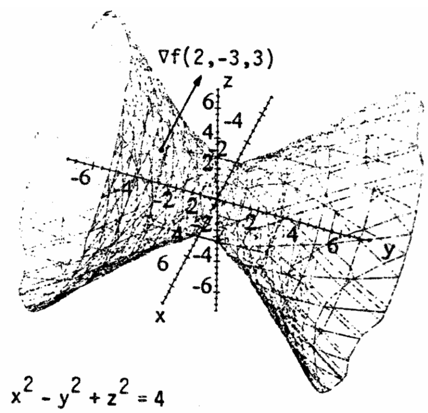
$= \lim_{s \to 0} \frac{s\sqrt{u_1^2 + u_2^2 + u_3^2}}{s} = \lim_{s \to 0} |\mathbf{u}| = 1$;

however, $\nabla f = \frac{x}{\sqrt{x^2+y^2+z^2}}\mathbf{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\mathbf{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\mathbf{k}$ fails to exist at the origin $(0,0,0)$

101. Let $f(x,y,z) = xy + z - 2 \Rightarrow \nabla f = y\mathbf{i} + x\mathbf{j} + \mathbf{k}$. At $(1,1,1)$, we have $\nabla f = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow$ the normal line is $x = 1 + t$, $y = 1 + t$, $z = 1 + t$, so at $t = -1 \Rightarrow x = 0$, $y = 0$, $z = 0$ and the normal line passes through the origin.

102. (b) $f(x,y,z) = x^2 - y^2 + z^2 = 4$
$\Rightarrow \nabla f = 2x\mathbf{i} - 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow$ at $(2,-3,3)$
the gradient is $\nabla f = 4\mathbf{i} + 6\mathbf{j} + 6\mathbf{k}$ which is normal to the surface

(c) Tangent plane: $4x + 6y + 6z = 8$ or
$2x + 3y + 3z = 4$
Normal line: $x = 2 + 4t$, $y = -3 + 6t$, $z = 3 + 6t$



CHAPTER 14 ADDITIONAL AND ADVANCED EXERCISES

1. By definition, $f_{xy}(0,0) = \lim_{h \to 0} \frac{f_x(0,h) - f_x(0,0)}{h}$ so we need to calculate the first partial derivatives in the numerator. For $(x,y) \neq (0,0)$ we calculate $f_x(x,y)$ by applying the differentiation rules to the formula for $f(x,y)$: $f_x(x,y) = \frac{x^2y - y^3}{x^2+y^2} + (xy)\frac{(x^2+y^2)(2x) - (x^2-y^2)(2x)}{(x^2+y^2)^2} = \frac{x^2y - y^3}{x^2+y^2} + \frac{4x^2y^3}{(x^2+y^2)^2} \Rightarrow f_x(0,h) = -\frac{h^3}{h^2} = -h$. For $(x,y) = (0,0)$ we apply the definition: $f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{0-0}{h} = 0$. Then by definition $f_{xy}(0,0) = \lim_{h \to 0} \frac{-h-0}{h} = -1$. Similarly, $f_{yx}(0,0) = \lim_{h \to 0} \frac{f_y(h,0) - f_y(0,0)}{h}$, so for $(x,y) \neq (0,0)$ we have $f_y(x,y) = \frac{x^3 - xy^2}{x^2+y^2} - \frac{4x^3y^2}{(x^2+y^2)^2} \Rightarrow f_y(h,0) = \frac{h^3}{h^2} = h$; for $(x,y) = (0,0)$ we obtain $f_y(0,0) = \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h}$ $= \lim_{h \to 0} \frac{0-0}{h} = 0$. Then by definition $f_{yx}(0,0) = \lim_{h \to 0} \frac{h-0}{h} = 1$. Note that $f_{xy}(0,0) \neq f_{yx}(0,0)$ in this case.

2. $\frac{\partial w}{\partial x} = 1 + e^x \cos y \Rightarrow w = x + e^x \cos y + g(y)$; $\frac{\partial w}{\partial y} = -e^x \sin y + g'(y) = 2y - e^x \sin y \Rightarrow g'(y) = 2y$ $\Rightarrow g(y) = y^2 + C$; $w = \ln 2$ when $x = \ln 2$ and $y = 0 \Rightarrow \ln 2 = \ln 2 + e^{\ln 2}\cos 0 + 0^2 + C \Rightarrow 0 = 2 + C$ $\Rightarrow C = -2$. Thus, $w = x + e^x \cos y + g(y) = x + e^x \cos y + y^2 - 2$.

3. Substitution of $u + u(x)$ and $v = v(x)$ in $g(u,v)$ gives $g(u(x), v(x))$ which is a function of the independent variable x. Then, $g(u,v) = \int_u^v f(t)\,dt \Rightarrow \frac{dg}{dx} = \frac{\partial g}{\partial u}\frac{du}{dx} + \frac{\partial g}{\partial v}\frac{dv}{dx} = \left(\frac{\partial}{\partial u}\int_u^v f(t)\,dt\right)\frac{du}{dx} + \left(\frac{\partial}{\partial v}\int_u^v f(t)\,dt\right)\frac{dv}{dx}$ $= \left(-\frac{\partial}{\partial u}\int_v^u f(t)\,dt\right)\frac{du}{dx} + \left(\frac{\partial}{\partial v}\int_u^v f(t)\,dt\right)\frac{dv}{dx} = -f(u(x))\frac{du}{dx} + f(v(x))\frac{dv}{dx} = f(v(x))\frac{dv}{dx} - f(u(x))\frac{du}{dx}$

4. Applying the chain rules, $f_x = \frac{df}{dr}\frac{\partial r}{\partial x} \Rightarrow f_{xx} = \left(\frac{d^2f}{dr^2}\right)\left(\frac{\partial r}{\partial x}\right)^2 + \frac{df}{dr}\frac{\partial^2 r}{\partial x^2}$. Similarly, $f_{yy} = \left(\frac{d^2f}{dr^2}\right)\left(\frac{\partial r}{\partial y}\right)^2 + \frac{df}{dr}\frac{\partial^2 r}{\partial y^2}$ and $f_{zz} = \left(\frac{d^2f}{dr^2}\right)\left(\frac{\partial r}{\partial z}\right)^2 + \frac{df}{dr}\frac{\partial^2 r}{\partial z^2}$. Moreover, $\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}} \Rightarrow \frac{\partial^2 r}{\partial x^2} = \frac{y^2+z^2}{\left(\sqrt{x^2+y^2+z^2}\right)^3}$; $\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2+z^2}}$ $\Rightarrow \frac{\partial^2 r}{\partial y^2} = \frac{x^2+z^2}{\left(\sqrt{x^2+y^2+z^2}\right)^3}$; and $\frac{\partial r}{\partial z} = \frac{z}{\sqrt{x^2+y^2+z^2}} \Rightarrow \frac{\partial^2 r}{\partial z^2} = \frac{x^2+y^2}{\left(\sqrt{x^2+y^2+z^2}\right)^3}$. Next, $f_{xx} + f_{yy} + f_{zz} = 0$

$\Rightarrow \left(\frac{d^2f}{dr^2}\right)\left(\frac{x^2}{x^2+y^2+z^2}\right) + \left(\frac{df}{dr}\right)\left(\frac{y^2+z^2}{\left(\sqrt{x^2+y^2+z^2}\right)^3}\right) + \left(\frac{d^2f}{dr^2}\right)\left(\frac{y^2}{x^2+y^2+z^2}\right) + \left(\frac{df}{dr}\right)\left(\frac{x^2+z^2}{\left(\sqrt{x^2+y^2+z^2}\right)^3}\right)$

$+ \left(\frac{d^2f}{dr^2}\right)\left(\frac{z^2}{x^2+y^2+z^2}\right) + \left(\frac{df}{dr}\right)\left(\frac{x^2+y^2}{\left(\sqrt{x^2+y^2+z^2}\right)^3}\right) = 0 \Rightarrow \frac{d^2f}{dr^2} + \left(\frac{2}{\sqrt{x^2+y^2+z^2}}\right)\frac{df}{dr} = 0 \Rightarrow \frac{d^2f}{dr^2} + \frac{2}{r}\frac{df}{dr} = 0$

$\Rightarrow \frac{d}{dr}(f') = \left(-\frac{2}{r}\right)f'$, where $f' = \frac{df}{dr} \Rightarrow \frac{df'}{f'} = -\frac{2\,dr}{r} \Rightarrow \ln f' = -2\ln r + \ln C \Rightarrow f' = Cr^{-2}$, or $\frac{df}{dr} = Cr^{-2} \Rightarrow f(r) = -\frac{C}{r} + b = \frac{a}{r} + b$ for some constants a and b (setting $a = -C$)

5. (a) Let $u = tx$, $v = ty$, and $w = f(u,v) = f(u(t,x), v(t,y)) = f(tx,ty) = t^n f(x,y)$, where t, x, and y are independent variables. Then $nt^{n-1}f(x,y) = \frac{\partial w}{\partial t} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial t} = x\frac{\partial w}{\partial u} + y\frac{\partial w}{\partial v}$. Now,
$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x} = \left(\frac{\partial w}{\partial u}\right)(t) + \left(\frac{\partial w}{\partial v}\right)(0) = t\frac{\partial w}{\partial u} \Rightarrow \frac{\partial w}{\partial u} = \left(\frac{1}{t}\right)\left(\frac{\partial w}{\partial x}\right)$. Likewise,
$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y} = \left(\frac{\partial w}{\partial u}\right)(0) + \left(\frac{\partial w}{\partial v}\right)(t) \Rightarrow \frac{\partial w}{\partial v} = \left(\frac{1}{t}\right)\left(\frac{\partial w}{\partial y}\right)$. Therefore,
$nt^{n-1}f(x,y) = x\frac{\partial w}{\partial u} + y\frac{\partial w}{\partial v} = \left(\frac{x}{t}\right)\left(\frac{\partial w}{\partial x}\right) + \left(\frac{y}{t}\right)\left(\frac{\partial w}{\partial y}\right)$. When $t = 1$, $u = x$, $v = y$, and $w = f(x,y)$
$\Rightarrow \frac{\partial w}{\partial x} = \frac{\partial f}{\partial x}$ and $\frac{\partial w}{\partial y} = \frac{\partial f}{\partial x} \Rightarrow nf(x,y) = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}$, as claimed.

(b) From part (a), $nt^{n-1}f(x,y) = x\frac{\partial w}{\partial u} + y\frac{\partial w}{\partial v}$. Differentiating with respect to t again we obtain
$n(n-1)t^{n-2}f(x,y) = x\frac{\partial^2 w}{\partial u^2}\frac{\partial u}{\partial t} + x\frac{\partial^2 w}{\partial v \partial u}\frac{\partial v}{\partial t} + y\frac{\partial^2 w}{\partial u \partial v}\frac{\partial u}{\partial t} + y\frac{\partial^2 w}{\partial v^2}\frac{\partial v}{\partial t} = x^2\frac{\partial^2 w}{\partial u^2} + 2xy\frac{\partial^2 w}{\partial u \partial v} + y^2\frac{\partial^2 w}{\partial v^2}$.
Also from part (a), $\frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right) = \frac{\partial}{\partial x}\left(t\frac{\partial w}{\partial u}\right) = t\frac{\partial^2 w}{\partial u^2}\frac{\partial u}{\partial x} + t\frac{\partial^2 w}{\partial v \partial u}\frac{\partial v}{\partial x} = t^2\frac{\partial^2 w}{\partial u^2}$, $\frac{\partial^2 w}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\right)$
$= \frac{\partial}{\partial y}\left(t\frac{\partial w}{\partial v}\right) = t\frac{\partial^2 w}{\partial u \partial v}\frac{\partial u}{\partial y} + t\frac{\partial^2 w}{\partial v^2}\frac{\partial v}{\partial y} = t^2\frac{\partial^2 w}{\partial v^2}$, and $\frac{\partial^2 w}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right) = \frac{\partial}{\partial y}\left(t\frac{\partial w}{\partial u}\right) = t\frac{\partial^2 w}{\partial u^2}\frac{\partial u}{\partial y} + t\frac{\partial^2 w}{\partial v \partial u}\frac{\partial v}{\partial y}$
$= t^2\frac{\partial^2 w}{\partial v \partial u} \Rightarrow \left(\frac{1}{t^2}\right)\frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial u^2}$, $\left(\frac{1}{t^2}\right)\frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial v^2}$, and $\left(\frac{1}{t^2}\right)\frac{\partial^2 w}{\partial y \partial x} = \frac{\partial^2 w}{\partial v \partial u}$
$\Rightarrow n(n-1)t^{n-2}f(x,y) = \left(\frac{x^2}{t^2}\right)\left(\frac{\partial^2 w}{\partial x^2}\right) + \left(\frac{2xy}{t^2}\right)\left(\frac{\partial^2 w}{\partial y \partial x}\right) + \left(\frac{y^2}{t^2}\right)\left(\frac{\partial^2 w}{\partial y^2}\right)$ for $t \neq 0$. When $t = 1$, $w = f(x,y)$ and we have $n(n-1)f(x,y) = x^2\left(\frac{\partial^2 f}{\partial x^2}\right) + 2xy\left(\frac{\partial^2 f}{\partial x \partial y}\right) + y^2\left(\frac{\partial^2 f}{\partial y^2}\right)$ as claimed.

6. (a) $\lim\limits_{r \to 0} \frac{\sin 6r}{6r} = \lim\limits_{t \to 0} \frac{\sin t}{t} = 1$, where $t = 6r$

(b) $f_r(0,0) = \lim\limits_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim\limits_{h \to 0} \frac{\left(\frac{\sin 6h}{6h}\right) - 1}{h} = \lim\limits_{h \to 0} \frac{\sin 6h - 6h}{6h^2} = \lim\limits_{h \to 0} \frac{6\cos 6h - 6}{12h}$
$= \lim\limits_{h \to 0} \frac{-36 \sin 6h}{12} = 0$ (applying l'Hôpital's rule twice)

(c) $f_\theta(r,\theta) = \lim\limits_{h \to 0} \frac{f(r,\theta+h) - f(r,\theta)}{h} = \lim\limits_{h \to 0} \frac{\left(\frac{\sin 6r}{6r}\right) - \left(\frac{\sin 6r}{6r}\right)}{h} = \lim\limits_{h \to 0} \frac{0}{h} = 0$

7. (a) $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \Rightarrow r = |\mathbf{r}| = \sqrt{x^2+y^2+z^2}$ and $\nabla r = \frac{x}{\sqrt{x^2+y^2+z^2}}\mathbf{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\mathbf{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\mathbf{k} = \frac{\mathbf{r}}{r}$

(b) $r^n = \left(\sqrt{x^2+y^2+z^2}\right)^n$
$\Rightarrow \nabla(r^n) = nx\left(x^2+y^2+z^2\right)^{(n/2)-1}\mathbf{i} + ny\left(x^2+y^2+z^2\right)^{(n/2)-1}\mathbf{j} + nz\left(x^2+y^2+z^2\right)^{(n/2)-1}\mathbf{k} = nr^{n-2}\mathbf{r}$

(c) Let $n = 2$ in part (b). Then $\frac{1}{2}\nabla(r^2) = \mathbf{r} \Rightarrow \nabla\left(\frac{1}{2}r^2\right) = \mathbf{r} \Rightarrow \frac{r^2}{2} = \frac{1}{2}\left(x^2+y^2+z^2\right)$ is the function.

(d) $d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k} \Rightarrow \mathbf{r} \cdot d\mathbf{r} = x\,dx + y\,dy + z\,dz$, and $dr = r_x\,dx + r_y\,dy + r_z\,dz = \frac{x}{r}dx + \frac{y}{r}dy + \frac{z}{r}dz$
$\Rightarrow r\,dr = x\,dx + y\,dy + z\,dz = \mathbf{r} \cdot d\mathbf{r}$

(e) $\mathbf{A} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k} \Rightarrow \mathbf{A} \cdot \mathbf{r} = ax + by + cz \Rightarrow \nabla(\mathbf{A} \cdot \mathbf{r}) = a\mathbf{i} + b\mathbf{j} + c\mathbf{k} = \mathbf{A}$

8. $f(g(t),h(t)) = c \Rightarrow 0 = \frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}\right) \cdot \left(\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}\right)$, where $\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}$ is the tangent vector
$\Rightarrow \nabla f$ is orthogonal to the tangent vector

9. $f(x,y,z) = xz^2 - yz + \cos xy - 1 \Rightarrow \nabla f = (z^2 - y\sin xy)\mathbf{i} + (-z - x\sin xy)\mathbf{j} + (2xz - y)\mathbf{k} \Rightarrow \nabla f(0,0,1) = \mathbf{i} - \mathbf{j}$
$\Rightarrow$ the tangent plane is $x - y = 0$; $\mathbf{r} = (\ln t)\mathbf{i} + (t\ln t)\mathbf{j} + t\mathbf{k} \Rightarrow \mathbf{r}' = \left(\frac{1}{t}\right)\mathbf{i} + (\ln t + 1)\mathbf{j} + \mathbf{k}$; $x = y = 0$, $z = 1$
$\Rightarrow t = 1 \Rightarrow \mathbf{r}'(1) = \mathbf{i} + \mathbf{j} + \mathbf{k}$. Since $(\mathbf{i} + \mathbf{j} + \mathbf{k}) \cdot (\mathbf{i} - \mathbf{j}) = \mathbf{r}'(1) \cdot \nabla f = 0$, $\mathbf{r}$ is parallel to the plane, and $\mathbf{r}(1) = 0\mathbf{i} + 0\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{r}$ is contained in the plane.

10. Let $f(x,y,z) = x^3 + y^3 + z^3 - xyz \Rightarrow \nabla f = (3x^2 - yz)\,\mathbf{i} + (3y^2 - xz)\,\mathbf{j} + (3z^2 - xy)\,\mathbf{k} \Rightarrow \nabla f(0,-1,1) = \mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$ $\Rightarrow$ the tangent plane is $x + 3y + 3z = 0$; $\mathbf{r} = \left(\frac{t^3}{4} - 2\right)\mathbf{i} + \left(\frac{4}{t} - 3\right)\mathbf{j} + (\cos(t-2))\,\mathbf{k}$ $\Rightarrow \mathbf{r}' = \left(\frac{3t^2}{4}\right)\mathbf{i} - \left(\frac{4}{t^2}\right)\mathbf{j} - (\sin(t-2))\,\mathbf{k}$; $x = 0, y = -1, z = 1 \Rightarrow t = 2 \Rightarrow \mathbf{r}'(2) = 3\mathbf{i} - \mathbf{j}$. Since $\mathbf{r}'(2) \cdot \nabla f = 0 \Rightarrow \mathbf{r}$ is parallel to the plane, and $\mathbf{r}(2) = -\mathbf{i} + \mathbf{k} \Rightarrow \mathbf{r}$ is contained in the plane.

11. $\frac{\partial z}{\partial x} = 3x^2 - 9y = 0$ and $\frac{\partial z}{\partial y} = 3y^2 - 9x = 0 \Rightarrow y = \frac{1}{3}x^2$ and $3\left(\frac{1}{3}x^2\right)^2 - 9x = 0 \Rightarrow \frac{1}{3}x^4 - 9x = 0$ $\Rightarrow x(x^3 - 27) = 0 \Rightarrow x = 0$ or $x = 3$. Now $x = 0 \Rightarrow y = 0$ or $(0,0)$ and $x = 3 \Rightarrow y = 3$ or $(3,3)$. Next $\frac{\partial^2 z}{\partial x^2} = 6x$, $\frac{\partial^2 z}{\partial y^2} = 6y$, and $\frac{\partial^2 z}{\partial x \partial y} = -9$. For $(0,0)$, $\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = -81 \Rightarrow$ no extremum (a saddle point), and for $(3,3)$, $\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = 243 > 0$ and $\frac{\partial^2 z}{\partial x^2} = 18 > 0 \Rightarrow$ a local minimum.

12. $f(x,y) = 6xye^{-(2x+3y)} \Rightarrow f_x(x,y) = 6y(1-2x)e^{-(2x+3y)} = 0$ and $f_y(x,y) = 6x(1-3y)e^{-(2x+3y)} = 0 \Rightarrow x = 0$ and $y = 0$, or $x = \frac{1}{2}$ and $y = \frac{1}{3}$. The value $f(0,0) = 0$ is on the boundary, and $f\left(\frac{1}{2}, \frac{1}{3}\right) = \frac{1}{e^2}$. On the positive y-axis, $f(0,y) = 0$, and on the positive x-axis, $f(x,0) = 0$. As $x \to \infty$ or $y \to \infty$ we see that $f(x,y) \to 0$. Thus the absolute maximum of f in the closed first quadrant is $\frac{1}{e^2}$ at the point $\left(\frac{1}{2}, \frac{1}{3}\right)$.

13. Let $f(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 \Rightarrow \nabla f = \frac{2x}{a^2}\mathbf{i} + \frac{2y}{b^2}\mathbf{j} + \frac{2z}{c^2}\mathbf{k} \Rightarrow$ an equation of the plane tangent at the point $P_0(x_0, y_0, y_0)$ is $\left(\frac{2x_0}{a^2}\right)x + \left(\frac{2y_0}{b^2}\right)y + \left(\frac{2z_0}{c^2}\right)z = \frac{2x_0^2}{a^2} + \frac{2y_0^2}{b^2} + \frac{2z_0^2}{c^2} = 2$ or $\left(\frac{x_0}{a^2}\right)x + \left(\frac{y_0}{b^2}\right)y + \left(\frac{z_0}{c^2}\right)z = 1$. The intercepts of the plane are $\left(\frac{a^2}{x_0}, 0, 0\right)$, $\left(0, \frac{b^2}{y_0}, 0\right)$ and $\left(0, 0, \frac{c^2}{z_0}\right)$. The volume of the tetrahedron formed by the plane and the coordinate planes is $V = \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{a^2}{x_0}\right)\left(\frac{b^2}{y_0}\right)\left(\frac{c^2}{z_0}\right) \Rightarrow$ we need to maximize $V(x,y,z) = \frac{(abc)^2}{6}(xyz)^{-1}$ subject to the constraint $f(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Thus, $\left[-\frac{(abc)^2}{6}\right]\left(\frac{1}{x^2yz}\right) = \frac{2x}{a^2}\lambda$, $\left[-\frac{(abc)^2}{6}\right]\left(\frac{1}{xy^2z}\right) = \frac{2y}{b^2}\lambda$, and $\left[-\frac{(abc)^2}{6}\right]\left(\frac{1}{xyz^2}\right) = \frac{2z}{c^2}\lambda$. Multiply the first equation by a^2yz, the second by b^2xz, and the third by c^2xy. Then equate the first and second $\Rightarrow a^2y^2 = b^2x^2 \Rightarrow y = \frac{b}{a}x, x > 0$; equate the first and third $\Rightarrow a^2z^2 = c^2x^2 \Rightarrow z = \frac{c}{a}x, x > 0$; substitute into $f(x,y,z) = 0 \Rightarrow x = \frac{a}{\sqrt{3}} \Rightarrow y = \frac{b}{\sqrt{3}} \Rightarrow z = \frac{c}{\sqrt{3}} \Rightarrow V = \frac{\sqrt{3}}{2}abc$.

14. $2(x-u) = -\lambda$, $2(y-v) = \lambda$, $-2(x-u) = \mu$, and $-2(y-v) = -2\mu v \Rightarrow x - u = v - y$, $x - u = -\frac{\mu}{2}$, and $y - v = \mu v \Rightarrow x - u = -\mu v = -\frac{\mu}{2} \Rightarrow v = \frac{1}{2}$ or $\mu = 0$.

CASE 1: $\mu = 0 \Rightarrow x = u, y = v$, and $\lambda = 0$; then $y = x + 1 \Rightarrow v = u + 1$ and $v^2 = u \Rightarrow v = v^2 + 1$ $\Rightarrow v^2 - v + 1 = 0 \Rightarrow v = \frac{1 \pm \sqrt{1-4}}{2} \Rightarrow$ no real solution.

CASE 2: $v = \frac{1}{2}$ and $u = v^2 \Rightarrow u = \frac{1}{4}$; $x - \frac{1}{4} = \frac{1}{2} - y$ and $y = x + 1 \Rightarrow x - \frac{1}{4} = -x - \frac{1}{2} \Rightarrow 2x = -\frac{1}{4} \Rightarrow x = -\frac{1}{8}$ $\Rightarrow y = \frac{7}{8}$. Then $f\left(-\frac{1}{8}, \frac{7}{8}, \frac{1}{4}, \frac{1}{2}\right) = \left(-\frac{1}{8} - \frac{1}{4}\right)^2 + \left(\frac{7}{8} - \frac{1}{2}\right)^2 = 2\left(\frac{3}{8}\right)^2 \Rightarrow$ the minimum distance is $\frac{3}{8}\sqrt{2}$. (Notice that f has no maximum value.)

15. Let (x_0, y_0) be any point in R. We must show $\lim_{(x,y) \to (x_0,y_0)} f(x,y) = f(x_0,y_0)$ or, equivalently that $\lim_{(h,k) \to (0,0)} |f(x_0+h, y_0+k) - f(x_0,y_0)| = 0$. Consider $f(x_0+h, y_0+k) - f(x_0,y_0)$ $= [f(x_0+h, y_0+k) - f(x_0, y_0+k)] + [f(x_0, y_0+k) - f(x_0,y_0)]$. Let $F(x) = f(x, y_0+k)$ and apply the Mean Value Theorem: there exists ξ with $x_0 < \xi < x_0 + h$ such that $F'(\xi)h = F(x_0+h) - F(x_0) \Rightarrow hf_x(\xi, y_0+k)$ $= f(x_0+h, y_0+k) - f(x_0, y_0+k)$. Similarly, $k\,f_y(x_0,\eta) = f(x_0, y_0+k) - f(x_0,y_0)$ for some η with $y_0 < \eta < y_0 + k$. Then $|f(x_0+h, y_0+k) - f(x_0,y_0)| \le |hf_x(\xi, y_0+k)| + |kf_y(x_0,\eta)|$. If M, N are positive real numbers such that $|f_x| \le M$ and $|f_y| \le N$ for all (x,y) in the xy-plane, then $|f(x_0+h, y_0+k) - f(x_0,y_0)|$ $\le M|h| + N|k|$. As $(h,k) \to 0$, $|f(x_0+h, y_0+k) - f(x_0,y_0)| \to 0 \Rightarrow \lim_{(h,k) \to (0,0)} |f(x_0+h, y_0+k) - f(x_0,y_0)|$ $= 0 \Rightarrow$ f is continuous at (x_0, y_0).

16. At extreme values, ∇f and $\mathbf{v} = \frac{d\mathbf{r}}{dt}$ are orthogonal because $\frac{df}{dt} = \nabla f \cdot \frac{d\mathbf{r}}{dt} = 0$ by the First Derivative Theorem for Local Extreme Values.

17. $\frac{\partial f}{\partial x} = 0 \Rightarrow f(x,y) = h(y)$ is a function of y only. Also, $\frac{\partial g}{\partial y} = \frac{\partial f}{\partial x} = 0 \Rightarrow g(x,y) = k(x)$ is a function of x only. Moreover, $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} \Rightarrow h'(y) = k'(x)$ for all x and y. This can happen only if $h'(y) = k'(x) = c$ is a constant. Integration gives $h(y) = cy + c_1$ and $k(x) = cx + c_2$, where c_1 and c_2 are constants. Therefore $f(x,y) = cy + c_1$ and $g(x,y) = cx + c_2$. Then $f(1,2) = g(1,2) = 5 \Rightarrow 5 = 2c + c_1 = c + c_2$, and $f(0,0) = 4 \Rightarrow c_1 = 4 \Rightarrow c = \frac{1}{2}$ $\Rightarrow c_2 = \frac{9}{2}$. Thus, $f(x,y) = \frac{1}{2}y + 4$ and $g(x,y) = \frac{1}{2}x + \frac{9}{2}$.

18. Let $g(x,y) = D_{\mathbf{u}}f(x,y) = f_x(x,y)a + f_y(x,y)b$. Then $D_{\mathbf{u}}g(x,y) = g_x(x,y)a + g_y(x,y)b$ $= f_{xx}(x,y)a^2 + f_{yx}(x,y)ab + f_{xy}(x,y)ba + f_{yy}(x,y)b^2 = f_{xx}(x,y)a^2 + 2f_{xy}(x,y)ab + f_{yy}(x,y)b^2$.

19. Since the particle is heat-seeking, at each point (x, y) it moves in the direction of maximal temperature increase, that is in the direction of $\nabla T(x,y) = (e^{-2y}\sin x)\,\mathbf{i} + (2e^{-2y}\cos x)\,\mathbf{j}$. Since $\nabla T(x,y)$ is parallel to the particle's velocity vector, it is tangent to the path $y = f(x)$ of the particle $\Rightarrow f'(x) = \frac{2e^{-2y}\cos x}{e^{-2y}\sin x} = 2\cot x$. Integration gives $f(x) = 2\ln|\sin x| + C$ and $f\left(\frac{\pi}{4}\right) = 0 \Rightarrow 0 = 2\ln\left|\sin\frac{\pi}{4}\right| + C \Rightarrow C = -2\ln\frac{\sqrt{2}}{2} = \ln\left(\frac{2}{\sqrt{2}}\right)^2$ $= \ln 2$. Therefore, the path of the particle is the graph of $y = 2\ln|\sin x| + \ln 2$.

20. The line of travel is $x = t$, $y = t$, $z = 30 - 5t$, and the bullet hits the surface $z = 2x^2 + 3y^2$ when $30 - 5t = 2t^2 + 3t^2 \Rightarrow t^2 + t - 6 = 0 \Rightarrow (t+3)(t-2) = 0 \Rightarrow t = 2$ (since $t > 0$). Thus the bullet hits the surface at the point (2, 2, 20). Now, the vector $4x\mathbf{i} + 6y\mathbf{j} - \mathbf{k}$ is normal to the surface at any (x, y, z), so that $\mathbf{n} = 8\mathbf{i} + 12\mathbf{j} - \mathbf{k}$ is normal to the surface at (2, 2, 20). If $\mathbf{v} = \mathbf{i} + \mathbf{j} - 5\mathbf{k}$, then the velocity of the particle after the ricochet is $\mathbf{w} = \mathbf{v} - 2\,\text{proj}_{\mathbf{n}}\,\mathbf{v} = \mathbf{v} - \left(\frac{2\mathbf{v}\cdot\mathbf{n}}{|\mathbf{n}|^2}\right)\mathbf{n} = \mathbf{v} - \left(\frac{2\cdot 25}{209}\right)\mathbf{n} = (\mathbf{i} + \mathbf{j} - 5\mathbf{k}) - \left(\frac{400}{209}\mathbf{i} + \frac{600}{209}\mathbf{j} - \frac{50}{209}\mathbf{k}\right)$ $= -\frac{191}{209}\mathbf{i} - \frac{391}{209}\mathbf{j} - \frac{995}{209}\mathbf{k}$.

21. (a) $\mathbf{k}$ is a vector normal to $z = 10 - x^2 - y^2$ at the point (0, 0, 10). So directions tangential to S at (0, 0, 10) will be unit vectors $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$. Also, $\nabla T(x,y,z) = (2xy + 4)\,\mathbf{i} + (x^2 + 2yz + 14)\,\mathbf{j} + (y^2 + 1)\,\mathbf{k}$ $\Rightarrow \nabla T(0,0,10) = 4\mathbf{i} + 14\mathbf{j} + \mathbf{k}$. We seek the unit vector $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$ such that $D_{\mathbf{u}}T(0,0,10)$ $= (4\mathbf{i} + 14\mathbf{j} + \mathbf{k}) \cdot (a\mathbf{i} + b\mathbf{j}) = (4\mathbf{i} + 14\mathbf{j}) \cdot (a\mathbf{i} + b\mathbf{j})$ is a maximum. The maximum will occur when $a\mathbf{i} + b\mathbf{j}$ has the same direction as $4\mathbf{i} + 14\mathbf{j}$, or $\mathbf{u} = \frac{1}{\sqrt{53}}(2\mathbf{i} + 7\mathbf{j})$.

(b) A vector normal to S at (1, 1, 8) is $\mathbf{n} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$. Now, $\nabla T(1,1,8) = 6\mathbf{i} + 31\mathbf{j} + 2\mathbf{k}$ and we seek the unit vector $\mathbf{u}$ such that $D_{\mathbf{u}}T(1,1,8) = \nabla T \cdot \mathbf{u}$ has its largest value. Now write $\nabla T = \mathbf{v} + \mathbf{w}$, where $\mathbf{v}$ is parallel to ∇T and $\mathbf{w}$ is orthogonal to ∇T. Then $D_{\mathbf{u}}T = \nabla T \cdot \mathbf{u} = (\mathbf{v} + \mathbf{w}) \cdot \mathbf{u} = \mathbf{v} \cdot \mathbf{u} + \mathbf{w} \cdot \mathbf{u} = \mathbf{w} \cdot \mathbf{u}$. Thus $D_{\mathbf{u}}T(1,1,8)$ is a maximum when $\mathbf{u}$ has the same direction as $\mathbf{w}$. Now, $\mathbf{w} = \nabla T - \left(\frac{\nabla T\cdot\mathbf{n}}{|\mathbf{n}|^2}\right)\mathbf{n}$ $= (6\mathbf{i} + 31\mathbf{j} + 2\mathbf{k}) - \left(\frac{12+62+2}{4+4+1}\right)(2\mathbf{i} + 2\mathbf{j} + \mathbf{k}) = \left(6 - \frac{152}{9}\right)\mathbf{i} + \left(31 - \frac{152}{9}\right)\mathbf{j} + \left(2 - \frac{76}{9}\right)\mathbf{k}$ $= -\frac{98}{9}\mathbf{i} + \frac{127}{9}\mathbf{j} - \frac{58}{9}\mathbf{k} \Rightarrow \mathbf{u} = \frac{\mathbf{w}}{|\mathbf{w}|} = -\frac{1}{\sqrt{29{,}097}}(98\mathbf{i} - 127\mathbf{j} + 58\mathbf{k})$.

22. Suppose the surface (boundary) of the mineral deposit is the graph of $z = f(x,y)$ (where the z-axis points up into the air). Then $-\frac{\partial f}{\partial x}\mathbf{i} - \frac{\partial f}{\partial y}\mathbf{j} + \mathbf{k}$ is an outer normal to the mineral deposit at (x, y) and $\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$ points in the direction of steepest ascent of the mineral deposit. This is in the direction of the vector $\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$ at (0, 0) (the location of the 1st borehole) that the geologists should drill their fourth borehole. To approximate this vector we use the fact that (0, 0, −1000), (0, 100, −950), and (100, 0, −1025) lie on the graph of $z = f(x,y)$. The plane containing these three points is a good approximation to the tangent plane to $z = f(x,y)$ at the point (0, 0, 0). A normal to this plane is $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 100 & 50 \\ 100 & 0 & -25 \end{vmatrix}$

$= -2500\mathbf{i} + 5000\mathbf{j} - 10{,}000\mathbf{k}$, or $-\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$. So at $(0, 0)$ the vector $\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$ is approximately $-\mathbf{i} + 2\mathbf{j}$. Thus the geologists should drill their fourth borehole in the direction of $\frac{1}{\sqrt{5}}(-\mathbf{i} + 2\mathbf{j})$ from the first borehole.

23. $w = e^{rt} \sin \pi x \Rightarrow w_t = re^{rt} \sin \pi x$ and $w_x = \pi e^{rt} \cos \pi x \Rightarrow w_{xx} = -\pi^2 e^{rt} \sin \pi x$; $w_{xx} = \frac{1}{c^2} w_t$, where c^2 is the positive constant determined by the material of the rod $\Rightarrow -\pi^2 e^{rt} \sin \pi x = \frac{1}{c^2}(re^{rt} \sin \pi x)$ $\Rightarrow (r + c^2\pi^2) e^{rt} \sin \pi x = 0 \Rightarrow r = -c^2\pi^2 \Rightarrow w = e^{-c^2\pi^2 t} \sin \pi x$

24. $w = e^{rt} \sin kx \Rightarrow w_t = re^{rt} \sin kx$ and $w_x = ke^{rt} \cos kx \Rightarrow w_{xx} = -k^2 e^{rt} \sin kx$; $w_{xx} = \frac{1}{c^2} w_t$ $\Rightarrow -k^2 e^{rt} \sin kx = \frac{1}{c^2}(re^{rt} \sin kx) \Rightarrow (r + c^2k^2) e^{rt} \sin kx = 0 \Rightarrow r = -c^2k^2 \Rightarrow w = e^{-c^2k^2 t} \sin kx$.
Now, $w(L, t) = 0 \Rightarrow e^{-c^2k^2 t} \sin kL = 0 \Rightarrow kL = n\pi$ for n an integer $\Rightarrow k = \frac{n\pi}{L} \Rightarrow w = e^{-c^2 n^2 \pi^2 t/L^2} \sin\left(\frac{n\pi}{L} x\right)$.
As $t \to \infty$, $w \to 0$ since $\left|\sin\left(\frac{n\pi}{L} x\right)\right| \le 1$ and $e^{-c^2 n^2 \pi^2 t/L^2} \to 0$.

CHAPTER 15 MULTIPLE INTEGRALS

15.1 DOUBLE AND ITERATED INTEGRALS OVER RECTANGLES

1. $\int_1^2 \int_0^4 2xy\, dy\, dx = \int_1^2 [x\,y^2]_0^4\, dx = \int_1^2 16x\, dx = [8\,x^2]_1^2 = 24$

2. $\int_0^2 \int_{-1}^1 (x - y)\, dy\, dx = \int_0^2 \left[xy - \frac{1}{2}y^2\right]_{-1}^{1} dx = \int_0^2 2x\; dx = [x^2]_0^2 = 4$

3. $\int_{-1}^0 \int_{-1}^1 (x + y + 1)\, dx\, dy = \int_{-1}^0 \left[\frac{x^2}{2} + yx + x\right]_{-1}^{1} dy = \int_{-1}^0 (2y + 2)\, dy = [y^2 + 2y]_{-1}^{0} = 1$

4. $\int_0^1 \int_0^1 \left(1 - \frac{x^2 + y^2}{2}\right) dx\, dy = \int_0^1 \left[x - \frac{x^3}{6} - \frac{x\,y^2}{2}\right]_0^1 dy = \int_0^1 \left(\frac{5}{6} - \frac{y^2}{2}\right) dy = \left[\frac{5}{6}y - \frac{y^3}{6}\right]_0^1 = \frac{2}{3}$

5. $\int_0^3 \int_0^2 (4 - y^2)\, dy\, dx = \int_0^3 \left[4y - \frac{y^3}{3}\right]_0^2 dx = \int_0^3 \frac{16}{3}\, dx = \left[\frac{16}{3}\,x\right]_0^3 = 16$

6. $\int_0^3 \int_{-2}^0 (x^2 y - 2xy)\, dy\, dx = \int_0^3 \left[\frac{x^2 y^2}{2} - xy^2\right]_{-2}^{0} dx = \int_0^3 (4x - 2x^2)\, dx = \left[2x^2 - \frac{2x^3}{3}\right]_0^3 = 0$

7. $\int_0^1 \int_0^1 \frac{y}{1 + x\,y}\, dx\, dy = \int_0^1 [\ln|1 + x\,y|]_0^1\, dy = \int_0^1 \ln|1 + y| dy = [y \ln|1 + y| - y + \ln|1 + y|]_0^1 = 2\ln 2 - 1$

8. $\int_1^4 \int_0^4 \left(\frac{x}{2} + \sqrt{y}\right) dx\, dy = \int_1^4 \left[\frac{1}{4}x^2 + x\sqrt{y}\right]_0^4 dy = \int_1^4 \left(4 + 4\,y^{1/2}\right) dy = \left[4y + \frac{8}{3}y^{3/2}\right]_1^4 = \frac{92}{3}$

9. $\int_0^{\ln 2} \int_1^{\ln 5} e^{2x+y}\, dy\, dx = \int_0^{\ln 2} [e^{2x+y}]_1^{\ln 5}\, dx = \int_0^{\ln 2} (5e^{2x} - e^{2x+1})\, dx = \left[\frac{5}{2}e^{2x} - \frac{1}{2}e^{2x+1}\right]_0^{\ln 2} = \frac{3}{2}(5 - e)$

10. $\int_0^1 \int_1^2 x\,y\,e^x\, dy\, dx = \int_0^1 \left[\frac{1}{2}x\,y^2 e^x\right]_1^2 dx = \int_0^1 \frac{3}{2}x\,e^x\, dx = \left[\frac{3}{2}x\,e^x - \frac{3}{2}e^x\right]_0^1 = \frac{3}{2}$

11. $\int_{-1}^2 \int_0^{\pi/2} y \sin x\; dx\, dy = \int_{-1}^2 [-y\cos x]_0^{\pi/2}\, dy = \int_{-1}^2 y\, dy = \left[\frac{1}{2}y^2\right]_{-1}^2 = \frac{3}{2}$

12. $\int_\pi^{2\pi} \int_0^\pi (\sin x + \cos y)\, dx\, dy = \int_\pi^{2\pi} [-\cos x + x\cos y]_0^\pi\, dy = \int_\pi^{2\pi} (2 + \pi\cos y)\, dy = [2y + \pi\sin y]_\pi^{2\pi} = 2\pi$

13. $\iint_R (6\,y^2 - 2\,x) dA = \int_0^1 \int_0^2 (6\,y^2 - 2\,x)\, dy\, dx = \int_0^1 [2\,y^3 - 2\,x\,y]_0^2\, dx = \int_0^1 (16 - 4\,x)\, dx = [16\,x - 2\,x^2]_0^1 = 14$

14. $\iint_R \frac{\sqrt{x}}{y^2} dA = \int_0^4 \int_1^2 \frac{\sqrt{x}}{y^2}\, dy\, dx = \int_0^4 \left[-\frac{\sqrt{x}}{y}\right]_1^2 dx = \int_0^4 \frac{1}{2}x^{1/2}\, dx = \left[\frac{1}{3}x^{3/2}\right]_0^4 = \frac{8}{3}$

15. $\iint_R x\,y\cos y\, dA = \int_{-1}^1 \int_0^\pi x\,y\cos y\; dy\, dx = \int_{-1}^1 [x\,y\sin y + x\cos y]_0^\pi\, dx = \int_{-1}^1 (-2x)\, dx = [-x^2]_{-1}^1 = 0$

16. $\iint_R y\sin(x + y)\, dA = \int_{-\pi}^0 \int_0^\pi y\sin(x + y)\; dy\, dx = \int_{-\pi}^0 [-y\cos(x + y) + \sin(x + y)]_0^\pi\, dx$

$= \int_{-\pi}^0 (\sin(x + \pi) - \pi\cos(x + \pi) - \sin x) dx = [-\cos(x + \pi) - \pi\sin(x + \pi) + \cos x]_{-\pi}^0 = 4$

17. $\iint_R e^{x-y} dA = \int_0^{\ln 2}\int_0^{\ln 2} e^{x-y}\, dy\, dx = \int_0^{\ln 2} \left[-e^{x-y}\right]_0^{\ln 2} dx = \int_0^{\ln 2} \left(-e^{x-\ln 2} + e^x\right) dx = \left[-e^{x-\ln 2} + e^x\right]_0^{\ln 2} = \frac{1}{2}$

18. $\iint_R x\, y\, e^{x y^2} dA = \int_0^2\int_0^1 x\, y\, e^{x y^2}\, dy\, dx = \int_0^2 \left[\frac{1}{2} e^{x y^2}\right]_0^1 dx = \int_0^2 \left(\frac{1}{2}e^x - \frac{1}{2}\right) dx = \left[\frac{1}{2}e^x - \frac{1}{2}x\right]_0^2 = \frac{1}{2}(e^2 - 3)$

19. $\iint_R \frac{x y^3}{x^2+1} dA = \int_0^1\int_0^2 \frac{x y^3}{x^2+1}\, dy\, dx = \int_0^1 \left[\frac{x y^4}{4(x^2+1)}\right]_0^2 dx = \int_0^1 \frac{4x}{x^2+1}\, dx = \left[2\ln|x^2+1|\right]_0^1 = 2\ln 2$

20. $\iint_R \frac{y}{x^2y^2+1} dA = \int_0^1\int_0^1 \frac{y}{(x y)^2+1}\, dx\, dy = \int_0^1 \left[\tan^{-1}(x y)\right]_0^1 dy = \int_0^1 \tan^{-1} y\, dy = \left[y \tan^{-1} y - \frac{1}{2}\ln|1+y^2|\right]_0^1 = \frac{\pi}{4} - \frac{1}{2}\ln 2$

21. $\int_1^2\int_1^2 \frac{1}{xy}\, dy\, dx = \int_1^2 \frac{1}{x}(\ln 2 - \ln 1)\, dx = (\ln 2)\int_1^2 \frac{1}{x}\, dx = (\ln 2)^2$

22. $\int_0^1\int_0^\pi y \cos xy\, dx\, dy = \int_0^1 \left[\sin xy\right]_0^\pi dy = \int_0^1 \sin \pi y\, dy = \left[-\frac{1}{\pi}\cos \pi y\right]_0^1 = -\frac{1}{\pi}(-1-1) = \frac{2}{\pi}$

23. $V = \iint_R f(x,y)\, dA = \int_{-1}^1\int_{-1}^1 (x^2+y^2)\, dy\, dx = \int_{-1}^1 \left[x^2 y + \frac{1}{3}y^3\right]_{-1}^1 dx = \int_{-1}^1 \left(2x^2 + \frac{2}{3}\right) dx = \left[\frac{2}{3}x^3 + \frac{2}{3}x\right]_{-1}^1 = \frac{8}{3}$

24. $V = \iint_R f(x,y)\, dA = \int_0^2\int_0^2 (16 - x^2 - y^2)\, dy\, dx = \int_0^2 \left[16y - x^2 y - \frac{1}{3}y^3\right]_0^2 dx = \int_0^2 \left(\frac{88}{3} - 2x^2\right) dx = \left[\frac{88}{3}x - \frac{2}{3}x^3\right]_0^2$
$= \frac{160}{3}$

25. $V = \iint_R f(x,y)\, dA = \int_0^1\int_0^1 (2 - x - y)\, dy\, dx = \int_0^1 \left[2y - xy - \frac{1}{2}y^2\right]_0^1 dx = \int_0^1 \left(\frac{3}{2} - x\right) dx = \left[\frac{3}{2}x - \frac{1}{2}x^2\right]_0^1 = 1$

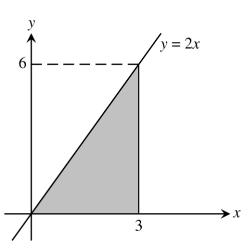
26. $V = \iint_R f(x,y)\, dA = \int_0^4\int_0^2 \frac{y}{2}\, dy\, dx = \int_0^4 \left[\frac{y^2}{4}\right]_0^2 dx = \int_0^4 1\, dx = [x]_0^4 = 4$

27. $V = \iint_R f(x,y)\, dA = \int_0^{\pi/2}\int_0^{\pi/4} 2\sin x \cos y\, dy\, dx = \int_0^{\pi/2} \left[2\sin x \sin y\right]_0^{\pi/4} dx = \int_0^{\pi/2} \left(\sqrt{2}\sin x\right) dx = \left[-\sqrt{2}\cos x\right]_0^{\pi/2}$
$= \sqrt{2}$

28. $V = \iint_R f(x,y)\, dA = \int_0^1\int_0^2 (4 - y^2)\, dy\, dx = \int_0^1 \left[4y - \frac{1}{3}y^3\right]_0^2 dx = \int_0^1 \left(\frac{16}{3}\right) dx = \left[\frac{16}{3}x\right]_0^1 = \frac{16}{3}$

15.2 DOUBLE INTEGRALS OVER GENERAL REGIONS

1.



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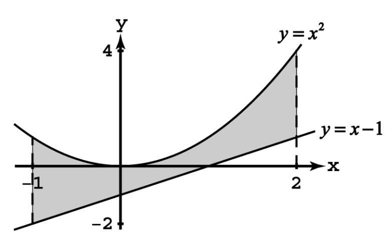
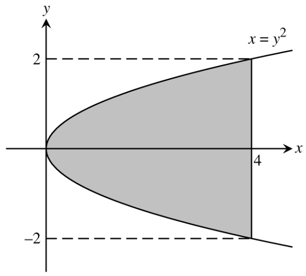
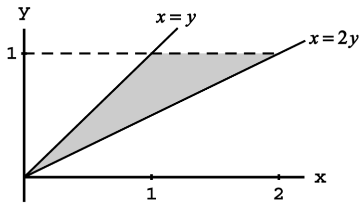
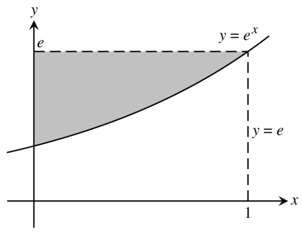
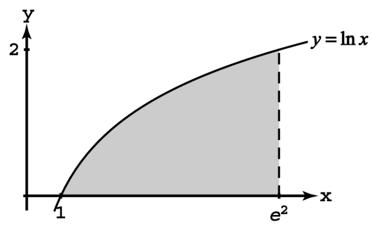
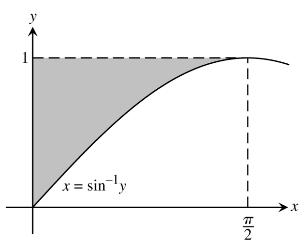


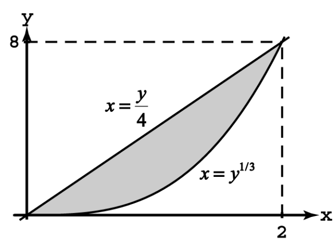
3. 

4. 

5. 

6. 

7. 

8. 

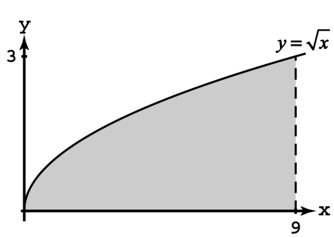
9. (a) $\int_0^2 \int_{x^3}^8 dy\,dx$ (b) $\int_0^8 \int_0^{y^{1/3}} dx\,dy$

10. (a) $\int_0^3 \int_0^{2x} dy\,dx$ (b) $\int_0^6 \int_{y/2}^3 dx\,dy$

11. (a) $\int_0^3 \int_{x^2}^{3x} dy\,dx$ (b) $\int_0^9 \int_{y/3}^{\sqrt{y}} dx\,dy$

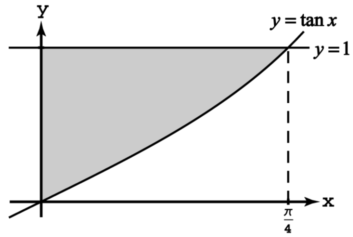
12. (a) $\int_0^2 \int_1^{e^x} dy\,dx$ (b) $\int_1^{e^2} \int_{\ln y}^2 dx\,dy$

13. (a) $\int_0^9 \int_0^{\sqrt{x}} dy\,dx$
(b) $\int_0^3 \int_{y^2}^9 dx\,dy$



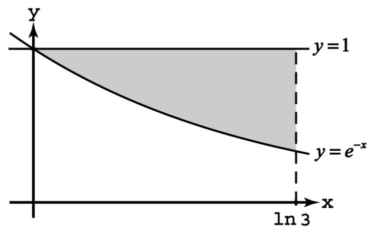
14. (a) $\int_0^{\pi/4}\int_{\tan x}^{1} dy\,dx$

(b) $\int_0^{1}\int_0^{\tan^{-1} y} dx\,dy$



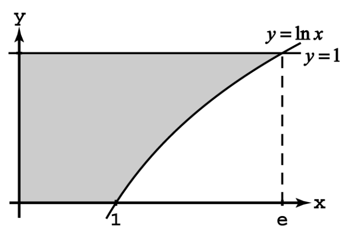
15. (a) $\int_0^{\ln 3}\int_{e^{-x}}^{1} dy\,dx$

(b) $\int_{1/3}^{1}\int_{-\ln y}^{\ln 3} dx\,dy$



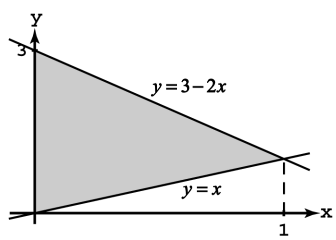
16. (a) $\int_0^{1}\int_0^{1} dy\,dx + \int_1^{e}\int_{\ln x}^{1} dy\,dx$

(b) $\int_0^{1}\int_0^{e^y} dx\,dy$



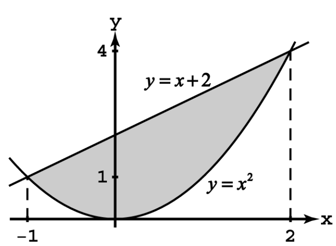
17. (a) $\int_0^{1}\int_x^{3-2x} dy\,dx$

(b) $\int_0^{1}\int_0^{y} dx\,dy + \int_1^{3}\int_0^{(3-y)/2} dx\,dy$



18. (a) $\int_{-1}^{2}\int_{x^2}^{x+2} dy\,dx$

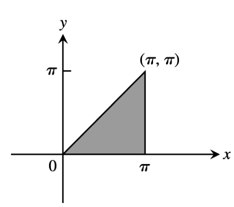
(b) $\int_0^{1}\int_{-\sqrt{y}}^{\sqrt{y}} dx\,dy + \int_1^{3}\int_{y-2}^{\sqrt{y}} dx\,dy$



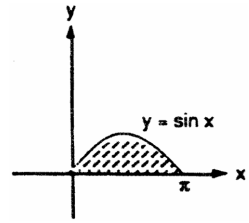
19. $\int_0^{\pi}\int_0^{x} (x \sin y)\,dy\,dx = \int_0^{\pi} [-x \cos y]_0^x\,dx$

$= \int_0^{\pi} (x - x\cos x)\,dx = \left[\frac{x^2}{2} - (\cos x + x \sin x)\right]_0^{\pi}$

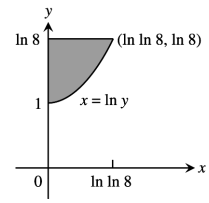
$= \frac{\pi^2}{2} + 2$



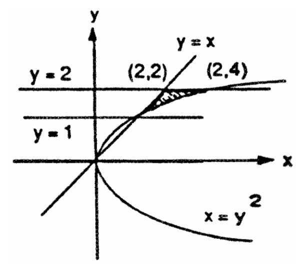
20. $\int_0^{\pi}\int_0^{\sin x} y\,dy\,dx = \int_0^{\pi}\left[\frac{y^2}{2}\right]_0^{\sin x} dx = \int_0^{\pi}\frac{1}{2}\sin^2 x\,dx$
$= \frac{1}{4}\int_0^{\pi}(1-\cos 2x)\,dx = \frac{1}{4}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi} = \frac{\pi}{4}$



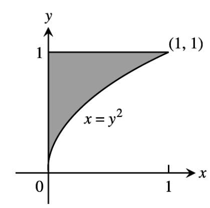
21. $\int_1^{\ln 8}\int_0^{\ln y} e^{x+y}\,dx\,dy = \int_1^{\ln 8}\left[e^{x+y}\right]_0^{\ln y} dy = \int_1^{\ln 8}\left(ye^y - e^y\right) dy$
$= \left[(y-1)e^y - e^y\right]_1^{\ln 8} = 8(\ln 8 - 1) - 8 + e$
$= 8\ln 8 - 16 + e$



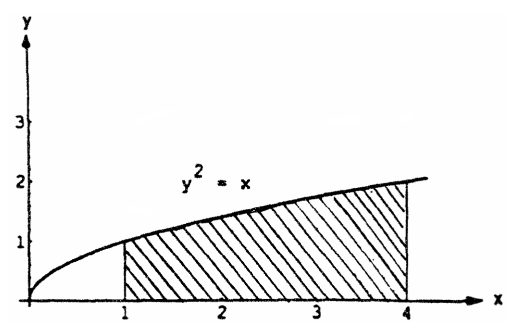
22. $\int_1^2\int_y^{y^2} dx\,dy = \int_1^2 (y^2 - y)\,dy = \left[\frac{y^3}{3} - \frac{y^2}{2}\right]_1^2$
$= \left(\frac{8}{3} - 2\right) - \left(\frac{1}{3} - \frac{1}{2}\right) = \frac{7}{3} - \frac{3}{2} = \frac{5}{6}$



23. $\int_0^1\int_0^{y^2} 3y^3e^{xy}\,dx\,dy = \int_0^1\left[3y^2e^{xy}\right]_0^{y^2} dy$
$= \int_0^1\left(3y^2e^{y^3} - 3y^2\right) dy = \left[e^{y^3} - y^3\right]_0^1 = e - 2$



24. $\int_1^4\int_0^{\sqrt{x}} \frac{3}{2}e^{y/\sqrt{x}}\,dy\,dx = \int_1^4\left[\frac{3}{2}\sqrt{x}\,e^{y/\sqrt{x}}\right]_0^{\sqrt{x}} dx$
$= \frac{3}{2}(e-1)\int_1^4\sqrt{x}\,dx = \left[\frac{3}{2}(e-1)\left(\frac{2}{3}\right)x^{3/2}\right]_1^4 = 7(e-1)$



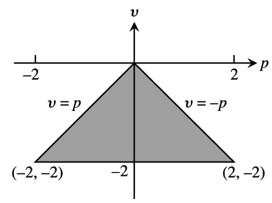
25. $\int_1^2\int_x^{2x} \frac{x}{y}\,dy\,dx = \int_1^2\left[x\ln y\right]_x^{2x} dx = (\ln 2)\int_1^2 x\,dx = \frac{3}{2}\ln 2$

26. $\int_0^1\int_0^{1-x}(x^2+y^2)\,dy\,dx = \int_0^1\left[x^2y + \frac{y^3}{3}\right]_0^{1-x} dx = \int_0^1\left[x^2(1-x) + \frac{(1-x)^3}{3}\right] dx = \int_0^1\left[x^2 - x^3 + \frac{(1-x)^3}{3}\right] dx$
$= \left[\frac{x^3}{3} - \frac{x^4}{4} - \frac{(1-x)^4}{12}\right]_0^1 = \left(\frac{1}{3} - \frac{1}{4} - 0\right) - \left(0 - 0 - \frac{1}{12}\right) = \frac{1}{6}$

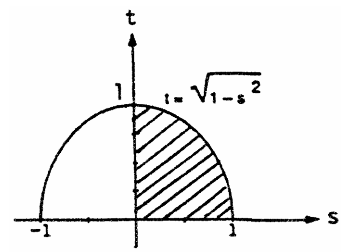
27. $\int_0^1\int_0^{1-u}\left(v - \sqrt{u}\right) dv\,du = \int_0^1\left[\frac{v^2}{2} - v\sqrt{u}\right]_0^{1-u} du = \int_0^1\left[\frac{1-2u+u^2}{2} - \sqrt{u}(1-u)\right] du$
$= \int_0^1\left(\frac{1}{2} - u + \frac{u^2}{2} - u^{1/2} + u^{3/2}\right) du = \left[\frac{u}{2} - \frac{u^2}{2} + \frac{u^3}{6} - \frac{2}{3}u^{3/2} + \frac{2}{5}u^{5/2}\right]_0^1 = \frac{1}{2} - \frac{1}{2} + \frac{1}{6} - \frac{2}{3} + \frac{2}{5} = -\frac{1}{2} + \frac{2}{5} = -\frac{1}{10}$

28. $\int_1^2 \int_0^{\ln t} e^s \ln t \, ds \, dt = \int_1^2 \left[e^s \ln t\right]_0^{\ln t} dt = \int_1^2 (t \ln t - \ln t) \, dt = \left[\frac{t^2}{2} \ln t - \frac{t^2}{4} - t \ln t + t\right]_1^2$
$= (2 \ln 2 - 1 - 2 \ln 2 + 2) - \left(-\frac{1}{4} + 1\right) = \frac{1}{4}$

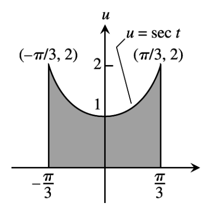
29. $\int_{-2}^0 \int_v^{-v} 2 \, dp \, dv = 2\int_{-2}^0 [p]_v^{-v} \, dv = 2\int_{-2}^0 -2v \, dv$
$= -2\left[v^2\right]_{-2}^0 = 8$



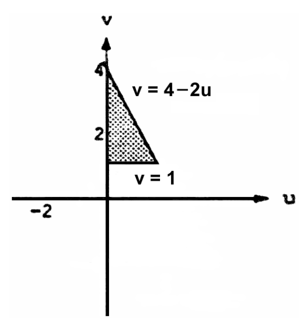
30. $\int_0^1 \int_0^{\sqrt{1-s^2}} 8t \, dt \, ds = \int_0^1 \left[4t^2\right]_0^{\sqrt{1-s^2}} ds$
$= \int_0^1 4\left(1 - s^2\right) ds = 4\left[s - \frac{s^3}{3}\right]_0^1 = \frac{8}{3}$



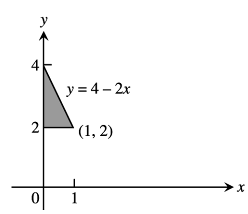
31. $\int_{-\pi/3}^{\pi/3} \int_0^{\sec t} 3 \cos t \, du \, dt = \int_{-\pi/3}^{\pi/3} [(3 \cos t)u]_0^{\sec t}$
$= \int_{-\pi/3}^{\pi/3} 3 \, dt = 2\pi$



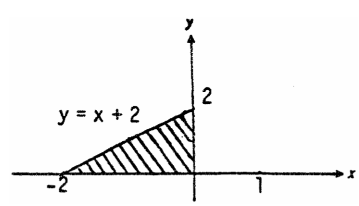
32. $\int_0^{3/2} \int_1^{4-2u} \frac{4-2u}{v^2} \, dv \, du = \int_0^{3/2} \left[\frac{2u-4}{v}\right]_1^{4-2u} du$
$= \int_0^{3/2} (3 - 2u) \, du = \left[3u - u^2\right]_0^{3/2} = \frac{9}{2}$



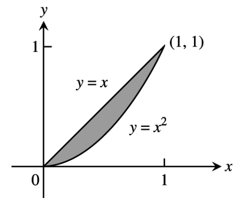
33. $\int_2^4 \int_0^{(4-y)/2} dx \, dy$



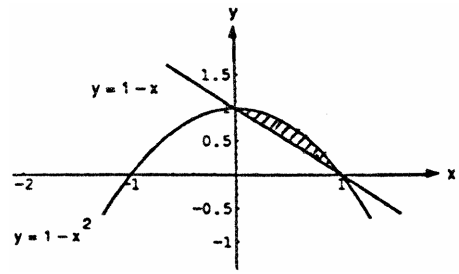
34. $\int_{-2}^0 \int_0^{x+2} dy \, dx$



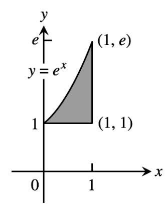
35. $\int_0^1 \int_{x^2}^{x} dy\,dx$



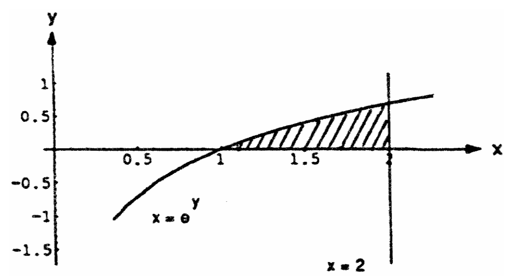
36. $\int_0^1 \int_{1-y}^{\sqrt{1-y}} dx\,dy$



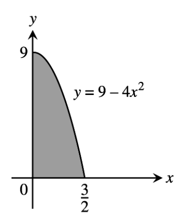
37. $\int_1^e \int_{\ln y}^{1} dx\,dy$



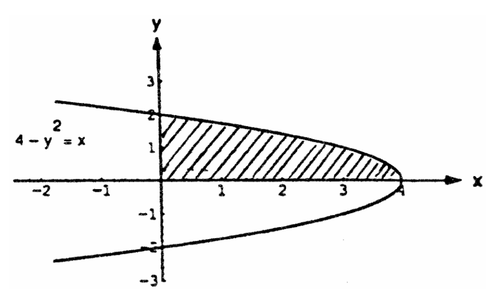
38. $\int_1^2 \int_0^{\ln x} dy\,dx$



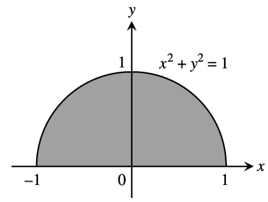
39. $\int_0^9 \int_0^{\frac{1}{2}\sqrt{9-y}} 16x\,dx\,dy$



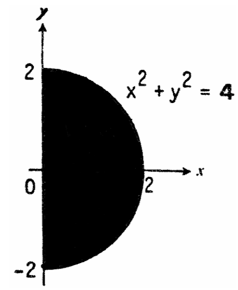
40. $\int_0^4 \int_0^{\sqrt{4-x}} y\,dy\,dx$



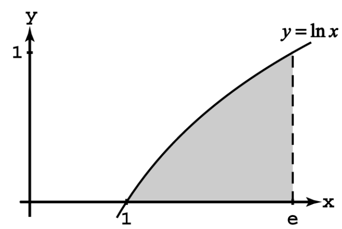
41. $\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} 3y\,dy\,dx$



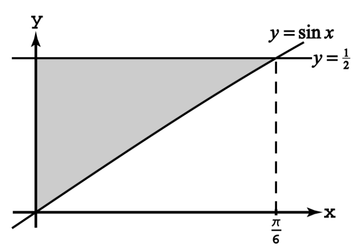
42. $\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}} 6x\,dx\,dy$



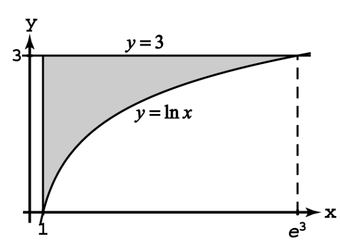
43. $\int_{0}^{1}\int_{e^y}^{e} x\,y\,dx\,dy$



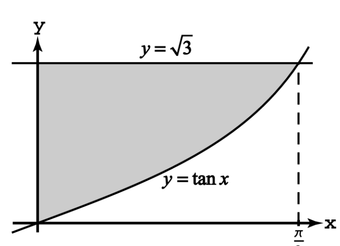
44. $\int_{0}^{1/2}\int_{0}^{\sin^{-1}y} x\,y^2\,dx\,dy$



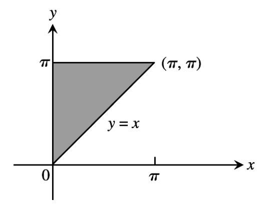
45. $\int_{1}^{e^3}\int_{\ln x}^{3} (x+y)dy\,dx$



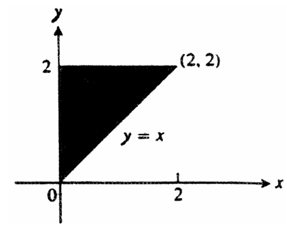
46. $\int_{0}^{\pi/3}\int_{\tan x}^{\sqrt{3}} \sqrt{x\,y}\,dy\,dx$



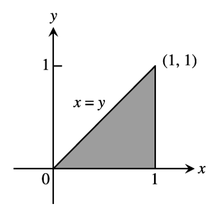
47. $\int_0^{\pi}\int_x^{\pi} \frac{\sin y}{y}\, dy\, dx = \int_0^{\pi}\int_0^{y} \frac{\sin y}{y}\, dx\, dy = \int_0^{\pi} \sin y\, dy = 2$



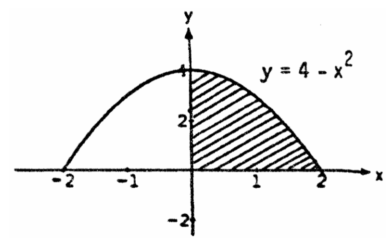
48. $\int_0^2\int_x^2 2y^2 \sin xy\, dy\, dx = \int_0^2\int_0^y 2y^2 \sin xy\, dx\, dy$
$= \int_0^2 [-2y \cos xy]_0^y\, dy = \int_0^2 (-2y \cos y^2 + 2y)\, dy$
$= [-\sin y^2 + y^2]_0^2 = 4 - \sin 4$



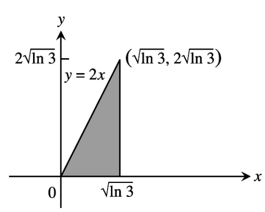
49. $\int_0^1\int_y^1 x^2 e^{xy}\, dx\, dy = \int_0^1\int_0^x x^2 e^{xy}\, dy\, dx = \int_0^1 [xe^{xy}]_0^x\, dx$
$= \int_0^1 (xe^{x^2} - x)\, dx = \left[\frac{1}{2} e^{x^2} - \frac{x^2}{2}\right]_0^1 = \frac{e-2}{2}$



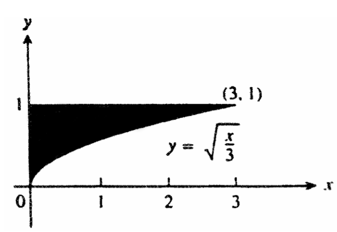
50. $\int_0^2\int_0^{4-x^2} \frac{xe^{2y}}{4-y}\, dy\, dx = \int_0^4\int_0^{\sqrt{4-y}} \frac{xe^{2y}}{4-y}\, dx\, dy$
$= \int_0^4 \left[\frac{x^2 e^{2y}}{2(4-y)}\right]_0^{\sqrt{4-y}} dy = \int_0^4 \frac{e^{2y}}{2}\, dy = \left[\frac{e^{2y}}{4}\right]_0^4 = \frac{e^8 - 1}{4}$



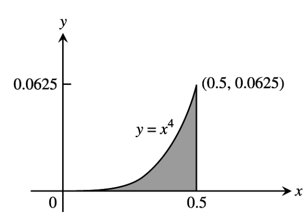
51. $\int_0^{2\sqrt{\ln 3}}\int_{y/2}^{\sqrt{\ln 3}} e^{x^2}\, dx\, dy = \int_0^{\sqrt{\ln 3}}\int_0^{2x} e^{x^2}\, dy\, dx$
$= \int_0^{\sqrt{\ln 3}} 2xe^{x^2}\, dx = [e^{x^2}]_0^{\sqrt{\ln 3}} = e^{\ln 3} - 1 = 2$



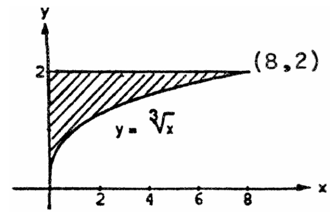
52. $\int_0^3\int_{\sqrt{x/3}}^1 e^{y^3}\, dy\, dx = \int_0^1\int_0^{3y^2} e^{y^3}\, dx\, dy$
$= \int_0^1 3y^2 e^{y^3}\, dy = [e^{y^3}]_0^1 = e - 1$



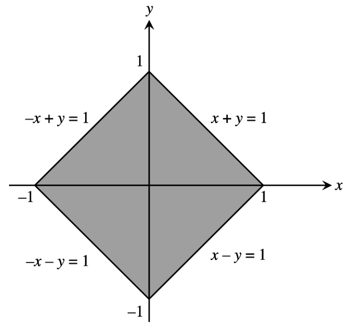
53. $\int_0^{1/16}\int_{y^{1/4}}^{1/2} \cos(16\pi x^5)\, dx\, dy = \int_0^{1/2}\int_0^{x^4} \cos(16\pi x^5)\, dy\, dx$
$= \int_0^{1/2} x^4 \cos(16\pi x^5)\, dx = \left[\frac{\sin(16\pi x^5)}{80\pi}\right]_0^{1/2} = \frac{1}{80\pi}$



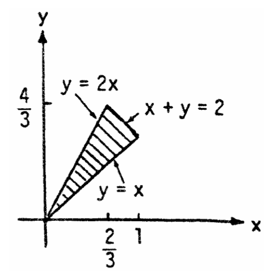
54. $\int_0^8 \int_{\sqrt[3]{x}}^2 \frac{1}{y^4+1}\,dy\,dx = \int_0^2 \int_0^{y^3} \frac{1}{y^4+1}\,dx\,dy$
$= \int_0^2 \frac{y^3}{y^4+1}\,dy = \frac{1}{4}\left[\ln\left(y^4+1\right)\right]_0^2 = \frac{\ln 17}{4}$



55. $\iint_R \left(y - 2x^2\right) dA$
$= \int_{-1}^{0} \int_{-x-1}^{x+1} \left(y - 2x^2\right) dy\,dx + \int_0^1 \int_{x-1}^{1-x} \left(y - 2x^2\right) dy\,dx$
$= \int_{-1}^{0} \left[\frac{1}{2}y^2 - 2x^2y\right]_{-x-1}^{x+1} dx + \int_0^1 \left[\frac{1}{2}y^2 - 2x^2y\right]_{x-1}^{1-x} dx$
$= \int_{-1}^{0} \left[\frac{1}{2}(x+1)^2 - 2x^2(x+1) - \frac{1}{2}(-x-1)^2 + 2x^2(-x-1)\right] dx$
$+ \int_0^1 \left[\frac{1}{2}(1-x)^2 - 2x^2(1-x) - \frac{1}{2}(x-1)^2 + 2x^2(x-1)\right] dx$
$= -4\int_{-1}^{0} \left(x^3 + x^2\right) dx + 4\int_0^1 \left(x^3 - x^2\right) dx$
$= -4\left[\frac{x^4}{4} + \frac{x^3}{3}\right]_{-1}^{0} + 4\left[\frac{x^4}{4} - \frac{x^3}{3}\right]_0^1 = 4\left[\frac{(-1)^4}{4} + \frac{(-1)^3}{3}\right] + 4\left(\frac{1}{4} - \frac{1}{3}\right) = 8\left(\frac{3}{12} - \frac{4}{12}\right) = -\frac{8}{12} = -\frac{2}{3}$



56. $\iint_R xy\,dA = \int_0^{2/3} \int_x^{2x} xy\,dy\,dx + \int_{2/3}^{1} \int_x^{2-x} xy\,dy\,dx$
$= \int_0^{2/3} \left[\frac{1}{2}xy^2\right]_x^{2x} dx + \int_{2/3}^{1} \left[\frac{1}{2}xy^2\right]_x^{2-x} dx$
$= \int_0^{2/3} \left(2x^3 - \frac{1}{2}x^3\right) dx + \int_{2/3}^{1} \left[\frac{1}{2}x(2-x)^2 - \frac{1}{2}x^3\right] dx$
$= \int_0^{2/3} \frac{3}{2}x^3\,dx + \int_{2/3}^{1} \left(2x - x^2\right) dx$
$= \left[\frac{3}{8}x^4\right]_0^{2/3} + \left[x^2 - \frac{2}{3}x^3\right]_{2/3}^{1} = \left(\frac{3}{8}\right)\left(\frac{16}{81}\right) + \left(1 - \frac{2}{3}\right) - \left[\frac{4}{9} - \left(\frac{2}{3}\right)\left(\frac{8}{27}\right)\right] = \frac{6}{81} + \frac{27}{81} - \left(\frac{36}{81} - \frac{16}{81}\right) = \frac{13}{81}$



57. $V = \int_0^1 \int_x^{2-x} \left(x^2 + y^2\right) dy\,dx = \int_0^1 \left[x^2y + \frac{y^3}{3}\right]_x^{2-x} dx = \int_0^1 \left[2x^2 - \frac{7x^3}{3} + \frac{(2-x)^3}{3}\right] dx = \left[\frac{2x^3}{3} - \frac{7x^4}{12} - \frac{(2-x)^4}{12}\right]_0^1$
$= \left(\frac{2}{3} - \frac{7}{12} - \frac{1}{12}\right) - \left(0 - 0 - \frac{16}{12}\right) = \frac{4}{3}$

58. $V = \int_{-2}^{1} \int_x^{2-x^2} x^2\,dy\,dx = \int_{-2}^{1} \left[x^2y\right]_x^{2-x^2} dx = \int_{-2}^{1} \left(2x^2 - x^4 - x^3\right) dx = \left[\frac{2}{3}x^3 - \frac{1}{5}x^5 - \frac{1}{4}x^4\right]_{-2}^{1}$
$= \left(\frac{2}{3} - \frac{1}{5} - \frac{1}{4}\right) - \left(-\frac{16}{3} + \frac{32}{5} - \frac{16}{4}\right) = \left(\frac{40}{60} - \frac{12}{60} - \frac{15}{60}\right) - \left(-\frac{320}{60} + \frac{384}{60} - \frac{240}{60}\right) = \frac{189}{60} = \frac{63}{20}$

59. $V = \int_{-4}^{1} \int_{3x}^{4-x^2} (x+4)\,dy\,dx = \int_{-4}^{1} \left[xy + 4y\right]_{3x}^{4-x^2} dx = \int_{-4}^{1} \left[x\left(4 - x^2\right) + 4\left(4 - x^2\right) - 3x^2 - 12x\right] dx$
$= \int_{-4}^{1} \left(-x^3 - 7x^2 - 8x + 16\right) dx = \left[-\frac{1}{4}x^4 - \frac{7}{3}x^3 - 4x^2 + 16x\right]_{-4}^{1} = \left(-\frac{1}{4} - \frac{7}{3} + 12\right) - \left(\frac{64}{3} - 64\right) = \frac{157}{3} - \frac{1}{4} = \frac{625}{12}$

60. $V = \int_0^2 \int_0^{\sqrt{4-x^2}} (3 - y)\,dy\,dx = \int_0^2 \left[3y - \frac{y^2}{2}\right]_0^{\sqrt{4-x^2}} dx = \int_0^2 \left[3\sqrt{4 - x^2} - \left(\frac{4-x^2}{2}\right)\right] dx$
$= \left[\frac{3}{2}x\sqrt{4 - x^2} + 6\sin^{-1}\left(\frac{x}{2}\right) - 2x + \frac{x^3}{6}\right]_0^2 = 6\left(\frac{\pi}{2}\right) - 4 + \frac{8}{6} = 3\pi - \frac{16}{6} = \frac{9\pi - 8}{3}$

61. $V = \int_0^2 \int_0^3 \left(4 - y^2\right) dx\,dy = \int_0^2 \left[4x - y^2x\right]_0^3 dy = \int_0^2 \left(12 - 3y^2\right) dy = \left[12y - y^3\right]_0^2 = 24 - 8 = 16$

62. $V=\int_0^2\int_0^{4-x^2}(4-x^2-y)\,dy\,dx=\int_0^2\left[(4-x^2)\,y-\frac{y^2}{2}\right]_0^{4-x^2}dx=\int_0^2\frac{1}{2}(4-x^2)^2\,dx=\int_0^2\left(8-4x^2+\frac{x^4}{2}\right)dx$
$=\left[8x-\frac{4}{3}x^3+\frac{1}{10}x^5\right]_0^2=16-\frac{32}{3}+\frac{32}{10}=\frac{480-320+96}{30}=\frac{128}{15}$

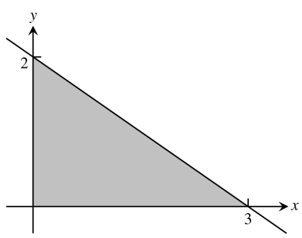
63. $V=\int_0^2\int_0^{2-x}(12-3y^2)\,dy\,dx=\int_0^2[12y-y^3]_0^{2-x}\,dx=\int_0^2[24-12x-(2-x)^3]\,dx=\left[24x-6x^2+\frac{(2-x)^4}{4}\right]_0^2=20$

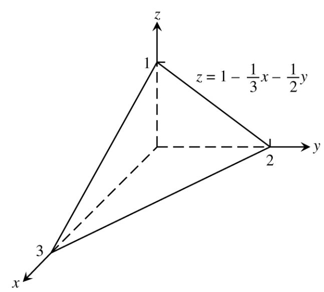
64. $V=\int_{-1}^0\int_{-x-1}^{x+1}(3-3x)\,dy\,dx+\int_0^1\int_{x-1}^{1-x}(3-3x)\,dy\,dx=6\int_{-1}^0(1-x^2)\,dx+6\int_0^1(1-x)^2\,dx=4+2=6$

65. $V=\int_1^2\int_{-1/x}^{1/x}(x+1)\,dy\,dx=\int_1^2[xy+y]_{-1/x}^{1/x}\,dx=\int_1^2\left[1+\frac{1}{x}-\left(-1-\frac{1}{x}\right)\right]dx=2\int_1^2\left(1+\frac{1}{x}\right)dx=2\,[x+\ln x]_1^2$
$=2(1+\ln 2)$

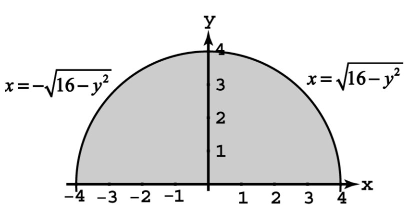
66. $V=4\int_0^{\pi/3}\int_0^{\sec x}(1+y^2)\,dy\,dx=4\int_0^{\pi/3}\left[y+\frac{y^3}{3}\right]_0^{\sec x}dx=4\int_0^{\pi/3}\left(\sec x+\frac{\sec^3 x}{3}\right)dx$
$=\frac{2}{3}\,[7\ln|\sec x+\tan x|+\sec x\tan x]_0^{\pi/3}=\frac{2}{3}\left[7\ln\left(2+\sqrt{3}\right)+2\sqrt{3}\right]$

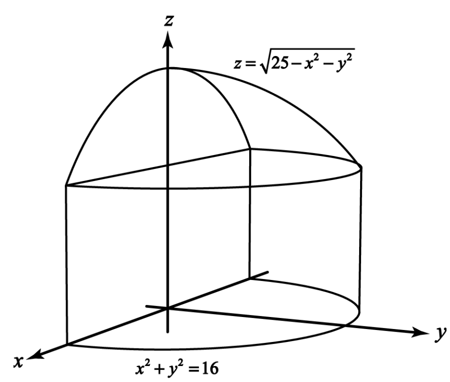
67.





68.





69. $\int_1^\infty\int_{e^{-x}}^1\frac{1}{x^3y}\,dy\,dx=\int_1^\infty\left[\frac{\ln y}{x^3}\right]_{e^{-x}}^1dx=\int_1^\infty-\left(\frac{-x}{x^3}\right)dx=-\lim_{b\to\infty}\left[\frac{1}{x}\right]_1^b=-\lim_{b\to\infty}\left(\frac{1}{b}-1\right)=1$

70. $\int_{-1}^1\int_{-1/\sqrt{1-x^2}}^{1/\sqrt{1-x^2}}(2y+1)\,dy\,dx=\int_{-1}^1[y^2+y]\Big|_{-1/(1-x^2)^{1/2}}^{1/(1-x^2)^{1/2}}dx=\int_{-1}^1\frac{2}{\sqrt{1-x^2}}\,dx=4\lim_{b\to 1^-}[\sin^{-1}x]_0^b$
$=4\lim_{b\to 1^-}[\sin^{-1}b-0]=2\pi$

71. $\int_{-\infty}^\infty\int_{-\infty}^\infty\frac{1}{(x^2+1)(y^2+1)}dx\,dy=2\int_0^\infty\left(\frac{2}{y^2+1}\right)\left(\lim_{b\to\infty}\tan^{-1}b-\tan^{-1}0\right)dy=2\pi\lim_{b\to\infty}\int_0^b\frac{1}{y^2+1}\,dy$
$=2\pi\left(\lim_{b\to\infty}\tan^{-1}b-\tan^{-1}0\right)=(2\pi)\left(\frac{\pi}{2}\right)=\pi^2$

72. $\int_0^{\infty}\int_0^{\infty} xe^{-(x+2y)}\,dx\,dy = \int_0^{\infty} e^{-2y} \lim_{b\to\infty} \left[-xe^{-x} - e^{-x}\right]_0^b\,dy = \int_0^{\infty} e^{-2y} \lim_{b\to\infty} \left(-be^{-b} - e^{-b} + 1\right)dy$
$= \int_0^{\infty} e^{-2y}\,dy = \frac{1}{2} \lim_{b\to\infty} \left(-e^{-2b} + 1\right) = \frac{1}{2}$

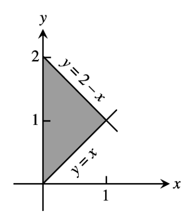
73. $\iint_R f(x,y)\,dA \approx \frac{1}{4}\, f\left(-\frac{1}{2}, 0\right) + \frac{1}{8}\, f(0,0) + \frac{1}{8}\, f\left(\frac{1}{4}, 0\right) = \frac{1}{4}\left(-\frac{1}{2}\right) + \frac{1}{8}\left(0 + \frac{1}{4}\right) = -\frac{3}{32}$

74. $\iint_R f(x,y)\,dA \approx \frac{1}{4}\left[f\left(\frac{7}{4}, \frac{11}{4}\right) + f\left(\frac{9}{4}, \frac{11}{4}\right) + f\left(\frac{7}{4}, \frac{13}{4}\right) + f\left(\frac{9}{4}, \frac{13}{4}\right)\right] = \frac{1}{16}(29 + 31 + 33 + 35) = \frac{128}{16} = 8$

75. The ray $\theta = \frac{\pi}{6}$ meets the circle $x^2 + y^2 = 4$ at the point $\left(\sqrt{3}, 1\right) \Rightarrow$ the ray is represented by the line $y = \frac{x}{\sqrt{3}}$. Thus,
$\iint_R f(x,y)\,dA = \int_0^{\sqrt{3}}\int_{x/\sqrt{3}}^{\sqrt{4-x^2}} \sqrt{4-x^2}\,dy\,dx = \int_0^{\sqrt{3}} \left[\left(4 - x^2\right) - \frac{x}{\sqrt{3}}\sqrt{4 - x^2}\right] dx = \left[4x - \frac{x^3}{3} + \frac{\left(4-x^2\right)^{3/2}}{3\sqrt{3}}\right]_0^{\sqrt{3}} = \frac{20\sqrt{3}}{9}$

76. $\int_2^{\infty}\int_0^2 \frac{1}{(x^2-x)\,(y-1)^{2/3}}\,dy\,dx = \int_2^{\infty} \left[\frac{3(y-1)^{1/3}}{(x^2-x)}\right]_0^2 dx = \int_2^{\infty} \left(\frac{3}{x^2-x} + \frac{3}{x^2-x}\right) dx = 6 \int_2^{\infty} \frac{dx}{x(x-1)}$
$= 6 \lim_{b\to\infty} \int_2^b \left(\frac{1}{x-1} - \frac{1}{x}\right) dx = 6 \lim_{b\to\infty} \left[\ln(x-1) - \ln x\right]_2^b = 6 \lim_{b\to\infty} \left[\ln(b-1) - \ln b - \ln 1 + \ln 2\right]$
$= 6\left[\lim_{b\to\infty} \ln\left(1 - \frac{1}{b}\right) + \ln 2\right] = 6 \ln 2$

77. $V = \int_0^1\int_x^{2-x} \left(x^2 + y^2\right) dy\,dx = \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_x^{2-x} dx$
$= \int_0^1 \left[2x^2 - \frac{7x^3}{3} + \frac{(2-x)^3}{3}\right] dx = \left[\frac{2x^3}{3} - \frac{7x^4}{12} - \frac{(2-x)^4}{12}\right]_0^1$
$= \left(\frac{2}{3} - \frac{7}{12} - \frac{1}{12}\right) - \left(0 - 0 - \frac{16}{12}\right) = \frac{4}{3}$



78. $\int_0^2 \left(\tan^{-1} \pi x - \tan^{-1} x\right) dx = \int_0^2\int_x^{\pi x} \frac{1}{1+y^2}\,dy\,dx = \int_0^2\int_{y/\pi}^{y} \frac{1}{1+y^2}\,dx\,dy + \int_2^{2\pi}\int_{y/\pi}^{2} \frac{1}{1+y^2}\,dx\,dy$
$= \int_0^2 \frac{\left(1-\frac{1}{\pi}\right)y}{1+y^2}\,dy + \int_2^{2\pi} \frac{\left(2-\frac{y}{\pi}\right)}{1+y^2}\,dy = \left(\frac{\pi-1}{2\pi}\right)\left[\ln\left(1+y^2\right)\right]_0^2 + \left[2\tan^{-1} y + \frac{1}{2\pi}\ln\left(1+y^2\right)\right]_2^{2\pi}$
$= \left(\frac{\pi-1}{2\pi}\right)\ln 5 + 2\tan^{-1} 2\pi - \frac{1}{2\pi}\ln\left(1 + 4\pi^2\right) - 2\tan^{-1} 2 + \frac{1}{2\pi}\ln 5$
$= 2\tan^{-1} 2\pi - 2\tan^{-1} 2 - \frac{1}{2\pi}\ln\left(1 + 4\pi^2\right) + \frac{\ln 5}{2}$

79. To maximize the integral, we want the domain to include all points where the integrand is positive and to exclude all points where the integrand is negative. These criteria are met by the points (x, y) such that $4 - x^2 - 2y^2 \geq 0$ or $x^2 + 2y^2 \leq 4$, which is the ellipse $x^2 + 2y^2 = 4$ together with its interior.

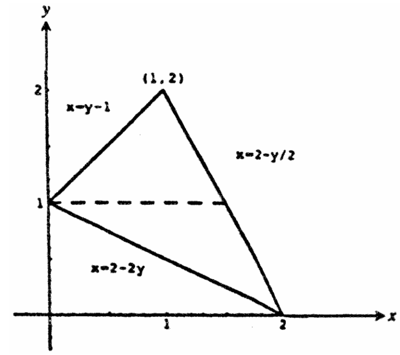
80. To minimize the integral, we want the domain to include all points where the integrand is negative and to exclude all points where the integrand is positive. These criteria are met by the points (x, y) such that $x^2 + y^2 - 9 \leq 0$ or $x^2 + y^2 \leq 9$, which is the closed disk of radius 3 centered at the origin.

81. No, it is not possible. By Fubini's theorem, the two orders of integration must give the same result.

82. One way would be to partition R into two triangles with the line y = 1. The integral of f over R could then be written as a sum of integrals that could be evaluated by integrating first with respect to x and then with respect to y:

$$\iint_R f(x, y)\, dA = \int_0^1 \int_{2-2y}^{2-(y/2)} f(x, y)\, dx\, dy + \int_1^2 \int_{y-1}^{2-(y/2)} f(x, y)\, dx\, dy.$$

Partitioning R with the line x = 1 would let us write the integral of f over R as a sum of iterated integrals with order dy dx.



83. $\int_{-b}^{b}\int_{-b}^{b} e^{-x^2-y^2}\, dx\, dy = \int_{-b}^{b}\int_{-b}^{b} e^{-y^2} e^{-x^2}\, dx\, dy = \int_{-b}^{b} e^{-y^2}\left(\int_{-b}^{b} e^{-x^2}\, dx\right) dy = \left(\int_{-b}^{b} e^{-x^2}\, dx\right)\left(\int_{-b}^{b} e^{-y^2}\, dy\right)$

$= \left(\int_{-b}^{b} e^{-x^2}\, dx\right)^2 = \left(2\int_0^b e^{-x^2}\, dx\right)^2 = 4\left(\int_0^b e^{-x^2}\, dx\right)^2$; taking limits as $b \to \infty$ gives the stated result.

84. $\int_0^1 \int_0^3 \frac{x^2}{(y-1)^{2/3}}\, dy\, dx = \int_0^3 \int_0^1 \frac{x^2}{(y-1)^{2/3}}\, dx\, dy = \int_0^3 \frac{1}{(y-1)^{2/3}}\left[\frac{x^3}{3}\right]_0^1 dy = \frac{1}{3}\int_0^3 \frac{dy}{(y-1)^{2/3}}$

$= \frac{1}{3}\lim_{b \to 1^-}\int_0^b \frac{dy}{(y-1)^{2/3}} + \frac{1}{3}\lim_{b \to 1^+}\int_b^3 \frac{dy}{(y-1)^{2/3}} = \lim_{b \to 1^-}\left[(y-1)^{1/3}\right]_0^b + \lim_{b \to 1^+}\left[(y-1)^{1/3}\right]_b^3$

$= \left[\lim_{b \to 1^-}(b-1)^{1/3} - (-1)^{1/3}\right] - \left[\lim_{b \to 1^+}(b-1)^{1/3} - (2)^{1/3}\right] = (0+1) - \left(0 - \sqrt[3]{2}\right) = 1 + \sqrt[3]{2}$

85-88. Example CAS commands:

Maple:

```
f := (x,y) -> 1/x/y;
q1 := Int( Int( f(x,y), y=1..x ), x=1..3 );
evalf( q1 );
value( q1 );
evalf( value(q1) );
```

89-94. Example CAS commands:

Maple:

```
f := (x,y) -> exp(x^2);
c,d := 0,1;
g1 := y ->2*y;
g2 := y -> 4;
q5 := Int( Int( f(x,y), x=g1(y)..g2(y) ), y=c..d );
value( q5 );
plot3d( 0, x=g1(y)..g2(y), y=c..d, color=pink, style=patchnogrid, axes=boxed, orientation=[-90,0],
        scaling=constrained, title="#89 (Section 15.2)" );
r5 := Int( Int( f(x,y), y=0..x/2 ), x=0..2 ) + Int( Int( f(x,y), y=0..1 ), x=2..4 );
value( r5);
value( q5-r5 );
```

85-94. Example CAS commands:

Mathematica: (functions and bounds will vary)

You can integrate using the built-in integral signs or with the command **Integrate**. In the **Integrate** command, the integration begins with the variable on the right. (In this case, y going from 1 to x).

```
Clear[x, y, f]
f[x_, y_]:= 1 / (x y)
Integrate[f[x, y], {x, 1, 3}, {y, 1, x}]
```

To reverse the order of integration, it is best to first plot the region over which the integration extends. This can be done with ImplicitPlot and all bounds involving both x and y can be plotted. A graphics package must be loaded. Remember to use the double equal sign for the equations of the bounding curves.

```
Clear[x, y, f]
<<Graphics`ImplicitPlot`
ImplicitPlot[{x==2y, x==4, y==0, y==1},{x, 0, 4.1}, {y, 0, 1.1}];
f[x_, y_]:=Exp[x^2]
Integrate[f[x, y], {x, 0, 2}, {y, 0, x/2}] + Integrate[f[x, y], {x, 2, 4}, {y, 0, 1}]
```

To get a numerical value for the result, use the numerical integrator, **NIntegrate**. Verify that this equals the original.

```
Integrate[f[x, y], {x, 0, 2}, {y, 0, x/2}] + NIntegrate[f[x, y], {x, 2, 4}, {y, 0, 1}]
NIntegrate[f[x, y], {y, 0, 1},{x, 2y, 4}]
```

Another way to show a region is with the FilledPlot command. This assumes that functions are given as y = f(x).

```
Clear[x, y, f]
<<Graphics`FilledPlot`
FilledPlot[{x^2, 9},{x, 0,3}, AxesLabels → {x, y}];
f[x_, y_]:= x Cos[y^2]
Integrate[f[x, y], {y, 0, 9}, {x, 0, Sqrt[y]}]
```

85. $\int_1^3 \int_1^x \frac{1}{xy}\, dy\, dx \approx 0.603$

86. $\int_0^1 \int_0^1 e^{-(x^2+y^2)}\, dy\, dx \approx 0.558$

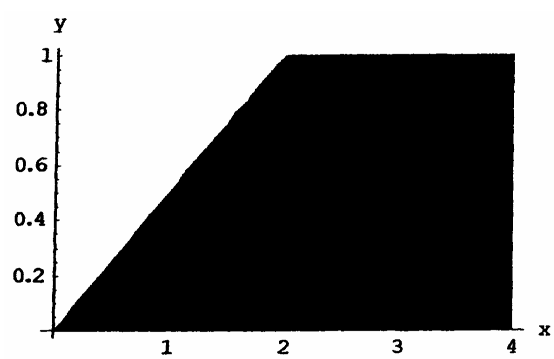
87. $\int_0^1 \int_0^1 \tan^{-1} xy\, dy\, dx \approx 0.233$

88. $\int_{-1}^1 \int_0^{\sqrt{1-x^2}} 3\sqrt{1-x^2-y^2}\, dy\, dx \approx 3.142$

89. Evaluate the integrals:

$$\int_0^1 \int_{2y}^4 e^{x^2}\, dx\, dy$$
$$= \int_0^2 \int_0^{x/2} e^{x^2}\, dy\, dx + \int_2^4 \int_0^1 e^{x^2}\, dy\, dx$$
$$= -\frac{1}{4} + \frac{1}{4}\left(e^4 - 2\sqrt{\pi}\,\mathrm{erfi}(2) + 2\sqrt{\pi}\,\mathrm{erfi}(4)\right)$$
$$\approx 1.1494 \times 10^6$$

The following graph was generated using Mathematica.

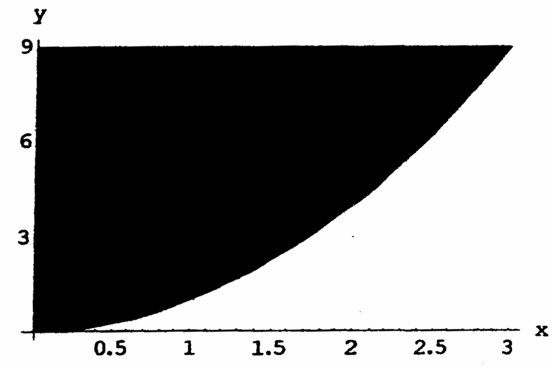


90. Evaluate the integrals:

$\int_0^3 \int_{x^2}^9 x \cos(y^2) dy\, dx = \int_0^9 \int_0^{\sqrt{y}} x \cos(y^2) dx\, dy$

$= \frac{\sin(81)}{4} \approx -0.157472$

The following graph was generated using Mathematica.

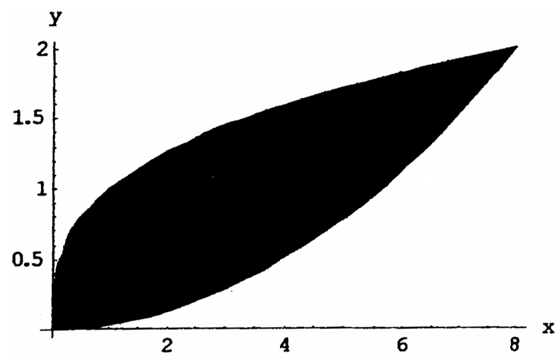


91. Evaluate the integrals:

$\int_0^2 \int_{y^3}^{4\sqrt{2y}} (x^2y - xy^2) dx\, dy = \int_0^8 \int_{x^2/32}^{\sqrt[3]{x}} (x^2y - xy^2) dy\, dx$

$= \frac{67,520}{693} \approx 97.4315$

The following graph was generated using Mathematica.

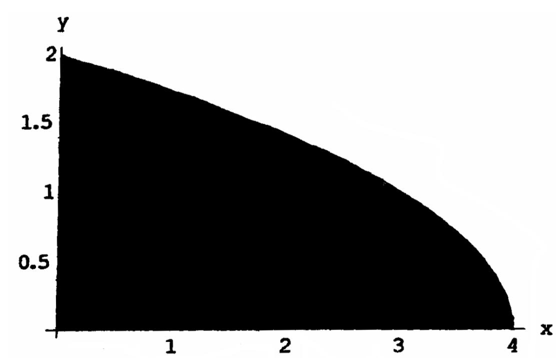


92. Evaluate the integrals:

$\int_0^2 \int_0^{4-y^2} e^{xy}\, dx\, dy = \int_0^4 \int_0^{\sqrt{4-x}} e^{xy}\, dy\, dx$

≈ 20.5648

The following graph was generated using Mathematica.



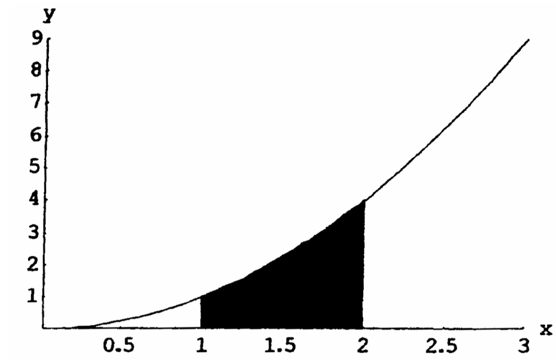
93. Evaluate the integrals:

$\int_1^2 \int_0^{x^2} \frac{1}{x+y}\, dy\, dx$

$= \int_0^1 \int_1^2 \frac{1}{x+y}\, dx\, dy + \int_1^4 \int_{\sqrt{y}}^2 \frac{1}{x+y}\, dx\, dy$

$-1 + \ln\left(\frac{27}{4}\right) \approx 0.909543$

The following graph was generated using Mathematica.



94. Evaluate the integrals:

$\int_1^2 \int_{y^3}^8 \frac{1}{\sqrt{x^2+y^2}}\, dx\, dy = \int_1^8 \int_1^{\sqrt[3]{x}} \frac{1}{\sqrt{x^2+y^2}}\, dy\, dx$

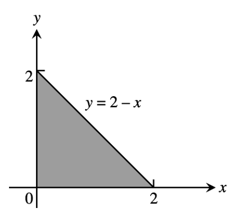
≈ 0.866649

The following graph was generated using Mathematica.

15.3 AREA BY DOUBLE INTEGRATION

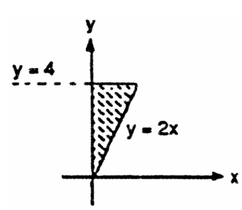
1. $\int_0^2 \int_0^{2-x} dy\, dx = \int_0^2 (2-x)\, dx = \left[2x - \frac{x^2}{2}\right]_0^2 = 2,$

or $\int_0^2 \int_0^{2-y} dx\, dy = \int_0^2 (2-y)\, dy = 2$



2. $\int_0^2 \int_{2x}^4 dy\, dx = \int_0^2 (4-2x)\, dx = \left[4x - x^2\right]_0^2 = 4,$

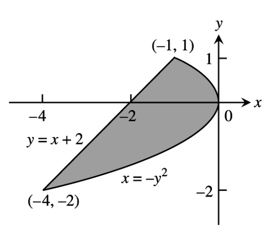
or $\int_0^4 \int_0^{y/2} dx\, dy = \int_0^4 \frac{y}{2}\, dy = 4$



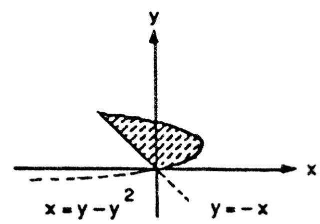
3. $\int_{-2}^1 \int_{y-2}^{-y^2} dx\, dy = \int_{-2}^1 (-y^2 - y + 2)\, dy$

$= \left[-\frac{y^3}{3} - \frac{y^2}{2} + 2y\right]_{-2}^1$

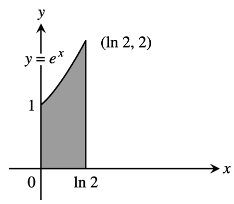
$= \left(-\frac{1}{3} - \frac{1}{2} + 2\right) - \left(\frac{8}{3} - 2 - 4\right) = \frac{9}{2}$



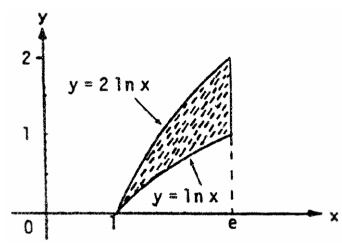
4. $\int_0^2 \int_{-y}^{y-y^2} dx\,dy = \int_0^2 (2y - y^2)\,dy = \left[y^2 - \frac{y^3}{3}\right]_0^2$
$= 4 - \frac{8}{3} = \frac{4}{3}$



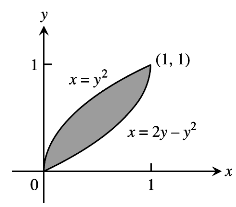
5. $\int_0^{\ln 2} \int_0^{e^x} dy\,dx = \int_0^{\ln 2} e^x\,dx = [e^x]_0^{\ln 2} = 2 - 1 = 1$



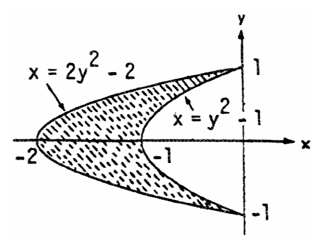
6. $\int_1^e \int_{\ln x}^{2\ln x} dy\,dx = \int_1^e \ln x\,dx = [x \ln x - x]_1^e$
$= (e - e) - (0 - 1) = 1$



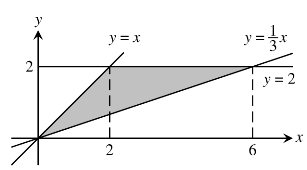
7. $\int_0^1 \int_{y^2}^{2y-y^2} dx\,dy = \int_0^1 (2y - 2y^2)\,dy = \left[y^2 - \frac{2}{3}y^3\right]_0^1$
$= \frac{1}{3}$



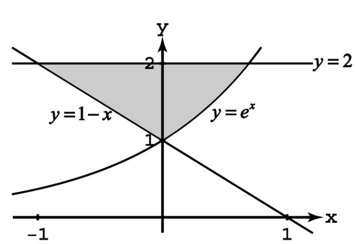
8. $\int_{-1}^1 \int_{2y^2-2}^{y^2-1} dx\,dy = \int_{-1}^1 (y^2 - 1 - 2y^2 + 2)\,dy$
$= \int_{-1}^1 (1 - y^2)\,dy = \left[y - \frac{y^3}{3}\right]_{-1}^1 = \frac{4}{3}$



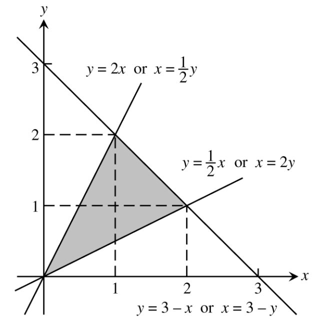
9. $\int_0^2 \int_y^{3y} 1\,dx\,dy = \int_0^2 [x]_y^{3y} dy$
$= \int_0^2 (2y)\,dy = [y^2]_0^2 = 4$



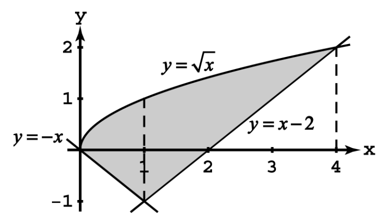
10. $\int_1^2 \int_{1-y}^{\ln y} 1\,dx\,dy = \int_0^2 [x]_{1-y}^{\ln y} dy$
$= \int_1^2 (\ln y - 1 + y)\,dy = \left[y \ln y - 2y + \frac{y^2}{2}\right]_1^2$
$= 2 \ln 2 - \frac{1}{2}$



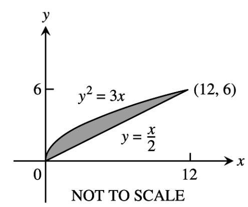
11. $\int_0^1 \int_{x/2}^{2x} 1\, dy\, dx + \int_1^2 \int_{x/2}^{3-x} 1\, dy\, dx$
$= \int_0^1 [y]_{x/2}^{2x} dx + \int_1^2 [y]_{x/2}^{3-x} dx$
$= \int_0^1 \left(\frac{3}{2}x\right) dx + \int_1^2 \left(3 - \frac{3}{2}x\right) dx$
$= \left[\frac{3}{4}x^2\right]_0^1 + \left[3x - \frac{3}{4}x^2\right]_1^2 = \frac{3}{2}$



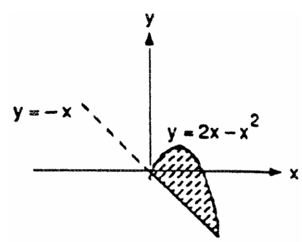
12. $\int_0^1 \int_{-x}^{\sqrt{x}} 1\, dy\, dx + \int_1^4 \int_{x-2}^{\sqrt{x}} 1\, dy\, dx$
$= \int_0^1 [y]_{-x}^{\sqrt{x}} dx + \int_1^4 [y]_{x-2}^{\sqrt{x}} dx$
$= \int_0^1 \left(\sqrt{x} + x\right) dx + \int_1^4 \left(\sqrt{x} - x + 2\right) dx$
$= \left[\frac{2}{3}x^{3/2} + \frac{1}{2}x^2\right]_0^1 + \left[\frac{2}{3}x^{3/2} - \frac{1}{2}x^2 + 2x\right]_1^4 = \frac{13}{3}$



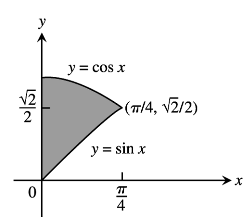
13. $\int_0^6 \int_{y^2/3}^{2y} dx\, dy = \int_0^6 \left(2y - \frac{y^2}{3}\right) dy = \left[y^2 - \frac{y^3}{9}\right]_0^6$
$= 36 - \frac{216}{9} = 12$



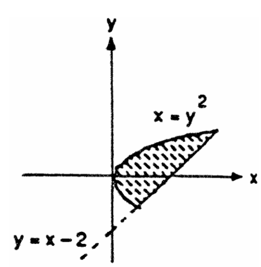
14. $\int_0^3 \int_{-x}^{2x-x^2} dy\, dx = \int_0^3 (3x - x^2)\, dx = \left[\frac{3}{2}x^2 - \frac{1}{3}x^3\right]_0^3$
$= \frac{27}{2} - 9 = \frac{9}{2}$



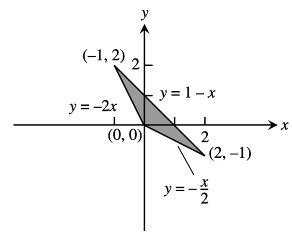
15. $\int_0^{\pi/4} \int_{\sin x}^{\cos x} dy\, dx$
$= \int_0^{\pi/4} (\cos x - \sin x)\, dx = [\sin x + \cos x]_0^{\pi/4}$
$= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (0 + 1) = \sqrt{2} - 1$



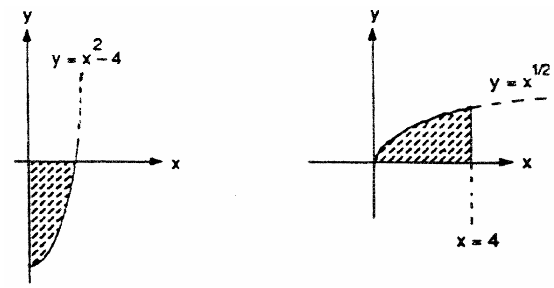
16. $\int_{-1}^2 \int_{y^2}^{y+2} dx\, dy = \int_{-1}^2 (y + 2 - y^2)\, dy = \left[\frac{y^2}{2} + 2y - \frac{y^3}{3}\right]_{-1}^2$
$= \left(2 + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right) = 5 - \frac{1}{2} = \frac{9}{2}$



17. $\int_{-1}^{0}\int_{-2x}^{1-x} dy\,dx + \int_{0}^{2}\int_{-x/2}^{1-x} dy\,dx$
$= \int_{-1}^{0}(1+x)\,dx + \int_{0}^{2}\left(1-\frac{x}{2}\right)dx$
$= \left[x+\frac{x^2}{2}\right]_{-1}^{0} + \left[x-\frac{x^2}{4}\right]_{0}^{2} = -\left(-1+\frac{1}{2}\right)+(2-1) = \frac{3}{2}$



18. $\int_{0}^{2}\int_{x^2-4}^{0} dy\,dx + \int_{0}^{4}\int_{0}^{\sqrt{x}} dy\,dx$
$= \int_{0}^{2}(4-x^2)\,dx + \int_{0}^{4} x^{1/2}\,dx$
$= \left[4x-\frac{x^3}{3}\right]_{0}^{2} + \left[\frac{2}{3}x^{3/2}\right]_{0}^{4} = \left(8-\frac{8}{3}\right)+\frac{16}{3} = \frac{32}{3}$



19. (a) average $= \frac{1}{\pi^2}\int_{0}^{\pi}\int_{0}^{\pi}\sin(x+y)\,dy\,dx = \frac{1}{\pi^2}\int_{0}^{\pi}\left[-\cos(x+y)\right]_{0}^{\pi}dx = \frac{1}{\pi^2}\int_{0}^{\pi}\left[-\cos(x+\pi)+\cos x\right]dx$
$= \frac{1}{\pi^2}\left[-\sin(x+\pi)+\sin x\right]_{0}^{\pi} = \frac{1}{\pi^2}\left[(-\sin 2\pi+\sin\pi)-(-\sin\pi+\sin 0)\right] = 0$

(b) average $= \frac{1}{\left(\frac{\pi^2}{2}\right)}\int_{0}^{\pi}\int_{0}^{\pi/2}\sin(x+y)\,dy\,dx = \frac{2}{\pi^2}\int_{0}^{\pi}\left[-\cos(x+y)\right]_{0}^{\pi/2}dx = \frac{2}{\pi^2}\int_{0}^{\pi}\left[-\cos\left(x+\frac{\pi}{2}\right)+\cos x\right]dx$
$= \frac{2}{\pi^2}\left[-\sin\left(x+\frac{\pi}{2}\right)+\sin x\right]_{0}^{\pi} = \frac{2}{\pi^2}\left[\left(-\sin\frac{3\pi}{2}+\sin\pi\right)-\left(-\sin\frac{\pi}{2}+\sin 0\right)\right] = \frac{4}{\pi^2}$

20. average value over the square $= \int_{0}^{1}\int_{0}^{1} xy\,dy\,dx = \int_{0}^{1}\left[\frac{xy^2}{2}\right]_{0}^{1}dx = \int_{0}^{1}\frac{x}{2}\,dx = \frac{1}{4} = 0.25;$

average value over the quarter circle $= \frac{1}{\left(\frac{\pi}{4}\right)}\int_{0}^{1}\int_{0}^{\sqrt{1-x^2}} xy\,dy\,dx = \frac{4}{\pi}\int_{0}^{1}\left[\frac{xy^2}{2}\right]_{0}^{\sqrt{1-x^2}}dx$
$= \frac{2}{\pi}\int_{0}^{1}(x-x^3)\,dx = \frac{2}{\pi}\left[\frac{x^2}{2}-\frac{x^4}{4}\right]_{0}^{1} = \frac{1}{2\pi} \approx 0.159$. The average value over the square is larger.

21. average height $= \frac{1}{4}\int_{0}^{2}\int_{0}^{2}(x^2+y^2)\,dy\,dx = \frac{1}{4}\int_{0}^{2}\left[x^2y+\frac{y^3}{3}\right]_{0}^{2}dx = \frac{1}{4}\int_{0}^{2}\left(2x^2+\frac{8}{3}\right)dx = \frac{1}{2}\left[\frac{x^3}{3}+\frac{4x}{3}\right]_{0}^{2} = \frac{8}{3}$

22. average $= \frac{1}{(\ln 2)^2}\int_{\ln 2}^{2\ln 2}\int_{\ln 2}^{2\ln 2}\frac{1}{xy}\,dy\,dx = \frac{1}{(\ln 2)^2}\int_{\ln 2}^{2\ln 2}\left[\frac{\ln y}{x}\right]_{\ln 2}^{2\ln 2}dx$
$= \frac{1}{(\ln 2)^2}\int_{\ln 2}^{2\ln 2}\frac{1}{x}(\ln 2+\ln\ln 2-\ln\ln 2)\,dx = \left(\frac{1}{\ln 2}\right)\int_{\ln 2}^{2\ln 2}\frac{dx}{x} = \left(\frac{1}{\ln 2}\right)\left[\ln x\right]_{\ln 2}^{2\ln 2}$
$= \left(\frac{1}{\ln 2}\right)(\ln 2+\ln\ln 2-\ln\ln 2) = 1$

23. $\int_{-5}^{5}\int_{-2}^{0}\frac{10{,}000e^y}{1+\frac{|x|}{2}}\,dy\,dx = 10{,}000\left(1-e^{-2}\right)\int_{-5}^{5}\frac{dx}{1+\frac{|x|}{2}} = 10{,}000\left(1-e^{-2}\right)\left[\int_{-5}^{0}\frac{dx}{1-\frac{x}{2}}+\int_{0}^{5}\frac{dx}{1+\frac{x}{2}}\right]$
$= 10{,}000\left(1-e^{-2}\right)\left[-2\ln\left(1-\frac{x}{2}\right)\right]_{-5}^{0} + 10{,}000\left(1-e^{-2}\right)\left[2\ln\left(1+\frac{x}{2}\right)\right]_{0}^{5}$
$= 10{,}000\left(1-e^{-2}\right)\left[2\ln\left(1+\frac{5}{2}\right)\right] + 10{,}000\left(1-e^{-2}\right)\left[2\ln\left(1+\frac{5}{2}\right)\right] = 40{,}000\left(1-e^{-2}\right)\ln\left(\frac{7}{2}\right) \approx 43{,}329$

24. $\int_{0}^{1}\int_{y^2}^{2y-y^2}100(y+1)\,dx\,dy = \int_{0}^{1}\left[100(y+1)x\right]_{y^2}^{2y-y^2}dy = \int_{0}^{1}100(y+1)\left(2y-2y^2\right)dy = 200\int_{0}^{1}\left(y-y^3\right)dy$
$= 200\left[\frac{y^2}{2}-\frac{y^4}{4}\right]_{0}^{1} = (200)\left(\frac{1}{4}\right) = 50$

25. Let (x_i, y_i) be the location of the weather station in county i for $i = 1, \ldots, 254$. The average temperature in Texas at time t_0 is approximately $\frac{\sum_{i=1}^{254} T(x_i,y_i)\,\Delta_i A}{A}$, where $T(x_i, y_i)$ is the temperature at time t_0 at the weather station in county i, $\Delta_i A$ is the area of county i, and A is the area of Texas.

26. Let $y = f(x)$ be a nonnegative, continuous function on [a, b], then $A = \iint_R dA = \int_a^b \int_0^{f(x)} dy\,dx = \int_a^b [y]_0^{f(x)}\,dx = \int_a^b f(x)\,dx$

15.4 DOUBLE INTEGRALS IN POLAR FORM

1. $x^2 + y^2 = 9^2 \Rightarrow r = 9 \Rightarrow \frac{\pi}{2} \le \theta \le 2\pi,\ 0 \le r \le 9$

2. $x^2 + y^2 = 1^2 \Rightarrow r = 1,\ x^2 + y^2 = 4^2 \Rightarrow r = 4 \Rightarrow -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},\ 1 \le r \le 4$

3. $y = x \Rightarrow \theta = \frac{\pi}{4},\ y = -x \Rightarrow \theta = \frac{3\pi}{4},\ y = 1 \Rightarrow r = \csc\theta \Rightarrow \frac{\pi}{4} \le \theta \le \frac{3\pi}{4},\ 0 \le r \le \csc\theta$

4. $x = 1 \Rightarrow r = \sec\theta,\ y = \sqrt{3}x \Rightarrow \theta = \frac{\pi}{3} \Rightarrow 0 \le \theta \le \frac{\pi}{3},\ 0 \le r \le \sec\theta$

5. $x^2 + y^2 = 1^2 \Rightarrow r = 1,\ x = 2\sqrt{3} \Rightarrow r = 2\sqrt{3}\sec\theta,\ y = 2 \Rightarrow r = 2\csc\theta;\ 2\sqrt{3}\sec\theta = 2\csc\theta \Rightarrow \theta = \frac{\pi}{6}$
$\Rightarrow 0 \le \theta \le \frac{\pi}{6},\ 1 \le r \le 2\sqrt{3}\sec\theta;\ \frac{\pi}{6} \le \theta \le \frac{\pi}{2},\ 1 \le r \le 2\sqrt{3}\csc\theta$

6. $x^2 + y^2 = 2^2 \Rightarrow r = 2,\ x = 1 \Rightarrow r = \sec\theta;\ 2 = \sec\theta \Rightarrow \theta = \frac{\pi}{3}$ or $\theta = -\frac{\pi}{3} \Rightarrow -\frac{\pi}{3} \le \theta \le \frac{\pi}{3},\ \sec\theta \le r \le 2$

7. $x^2 + y^2 = 2x \Rightarrow r = 2\cos\theta \Rightarrow -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},\ 0 \le r \le 2\cos\theta$

8. $x^2 + y^2 = 2y \Rightarrow r = 2\sin\theta \Rightarrow 0 \le \theta \le \pi,\ 0 \le r \le 2\sin\theta$

9. $\int_{-1}^{1}\int_0^{\sqrt{1-x^2}} dy\,dx = \int_0^{\pi}\int_0^1 r\,dr\,d\theta = \frac{1}{2}\int_0^{\pi} d\theta = \frac{\pi}{2}$

10. $\int_0^1\int_0^{\sqrt{1-y^2}} (x^2 + y^2)\,dx\,dy = \int_0^{\pi/2}\int_0^1 r^3\,dr\,d\theta = \frac{1}{4}\int_0^{\pi/2} d\theta = \frac{\pi}{8}$

11. $\int_0^2\int_0^{\sqrt{4-y^2}} (x^2 + y^2)\,dx\,dy = \int_0^{\pi/2}\int_0^2 r^3\,dr\,d\theta = 4\int_0^{\pi/2} d\theta = 2\pi$

12. $\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\,dx = \int_0^{2\pi}\int_0^a r\,dr\,d\theta = \frac{a^2}{2}\int_0^{2\pi} d\theta = \pi a^2$

13. $\int_0^6\int_0^y x\,dx\,dy = \int_{\pi/4}^{\pi/2}\int_0^{6\csc\theta} r^2\cos\theta\,dr\,d\theta = 72\int_{\pi/4}^{\pi/2}\cot\theta\csc^2\theta\,d\theta = -36\left[\cot^2\theta\right]_{\pi/4}^{\pi/2} = 36$

14. $\int_0^2\int_0^x y\,dy\,dx = \int_0^{\pi/4}\int_0^{2\sec\theta} r^2\sin\theta\,dr\,d\theta = \frac{8}{3}\int_0^{\pi/4}\tan\theta\sec^2\theta\,d\theta = \frac{4}{3}$

15. $\int_1^{\sqrt{3}}\int_1^x dy\,dx = \int_{\pi/6}^{\pi/4}\int_{\csc\theta}^{\sqrt{3}\sec\theta} r\,dr\,d\theta = \int_{\pi/6}^{\pi/4}\left(\frac{3}{2}\sec^2\theta - \frac{1}{2}\csc^2\theta\right)d\theta = \left[\frac{3}{2}\tan\theta + \frac{1}{2}\cot\theta\right]_{\pi/6}^{\pi/4} = 2 - \sqrt{3}$

16. $\int_{\sqrt{2}}^2\int_{\sqrt{4-y^2}}^{y} dy\,dx = \int_{\pi/4}^{\pi/2}\int_2^{2\csc\theta} r\,dr\,d\theta = \int_{\pi/6}^{\pi/4}(2\csc^2\theta - 2)\,d\theta = \left[-2\cot\theta - \frac{1}{2}\theta\right]_{\pi/4}^{\pi/2} = 2 - \frac{\pi}{2}$

17. $\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{0} \frac{2}{1+\sqrt{x^2+y^2}}\,dy\,dx = \int_{\pi}^{3\pi/2}\int_0^1 \frac{2r}{1+r}\,dr\,d\theta = 2\int_{\pi}^{3\pi/2}\int_0^1 \left(1-\frac{1}{1+r}\right) dr\,d\theta = 2\int_{\pi}^{3\pi/2} (1-\ln 2)\,d\theta$
$= (1-\ln 2)\pi$

18. $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{(1+x^2+y^2)^2}\,dy\,dx = 4\int_0^{\pi/2}\int_0^1 \frac{2r}{(1+r^2)^2}\,dr\,d\theta = 4\int_0^{\pi/2}\left[-\frac{1}{1+r^2}\right]_0^1 d\theta = 2\int_0^{\pi/2} d\theta = \pi$

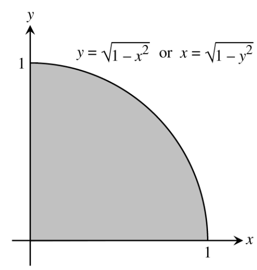
19. $\int_0^{\ln 2}\int_0^{\sqrt{(\ln 2)^2-y^2}} e^{\sqrt{x^2+y^2}}\,dx\,dy = \int_0^{\pi/2}\int_0^{\ln 2} re^r\,dr\,d\theta = \int_0^{\pi/2}(2\ln 2-1)\,d\theta = \frac{\pi}{2}(2\ln 2-1)$

20. $\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln(x^2+y^2+1)\,dx\,dy = 4\int_0^{\pi/2}\int_0^1 \ln(r^2+1)\,r\,dr\,d\theta = 2\int_0^{\pi/2}(\ln 4-1)\,d\theta = \pi(\ln 4-1)$

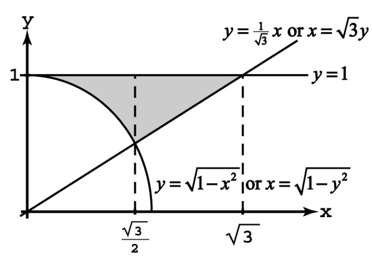
21. $\int_0^1\int_x^{\sqrt{2-x^2}} (x+2y)\,dy\,dx = \int_{\pi/4}^{\pi/2}\int_0^{\sqrt{2}} (r\cos\theta+2r\sin\theta)\,r\,dr\,d\theta = \int_{\pi/4}^{\pi/2}\left[\frac{r^3}{3}\cos\theta+\frac{2r^3}{3}\sin\theta\right]_0^{\sqrt{2}} d\theta$
$= \int_{\pi/4}^{\pi/2}\left(\frac{2\sqrt{2}}{3}\cos\theta+\frac{4\sqrt{2}}{3}\sin\theta\right)d\theta = \left[\frac{2\sqrt{2}}{3}\sin\theta-\frac{4\sqrt{2}}{3}\cos\theta\right]_{\pi/4}^{\pi/2} = \frac{2(1+\sqrt{2})}{3}$

22. $\int_1^2\int_0^{\sqrt{2x-x^2}} \frac{1}{(x^2+y^2)^2}\,dy\,dx = \int_0^{\pi/4}\int_{\sec\theta}^{2\cos\theta} \frac{1}{r^4}\,r\,dr\,d\theta = \int_0^{\pi/4}\left[-\frac{1}{2r^2}\right]_{\sec\theta}^{2\cos\theta} d\theta = = \int_0^{\pi/4}\left(\frac{1}{2}\cos^2\theta-\frac{1}{8}\sec^2\theta\right)d\theta$
$= \left[\frac{1}{4}\theta+\frac{1}{8}\sin 2\theta-\frac{1}{8}\tan\theta\right]_0^{\pi/4} = \frac{\pi}{16}$

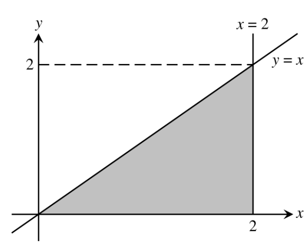
23. $\int_0^1\int_0^{\sqrt{1-x^2}} x\,y\,dy\,dx$ or
$\int_0^1\int_0^{\sqrt{1-y^2}} x\,y\,dx\,dy$



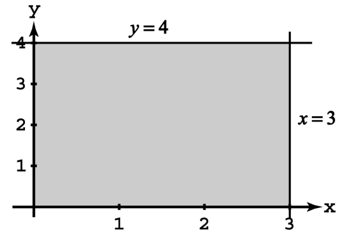
24. $\int_{1/2}^{1}\int_{\sqrt{1-y^2}}^{\sqrt{3}y} x\,dx\,dy$ or
$\int_0^{\sqrt{3}/2}\int_{\sqrt{1-x^2}}^{1} x\,dy\,dx + \int_{\sqrt{3}/2}^{\sqrt{3}}\int_{x/\sqrt{3}}^{1} x\,dy\,dx$



25. $\int_0^2\int_0^x y^2(x^2+y^2)\,dy\,dx$ or
$\int_0^2\int_y^2 y^2(x^2+y^2)\,dx\,dy$



26. $\int_0^3\int_0^4 (x^2+y^2)^3\,dy\,dx$ or
$\int_0^4\int_0^3 (x^2+y^2)^3\,dx\,dy$



27. $\int_0^{\pi/2}\int_0^{2\sqrt{2-\sin 2\theta}} r\,dr\,d\theta = 2\int_0^{\pi/2}(2-\sin 2\theta)\,d\theta = 2(\pi-1)$

28. $A = 2\int_0^{\pi/2}\int_1^{1+\cos\theta} r\,dr\,d\theta = \int_0^{\pi/2}(2\cos\theta+\cos^2\theta)\,d\theta = \frac{8+\pi}{4}$

29. $A = 2\int_0^{\pi/6}\int_0^{12\cos 3\theta} r\,dr\,d\theta = 144\int_0^{\pi/6}\cos^2 3\theta\,d\theta = 12\pi$

30. $A = \int_0^{2\pi}\int_0^{4\theta/3} r\,dr\,d\theta = \frac{8}{9}\int_0^{2\pi}\theta^2\,d\theta = \frac{64\pi^3}{27}$

31. $A = \int_0^{\pi/2}\int_0^{1+\sin\theta} r\,dr\,d\theta = \frac{1}{2}\int_0^{\pi/2}\left(\frac{3}{2}+2\sin\theta-\frac{\cos 2\theta}{2}\right)d\theta = \frac{3\pi}{8}+1$

32. $A = 4\int_0^{\pi/2}\int_0^{1-\cos\theta} r\,dr\,d\theta = 2\int_0^{\pi/2}\left(\frac{3}{2}-2\cos\theta+\frac{\cos 2\theta}{2}\right)d\theta = \frac{3\pi}{2}-4$

33. $\text{average} = \frac{4}{\pi a^2}\int_0^{\pi/2}\int_0^{a} r\sqrt{a^2-r^2}\,dr\,d\theta = \frac{4}{3\pi a^2}\int_0^{\pi/2} a^3\,d\theta = \frac{2a}{3}$

34. $\text{average} = \frac{4}{\pi a^2}\int_0^{\pi/2}\int_0^{a} r^2\,dr\,d\theta = \frac{4}{3\pi a^2}\int_0^{\pi/2} a^3\,d\theta = \frac{2a}{3}$

35. $\text{average} = \frac{1}{\pi a^2}\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\,dy\,dx = \frac{1}{\pi a^2}\int_0^{2\pi}\int_0^{a} r^2\,dr\,d\theta = \frac{a}{3\pi}\int_0^{2\pi} d\theta = \frac{2a}{3}$

36. $\text{average} = \frac{1}{\pi}\iint_R [(1-x)^2+y^2]\,dy\,dx = \frac{1}{\pi}\int_0^{2\pi}\int_0^{1}[(1-r\cos\theta)^2+r^2\sin^2\theta]\,r\,dr\,d\theta$
$= \frac{1}{\pi}\int_0^{2\pi}\int_0^{1}(r^3-2r^2\cos\theta+r)\,dr\,d\theta = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{3}{4}-\frac{2\cos\theta}{3}\right)d\theta = \frac{1}{\pi}\left[\frac{3}{4}\theta-\frac{2\sin\theta}{3}\right]_0^{2\pi} = \frac{3}{2}$

37. $\int_0^{2\pi}\int_1^{\sqrt{e}}\left(\frac{\ln r^2}{r}\right) r\,dr\,d\theta = \int_0^{2\pi}\int_1^{\sqrt{e}} 2\ln r\,dr\,d\theta = 2\int_0^{2\pi}[r\ln r - r]_1^{e^{1/2}}\,d\theta = 2\int_0^{2\pi}\sqrt{e}\left[\left(\frac{1}{2}-1\right)+1\right]d\theta = 2\pi\left(2-\sqrt{e}\right)$

38. $\int_0^{2\pi}\int_1^{e}\left(\frac{\ln r^2}{r}\right)dr\,d\theta = \int_0^{2\pi}\int_1^{e}\left(\frac{2\ln r}{r}\right)dr\,d\theta = \int_0^{2\pi}[(\ln r)^2]_1^{e}\,d\theta = \int_0^{2\pi}d\theta = 2\pi$

39. $V = 2\int_0^{\pi/2}\int_1^{1+\cos\theta} r^2\cos\theta\,dr\,d\theta = \frac{2}{3}\int_0^{\pi/2}(3\cos^2\theta+3\cos^3\theta+\cos^4\theta)\,d\theta$
$= \frac{2}{3}\left[\frac{15\theta}{8}+\sin 2\theta+3\sin\theta-\sin^3\theta+\frac{\sin 4\theta}{32}\right]_0^{\pi/2} = \frac{4}{3}+\frac{5\pi}{8}$

40. $V = 4\int_0^{\pi/4}\int_0^{\sqrt{2\cos 2\theta}} r\sqrt{2-r^2}\,dr\,d\theta = -\frac{4}{3}\int_0^{\pi/4}\left[(2-2\cos 2\theta)^{3/2}-2^{3/2}\right]d\theta$
$= \frac{2\pi\sqrt{2}}{3}-\frac{32}{3}\int_0^{\pi/4}(1-\cos^2\theta)\sin\theta\,d\theta = \frac{2\pi\sqrt{2}}{3}-\frac{32}{3}\left[\frac{\cos^3\theta}{3}-\cos\theta\right]_0^{\pi/4} = \frac{6\pi\sqrt{2}+40\sqrt{2}-64}{9}$

41. (a) $I^2 = \int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\,dx\,dy = \int_0^{\pi/2}\int_0^{\infty}\left(e^{-r^2}\right) r\,dr\,d\theta = \int_0^{\pi/2}\left[\lim_{b\to\infty}\int_0^{b} re^{-r^2}\,dr\right]d\theta$

$= -\frac{1}{2}\int_0^{\pi/2}\lim_{b\to\infty}\left(e^{-b^2}-1\right)d\theta = \frac{1}{2}\int_0^{\pi/2}d\theta = \frac{\pi}{4} \Rightarrow I = \frac{\sqrt{\pi}}{2}$

(b) $\lim_{x\to\infty}\int_0^{x}\frac{2e^{-t^2}}{\sqrt{\pi}}\,dt = \frac{2}{\sqrt{\pi}}\int_0^{\infty}e^{-t^2}\,dt = \left(\frac{2}{\sqrt{\pi}}\right)\left(\frac{\sqrt{\pi}}{2}\right) = 1$, from part (a)

42. $\int_0^{\infty}\int_0^{\infty}\frac{1}{(1+x^2+y^2)^2}\,dx\,dy = \int_0^{\pi/2}\int_0^{\infty}\frac{r}{(1+r^2)^2}\,dr\,d\theta = \frac{\pi}{2}\lim_{b\to\infty}\int_0^{b}\frac{r}{(1+r^2)^2}\,dr = \frac{\pi}{4}\lim_{b\to\infty}\left[-\frac{1}{1+r^2}\right]_0^b$

$= \frac{\pi}{4}\lim_{b\to\infty}\left(1-\frac{1}{1+b^2}\right) = \frac{\pi}{4}$

43. Over the disk $x^2+y^2 \le \frac{3}{4}$: $\iint_R \frac{1}{1-x^2-y^2}\,dA = \int_0^{2\pi}\int_0^{\sqrt{3}/2}\frac{r}{1-r^2}\,dr\,d\theta = \int_0^{2\pi}\left[-\frac{1}{2}\ln\left(1-r^2\right)\right]_0^{\sqrt{3}/2}d\theta$

$= \int_0^{2\pi}\left(-\frac{1}{2}\ln\frac{1}{4}\right)d\theta = (\ln 2)\int_0^{2\pi}d\theta = \pi\ln 4$

Over the disk $x^2+y^2 \le 1$: $\iint_R \frac{1}{1-x^2-y^2}\,dA = \int_0^{2\pi}\int_0^{1}\frac{r}{1-r^2}\,dr\,d\theta = \int_0^{2\pi}\left[\lim_{a\to 1^-}\int_0^{a}\frac{r}{1-r^2}\,dr\right]d\theta$

$= \int_0^{2\pi}\lim_{a\to 1^-}\left[-\frac{1}{2}\ln\left(1-a^2\right)\right]d\theta = 2\pi\cdot\lim_{a\to 1^-}\left[-\frac{1}{2}\ln\left(1-a^2\right)\right] = 2\pi\cdot\infty$, so the integral does not exist over $x^2+y^2 \le 1$

44. The area in polar coordinates is given by $A = \int_{\alpha}^{\beta}\int_0^{f(\theta)} r\,dr\,d\theta = \int_{\alpha}^{\beta}\left[\frac{r^2}{2}\right]_0^{f(\theta)}d\theta = \frac{1}{2}\int_{\alpha}^{\beta}f^2(\theta)\,d\theta = \int_{\alpha}^{\beta}\frac{1}{2}r^2\,d\theta$, where $r = f(\theta)$

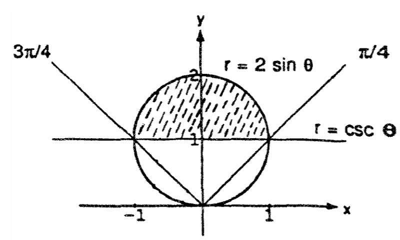
45. average $= \frac{1}{\pi a^2}\int_0^{2\pi}\int_0^{a}\left[(r\cos\theta-h)^2+r^2\sin^2\theta\right]r\,dr\,d\theta = \frac{1}{\pi a^2}\int_0^{2\pi}\int_0^{a}\left(r^3-2r^2h\cos\theta+rh^2\right)dr\,d\theta$

$= \frac{1}{\pi a^2}\int_0^{2\pi}\left(\frac{a^4}{4}-\frac{2a^3h\cos\theta}{3}+\frac{a^2h^2}{2}\right)d\theta = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{a^2}{4}-\frac{2ah\cos\theta}{3}+\frac{h^2}{2}\right)d\theta = \frac{1}{\pi}\left[\frac{a^2\theta}{4}-\frac{2ah\sin\theta}{3}+\frac{h^2\theta}{2}\right]_0^{2\pi}$

$= \frac{1}{2}\left(a^2+2h^2\right)$

46. $A = \int_{\pi/4}^{3\pi/4}\int_{\csc\theta}^{2\sin\theta} r\,dr\,d\theta = \frac{1}{2}\int_{\pi/4}^{3\pi/4}\left(4\sin^2\theta-\csc^2\theta\right)d\theta$

$= \frac{1}{2}\left[2\theta-\sin 2\theta+\cot\theta\right]_{\pi/4}^{3\pi/4} = \frac{\pi}{2}$



47-50. Example CAS commands:

Maple:

```
f := (x,y) -> y/(x^2+y^2);
    a,b := 0,1;
    f1 := x -> x;
    f2 := x -> 1;
    plot3d( f(x,y), y=f1(x)..f2(x), x=a..b, axes=boxed, style=patchnogrid, shading=zhue, orientation=[0,180], title="#47(a)
        (Section 15.4)" );                                            # (a)
    q1 := eval( x=a, [x=r*cos(theta),y=r*sin(theta)] );               # (b)
    q2 := eval( x=b, [x=r*cos(theta),y=r*sin(theta)] );
    q3 := eval( y=f1(x), [x=r*cos(theta),y=r*sin(theta)] );
    q4 := eval( y=f2(x), [x=r*cos(theta),y=r*sin(theta)] );
    theta1 := solve( q3, theta );
```

```
theta2 := solve( q1, theta );
r1 := 0;
r2 := solve( q4, r );
plot3d(0,r=r1..r2, theta=theta1..theta2, axes=boxed, style=patchnogrid, shading=zhue, orientation=[-90,0],
        title="#47(c) (Section 15.4)" );
fP := simplify(eval( f(x,y), [x=r*cos(theta),y=r*sin(theta)] ));                # (d)
q5 := Int( Int( fP*r, r=r1..r2 ), theta=theta1..theta2 );
value( q5 );
```

Mathematica: (functions and bounds will vary)

For 47 and 48, begin by drawing the region of integration with the **FilledPlot** command.

```
Clear[x, y, r, t]
<<Graphics`FilledPlot`
FilledPlot[{x, 1}, {x, 0, 1}, AspectRatio → 1, AxesLabel → {x,y}];
```

The picture demonstrates that r goes from 0 to the line y=1 or r = 1/ Sin[t], while t goes from π/4 to π/2.

```
f:= y / (x^2 + y^2)
topolar={x → r Cos[t], y → r Sin[t]};
fp= f/.topolar //Simplify
Integrate[r fp, {t, π/4, π/2}, {r, 0, 1/Sin[t]}]
```

For 49 and 50, drawing the region of integration with the ImplicitPlot command.

```
Clear[x, y]
<<Graphics`ImplicitPlot`
ImplicitPlot[{x==y, x==2 − y, y==0, y==1}, {x, 0, 2.1}, {y, 0, 1.1}];
```

The picture shows that as t goes from 0 to π/4, r goes from 0 to the line x=2 − y. **Solve** will find the bound for r.

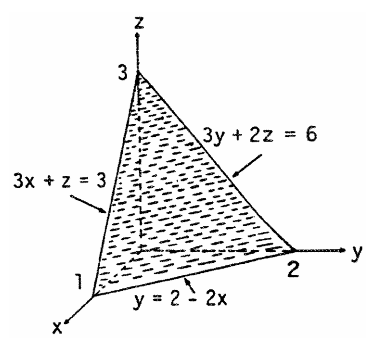
```
bdr=Solve[r Cos[t]==2 − r Sin[t], r]//Simplify
f:=Sqrt[x + y]
topolar={x → r Cos[t], y →  r Sin[t]};
fp= f/.topolar //Simplify
Integrate[r fp, {t, 0, π/4}, {r, 0, bdr[[1, 1, 2]]}]
```

15.5 TRIPLE INTEGRALS IN RECTANGULAR COORDINATES

1. $\int_0^1\int_0^{1-x}\int_{x+z}^1 F(x,y,z)\,dy\,dz\,dx = \int_0^1\int_0^{1-x}\int_{x+z}^1 dy\,dz\,dx = \int_0^1\int_0^{1-x}(1-x-z)\,dz\,dx$
$= \int_0^1 \left[(1-x) - x(1-x) - \frac{(1-x)^2}{2}\right]dx = \int_0^1 \frac{(1-x)^2}{2}\,dx = \left[-\frac{(1-x)^3}{6}\right]_0^1 = \frac{1}{6}$

2. $\int_0^1\int_0^2\int_0^3 dz\,dy\,dx = \int_0^1\int_0^2 3\,dy\,dx = \int_0^1 6\,dx = 6,\ \int_0^2\int_0^1\int_0^3 dz\,dx\,dy,\ \int_0^3\int_0^2\int_0^1 dx\,dy\,dz,\ \int_0^2\int_0^3\int_0^1 dx\,dz\,dy,$
$\int_0^3\int_0^1\int_0^2 dy\,dx\,dz,\ \int_0^1\int_0^3\int_0^2 dy\,dz\,dx$

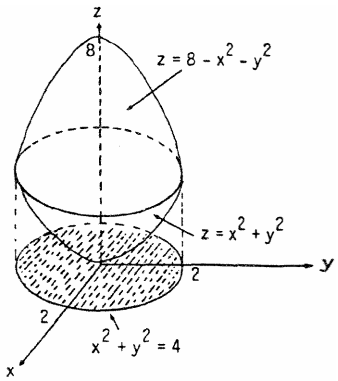
3. $\int_0^1\int_0^{2-2x}\int_0^{3-3x-3y/2} dz\,dy\,dx$
$= \int_0^1\int_0^{2-2x}\left(3-3x-\frac{3}{2}y\right)dy\,dx$
$= \int_0^1\left[3(1-x)\cdot 2(1-x) - \frac{3}{4}\cdot 4(1-x)^2\right]dx$
$= 3\int_0^1 (1-x)^2\,dx = \left[-(1-x)^3\right]_0^1 = 1,$
$\int_0^2\int_0^{1-y/2}\int_0^{3-3x-3y/2} dz\,dx\,dy,\ \int_0^1\int_0^{3-3x}\int_0^{2-2x-2z/3} dy\,dz\,dx,$



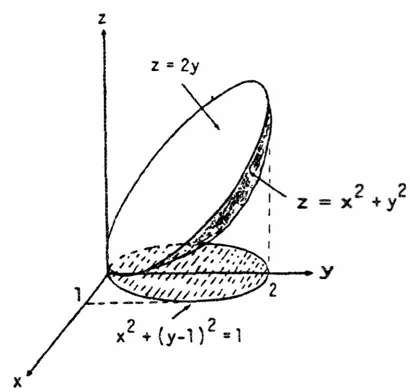
$\int_0^3\int_0^{1-z/3}\int_0^{2-2x-2z/3} dy\,dx\,dz$, $\int_0^2\int_0^{3-3y/2}\int_0^{1-y/2-z/3} dx\,dz\,dy$, $\int_0^3\int_0^{2-2z/3}\int_0^{1-y/2-z/3} dx\,dy\,dz$

4. $\int_0^2\int_0^3\int_0^{\sqrt{4-x^2}} dz\,dy\,dx = \int_0^2\int_0^3 \sqrt{4-x^2}\,dy\,dx = \int_0^2 3\sqrt{4-x^2}\,dx = \frac{3}{2}\left[x\sqrt{4-x^2} + 4\sin^{-1}\frac{x}{2}\right]_0^2 = 6\sin^{-1} 1 = 3\pi$,
$\int_0^3\int_0^2\int_0^{\sqrt{4-x^2}} dz\,dx\,dy$, $\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^3 dy\,dz\,dx$, $\int_0^2\int_0^{\sqrt{4-z^2}}\int_0^3 dy\,dx\,dz$, $\int_0^2\int_0^3\int_0^{\sqrt{4-z^2}} dx\,dy\,dz$, $\int_0^3\int_0^2\int_0^{\sqrt{4-z^2}} dx\,dz\,dy$

5. $\int_{-2}^2\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{x^2+y^2}^{8-x^2-y^2} dz\,dy\,dx = 4\int_0^2\int_0^{\sqrt{4-x^2}}\int_{x^2+y^2}^{8-x^2-y^2} dz\,dy\,dx$
$= 4\int_0^2\int_0^{\sqrt{4-x^2}} [8 - 2(x^2+y^2)]\,dy\,dx$
$= 8\int_0^2\int_0^{\sqrt{4-x^2}} (4 - x^2 - y^2)\,dy\,dx$
$= 8\int_0^{\pi/2}\int_0^2 (4 - r^2)\,r\,dr\,d\theta = 8\int_0^{\pi/2}\left[2r^2 - \frac{r^4}{4}\right]_0^2 d\theta$
$= 32\int_0^{\pi/2} d\theta = 32\left(\frac{\pi}{2}\right) = 16\pi$,
$\int_{-2}^2\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{x^2+y^2}^{8-x^2-y^2} dz\,dx\,dy$,
$\int_{-2}^2\int_{y^2}^4\int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} dx\,dz\,dy + \int_{-2}^2\int_4^{8-y^2}\int_{-\sqrt{8-z-y^2}}^{\sqrt{8-z-y^2}} dx\,dz\,dy$,
$\int_0^4\int_{-\sqrt{z}}^{\sqrt{z}}\int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} dx\,dy\,dz + \int_4^8\int_{-\sqrt{8-z}}^{\sqrt{8-z}}\int_{-\sqrt{8-z-y^2}}^{\sqrt{8-z-y^2}} dx\,dy\,dz$, $\int_{-2}^2\int_{x^2}^4\int_{-\sqrt{z-x^2}}^{\sqrt{z-x^2}} dy\,dz\,dx + \int_{-2}^2\int_4^{8-x^2}\int_{-\sqrt{8-z-x^2}}^{\sqrt{8-z-x^2}} dy\,dz\,dx$,
$\int_0^4\int_{-\sqrt{z}}^{\sqrt{z}}\int_{-\sqrt{z-x^2}}^{\sqrt{z-x^2}} dy\,dx\,dz + \int_4^8\int_{-\sqrt{8-z}}^{\sqrt{8-z}}\int_{-\sqrt{8-z-x^2}}^{\sqrt{8-z-x^2}} dy\,dx\,dz$



6. The projection of D onto the xy-plane has the boundary $x^2 + y^2 = 2y \Rightarrow x^2 + (y-1)^2 = 1$, which is a circle. Therefore the two integrals are:
$\int_0^2\int_{-\sqrt{2y-y^2}}^{\sqrt{2y-y^2}}\int_{x^2+y^2}^{2y} dz\,dx\,dy$ and $\int_{-1}^1\int_{1-\sqrt{1-x^2}}^{1+\sqrt{1-x^2}}\int_{x^2+y^2}^{2y} dz\,dy\,dx$



7. $\int_0^1\int_0^1\int_0^1 (x^2+y^2+z^2)\,dz\,dy\,dx = \int_0^1\int_0^1 \left(x^2+y^2+\frac{1}{3}\right) dy\,dx = \int_0^1 \left(x^2+\frac{2}{3}\right) dx = 1$

8. $\int_0^{\sqrt{2}}\int_0^{3y}\int_{x^2+3y^2}^{8-x^2-y^2} dz\,dx\,dy = \int_0^{\sqrt{2}}\int_0^{3y} (8 - 2x^2 - 4y^2)\,dx\,dy = \int_0^{\sqrt{2}} \left[8x - \frac{2}{3}x^3 - 4xy^2\right]_0^{3y} dy$
$= \int_0^{\sqrt{2}} (24y - 18y^3 - 12y^3)\,dy = \left[12y^2 - \frac{15}{2}y^4\right]_0^{\sqrt{2}} = 24 - 30 = -6$

9. $\int_1^e\int_1^{e^2}\int_1^{e^3} \frac{1}{xyz}\,dx\,dy\,dz = \int_1^e\int_1^{e^2}\left[\frac{\ln x}{yz}\right]_1^{e^3} dy\,dz = \int_1^e\int_1^{e^2} \frac{3}{yz}\,dy\,dz = \int_1^e \left[\frac{\ln y}{z}\right]_1^{e^2} dz = \int_1^e \frac{6}{z}\,dz = 6$

10. $\int_0^1\int_0^{3-3x}\int_0^{3-3x-y} dz\,dy\,dx = \int_0^1\int_0^{3-3x} (3 - 3x - y)\,dy\,dx = \int_0^1 \left[(3-3x)^2 - \frac{1}{2}(3-3x)^2\right] dx = \frac{9}{2}\int_0^1 (1-x)^2\,dx$
$= -\frac{3}{2}\left[(1-x)^3\right]_0^1 = \frac{3}{2}$

11. $\int_0^{\pi/6}\int_0^1\int_{-2}^3 y\sin z\,dx\,dy\,dz = \int_0^{\pi/6}\int_0^1 5y\sin z\,dy\,dz = \frac{5}{2}\int_0^{\pi/6}\sin z\,dz = \frac{5\left(2-\sqrt{3}\right)}{4}$

12. $\int_{-1}^1\int_0^1\int_0^2 (x+y+z)\,dy\,dx\,dz = \int_{-1}^1\int_0^1 = \left[xy+\frac{1}{2}y^2+zy\right]_0^2 dx\,dz = \int_{-1}^1\int_0^1 (2x+2+2z)\,dx\,dz$
$= \int_{-1}^1 \left[x^2+2x+2zx\right]_0^1 dz = \int_{-1}^1 (3+2z)\,dz = \left[3z+z^2\right]_{-1}^1 = 6$

13. $\int_0^3\int_0^{\sqrt{9-x^2}}\int_0^{\sqrt{9-x^2}} dz\,dy\,dx = \int_0^3\int_0^{\sqrt{9-x^2}} \sqrt{9-x^2}\,dy\,dx = \int_0^3 (9-x^2)\,dx = \left[9x-\frac{x^3}{3}\right]_0^3 = 18$

14. $\int_0^2\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_0^{2x+y} dz\,dx\,dy = \int_0^2\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (2x+y)\,dx\,dy = \int_0^2 \left[x^2+xy\right]_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dy = \int_0^2 (4-y^2)^{1/2}(2y)\,dy$
$= \left[-\frac{2}{3}(4-y^2)^{3/2}\right]_0^2 = \frac{2}{3}(4)^{3/2} = \frac{16}{3}$

15. $\int_0^1\int_0^{2-x}\int_0^{2-x-y} dz\,dy\,dx = \int_0^1\int_0^{2-x} (2-x-y)\,dy\,dx = \int_0^1 \left[(2-x)^2-\frac{1}{2}(2-x)^2\right] dx = \frac{1}{2}\int_0^1 (2-x)^2\,dx$
$= \left[-\frac{1}{6}(2-x)^3\right]_0^1 = -\frac{1}{6}+\frac{8}{6} = \frac{7}{6}$

16. $\int_0^1\int_0^{1-x^2}\int_3^{4-x^2-y} x\,dz\,dy\,dx = \int_0^1\int_0^{1-x^2} x(1-x^2-y)\,dy\,dx = \int_0^1 x\left[(1-x^2)^2-\frac{1}{2}(1-x^2)\right] dx = \int_0^1 \frac{1}{2}x(1-x^2)^2\,dx$
$= \left[-\frac{1}{12}(1-x^2)^3\right]_0^1 = \frac{1}{12}$

17. $\int_0^\pi\int_0^\pi\int_0^\pi \cos(u+v+w)\,du\,dv\,dw = \int_0^\pi\int_0^\pi [\sin(w+v+\pi)-\sin(w+v)]\,dv\,dw$
$= \int_0^\pi [(-\cos(w+2\pi)+\cos(w+\pi))+(\cos(w+\pi)-\cos w)]\,dw$
$= [-\sin(w+2\pi)+\sin(w+\pi)-\sin w+\sin(w+\pi)]_0^\pi = 0$

18. $\int_0^1\int_1^{\sqrt{e}}\int_1^e s e^s \ln r\,\frac{(\ln t)^2}{t}\,dt\,dr\,ds = \int_0^1\int_1^{\sqrt{e}} (s e^s \ln r)\left[\frac{1}{3}(\ln t)^3\right]_1^e dr\,ds = \int_0^1\int_1^{\sqrt{e}} \frac{s e^s}{3}\ln r\,dr\,ds = \int_0^1 \frac{s e^s}{3}\left[r\ln r - r\right]_1^{\sqrt{e}} ds$
$= \frac{2-\sqrt{e}}{6}\int_0^1 s e^s ds = \frac{2-\sqrt{e}}{6}\left[s e^s - e^s\right]_0^1 = \frac{2-\sqrt{e}}{6}$

19. $\int_0^{\pi/4}\int_0^{\ln\sec v}\int_{-\infty}^{2t} e^x\,dx\,dt\,dv = \int_0^{\pi/4}\int_0^{\ln\sec v} \lim_{b\to-\infty}(e^{2t}-e^b)\,dt\,dv = \int_0^{\pi/4}\int_0^{\ln\sec v} e^{2t}\,dt\,dv = \int_0^{\pi/4}\left(\frac{1}{2}e^{2\ln\sec v}-\frac{1}{2}\right) dv$
$= \int_0^{\pi/4}\left(\frac{\sec^2 v}{2}-\frac{1}{2}\right) dv = \left[\frac{\tan v}{2}-\frac{v}{2}\right]_0^{\pi/4} = \frac{1}{2}-\frac{\pi}{8}$

20. $\int_0^7\int_0^2\int_0^{\sqrt{4-q^2}} \frac{q}{r+1}\,dp\,dq\,dr = \int_0^7\int_0^2 \frac{q\sqrt{4-q^2}}{r+1}\,dq\,dr = \int_0^7 \frac{1}{3(r+1)}\left[-(4-q^2)^{3/2}\right]_0^2 dr = \frac{8}{3}\int_0^7 \frac{1}{r+1}\,dr = \frac{8\ln 8}{3} = 8\ln 2$

21. (a) $\int_{-1}^1\int_0^{1-x^2}\int_{x^2}^{1-z} dy\,dz\,dx$ (b) $\int_0^1\int_{-\sqrt{1-z}}^{\sqrt{1-z}}\int_{x^2}^{1-z} dy\,dx\,dz$ (c) $\int_0^1\int_0^{1-z}\int_{-\sqrt{y}}^{\sqrt{y}} dx\,dy\,dz$
(d) $\int_0^1\int_0^{1-y}\int_{-\sqrt{y}}^{\sqrt{y}} dx\,dz\,dy$ (e) $\int_0^1\int_{-\sqrt{y}}^{\sqrt{y}}\int_0^{1-y} dz\,dx\,dy$

22. (a) $\int_0^1\int_0^1\int_{-1}^{-\sqrt{z}} dy\,dz\,dx$ (b) $\int_0^1\int_0^1\int_{-1}^{-\sqrt{z}} dy\,dx\,dz$ (c) $\int_0^1\int_{-1}^{-\sqrt{z}}\int_0^1 dx\,dy\,dz$
(d) $\int_{-1}^0\int_0^{y^2}\int_0^1 dx\,dz\,dy$ (e) $\int_{-1}^0\int_0^1\int_0^{y^2} dz\,dx\,dy$

23. $V = \int_0^1\int_{-1}^1\int_0^{y^2} dz\,dy\,dx = \int_0^1\int_{-1}^1 y^2\,dy\,dx = \frac{2}{3}\int_0^1 dx = \frac{2}{3}$

24. $V=\int_0^1\int_0^{1-x}\int_0^{2-2z} dy\,dz\,dx=\int_0^1\int_0^{1-x}(2-2z)\,dz\,dx=\int_0^1\left[2z-z^2\right]_0^{1-x} dx=\int_0^1\left(1-x^2\right) dx=\left[x-\frac{x^3}{3}\right]_0^1=\frac{2}{3}$

25. $V=\int_0^4\int_0^{\sqrt{4-x}}\int_0^{2-y} dz\,dy\,dx=\int_0^4\int_0^{\sqrt{4-x}}(2-y)\,dy\,dx=\int_0^4\left[2\sqrt{4-x}-\left(\frac{4-x}{2}\right)\right] dx$
$=\left[-\frac{4}{3}(4-x)^{3/2}+\frac{1}{4}(4-x)^2\right]_0^4=\frac{4}{3}(4)^{3/2}-\frac{1}{4}(16)=\frac{32}{3}-4=\frac{20}{3}$

26. $V=2\int_0^1\int_{-\sqrt{1-x^2}}^0\int_0^{-y} dz\,dy\,dx=-2\int_0^1\int_{-\sqrt{1-x^2}}^0 y\,dy\,dx=\int_0^1\left(1-x^2\right) dx=\frac{2}{3}$

27. $V=\int_0^1\int_0^{2-2x}\int_0^{3-3x-3y/2} dz\,dy\,dx=\int_0^1\int_0^{2-2x}\left(3-3x-\frac{3}{2}y\right) dy\,dx=\int_0^1\left[6(1-x)^2-\frac{3}{4}\cdot 4(1-x)^2\right] dx$
$=\int_0^1 3(1-x)^2\,dx=\left[-(1-x)^3\right]_0^1=1$

28. $V=\int_0^1\int_0^{1-x}\int_0^{\cos(\pi x/2)} dz\,dy\,dx=\int_0^1\int_0^{1-x}\cos\left(\frac{\pi x}{2}\right) dy\,dx=\int_0^1\left(\cos\frac{\pi x}{2}\right)(1-x)\,dx$
$=\int_0^1\cos\left(\frac{\pi x}{2}\right) dx-\int_0^1 x\cos\left(\frac{\pi x}{2}\right) dx=\left[\frac{2}{\pi}\sin\frac{\pi x}{2}\right]_0^1-\frac{4}{\pi^2}\int_0^{\pi/2} u\cos u\,du=\frac{2}{\pi}-\frac{4}{\pi^2}\left[\cos u+u\sin u\right]_0^{\pi/2}$
$=\frac{2}{\pi}-\frac{4}{\pi^2}\left(\frac{\pi}{2}-1\right)=\frac{4}{\pi^2}$

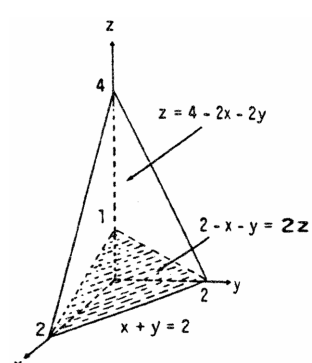
29. $V=8\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2}} dz\,dy\,dx=8\int_0^1\int_0^{\sqrt{1-x^2}}\sqrt{1-x^2}\,dy\,dx=8\int_0^1\left(1-x^2\right) dx=\frac{16}{3}$

30. $V=\int_0^2\int_0^{4-x^2}\int_0^{4-x^2-y} dz\,dy\,dx=\int_0^2\int_0^{4-x^2}\left(4-x^2-y\right) dy\,dx=\int_0^2\left[\left(4-x^2\right)^2-\frac{1}{2}\left(4-x^2\right)^2\right] dx$
$=\frac{1}{2}\int_0^2\left(4-x^2\right)^2 dx=\int_0^2\left(8-4x^2+\frac{x^4}{2}\right) dx=\frac{128}{15}$

31. $V=\int_0^4\int_0^{\left(\sqrt{16-y^2}\right)/2}\int_0^{4-y} dx\,dz\,dy=\int_0^4\int_0^{\left(\sqrt{16-y^2}\right)/2}(4-y)\,dz\,dy=\int_0^4\frac{\sqrt{16-y^2}}{2}(4-y)\,dy$
$=\int_0^4 2\sqrt{16-y^2}\,dy-\frac{1}{2}\int_0^4 y\sqrt{16-y^2}\,dy=\left[y\sqrt{16-y^2}+16\sin^{-1}\frac{y}{4}\right]_0^4+\left[\frac{1}{6}\left(16-y^2\right)^{3/2}\right]_0^4$
$=16\left(\frac{\pi}{2}\right)-\frac{1}{6}(16)^{3/2}=8\pi-\frac{32}{3}$

32. $V=\int_{-2}^2\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^{3-x} dz\,dy\,dx=\int_{-2}^2\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(3-x)\,dy\,dx=2\int_{-2}^2(3-x)\sqrt{4-x^2}\,dx$
$=3\int_{-2}^2 2\sqrt{4-x^2}\,dx-2\int_{-2}^2 x\sqrt{4-x^2}\,dx=3\left[x\sqrt{4-x^2}+4\sin^{-1}\frac{x}{2}\right]_{-2}^2+\left[\frac{2}{3}\left(4-x^2\right)^{3/2}\right]_{-2}^2$
$=12\sin^{-1}1-12\sin^{-1}(-1)=12\left(\frac{\pi}{2}\right)-12\left(-\frac{\pi}{2}\right)=12\pi$

33. $\int_0^2\int_0^{2-x}\int_{(2-x-y)/2}^{4-2x-2y} dz\,dy\,dx=\int_0^2\int_0^{2-x}\left(3-\frac{3x}{2}-\frac{3y}{2}\right) dy\,dx$
$=\int_0^2\left[3\left(1-\frac{x}{2}\right)(2-x)-\frac{3}{4}(2-x)^2\right] dx$
$=\int_0^2\left[6-6x+\frac{3x^2}{2}-\frac{3(2-x)^2}{4}\right] dx$
$=\left[6x-3x^2+\frac{x^3}{2}+\frac{(2-x)^3}{4}\right]_0^2=(12-12+4+0)-\frac{2^3}{4}=2$



34. $V = \int_0^4 \int_z^8 \int_z^{8-z} dx\,dy\,dz = \int_0^4 \int_z^8 (8-2z)\,dy\,dz = \int_0^4 (8-2z)(8-z)\,dz = \int_0^4 (64-24z+2z^2)\,dz$
$= \left[64z - 12z^2 + \frac{2}{3}z^3\right]_0^4 = \frac{320}{3}$

35. $V = 2\int_{-2}^2 \int_0^{\sqrt{4-x^2}/2} \int_0^{x+2} dz\,dy\,dx = 2\int_{-2}^2 \int_0^{\sqrt{4-x^2}/2} (x+2)\,dy\,dx = \int_{-2}^2 (x+2)\sqrt{4-x^2}\,dx$
$= \int_{-2}^2 2\sqrt{4-x^2}\,dx + \int_{-2}^2 x\sqrt{4-x^2}\,dx = \left[x\sqrt{4-x^2} + 4\sin^{-1}\frac{x}{2}\right]_{-2}^2 + \left[-\frac{1}{3}(4-x^2)^{3/2}\right]_{-2}^2$
$= 4\left(\frac{\pi}{2}\right) - 4\left(-\frac{\pi}{2}\right) = 4\pi$

36. $V = 2\int_0^1 \int_0^{1-y^2} \int_0^{x^2+y^2} dz\,dx\,dy = 2\int_0^1 \int_0^{1-y^2} (x^2+y^2)\,dx\,dy = 2\int_0^1 \left[\frac{x^3}{3} + xy^2\right]_0^{1-y^2} dy$
$= 2\int_0^1 (1-y^2)\left[\frac{1}{3}(1-y^2)^2 + y^2\right] dy = 2\int_0^1 (1-y^2)\left(\frac{1}{3} + \frac{1}{3}y^2 + \frac{1}{3}y^4\right) dy = \frac{2}{3}\int_0^1 (1-y^6)\,dy$
$= \frac{2}{3}\left[y - \frac{y^7}{7}\right]_0^1 = \left(\frac{2}{3}\right)\left(\frac{6}{7}\right) = \frac{4}{7}$

37. $\text{average} = \frac{1}{8}\int_0^2 \int_0^2 \int_0^2 (x^2+9)\,dz\,dy\,dx = \frac{1}{8}\int_0^2 \int_0^2 (2x^2+18)\,dy\,dx = \frac{1}{8}\int_0^2 (4x^2+36)\,dx = \frac{31}{3}$

38. $\text{average} = \frac{1}{2}\int_0^1 \int_0^1 \int_0^2 (x+y-z)\,dz\,dy\,dx = \frac{1}{2}\int_0^1 \int_0^1 (2x+2y-2)\,dy\,dx = \frac{1}{2}\int_0^1 (2x-1)\,dx = 0$

39. $\text{average} = \int_0^1 \int_0^1 \int_0^1 (x^2+y^2+z^2)\,dz\,dy\,dx = \int_0^1 \int_0^1 \left(x^2+y^2+\frac{1}{3}\right) dy\,dx = \int_0^1 \left(x^2+\frac{2}{3}\right) dx = 1$

40. $\text{average} = \frac{1}{8}\int_0^2 \int_0^2 \int_0^2 xyz\,dz\,dy\,dx = \frac{1}{4}\int_0^2 \int_0^2 xy\,dy\,dx = \frac{1}{2}\int_0^2 x\,dx = 1$

41. $\int_0^4 \int_0^1 \int_{2y}^2 \frac{4\cos(x^2)}{2\sqrt{z}}\,dx\,dy\,dz = \int_0^4 \int_0^2 \int_0^{x/2} \frac{4\cos(x^2)}{2\sqrt{z}}\,dy\,dx\,dz = \int_0^4 \int_0^2 \frac{x\cos(x^2)}{\sqrt{z}}\,dx\,dz = \int_0^4 \left(\frac{\sin 4}{2}\right) z^{-1/2}\,dz$
$= \left[(\sin 4)z^{1/2}\right]_0^4 = 2\sin 4$

42. $\int_0^1 \int_0^1 \int_{x^2}^1 12xz\,e^{zy^2}\,dy\,dx\,dz = \int_0^1 \int_0^1 \int_0^{\sqrt{y}} 12xz\,e^{zy^2}\,dx\,dy\,dz = \int_0^1 \int_0^1 6yz\,e^{zy^2}\,dy\,dz = \int_0^1 \left[3e^{zy^2}\right]_0^1 dz$
$= 3\int_0^1 (e^z - z)\,dz = 3\left[e^z - 1\right]_0^1 = 3e - 6$

43. $\int_0^1 \int_{\sqrt[3]{z}}^1 \int_0^{\ln 3} \frac{\pi e^{2x}\sin(\pi y^2)}{y^2}\,dx\,dy\,dz = \int_0^1 \int_{\sqrt[3]{z}}^1 \frac{4\pi\sin(\pi y^2)}{y^2}\,dy\,dz = \int_0^1 \int_0^{y^3} \frac{4\pi\sin(\pi y^2)}{y^2}\,dz\,dy$
$= \int_0^1 4\pi y\sin(\pi y^2)\,dy = \left[-2\cos(\pi y^2)\right]_0^1 = -2(-1) + 2(1) = 4$

44. $\int_0^2 \int_0^{4-x^2} \int_0^x \frac{\sin 2z}{4-z}\,dy\,dz\,dx = \int_0^2 \int_0^{4-x^2} \frac{x\sin 2z}{4-z}\,dz\,dx = \int_0^4 \int_0^{\sqrt{4-z}} \left(\frac{\sin 2z}{4-z}\right) x\,dx\,dz = \int_0^4 \left(\frac{\sin 2z}{4-z}\right)\frac{1}{2}(4-z)\,dz$
$= \left[-\frac{1}{4}\cos 2z\right]_0^4 = \left[-\frac{1}{4} + \frac{1}{2}\sin^2 z\right]_0^4 = \frac{\sin^2 4}{2}$

45. $\int_0^1 \int_0^{4-a-x^2} \int_a^{4-x^2-y} dz\,dy\,dx = \frac{4}{15} \Rightarrow \int_0^1 \int_0^{4-a-x^2} (4-x^2-y-a)\,dy\,dx = \frac{4}{15}$
$\Rightarrow \int_0^1 \left[(4-a-x^2)^2 - \frac{1}{2}(4-a-x^2)^2\right] dx = \frac{4}{15} \Rightarrow \frac{1}{2}\int_0^1 (4-a-x^2)^2\,dx = \frac{4}{15} \Rightarrow \int_0^1 [(4-a)^2 - 2x^2(4-a) + x^4]\,dx$
$= \frac{8}{15} \Rightarrow \left[(4-a)^2 x - \frac{2}{3}x^3(4-a) + \frac{x^5}{5}\right]_0^1 = \frac{8}{15} \Rightarrow (4-a)^2 - \frac{2}{3}(4-a) + \frac{1}{5} = \frac{8}{15} \Rightarrow 15(4-a)^2 - 10(4-a) - 5 = 0$
$\Rightarrow 3(4-a)^2 - 2(4-a) - 1 = 0 \Rightarrow [3(4-a)+1][(4-a)-1] = 0 \Rightarrow 4-a = -\frac{1}{3} \text{ or } 4-a = 1 \Rightarrow a = \frac{13}{3} \text{ or } a = 3$

46. The volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is $\frac{4abc\pi}{3}$ so that $\frac{4(1)(2)(c)\pi}{3}=8\pi \Rightarrow c=3$.

47. To minimize the integral, we want the domain to include all points where the integrand is negative and to exclude all points where it is positive. These criteria are met by the points (x, y, z) such that $4x^2+4y^2+z^2-4\le 0$ or $4x^2+4y^2+z^2\le 4$, which is a solid ellipsoid centered at the origin.

48. To maximize the integral, we want the domain to include all points where the integrand is positive and to exclude all points where it is negative. These criteria are met by the points (x, y, z) such that $1-x^2-y^2-z^2\ge 0$ or $x^2+y^2+z^2\le 1$, which is a solid sphere of radius 1 centered at the origin.

49-52. Example CAS commands:

Maple:

```
F := (x,y,z) -> x^2*y^2*z;
q1 := Int( Int( Int( F(x,y,z), y=-sqrt(1-x^2)..sqrt(1-x^2) ), x=-1..1 ), z=0..1 );
value( q1 );
```

Mathematica: (functions and bounds will vary)

```
Clear[f, x, y, z];
f:= x^2 y^2 z
Integrate[f, {x,−1,1}, {y,−Sqrt[1 − x^2], Sqrt[1 − x^2]}, {z, 0, 1}]
N[%]
topolar={x → r Cos[t], y → r Sin[t]};
fp= f/.topolar //Simplify
Integrate[r fp, {t, 0, 2π}, {r, 0, 1},{z, 0, 1}]
N[%]
```

15.6 MOMENTS AND CENTERS OF MASS

1. $M=\int_0^1\int_x^{2-x^2} 3\,dy\,dx = 3\int_0^1 (2-x^2-x)\,dx=\frac{7}{2}$; $M_y=\int_0^1\int_x^{2-x^2} 3x\,dy\,dx = 3\int_0^1 [xy]_x^{2-x^2}\,dx$
$=3\int_0^1 (2x-x^3-x^2)\,dx=\frac{5}{4}$; $M_x=\int_0^1\int_x^{2-x^2} 3y\,dy\,dx=\frac{3}{2}\int_0^1 [y^2]_x^{2-x^2}\,dx=\frac{3}{2}\int_0^1 (4-5x^2+x^4)\,dx=\frac{19}{5}$
$\Rightarrow\ \overline{x}=\frac{5}{14}$ and $\overline{y}=\frac{38}{35}$

2. $M=\delta\int_0^3\int_0^3 dy\,dx=\delta\int_0^3 3\,dx=9\delta$; $I_x=\delta\int_0^3\int_0^3 y^2\,dy\,dx=\delta\int_0^3\left[\frac{y^3}{3}\right]_0^3 dx=27\delta$;
$I_y=\delta\int_0^3\int_0^3 x^2\,dy\,dx=\delta\int_0^3 [x^2y]_0^3\,dx=\delta\int_0^3 3x^2\,dx=27\delta$

3. $M=\int_0^2\int_{y^2/2}^{4-y} dx\,dy=\int_0^2\left(4-y-\frac{y^2}{2}\right)dy=\frac{14}{3}$; $M_y=\int_0^2\int_{y^2/2}^{4-y} x\,dx\,dy=\frac{1}{2}\int_0^2 [x^2]_{y^2/2}^{4-y}\,dy$
$=\frac{1}{2}\int_0^2\left(16-8y+y^2-\frac{y^4}{4}\right)dy=\frac{128}{15}$; $M_x=\int_0^2\int_{y^2/2}^{4-y} y\,dx\,dy=\int_0^2\left(4y-y^2-\frac{y^3}{2}\right)dy=\frac{10}{3}$
$\Rightarrow\ \overline{x}=\frac{64}{35}$ and $\overline{y}=\frac{5}{7}$

4. $M=\int_0^3\int_0^{3-x} dy\,dx=\int_0^3 (3-x)\,dx=\frac{9}{2}$; $M_y=\int_0^3\int_0^{3-x} x\,dy\,dx=\int_0^3 [xy]_0^{3-x}\,dx=\int_0^3 (3x-x^2)\,dx=\frac{9}{2}$
$\Rightarrow\ \overline{x}=1$ and $\overline{y}=1$, by symmetry

5. $M=\int_0^a\int_0^{\sqrt{a^2-x^2}} dy\,dx=\frac{\pi a^2}{4}$; $M_y=\int_0^a\int_0^{\sqrt{a^2-x^2}} x\,dy\,dx=\int_0^a [xy]_0^{\sqrt{a^2-x^2}}\,dx=\int_0^a x\sqrt{a^2-x^2}\,dx=\frac{a^3}{3}$
$\Rightarrow\ \overline{x}=\overline{y}=\frac{4a}{3\pi}$, by symmetry

6. $M = \int_0^{\pi}\int_0^{\sin x} dy\,dx = \int_0^{\pi} \sin x\,dx = 2;\ M_x = \int_0^{\pi}\int_0^{\sin x} y\,dy\,dx = \frac{1}{2}\int_0^{\pi}\left[y^2\right]_0^{\sin x} dx = \frac{1}{2}\int_0^{\pi} \sin^2 x\,dx$
$= \frac{1}{4}\int_0^{\pi}(1-\cos 2x)\,dx = \frac{\pi}{4} \Rightarrow \overline{x} = \frac{\pi}{2}$ and $\overline{y} = \frac{\pi}{8}$

7. $I_x = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} y^2\,dy\,dx = \int_{-2}^{2}\left[\frac{y^3}{3}\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx = \frac{2}{3}\int_{-2}^{2}(4-x^2)^{3/2}\,dx = 4\pi;\ I_y = 4\pi$, by symmetry;
$I_o = I_x + I_y = 8\pi$

8. $I_y = \int_{\pi}^{2\pi}\int_0^{(\sin^2 x)/x^2} x^2\,dy\,dx = \int_{\pi}^{2\pi}(\sin^2 x - 0)\,dx = \frac{1}{2}\int_{\pi}^{2\pi}(1-\cos 2x)\,dx = \frac{\pi}{2}$

9. $M = \int_{-\infty}^{0}\int_0^{e^x} dy\,dx = \int_{-\infty}^{0} e^x\,dx = \lim_{b\to-\infty}\int_b^0 e^x\,dx = 1 - \lim_{b\to-\infty} e^b = 1;\ M_y = \int_{-\infty}^{0}\int_0^{e^x} x\,dy\,dx = \int_{-\infty}^{0} xe^x\,dx$
$= \lim_{b\to-\infty}\int_b^0 xe^x\,dx = \lim_{b\to-\infty}\left[xe^x - e^x\right]_b^0 = -1 - \lim_{b\to-\infty}(be^b - e^b) = -1;\ M_x = \int_{-\infty}^{0}\int_0^{e^x} y\,dy\,dx$
$= \frac{1}{2}\int_{-\infty}^{0} e^{2x}\,dx = \frac{1}{2}\lim_{b\to-\infty}\int_b^0 e^{2x}\,dx = \frac{1}{4} \Rightarrow \overline{x} = -1$ and $\overline{y} = \frac{1}{4}$

10. $M_y = \int_0^{\infty}\int_0^{e^{-x^2/2}} x\,dy\,dx = \lim_{b\to\infty}\int_0^b xe^{-x^2/2}\,dx = -\lim_{b\to\infty}\left[\frac{1}{e^{x^2/2}} - 1\right]_0^b = 1$

11. $M = \int_0^2\int_{-y}^{y-y^2}(x+y)\,dx\,dy = \int_0^2\left[\frac{x^2}{2} + xy\right]_{-y}^{y-y^2} dy = \int_0^2\left(\frac{y^4}{2} - 2y^3 + 2y^2\right)dy = \left[\frac{y^5}{10} - \frac{y^4}{2} + \frac{2y^3}{3}\right]_0^2 = \frac{8}{15}$;
$I_x = \int_0^2\int_{-y}^{y-y^2} y^2(x+y)\,dx\,dy = \int_0^2\left[\frac{x^2y^2}{2} + xy^3\right]_{-y}^{y-y^2} dy = \int_0^2\left(\frac{y^6}{2} - 2y^5 + 2y^4\right)dy = \frac{64}{105}$;

12. $M = \int_{-\sqrt{3}/2}^{\sqrt{3}/2}\int_{4y^2}^{\sqrt{12-4y^2}} 5x\,dx\,dy = 5\int_{-\sqrt{3}/2}^{\sqrt{3}/2}\left[\frac{x^2}{2}\right]_{4y^2}^{\sqrt{12-4y^2}} dy = \frac{5}{2}\int_{-\sqrt{3}/2}^{\sqrt{3}/2}(12 - 4y^2 - 16y^4)\,dy = 23\sqrt{3}$

13. $M = \int_0^1\int_x^{2-x}(6x+3y+3)\,dy\,dx = \int_0^1\left[6xy + \frac{3}{2}y^2 + 3y\right]_x^{2-x} dx = \int_0^1(12 - 12x^2)\,dx = 8$;
$M_y = \int_0^1\int_x^{2-x} x(6x+3y+3)\,dy\,dx = \int_0^1(12x - 12x^3)\,dx = 3;\ M_x = \int_0^1\int_x^{2-x} y(6x+3y+3)\,dy\,dx$
$= \int_0^1(14 - 6x - 6x^2 - 2x^3)\,dx = \frac{17}{2} \Rightarrow \overline{x} = \frac{3}{8}$ and $\overline{y} = \frac{17}{16}$

14. $M = \int_0^1\int_{y^2}^{2y-y^2}(y+1)\,dx\,dy = \int_0^1(2y - 2y^3)\,dy = \frac{1}{2};\ M_x = \int_0^1\int_{y^2}^{2y-y^2} y(y+1)\,dx\,dy = \int_0^1(2y^2 - 2y^4)\,dy = \frac{4}{15}$;
$M_y = \int_0^1\int_{y^2}^{2y-y^2} x(y+1)\,dx\,dy = \int_0^1(2y^2 - 2y^4)\,dy = \frac{4}{15} \Rightarrow \overline{x} = \frac{8}{15}$ and $\overline{y} = \frac{8}{15};\ I_x = \int_0^1\int_{y^2}^{2y-y^2} y^2(y+1)\,dx\,dy$
$= 2\int_0^1(y^3 - y^5)\,dy = \frac{1}{6}$

15. $M = \int_0^1\int_0^6(x+y+1)\,dx\,dy = \int_0^1(6y+24)\,dy = 27;\ M_x = \int_0^1\int_0^6 y(x+y+1)\,dx\,dy = \int_0^1 y(6y+24)\,dy = 14$;
$M_y = \int_0^1\int_0^6 x(x+y+1)\,dx\,dy = \int_0^1(18y+90)\,dy = 99 \Rightarrow \overline{x} = \frac{11}{3}$ and $\overline{y} = \frac{14}{27};\ I_y = \int_0^1\int_0^6 x^2(x+y+1)\,dx\,dy$
$= 216\int_0^1\left(\frac{y}{3} + \frac{11}{6}\right)dy = 432$

16. $M = \int_{-1}^{1}\int_{x^2}^{1}(y+1)\,dy\,dx = -\int_{-1}^{1}\left(\frac{x^4}{2} + x^2 - \frac{3}{2}\right)dx = \frac{32}{15};\ M_x = \int_{-1}^{1}\int_{x^2}^{1} y(y+1)\,dy\,dx = \int_{-1}^{1}\left(\frac{5}{6} - \frac{x^6}{3} - \frac{x^4}{2}\right)dx$
$= \frac{48}{35};\ M_y = \int_{-1}^{1}\int_{x^2}^{1} x(y+1)\,dy\,dx = \int_{-1}^{1}\left(\frac{3x}{2} - \frac{x^5}{2} - x^3\right)dx = 0 \Rightarrow \overline{x} = 0$ and $\overline{y} = \frac{9}{14};\ I_y = \int_{-1}^{1}\int_{x^2}^{1} x^2(y+1)\,dy\,dx$
$= \int_{-1}^{1}\left(\frac{3x^2}{2} - \frac{x^6}{2} - x^4\right)dx = \frac{16}{35}$

17. $M = \int_{-1}^{1}\int_{0}^{x^2}(7y+1)\,dy\,dx = \int_{-1}^{1}\left(\frac{7x^4}{2}+x^2\right)dx = \frac{31}{15}$; $M_x = \int_{-1}^{1}\int_{0}^{x^2} y(7y+1)\,dy\,dx = \int_{-1}^{1}\left(\frac{7x^6}{3}+\frac{x^4}{2}\right)dx = \frac{13}{15}$;
$M_y = \int_{-1}^{1}\int_{0}^{x^2} x(7y+1)\,dy\,dx = \int_{-1}^{1}\left(\frac{7x^5}{2}+x^3\right)dx = 0 \Rightarrow \overline{x} = 0$ and $\overline{y} = \frac{13}{31}$; $I_y = \int_{-1}^{1}\int_{0}^{x^2} x^2(7y+1)\,dy\,dx$
$= \int_{-1}^{1}\left(\frac{7x^6}{2}+x^4\right)dx = \frac{7}{5}$

18. $M = \int_{0}^{20}\int_{-1}^{1}\left(1+\frac{x}{20}\right)dy\,dx = \int_{0}^{20}\left(2+\frac{x}{10}\right)dx = 60$; $M_x = \int_{0}^{20}\int_{-1}^{1} y\left(1+\frac{x}{20}\right)dy\,dx = \int_{0}^{20}\left[\left(1+\frac{x}{20}\right)\left(\frac{y^2}{2}\right)\right]_{-1}^{1} dx = 0$;
$M_y = \int_{0}^{20}\int_{-1}^{1} x\left(1+\frac{x}{20}\right)dy\,dx = \int_{0}^{20}\left(2x+\frac{x^2}{10}\right)dx = \frac{2000}{3} \Rightarrow \overline{x} = \frac{100}{9}$ and $\overline{y} = 0$; $I_x = \int_{0}^{20}\int_{-1}^{1} y^2\left(1+\frac{x}{20}\right)dy\,dx$
$= \frac{2}{3}\int_{0}^{20}\left(1+\frac{x}{20}\right)dx = 20$

19. $M = \int_{0}^{1}\int_{-y}^{y}(y+1)\,dx\,dy = \int_{0}^{1}(2y^2+2y)\,dy = \frac{5}{3}$; $M_x = \int_{0}^{1}\int_{-y}^{y} y(y+1)\,dx\,dy = 2\int_{0}^{1}(y^3+y^2)\,dy = \frac{7}{6}$;
$M_y = \int_{0}^{1}\int_{-y}^{y} x(y+1)\,dx\,dy = \int_{0}^{1} 0\,dy = 0 \Rightarrow \overline{x} = 0$ and $\overline{y} = \frac{7}{10}$; $I_x = \int_{0}^{1}\int_{-y}^{y} y^2(y+1)\,dx\,dy = \int_{0}^{1}(2y^4+2y^3)\,dy$
$= \frac{9}{10}$; $I_y = \int_{0}^{1}\int_{-y}^{y} x^2(y+1)\,dx\,dy = \frac{1}{3}\int_{0}^{1}(2y^4+2y^3)\,dy = \frac{3}{10} \Rightarrow I_o = I_x + I_y = \frac{6}{5}$

20. $M = \int_{0}^{1}\int_{-y}^{y}(3x^2+1)\,dx\,dy = \int_{0}^{1}(2y^3+2y)\,dy = \frac{3}{2}$; $M_x = \int_{0}^{1}\int_{-y}^{y} y(3x^2+1)\,dx\,dy = \int_{0}^{1}(2y^4+2y^2)\,dy = \frac{16}{15}$;
$M_y = \int_{0}^{1}\int_{-y}^{y} x(3x^2+1)\,dx\,dy = 0 \Rightarrow \overline{x} = 0$ and $\overline{y} = \frac{32}{45}$; $I_x = \int_{0}^{1}\int_{-y}^{y} y^2(3x^2+1)\,dx\,dy = \int_{0}^{1}(2y^5+2y^3)\,dy = \frac{5}{6}$;
$I_y = \int_{0}^{1}\int_{-y}^{y} x^2(3x^2+1)\,dx\,dy = 2\int_{0}^{1}\left(\frac{3}{5}y^5+\frac{1}{3}y^3\right)dy = \frac{11}{30} \Rightarrow I_o = I_x + I_y = \frac{6}{5}$

21. $I_x = \int_{0}^{a}\int_{0}^{b}\int_{0}^{c}(y^2+z^2)\,dz\,dy\,dx = \int_{0}^{a}\int_{0}^{b}\left(cy^2+\frac{c^3}{3}\right)dy\,dx = \int_{0}^{a}\left(\frac{cb^3}{3}+\frac{c^3b}{3}\right)dx = \frac{abc\,(b^2+c^2)}{3}$
$= \frac{M}{3}(b^2+c^2)$ where $M = abc$; $I_y = \frac{M}{3}(a^2+c^2)$ and $I_z = \frac{M}{3}(a^2+b^2)$, by symmetry

22. The plane $z = \frac{4-2y}{3}$ is the top of the wedge $\Rightarrow I_x = \int_{-3}^{3}\int_{-2}^{4}\int_{-4/3}^{(4-2y)/3}(y^2+z^2)\,dz\,dy\,dx$
$= \int_{-3}^{3}\int_{-2}^{4}\left[\frac{8y^2}{3}-\frac{2y^3}{3}+\frac{8(2-y)^3}{81}+\frac{64}{81}\right]dy\,dx = \int_{-3}^{3}\frac{104}{3}\,dx = 208$; $I_y = \int_{-3}^{3}\int_{-2}^{4}\int_{-4/3}^{(4-2y)/3}(x^2+z^2)\,dz\,dy\,dx$
$= \int_{-3}^{3}\int_{-2}^{4}\left[\frac{(4-2y)^3}{81}+\frac{x^2(4-2y)}{3}+\frac{4x^2}{3}+\frac{64}{81}\right]dy\,dx = \int_{-3}^{3}\left(12x^2+\frac{32}{3}\right)dx = 280$;
$I_z = \int_{-3}^{3}\int_{-2}^{4}\int_{-4/3}^{(4-2y)/3}(x^2+y^2)\,dz\,dy\,dx = \int_{-3}^{3}\int_{-2}^{4}(x^2+y^2)\left(\frac{8}{3}-\frac{2y}{3}\right)dy\,dx = 12\int_{-3}^{3}(x^2+2)\,dx = 360$

23. $M = 4\int_{0}^{1}\int_{0}^{1}\int_{4y^2}^{4}dz\,dy\,dx = 4\int_{0}^{1}\int_{0}^{1}(4-4y^2)\,dy\,dx = 16\int_{0}^{1}\frac{2}{3}\,dx = \frac{32}{3}$; $M_{xy} = 4\int_{0}^{1}\int_{0}^{1}\int_{4y^2}^{4} z\,dz\,dy\,dx$
$= 2\int_{0}^{1}\int_{0}^{1}(16-16y^4)\,dy\,dx = \frac{128}{5}\int_{0}^{1}dx = \frac{128}{5} \Rightarrow \overline{z} = \frac{12}{5}$, and $\overline{x} = \overline{y} = 0$, by symmetry;
$I_x = 4\int_{0}^{1}\int_{0}^{1}\int_{4y^2}^{4}(y^2+z^2)\,dz\,dy\,dx = 4\int_{0}^{1}\int_{0}^{1}\left[\left(4y^2+\frac{64}{3}\right)-\left(4y^4+\frac{64y^6}{3}\right)\right]dy\,dx = 4\int_{0}^{1}\frac{1976}{105}\,dx = \frac{7904}{105}$;
$I_y = 4\int_{0}^{1}\int_{0}^{1}\int_{4y^2}^{4}(x^2+z^2)\,dz\,dy\,dx = 4\int_{0}^{1}\int_{0}^{1}\left[\left(4x^2+\frac{64}{3}\right)-\left(4x^2y^2+\frac{64y^6}{3}\right)\right]dy\,dx = 4\int_{0}^{1}\left(\frac{8}{3}x^2+\frac{128}{7}\right)dx$
$= \frac{4832}{63}$; $I_z = 4\int_{0}^{1}\int_{0}^{1}\int_{4y^2}^{4}(x^2+y^2)\,dz\,dy\,dx = 16\int_{0}^{1}\int_{0}^{1}(x^2-x^2y^2+y^2-y^4)\,dy\,dx$
$= 16\int_{0}^{1}\left(\frac{2x^2}{3}+\frac{2}{15}\right)dx = \frac{256}{45}$

24. (a) $M = \int_{-2}^{2}\int_{\left(-\sqrt{4-x^2}\right)/2}^{\left(\sqrt{4-x^2}\right)/2}\int_{0}^{2-x}dz\,dy\,dx = \int_{-2}^{2}\int_{\left(-\sqrt{4-x^2}\right)/2}^{\left(\sqrt{4-x^2}\right)/2}(2-x)\,dy\,dx = \int_{-2}^{2}(2-x)\left(\sqrt{4-x^2}\right)dx = 4\pi$;
$M_{yz} = \int_{-2}^{2}\int_{\left(-\sqrt{4-x^2}\right)/2}^{\left(\sqrt{4-x^2}\right)/2}\int_{0}^{2-x} x\,dz\,dy\,dx = \int_{-2}^{2}\int_{\left(-\sqrt{4-x^2}\right)/2}^{\left(\sqrt{4-x^2}\right)/2} x(2-x)\,dy\,dx = \int_{-2}^{2} x(2-x)\left(\sqrt{4-x^2}\right)dx = -2\pi$;

$M_{xz} = \int_{-2}^{2}\int_{\left(-\sqrt{4-x^2}\right)/2}^{\left(\sqrt{4-x^2}\right)/2}\int_0^{2-x} y\,dz\,dy\,dx = \int_{-2}^{2}\int_{\left(-\sqrt{4-x^2}\right)/2}^{\left(\sqrt{4-x^2}\right)/2} y(2-x)\,dy\,dx$

$= \frac{1}{2}\int_{-2}^{2}(2-x)\left[\frac{4-x^2}{4} - \frac{4-x^2}{4}\right] dx = 0 \Rightarrow \overline{x} = -\frac{1}{2}$ and $\overline{y} = 0$

(b) $M_{xy} = \int_{-2}^{2}\int_{\left(-\sqrt{4-x^2}\right)/2}^{\left(\sqrt{4-x^2}\right)/2}\int_0^{2-x} z\,dz\,dy\,dx = \frac{1}{2}\int_{-2}^{2}\int_{\left(-\sqrt{4-x^2}\right)/2}^{\left(\sqrt{4-x^2}\right)/2}(2-x)^2\,dy\,dx = \frac{1}{2}\int_{-2}^{2}(2-x)^2\left(\sqrt{4-x^2}\right) dx$

$= 5\pi \Rightarrow \overline{z} = \frac{5}{4}$

25. (a) $M = 4\int_0^2\int_0^{\sqrt{4-x^2}}\int_{x^2+y^2}^{4} dz\,dy\,dx = 4\int_0^{\pi/2}\int_0^2\int_{r^2}^4 r\,dz\,dr\,d\theta = 4\int_0^{\pi/2}\int_0^2 (4r - r^3)\,dr\,d\theta = 4\int_0^{\pi/2} 4\,d\theta = 8\pi;$

$M_{xy} = \int_0^{2\pi}\int_0^2\int_{r^2}^4 zr\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2 \frac{r}{2}(16 - r^4)\,dr\,d\theta = \frac{32}{3}\int_0^{2\pi} d\theta = \frac{64\pi}{3} \Rightarrow \overline{z} = \frac{8}{3}$, and $\overline{x} = \overline{y} = 0$, by symmetry

(b) $M = 8\pi \Rightarrow 4\pi = \int_0^{2\pi}\int_0^{\sqrt{c}}\int_{r^2}^{c} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^{\sqrt{c}}(cr - r^3)\,dr\,d\theta = \int_0^{2\pi}\frac{c^2}{4}\,d\theta = \frac{c^2\pi}{2} \Rightarrow c^2 = 8 \Rightarrow c = 2\sqrt{2}$, since $c > 0$

26. $M = 8$; $M_{xy} = \int_{-1}^{1}\int_3^5\int_{-1}^{1} z\,dz\,dy\,dx = \int_{-1}^{1}\int_3^5\left[\frac{z^2}{2}\right]_{-1}^{1} dy\,dx = 0$; $M_{yz} = \int_{-1}^{1}\int_3^5\int_{-1}^{1} x\,dz\,dy\,dx$

$= 2\int_{-1}^{1}\int_3^5 x\,dy\,dx = 4\int_{-1}^{1} x\,dx = 0$; $M_{xz} = \int_{-1}^{1}\int_3^5\int_{-1}^{1} y\,dz\,dy\,dx = 2\int_{-1}^{1}\int_3^5 y\,dy\,dx = 16\int_{-1}^{1} dx = 32$

$\Rightarrow \overline{x} = 0, \overline{y} = 4, \overline{z} = 0$; $I_x = \int_{-1}^{1}\int_3^5\int_{-1}^{1}(y^2 + z^2)\,dz\,dy\,dx = \int_{-1}^{1}\int_3^5\left(2y^2 + \frac{2}{3}\right) dy\,dx = \frac{2}{3}\int_{-1}^{1} 100\,dx = \frac{400}{3}$;

$I_y = \int_{-1}^{1}\int_3^5\int_{-1}^{1}(x^2 + z^2)\,dz\,dy\,dx = \int_{-1}^{1}\int_3^5\left(2x^2 + \frac{2}{3}\right) dy\,dx = \frac{4}{3}\int_{-1}^{1}(3x^2 + 1)\,dx = \frac{16}{3}$;

$I_z = \int_{-1}^{1}\int_3^5\int_{-1}^{1}(x^2 + y^2)\,dz\,dy\,dx = 2\int_{-1}^{1}\int_3^5(x^2 + y^2)\,dy\,dx = 2\int_{-1}^{1}\left(2x^2 + \frac{98}{3}\right) dx = \frac{400}{3}$

27. The plane $y + 2z = 2$ is the top of the wedge $\Rightarrow I_L = \int_{-2}^{2}\int_{-2}^{4}\int_{-1}^{(2-y)/2}\left[(y-6)^2 + z^2\right] dz\,dy\,dx$

$= \int_{-2}^{2}\int_{-2}^{4}\left[\frac{(y-6)^2(4-y)}{2} + \frac{(2-y)^3}{24} + \frac{1}{3}\right] dy\,dx$; let $t = 2 - y \Rightarrow I_L = 4\int_{-2}^{4}\left(\frac{13t^3}{24} + 5t^2 + 16t + \frac{49}{3}\right) dt = 1386$;

$M = \frac{1}{2}(3)(6)(4) = 36$

28. The plane $y + 2z = 2$ is the top of the wedge $\Rightarrow I_L = \int_{-2}^{2}\int_{-2}^{4}\int_{-1}^{(2-y)/2}\left[(x-4)^2 + y^2\right] dz\,dy\,dx$

$= \frac{1}{2}\int_{-2}^{2}\int_{-2}^{4}(x^2 - 8x + 16 + y^2)(4-y)\,dy\,dx = \int_{-2}^{2}(9x^2 - 72x + 162)\,dx = 696$; $M = \frac{1}{2}(3)(6)(4) = 36$

29. (a) $M = \int_0^2\int_0^{2-x}\int_0^{2-x-y} 2x\,dz\,dy\,dx = \int_0^2\int_0^{2-x}(4x - 2x^2 - 2xy)\,dy\,dx = \int_0^2(x^3 - 4x^2 + 4x)\,dx = \frac{4}{3}$

(b) $M_{xy} = \int_0^2\int_0^{2-x}\int_0^{2-x-y} 2xz\,dz\,dy\,dx = \int_0^2\int_0^{2-x} x(2-x-y)^2\,dy\,dx = \int_0^2 \frac{x(2-x)^3}{3}\,dx = \frac{8}{15}$; $M_{xz} = \frac{8}{15}$ by symmetry; $M_{yz} = \int_0^2\int_0^{2-x}\int_0^{2-x-y} 2x^2\,dz\,dy\,dx = \int_0^2\int_0^{2-x} 2x^2(2-x-y)\,dy\,dx = \int_0^2(2x - x^2)^2\,dx = \frac{16}{15}$

$\Rightarrow \overline{x} = \frac{4}{5}$, and $\overline{y} = \overline{z} = \frac{2}{5}$

30. (a) $M = \int_0^2\int_0^{\sqrt{x}}\int_0^{4-x^2} kxy\,dz\,dy\,dx = k\int_0^2\int_0^{\sqrt{x}} xy(4 - x^2)\,dy\,dx = \frac{k}{2}\int_0^2(4x^2 - x^4)\,dx = \frac{32k}{15}$

(b) $M_{yz} = \int_0^2\int_0^{\sqrt{x}}\int_0^{4-x^2} kx^2y\,dz\,dy\,dx = k\int_0^2\int_0^{\sqrt{x}} x^2y(4 - x^2)\,dy\,dx = \frac{k}{2}\int_0^2(4x^3 - x^5)\,dx = \frac{8k}{3}$

$\Rightarrow \overline{x} = \frac{5}{4}$; $M_{xz} = \int_0^2\int_0^{\sqrt{x}}\int_0^{4-x^2} kxy^2\,dz\,dy\,dx = k\int_0^2\int_0^{\sqrt{x}} xy^2(4 - x^2)\,dy\,dx = \frac{k}{3}\int_0^2\left(4x^{5/2} - x^{9/2}\right) dx$

$= \frac{256\sqrt{2}k}{231} \Rightarrow \overline{y} = \frac{40\sqrt{2}}{77}$; $M_{xy} = \int_0^2\int_0^{\sqrt{x}}\int_0^{4-x^2} kxyz\,dz\,dy\,dx = \int_0^2\int_0^{\sqrt{x}} xy(4 - x^2)^2\,dy\,dx$

$= \frac{k}{4}\int_0^2(16x^2 - 8x^4 + x^6)\,dx = \frac{256k}{105} \Rightarrow \overline{z} = \frac{8}{7}$

31. (a) $M = \int_0^1\int_0^1\int_0^1(x + y + z + 1)\,dz\,dy\,dx = \int_0^1\int_0^1\left(x + y + \frac{3}{2}\right) dy\,dx = \int_0^1(x + 2)\,dx = \frac{5}{2}$

(b) $M_{xy} = \int_0^1 \int_0^1 \int_0^1 z(x+y+z+1)\,dz\,dy\,dx = \frac{1}{2}\int_0^1 \int_0^1 \left(x+y+\frac{5}{3}\right) dy\,dx = \frac{1}{2}\int_0^1 \left(x+\frac{13}{6}\right) dx = \frac{4}{3}$
$\Rightarrow M_{xy} = M_{yz} = M_{xz} = \frac{4}{3}$, by symmetry $\Rightarrow \overline{x} = \overline{y} = \overline{z} = \frac{8}{15}$

(c) $I_z = \int_0^1 \int_0^1 \int_0^1 (x^2+y^2)(x+y+z+1)\,dz\,dy\,dx = \int_0^1 \int_0^1 (x^2+y^2)\left(x+y+\frac{3}{2}\right) dy\,dx$
$= \int_0^1 \left(x^3+2x^2+\frac{1}{3}x+\frac{3}{4}\right) dx = \frac{11}{6} \Rightarrow I_x = I_y = I_z = \frac{11}{6}$, by symmetry

32. The plane $y + 2z = 2$ is the top of the wedge.

(a) $M = \int_{-1}^1 \int_{-2}^4 \int_{-1}^{(2-y)/2} (x+1)\,dz\,dy\,dx = \int_{-1}^1 \int_{-2}^4 (x+1)\left(2-\frac{y}{2}\right) dy\,dx = 18$

(b) $M_{yz} = \int_{-1}^1 \int_{-2}^4 \int_{-1}^{(2-y)/2} x(x+1)\,dz\,dy\,dx = \int_{-1}^1 \int_{-2}^4 x(x+1)\left(2-\frac{y}{2}\right) dy\,dx = 6;$
$M_{xz} = \int_{-1}^1 \int_{-2}^4 \int_{-1}^{(2-y)/2} y(x+1)\,dz\,dy\,dx = \int_{-1}^1 \int_{-2}^4 y(x+1)\left(2-\frac{y}{2}\right) dy\,dx = 0;$
$M_{xy} = \int_{-1}^1 \int_{-2}^4 \int_{-1}^{(2-y)/2} z(x+1)\,dz\,dy\,dx = \frac{1}{2}\int_{-1}^1 \int_{-2}^4 (x+1)\left(\frac{y^2}{4}-y\right) dy\,dx = 0 \Rightarrow \overline{x} = \frac{1}{3}$, and $\overline{y} = \overline{z} = 0$

(c) $I_x = \int_{-1}^1 \int_{-2}^4 \int_{-1}^{(2-y)/2} (x+1)(y^2+z^2)\,dz\,dy\,dx = \int_{-1}^1 \int_{-2}^4 (x+1)\left[2y^2+\frac{1}{3}-\frac{y^3}{2}+\frac{1}{3}\left(1-\frac{y}{2}\right)^3\right] dy\,dx = 45;$
$I_y = \int_{-1}^1 \int_{-2}^4 \int_{-1}^{(2-y)/2} (x+1)(x^2+z^2)\,dz\,dy\,dx = \int_{-1}^1 \int_{-2}^4 (x+1)\left[2x^2+\frac{1}{3}-\frac{x^2y}{2}+\frac{1}{3}\left(1-\frac{y}{2}\right)^3\right] dy\,dx = 15;$
$I_z = \int_{-1}^1 \int_{-2}^4 \int_{-1}^{(2-y)/2} (x+1)(x^2+y^2)\,dz\,dy\,dx = \int_{-1}^1 \int_{-2}^4 (x+1)\left(2-\frac{y}{2}\right)(x^2+y^2)\,dy\,dx = 42$

33. $M = \int_0^1 \int_{z-1}^{1-z} \int_0^{\sqrt{z}} (2y+5)\,dy\,dx\,dz = \int_0^1 \int_{z-1}^{1-z} \left(z+5\sqrt{z}\right) dx\,dz = \int_0^1 2\left(z+5\sqrt{z}\right)(1-z)\,dz$
$= 2\int_0^1 \left(5z^{1/2}+z-5z^{3/2}-z^2\right) dz = 2\left[\frac{10}{3}z^{3/2}+\frac{1}{2}z^2-2z^{5/2}-\frac{1}{3}z^3\right]_0^1 = 2\left(\frac{9}{3}-\frac{3}{2}\right) = 3$

34. $M = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{2(x^2+y^2)}^{16-2(x^2+y^2)} \sqrt{x^2+y^2}\,dz\,dy\,dx = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{x^2+y^2}\,[16-4(x^2+y^2)]\,dy\,dx$
$= 4\int_0^{2\pi} \int_0^2 r(4-r^2)\,r\,dr\,d\theta = 4\int_0^{2\pi} \left[\frac{4r^3}{3}-\frac{r^5}{5}\right]_0^2 d\theta = 4\int_0^{2\pi} \frac{64}{15}\,d\theta = \frac{512\pi}{15}$

35. (a) $\overline{x} = \frac{M_{yz}}{M} = 0 \Rightarrow \iiint_R x\delta(x,y,z)\,dx\,dy\,dz = 0 \Rightarrow M_{yz} = 0$

(b) $I_L = \iiint_D |\mathbf{v}-h\mathbf{i}|^2\,dm = \iiint_D |(x-h)\,\mathbf{i}+y\mathbf{j}|^2\,dm = \iiint_D (x^2-2xh+h^2+y^2)\,dm$
$= \iiint_D (x^2+y^2)\,dm - 2h\iiint_D x\,dm + h^2\iiint_D dm = I_x - 0 + h^2m = I_{c.m.} + h^2m$

36. $I_L = I_{c.m.} + mh^2 = \frac{2}{5}ma^2 + ma^2 = \frac{7}{5}ma^2$

37. (a) $(\overline{x},\overline{y},\overline{z}) = \left(\frac{a}{2},\frac{b}{2},\frac{c}{2}\right) \Rightarrow I_z = I_{c.m.} + abc\left(\sqrt{\frac{a^2}{4}+\frac{b^2}{4}}\right)^2 \Rightarrow I_{c.m.} = I_z - \frac{abc\,(a^2+b^2)}{4}$
$= \frac{abc\,(a^2+b^2)}{3} - \frac{abc\,(a^2+b^2)}{4} = \frac{abc\,(a^2+b^2)}{12}$; $R_{c.m.} = \sqrt{\frac{I_{c.m.}}{M}} = \sqrt{\frac{a^2+b^2}{12}}$

(b) $I_L = I_{c.m.} + abc\left(\sqrt{\frac{a^2}{4}+\left(\frac{b}{2}-2b\right)^2}\right)^2 = \frac{abc\,(a^2+b^2)}{12} + \frac{abc\,(a^2+9b^2)}{4} = \frac{abc\,(4a^2+28b^2)}{12}$
$= \frac{abc\,(a^2+7b^2)}{3}$; $R_L = \sqrt{\frac{I_L}{M}} = \sqrt{\frac{a^2+7b^2}{3}}$

38. $M = \int_{-3}^3 \int_{-2}^4 \int_{-4/3}^{(4-2y)/3} dz\,dy\,dx = \int_{-3}^3 \int_{-2}^4 \frac{2}{3}(4-y)\,dy\,dx = \int_{-3}^3 \frac{2}{3}\left[4y-\frac{y^2}{2}\right]_{-2}^4 dx = 12\int_{-3}^3 dx = 72;$
$\overline{x} = \overline{y} = \overline{z} = 0$ from Exercise 22 $\Rightarrow I_x = I_{c.m.} + 72\left(\sqrt{0^2+0^2}\right)^2 = I_{c.m.} \Rightarrow I_L = I_{c.m.} + 72\left(\sqrt{16+\frac{16}{9}}\right)^2$
$= 208 + 72\left(\frac{160}{9}\right) = 1488$

15.7 TRIPLE INTEGRALS IN CYLINDRICAL AND SPHERICAL COORDINATES

1. $\int_0^{2\pi}\int_0^1\int_r^{\sqrt{2-r^2}} dz\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^1\left[r\left(2-r^2\right)^{1/2}-r^2\right] dr\, d\theta = \int_0^{2\pi}\left[-\frac{1}{3}\left(2-r^2\right)^{3/2}-\frac{r^3}{3}\right]_0^1 d\theta$
$= \int_0^{2\pi}\left(\frac{2^{3/2}}{3}-\frac{2}{3}\right) d\theta = \frac{4\pi\left(\sqrt{2}-1\right)}{3}$

2. $\int_0^{2\pi}\int_0^3\int_{r^2/3}^{\sqrt{18-r^2}} dz\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^3\left[r\left(18-r^2\right)^{1/2}-\frac{r^3}{3}\right] dr\, d\theta = \int_0^{2\pi}\left[-\frac{1}{3}\left(18-r^2\right)^{3/2}-\frac{r^4}{12}\right]_0^3 d\theta$
$= \frac{9\pi\left(8\sqrt{2}-7\right)}{2}$

3. $\int_0^{2\pi}\int_0^{\theta/2\pi}\int_0^{3+24r^2} dz\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^{\theta/2\pi}\left(3r+24r^3\right) dr\, d\theta = \int_0^{2\pi}\left[\frac{3}{2}r^2+6r^4\right]_0^{\theta/2\pi} d\theta = \frac{3}{2}\int_0^{2\pi}\left(\frac{\theta^2}{4\pi^2}+\frac{4\theta^4}{16\pi^4}\right) d\theta$
$= \frac{3}{2}\left[\frac{\theta^3}{12\pi^2}+\frac{\theta^5}{20\pi^4}\right]_0^{2\pi} = \frac{17\pi}{5}$

4. $\int_0^{\pi}\int_0^{\theta/\pi}\int_{-\sqrt{4-r^2}}^{3\sqrt{4-r^2}} z\, dz\, r\, dr\, d\theta = \int_0^{\pi}\int_0^{\theta/\pi}\frac{1}{2}\left[9\left(4-r^2\right)-\left(4-r^2\right)\right] r\, dr\, d\theta = 4\int_0^{\pi}\int_0^{\theta/\pi}\left(4r-r^3\right) dr\, d\theta$
$= 4\int_0^{\pi}\left[2r^2-\frac{r^4}{4}\right]_0^{\theta/\pi} = 4\int_0^{\pi}\left(\frac{2\theta^2}{\pi^2}-\frac{\theta^4}{4\pi^4}\right) d\theta = \frac{37\pi}{15}$

5. $\int_0^{2\pi}\int_0^1\int_r^{\left(2-r^2\right)^{-1/2}} 3\, dz\, r\, dr\, d\theta = 3\int_0^{2\pi}\int_0^1\left[r\left(2-r^2\right)^{-1/2}-r^2\right] dr\, d\theta = 3\int_0^{2\pi}\left[-\left(2-r^2\right)^{1/2}-\frac{r^3}{3}\right]_0^1 d\theta$
$= 3\int_0^{2\pi}\left(\sqrt{2}-\frac{4}{3}\right) d\theta = \pi\left(6\sqrt{2}-8\right)$

6. $\int_0^{2\pi}\int_0^1\int_{-1/2}^{1/2}\left(r^2\sin^2\theta+z^2\right) dz\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^1\left(r^3\sin^2\theta+\frac{r}{12}\right) dr\, d\theta = \int_0^{2\pi}\left(\frac{\sin^2\theta}{4}+\frac{1}{24}\right) d\theta = \frac{\pi}{3}$

7. $\int_0^{2\pi}\int_0^3\int_0^{z/3} r^3\, dr\, dz\, d\theta = \int_0^{2\pi}\int_0^3\frac{z^4}{324}\, dz\, d\theta = \int_0^{2\pi}\frac{3}{20}\, d\theta = \frac{3\pi}{10}$

8. $\int_{-1}^1\int_0^{2\pi}\int_0^{1+\cos\theta} 4r\, dr\, d\theta\, dz = \int_{-1}^1\int_0^{2\pi} 2(1+\cos\theta)^2\, d\theta\, dz = \int_{-1}^1 6\pi\, d\theta = 12\pi$

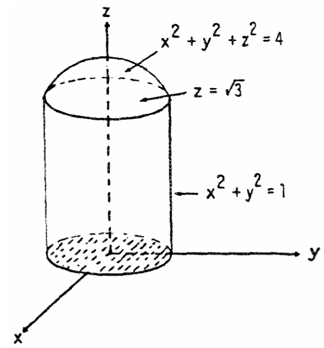
9. $\int_0^1\int_0^{\sqrt{z}}\int_0^{2\pi}\left(r^2\cos^2\theta+z^2\right) r\, d\theta\, dr\, dz = \int_0^1\int_0^{\sqrt{z}}\left[\frac{r^2\theta}{2}+\frac{r^2\sin 2\theta}{4}+z^2\theta\right]_0^{2\pi} r\, dr\, dz = \int_0^1\int_0^{\sqrt{z}}\left(\pi r^3+2\pi r z^2\right) dr\, dz$
$= \int_0^1\left[\frac{\pi r^4}{4}+\pi r^2 z^2\right]_0^{\sqrt{z}} dz = \int_0^1\left(\frac{\pi z^2}{4}+\pi z^3\right) dz = \left[\frac{\pi z^3}{12}+\frac{\pi z^4}{4}\right]_0^1 = \frac{\pi}{3}$

10. $\int_0^2\int_{r-2}^{\sqrt{4-r^2}}\int_0^{2\pi}(r\sin\theta+1)\, r\, d\theta\, dz\, dr = \int_0^2\int_{r-2}^{\sqrt{4-r^2}} 2\pi r\, dz\, dr = 2\pi\int_0^2\left[r\left(4-r^2\right)^{1/2}-r^2+2r\right] dr$
$= 2\pi\left[-\frac{1}{3}\left(4-r^2\right)^{3/2}-\frac{r^3}{3}+r^2\right]_0^2 = 2\pi\left[-\frac{8}{3}+4+\frac{1}{3}(4)^{3/2}\right] = 8\pi$

11. (a) $\int_0^{2\pi}\int_0^1\int_0^{\sqrt{4-r^2}} dz\, r\, dr\, d\theta$

(b) $\int_0^{2\pi}\int_0^{\sqrt{3}}\int_0^1 r\, dr\, dz\, d\theta + \int_0^{2\pi}\int_{\sqrt{3}}^2\int_0^{\sqrt{4-z^2}} r\, dr\, dz\, d\theta$

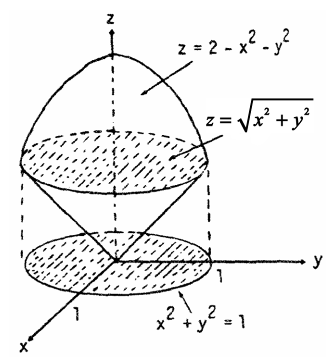
(c) $\int_0^1\int_0^{\sqrt{4-r^2}}\int_0^{2\pi} r\, d\theta\, dz\, dr$



12. (a) $\int_0^{2\pi}\int_0^1\int_r^{2-r^2} dz\, r\, dr\, d\theta$

(b) $\int_0^{2\pi}\int_0^1\int_0^z r\, dr\, dz\, d\theta + \int_0^{2\pi}\int_1^2\int_0^{\sqrt{2-z}} r\, dr\, dz\, d\theta$

(c) $\int_0^1\int_r^{2-r^2}\int_0^{2\pi} r\, d\theta\, dz\, dr$



13. $\int_{-\pi/2}^{\pi/2}\int_0^{\cos\theta}\int_0^{3r^2} f(r, \theta, z)\, dz\, r\, dr\, d\theta$

14. $\int_{-\pi/2}^{\pi/2}\int_0^1\int_0^{r\cos\theta} r^3\, dz\, dr\, d\theta = \int_{-\pi/2}^{\pi/2}\int_0^1 r^4 \cos\theta\, dr\, d\theta = \frac{1}{5}\int_{-\pi/2}^{\pi/2}\cos\theta\, d\theta = \frac{2}{5}$

15. $\int_0^{\pi}\int_0^{2\sin\theta}\int_0^{4-r\sin\theta} f(r, \theta, z)\, dz\, r\, dr\, d\theta$

16. $\int_{-\pi/2}^{\pi/2}\int_0^{3\cos\theta}\int_0^{5-r\cos\theta} f(r, \theta, z)\, dz\, r\, dr\, d\theta$

17. $\int_{-\pi/2}^{\pi/2}\int_1^{1+\cos\theta}\int_0^4 f(r, \theta, z)\, dz\, r\, dr\, d\theta$

18. $\int_{-\pi/2}^{\pi/2}\int_{\cos\theta}^{2\cos\theta}\int_0^{3-r\sin\theta} f(r, \theta, z)\, dz\, r\, dr\, d\theta$

19. $\int_0^{\pi/4}\int_0^{\sec\theta}\int_0^{2-r\sin\theta} f(r, \theta, z)\, dz\, r\, dr\, d\theta$

20. $\int_{\pi/4}^{\pi/2}\int_0^{\csc\theta}\int_0^{2-r\sin\theta} f(r, \theta, z)\, dz\, r\, dr\, d\theta$

21. $\int_0^{\pi}\int_0^{\pi}\int_0^{2\sin\phi} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \frac{8}{3}\int_0^{\pi}\int_0^{\pi}\sin^4\phi\, d\phi\, d\theta = \frac{8}{3}\int_0^{\pi}\left(\left[-\frac{\sin^3\phi\cos\phi}{4}\right]_0^{\pi} + \frac{3}{4}\int_0^{\pi}\sin^2\phi\, d\phi\right) d\theta$
$= 2\int_0^{\pi}\int_0^{\pi}\sin^2\phi\, d\phi\, d\theta = \int_0^{\pi}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi} d\theta = \int_0^{\pi}\pi\, d\theta = \pi^2$

22. $\int_0^{2\pi}\int_0^{\pi/4}\int_0^2 (\rho\cos\phi)\,\rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi/4} 4\cos\phi\sin\phi\, d\phi\, d\theta = \int_0^{2\pi}\left[2\sin^2\phi\right]_0^{\pi/4} d\theta = \int_0^{2\pi} d\theta = 2\pi$

23. $\int_0^{2\pi}\int_0^{\pi}\int_0^{(1-\cos\phi)/2} \rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \frac{1}{24}\int_0^{2\pi}\int_0^{\pi}(1-\cos\phi)^3\sin\phi\, d\phi\, d\theta = \frac{1}{96}\int_0^{2\pi}\left[(1-\cos\phi)^4\right]_0^{\pi} d\theta$
$= \frac{1}{96}\int_0^{2\pi}(2^4 - 0)\, d\theta = \frac{16}{96}\int_0^{2\pi} d\theta = \frac{1}{6}(2\pi) = \frac{\pi}{3}$

24. $\int_0^{3\pi/2}\int_0^{\pi}\int_0^1 5\rho^3\sin^3\phi\, d\rho\, d\phi\, d\theta = \frac{5}{4}\int_0^{3\pi/2}\int_0^{\pi}\sin^3\phi\, d\phi\, d\theta = \frac{5}{4}\int_0^{3\pi/2}\left(\left[-\frac{\sin^2\phi\cos\phi}{3}\right]_0^{\pi} + \frac{2}{3}\int_0^{\pi}\sin\phi\, d\phi\right) d\theta$
$= \frac{5}{6}\int_0^{3\pi/2}\left[-\cos\phi\right]_0^{\pi} d\theta = \frac{5}{3}\int_0^{3\pi/2} d\theta = \frac{5\pi}{2}$

25. $\int_0^{2\pi}\int_0^{\pi/3}\int_{\sec\phi}^{2} 3\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi/3}(8-\sec^3\phi)\sin\phi\,d\phi\,d\theta = \int_0^{2\pi}\left[-8\cos\phi-\frac{1}{2}\sec^2\phi\right]_0^{\pi/3}d\theta$
$= \int_0^{2\pi}\left[(-4-2)-\left(-8-\frac{1}{2}\right)\right]d\theta = \frac{5}{2}\int_0^{2\pi}d\theta = 5\pi$

26. $\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sec\phi}\rho^3\sin\phi\cos\phi\,d\rho\,d\phi\,d\theta = \frac{1}{4}\int_0^{2\pi}\int_0^{\pi/4}\tan\phi\sec^2\phi\,d\phi\,d\theta = \frac{1}{4}\int_0^{2\pi}\left[\frac{1}{2}\tan^2\phi\right]_0^{\pi/4}d\theta = \frac{1}{8}\int_0^{2\pi}d\theta = \frac{\pi}{4}$

27. $\int_0^2\int_{-\pi}^0\int_{\pi/4}^{\pi/2}\rho^3\sin 2\phi\,d\phi\,d\theta\,d\rho = \int_0^2\int_{-\pi}^0\rho^3\left[-\frac{\cos 2\phi}{2}\right]_{\pi/4}^{\pi/2}d\theta\,d\rho = \int_0^2\int_{-\pi}^0\frac{\rho^3}{2}\,d\theta\,d\rho = \int_0^2\frac{\rho^3\pi}{2}\,d\rho = \left[\frac{\pi\rho^4}{8}\right]_0^2 = 2\pi$

28. $\int_{\pi/6}^{\pi/3}\int_{\csc\phi}^{2\csc\phi}\int_0^{2\pi}\rho^2\sin\phi\,d\theta\,d\rho\,d\phi = 2\pi\int_{\pi/6}^{\pi/3}\int_{\csc\phi}^{2\csc\phi}\rho^2\sin\phi\,d\rho\,d\phi = \frac{2\pi}{3}\int_{\pi/6}^{\pi/3}\left[\rho^3\sin\phi\right]_{\csc\phi}^{2\csc\phi}d\phi = \frac{14\pi}{3}\int_{\pi/6}^{\pi/3}\csc^2\phi\,d\phi = \frac{28\pi}{3\sqrt{3}}$

29. $\int_0^1\int_0^{\pi}\int_0^{\pi/4}12\rho\sin^3\phi\,d\phi\,d\theta\,d\rho = \int_0^1\int_0^{\pi}\left(12\rho\left[\frac{-\sin^2\phi\cos\phi}{3}\right]_0^{\pi/4} + 8\rho\int_0^{\pi/4}\sin\phi\,d\phi\right)d\theta\,d\rho$
$= \int_0^1\int_0^{\pi}\left(-\frac{2\rho}{\sqrt{2}} - 8\rho\left[\cos\phi\right]_0^{\pi/4}\right)d\theta\,d\rho = \int_0^1\int_0^{\pi}\left(8\rho-\frac{10\rho}{\sqrt{2}}\right)d\theta\,d\rho = \pi\int_0^1\left(8\rho-\frac{10\rho}{\sqrt{2}}\right)d\rho = \pi\left[4\rho^2-\frac{5\rho^2}{\sqrt{2}}\right]_0^1$
$= \frac{\left(4\sqrt{2}-5\right)\pi}{\sqrt{2}}$

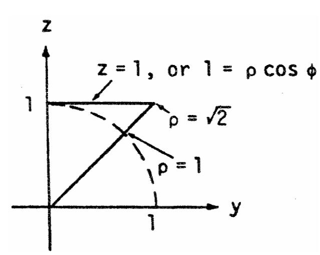
30. $\int_{\pi/6}^{\pi/2}\int_{-\pi/2}^{\pi/2}\int_{\csc\phi}^{2}5\rho^4\sin^3\phi\,d\rho\,d\theta\,d\phi = \int_{\pi/6}^{\pi/2}\int_{-\pi/2}^{\pi/2}(32-\csc^5\phi)\sin^3\phi\,d\theta\,d\phi = \int_{\pi/6}^{\pi/2}\int_{-\pi/2}^{\pi/2}(32\sin^3\phi-\csc^2\phi)\,d\theta\,d\phi$
$= \pi\int_{\pi/6}^{\pi/2}(32\sin^3\phi-\csc^2\phi)\,d\phi = \pi\left[-\frac{32\sin^2\phi\cos\phi}{3}\right]_{\pi/6}^{\pi/2} + \frac{64\pi}{3}\int_{\pi/6}^{\pi/2}\sin\phi\,d\phi + \pi\left[\cot\phi\right]_{\pi/6}^{\pi/2}$
$= \pi\left(\frac{32\sqrt{3}}{24}\right) - \frac{64\pi}{3}\left[\cos\phi\right]_{\pi/6}^{\pi/2} - \pi\left(\sqrt{3}\right) = \frac{\sqrt{3}}{3}\pi + \left(\frac{64\pi}{3}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{33\pi\sqrt{3}}{3} = 11\pi\sqrt{3}$

31. (a) $x^2+y^2=1 \Rightarrow \rho^2\sin^2\phi = 1$, and $\rho\sin\phi = 1 \Rightarrow \rho = \csc\phi$; thus
$\int_0^{2\pi}\int_0^{\pi/6}\int_0^2\rho^2\sin\phi\,d\rho\,d\phi\,d\theta + \int_0^{2\pi}\int_{\pi/6}^{\pi/2}\int_0^{\csc\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

(b) $\int_0^{2\pi}\int_1^2\int_{\pi/6}^{\sin^{-1}(1/\rho)}\rho^2\sin\phi\,d\phi\,d\rho\,d\theta + \int_0^{2\pi}\int_0^2\int_0^{\pi/6}\rho^2\sin\phi\,d\phi\,d\rho\,d\theta$

32. (a) $\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sec\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

(b) $\int_0^{2\pi}\int_0^1\int_0^{\pi/4}\rho^2\sin\phi\,d\phi\,d\rho\,d\theta$
$+ \int_0^{2\pi}\int_1^{\sqrt{2}}\int_{\cos^{-1}(1/\rho)}^{\pi/4}\rho^2\sin\phi\,d\phi\,d\rho\,d\theta$



33. $V = \int_0^{2\pi}\int_0^{\pi/2}\int_{\cos\phi}^2\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/2}(8-\cos^3\phi)\sin\phi\,d\phi\,d\theta$
$= \frac{1}{3}\int_0^{2\pi}\left[-8\cos\phi+\frac{\cos^4\phi}{4}\right]_0^{\pi/2}d\theta = \frac{1}{3}\int_0^{2\pi}\left(8-\frac{1}{4}\right)d\theta = \left(\frac{31}{12}\right)(2\pi) = \frac{31\pi}{6}$

34. $V = \int_0^{2\pi}\int_0^{\pi/2}\int_1^{1+\cos\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/2}(3\cos\phi+3\cos^2\phi+\cos^3\phi)\sin\phi\,d\phi\,d\theta$
$= \frac{1}{3}\int_0^{2\pi}\left[-\frac{3}{2}\cos^2\phi-\cos^3\phi-\frac{1}{4}\cos^4\phi\right]_0^{\pi/2}d\theta = \frac{1}{3}\int_0^{2\pi}\left(\frac{3}{2}+1+\frac{1}{4}\right)d\theta = \frac{11}{12}\int_0^{2\pi}d\theta = \left(\frac{11}{12}\right)(2\pi) = \frac{11\pi}{6}$

35. $V = \int_0^{2\pi}\int_0^{\pi}\int_0^{1-\cos\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{1}{3}\int_0^{2\pi}\int_0^{\pi}(1-\cos\phi)^3\sin\phi\,d\phi\,d\theta = \frac{1}{3}\int_0^{2\pi}\left[\frac{(1-\cos\phi)^4}{4}\right]_0^{\pi}d\theta$
$= \frac{1}{12}(2)^4\int_0^{2\pi}d\theta = \frac{4}{3}(2\pi) = \frac{8\pi}{3}$

36. $V = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{1-\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/2} (1-\cos\phi)^3 \sin\phi \, d\phi \, d\theta = \frac{1}{3}\int_0^{2\pi}\left[\frac{(1-\cos\phi)^4}{4}\right]_0^{\pi/2} d\theta$
$= \frac{1}{12}\int_0^{2\pi} d\theta = \frac{1}{12}(2\pi) = \frac{\pi}{6}$

37. $V = \int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^{2\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \frac{8}{3}\int_0^{2\pi}\int_{\pi/4}^{\pi/2} \cos^3\phi \sin\phi \, d\phi \, d\theta = \frac{8}{3}\int_0^{2\pi}\left[-\frac{\cos^4\phi}{4}\right]_{\pi/4}^{\pi/2} d\theta$
$= \left(\frac{8}{3}\right)\left(\frac{1}{16}\right)\int_0^{2\pi} d\theta = \frac{1}{6}(2\pi) = \frac{\pi}{3}$

38. $V = \int_0^{2\pi}\int_{\pi/3}^{\pi/2}\int_0^{2} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \frac{8}{3}\int_0^{2\pi}\int_{\pi/3}^{\pi/2} \sin\phi \, d\phi \, d\theta = \frac{8}{3}\int_0^{2\pi} [-\cos\phi]_{\pi/3}^{\pi/2} d\theta = \frac{4}{3}\int_0^{2\pi} d\theta = \frac{8\pi}{3}$

39. (a) $8\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{2} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

(b) $8\int_0^{\pi/2}\int_0^{2}\int_0^{\sqrt{4-r^2}} dz \, r \, dr \, d\theta$

(c) $8\int_0^{2}\int_0^{\sqrt{4-x^2}}\int_0^{\sqrt{4-x^2-y^2}} dz \, dy \, dx$

40. (a) $\int_0^{\pi/2}\int_0^{3/\sqrt{2}}\int_r^{\sqrt{9-r^2}} dz \, r \, dr \, d\theta$

(b) $\int_0^{\pi/2}\int_0^{\pi/4}\int_0^{3} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

(c) $\int_0^{\pi/2}\int_0^{\pi/4}\int_0^{3} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = 9\int_0^{\pi/2}\int_0^{\pi/4} \sin\phi \, d\phi \, d\theta = -9\int_0^{\pi/2}\left(\frac{1}{\sqrt{2}} - 1\right) d\theta = \frac{9\pi\left(2-\sqrt{2}\right)}{4}$

41. (a) $V = \int_0^{2\pi}\int_0^{\pi/3}\int_{\sec\phi}^{2} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

(b) $V = \int_0^{2\pi}\int_0^{\sqrt{3}}\int_1^{\sqrt{4-r^2}} dz \, r \, dr \, d\theta$

(c) $V = \int_{-\sqrt{3}}^{\sqrt{3}}\int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}}\int_1^{\sqrt{4-x^2-y^2}} dz \, dy \, dx$

(d) $V = \int_0^{2\pi}\int_0^{\sqrt{3}} \left[r(4-r^2)^{1/2} - r\right] dr \, d\theta = \int_0^{2\pi}\left[-\frac{(4-r^2)^{3/2}}{3} - \frac{r^2}{2}\right]_0^{\sqrt{3}} d\theta = \int_0^{2\pi}\left(-\frac{1}{3} - \frac{3}{2} + \frac{4^{3/2}}{3}\right) d\theta$
$= \frac{5}{6}\int_0^{2\pi} d\theta = \frac{5\pi}{3}$

42. (a) $I_z = \int_0^{2\pi}\int_0^{1}\int_0^{\sqrt{1-r^2}} r^2 \, dz \, r \, dr \, d\theta$

(b) $I_z = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{1} (\rho^2 \sin^2\phi)(\rho^2 \sin\phi) \, d\rho \, d\phi \, d\theta$, since $r^2 = x^2 + y^2 = \rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta = \rho^2 \sin^2\phi$

(c) $I_z = \int_0^{2\pi}\int_0^{\pi/2} \frac{1}{5}\sin^3\phi \, d\phi \, d\theta = \frac{1}{5}\int_0^{2\pi}\left(\left[-\frac{\sin^2\phi\cos\phi}{3}\right]_0^{\pi/2} + \frac{2}{3}\int_0^{\pi/2} \sin\phi \, d\phi\right) d\theta = \frac{2}{15}\int_0^{2\pi} [-\cos\phi]_0^{\pi/2} d\theta$
$= \frac{2}{15}(2\pi) = \frac{4\pi}{15}$

43. $V = 4\int_0^{\pi/2}\int_0^{1}\int_{r^4-1}^{4-4r^2} dz \, r \, dr \, d\theta = 4\int_0^{\pi/2}\int_0^{1} (5r - 4r^3 - r^5) \, dr \, d\theta = 4\int_0^{\pi/2}\left(\frac{5}{2} - 1 - \frac{1}{6}\right) d\theta = 4\int_0^{\pi/2} d\theta = \frac{8\pi}{3}$

44. $V = 4\int_0^{\pi/2}\int_0^{1}\int_{-\sqrt{1-r^2}}^{1-r} dz \, r \, dr \, d\theta = 4\int_0^{\pi/2}\int_0^{1}\left(r - r^2 + r\sqrt{1-r^2}\right) dr \, d\theta = 4\int_0^{\pi/2}\left[\frac{r^2}{2} - \frac{r^3}{3} - \frac{1}{3}(1-r^2)^{3/2}\right]_0^{1} d\theta$
$= 4\int_0^{\pi/2}\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{3}\right) d\theta = 2\int_0^{\pi/2} d\theta = 2\left(\frac{\pi}{2}\right) = \pi$

45. $V = \int_{3\pi/2}^{2\pi}\int_0^{3\cos\theta}\int_0^{-r\sin\theta} dz \, r \, dr \, d\theta = \int_{3\pi/2}^{2\pi}\int_0^{3\cos\theta} -r^2 \sin\theta \, dr \, d\theta = \int_{3\pi/2}^{2\pi} (-9\cos^3\theta)(\sin\theta) \, d\theta = \left[\frac{9}{4}\cos^4\theta\right]_{3\pi/2}^{2\pi}$
$= \frac{9}{4} - 0 = \frac{9}{4}$

46. $V = 2\int_{\pi/2}^{\pi}\int_0^{-3\cos\theta}\int_0^{r} dz \, r \, dr \, d\theta = 2\int_{\pi/2}^{\pi}\int_0^{-3\cos\theta} r^2 \, dr \, d\theta = \frac{2}{3}\int_{\pi/2}^{\pi} -27\cos^3\theta \, d\theta$
$= -18\left(\left[\frac{\cos^2\theta\sin\theta}{3}\right]_{\pi/2}^{\pi} + \frac{2}{3}\int_{\pi/2}^{\pi} \cos\theta \, d\theta\right) = -12\,[\sin\theta]_{\pi/2}^{\pi} = 12$

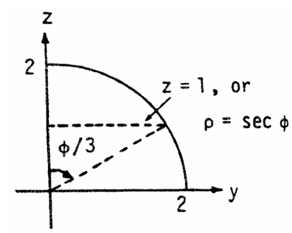
47. $V=\int_0^{\pi/2}\int_0^{\sin\theta}\int_0^{\sqrt{1-r^2}} dz\, r\, dr\, d\theta=\int_0^{\pi/2}\int_0^{\sin\theta} r\sqrt{1-r^2}\, dr\, d\theta=\int_0^{\pi/2}\left[-\frac{1}{3}\left(1-r^2\right)^{3/2}\right]_0^{\sin\theta} d\theta$
$=-\frac{1}{3}\int_0^{\pi/2}\left[\left(1-\sin^2\theta\right)^{3/2}-1\right] d\theta=-\frac{1}{3}\int_0^{\pi/2}\left(\cos^3\theta-1\right) d\theta=-\frac{1}{3}\left(\left[\frac{\cos^2\theta\sin\theta}{3}\right]_0^{\pi/2}+\frac{2}{3}\int_0^{\pi/2}\cos\theta\, d\theta\right)+\left[\frac{\theta}{3}\right]_0^{\pi/2}$
$=-\frac{2}{9}\left[\sin\theta\right]_0^{\pi/2}+\frac{\pi}{6}=\frac{-4+3\pi}{18}$

48. $V=\int_0^{\pi/2}\int_0^{\cos\theta}\int_0^{3\sqrt{1-r^2}} dz\, r\, dr\, d\theta=\int_0^{\pi/2}\int_0^{\cos\theta} 3r\sqrt{1-r^2}\, dr\, d\theta=\int_0^{\pi/2}\left[-\left(1-r^2\right)^{3/2}\right]_0^{\cos\theta} d\theta$
$=\int_0^{\pi/2}\left[-\left(1-\cos^2\theta\right)^{3/2}+1\right] d\theta=\int_0^{\pi/2}\left(1-\sin^3\theta\right) d\theta=\left[\theta+\frac{\sin^2\theta\cos\theta}{3}\right]_0^{\pi/2}-\frac{2}{3}\int_0^{\pi/2}\sin\theta\, d\theta$
$=\frac{\pi}{2}+\frac{2}{3}\left[\cos\theta\right]_0^{\pi/2}=\frac{\pi}{2}-\frac{2}{3}=\frac{3\pi-4}{6}$

49. $V=\int_0^{2\pi}\int_{\pi/3}^{2\pi/3}\int_0^{a}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta=\int_0^{2\pi}\int_{\pi/3}^{2\pi/3}\frac{a^3}{3}\sin\phi\, d\phi\, d\theta=\frac{a^3}{3}\int_0^{2\pi}\left[-\cos\phi\right]_{\pi/3}^{2\pi/3} d\theta=\frac{a^3}{3}\int_0^{2\pi}\left(\frac{1}{2}+\frac{1}{2}\right) d\theta=\frac{2\pi a^3}{3}$

50. $V=\int_0^{\pi/6}\int_0^{\pi/2}\int_0^{a}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta=\frac{a^3}{3}\int_0^{\pi/6}\int_0^{\pi/2}\sin\phi\, d\phi\, d\theta=\frac{a^3}{3}\int_0^{\pi/6} d\theta=\frac{a^3\pi}{18}$

51. $V=\int_0^{2\pi}\int_0^{\pi/3}\int_{\sec\phi}^{2}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$
$=\frac{1}{3}\int_0^{2\pi}\int_0^{\pi/3}\left(8\sin\phi-\tan\phi\sec^2\phi\right) d\phi\, d\theta$
$=\frac{1}{3}\int_0^{2\pi}\left[-8\cos\phi-\frac{1}{2}\tan^2\phi\right]_0^{\pi/3} d\theta$
$=\frac{1}{3}\int_0^{2\pi}\left[-4-\frac{1}{2}(3)+8\right] d\theta=\frac{1}{3}\int_0^{2\pi}\frac{5}{2}\, d\theta=\frac{5}{6}(2\pi)=\frac{5\pi}{3}$



52. $V=4\int_0^{\pi/2}\int_0^{\pi/4}\int_{\sec\phi}^{2\sec\phi}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta=\frac{4}{3}\int_0^{\pi/2}\int_0^{\pi/4}\left(8\sec^3\phi-\sec^3\phi\right)\sin\phi\, d\phi\, d\theta$
$=\frac{28}{3}\int_0^{\pi/2}\int_0^{\pi/4}\sec^3\phi\sin\phi\, d\phi\, d\theta=\frac{28}{3}\int_0^{\pi/2}\int_0^{\pi/4}\tan\phi\sec^2\phi\, d\phi\, d\theta=\frac{28}{3}\int_0^{\pi/2}\left[\frac{1}{2}\tan^2\phi\right]_0^{\pi/4} d\theta=\frac{14}{3}\int_0^{\pi/2} d\theta=\frac{7\pi}{3}$

53. $V=4\int_0^{\pi/2}\int_0^{1}\int_0^{r^2} dz\, r\, dr\, d\theta=4\int_0^{\pi/2}\int_0^{1} r^3\, dr\, d\theta=\int_0^{\pi/2} d\theta=\frac{\pi}{2}$

54. $V=4\int_0^{\pi/2}\int_0^{1}\int_{r^2}^{r^2+1} dz\, r\, dr\, d\theta=4\int_0^{\pi/2}\int_0^{1} r\, dr\, d\theta=2\int_0^{\pi/2} d\theta=\pi$

55. $V=8\int_0^{\pi/2}\int_1^{\sqrt{2}}\int_0^{r} dz\, r\, dr\, d\theta=8\int_0^{\pi/2}\int_1^{\sqrt{2}} r^2\, dr\, d\theta=8\left(\frac{2\sqrt{2}-1}{3}\right)\int_0^{\pi/2} d\theta=\frac{4\pi\left(2\sqrt{2}-1\right)}{3}$

56. $V=8\int_0^{\pi/2}\int_1^{\sqrt{2}}\int_0^{\sqrt{2-r^2}} dz\, r\, dr\, d\theta=8\int_0^{\pi/2}\int_1^{\sqrt{2}} r\sqrt{2-r^2}\, dr\, d\theta=8\int_0^{\pi/2}\left[-\frac{1}{3}\left(2-r^2\right)^{3/2}\right]_1^{\sqrt{2}} d\theta=\frac{8}{3}\int_0^{\pi/2} d\theta=\frac{4\pi}{3}$

57. $V=\int_0^{2\pi}\int_0^{2}\int_0^{4-r\sin\theta} dz\, r\, dr\, d\theta=\int_0^{2\pi}\int_0^{2}\left(4r-r^2\sin\theta\right) dr\, d\theta=8\int_0^{2\pi}\left(1-\frac{\sin\theta}{3}\right) d\theta=16\pi$

58. $V=\int_0^{2\pi}\int_0^{2}\int_0^{4-r\cos\theta-r\sin\theta} dz\, r\, dr\, d\theta=\int_0^{2\pi}\int_0^{2}\left[4r-r^2(\cos\theta+\sin\theta)\right] dr\, d\theta=\frac{8}{3}\int_0^{2\pi}(3-\cos\theta-\sin\theta)\, d\theta=16\pi$

59. The paraboloids intersect when $4x^2+4y^2=5-x^2-y^2 \Rightarrow x^2+y^2=1$ and $z=4$
$\Rightarrow V=4\int_0^{\pi/2}\int_0^{1}\int_{4r^2}^{5-r^2} dz\, r\, dr\, d\theta=4\int_0^{\pi/2}\int_0^{1}\left(5r-5r^3\right) dr\, d\theta=20\int_0^{\pi/2}\left[\frac{r^2}{2}-\frac{r^4}{4}\right]_0^{1} d\theta=5\int_0^{\pi/2} d\theta=\frac{5\pi}{2}$

60. The paraboloid intersects the xy-plane when $9 - x^2 - y^2 = 0 \Rightarrow x^2 + y^2 = 9 \Rightarrow V = 4 \int_0^{\pi/2} \int_1^3 \int_0^{9-r^2} dz\, r\, dr\, d\theta$
$= 4 \int_0^{\pi/2} \int_1^3 (9r - r^3)\, dr\, d\theta = 4 \int_0^{\pi/2} \left[\frac{9r^2}{2} - \frac{r^4}{4}\right]_1^3 d\theta = 4 \int_0^{\pi/2} \left(\frac{81}{4} - \frac{17}{4}\right) d\theta = 64 \int_0^{\pi/2} d\theta = 32\pi$

61. $V = 8 \int_0^{2\pi} \int_0^1 \int_0^{\sqrt{4-r^2}} dz\, r\, dr\, d\theta = 8 \int_0^{2\pi} \int_0^1 r\,(4 - r^2)^{1/2}\, dr\, d\theta = 8 \int_0^{2\pi} \left[-\frac{1}{3}(4 - r^2)^{3/2}\right]_0^1 d\theta$
$= -\frac{8}{3} \int_0^{2\pi} \left(3^{3/2} - 8\right) d\theta = \frac{4\pi\left(8 - 3\sqrt{3}\right)}{3}$

62. The sphere and paraboloid intersect when $x^2 + y^2 + z^2 = 2$ and $z = x^2 + y^2 \Rightarrow z^2 + z - 2 = 0$
$\Rightarrow (z + 2)(z - 1) = 0 \Rightarrow z = 1$ or $z = -2 \Rightarrow z = 1$ since $z \geq 0$. Thus, $x^2 + y^2 = 1$ and the volume is given by the triple integral $V = 4 \int_0^{\pi/2} \int_0^1 \int_{r^2}^{\sqrt{2-r^2}} dz\, r\, dr\, d\theta = 4 \int_0^{\pi/2} \int_0^1 \left[r\,(2 - r^2)^{1/2} - r^3\right] dr\, d\theta$
$= 4 \int_0^{\pi/2} \left[-\frac{1}{3}(2 - r^2)^{3/2} - \frac{r^4}{4}\right]_0^1 d\theta = 4 \int_0^{\pi/2} \left(\frac{2\sqrt{2}}{3} - \frac{7}{12}\right) d\theta = \frac{\pi\left(8\sqrt{2} - 7\right)}{6}$

63. average $= \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 \int_{-1}^1 r^2\, dz\, dr\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 2r^2\, dr\, d\theta = \frac{1}{3\pi} \int_0^{2\pi} d\theta = \frac{2}{3}$

64. average $= \frac{1}{\left(\frac{4\pi}{3}\right)} \int_0^{2\pi} \int_0^1 \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} r^2\, dz\, dr\, d\theta = \frac{3}{4\pi} \int_0^{2\pi} \int_0^1 2r^2\sqrt{1 - r^2}\, dr\, d\theta$
$= \frac{3}{2\pi} \int_0^{2\pi} \left[\frac{1}{8}\sin^{-1} r - \frac{1}{8} r\sqrt{1 - r^2}\,(1 - 2r^2)\right]_0^1 d\theta = \frac{3}{16\pi} \int_0^{2\pi} \left(\frac{\pi}{2} + 0\right) d\theta = \frac{3}{32} \int_0^{2\pi} d\theta = \left(\frac{3}{32}\right)(2\pi) = \frac{3\pi}{16}$

65. average $= \frac{1}{\left(\frac{4\pi}{3}\right)} \int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^3 \sin\phi\, d\rho\, d\phi\, d\theta = \frac{3}{16\pi} \int_0^{2\pi} \int_0^{\pi} \sin\phi\, d\phi\, d\theta = \frac{3}{8\pi} \int_0^{2\pi} d\theta = \frac{3}{4}$

66. average $= \frac{1}{\left(\frac{2\pi}{3}\right)} \int_0^{2\pi} \int_0^{\pi/2} \int_0^1 \rho^3 \cos\phi \sin\phi\, d\rho\, d\phi\, d\theta = \frac{3}{8\pi} \int_0^{2\pi} \int_0^{\pi/2} \cos\phi \sin\phi\, d\phi\, d\theta = \frac{3}{8\pi} \int_0^{2\pi} \left[\frac{\sin^2\phi}{2}\right]_0^{\pi/2} d\theta$
$= \frac{3}{16\pi} \int_0^{2\pi} d\theta = \left(\frac{3}{16\pi}\right)(2\pi) = \frac{3}{8}$

67. $M = 4 \int_0^{\pi/2} \int_0^1 \int_0^r dz\, r\, dr\, d\theta = 4 \int_0^{\pi/2} \int_0^1 r^2\, dr\, d\theta = \frac{4}{3} \int_0^{\pi/2} d\theta = \frac{2\pi}{3}$; $M_{xy} = \int_0^{2\pi} \int_0^1 \int_0^r z\, dz\, r\, dr\, d\theta$
$= \frac{1}{2} \int_0^{2\pi} \int_0^1 r^3\, dr\, d\theta = \frac{1}{8} \int_0^{2\pi} d\theta = \frac{\pi}{4} \Rightarrow \overline{z} = \frac{M_{xy}}{M} = \left(\frac{\pi}{4}\right)\left(\frac{3}{2\pi}\right) = \frac{3}{8}$, and $\overline{x} = \overline{y} = 0$, by symmetry

68. $M = \int_0^{\pi/2} \int_0^2 \int_0^r dz\, r\, dr\, d\theta = \int_0^{\pi/2} \int_0^2 r^2\, dr\, d\theta = \frac{8}{3} \int_0^{\pi/2} d\theta = \frac{4\pi}{3}$; $M_{yz} = \int_0^{\pi/2} \int_0^2 \int_0^r x\, dz\, r\, dr\, d\theta$
$= \int_0^{\pi/2} \int_0^2 r^3 \cos\theta\, dr\, d\theta = 4 \int_0^{\pi/2} \cos\theta\, d\theta = 4$; $M_{xz} = \int_0^{\pi/2} \int_0^2 \int_0^r y\, dz\, r\, dr\, d\theta = \int_0^{\pi/2} \int_0^2 r^3 \sin\theta\, dr\, d\theta$
$= 4 \int_0^{\pi/2} \sin\theta\, d\theta = 4$; $M_{xy} = \int_0^{\pi/2} \int_0^2 \int_0^r z\, dz\, r\, dr\, d\theta = \frac{1}{2} \int_0^{\pi/2} \int_0^2 r^3\, dr\, d\theta = 2 \int_0^{\pi/2} d\theta = \pi \Rightarrow \overline{x} = \frac{M_{yz}}{M} = \frac{3}{\pi}$,
$\overline{y} = \frac{M_{xz}}{M} = \frac{3}{\pi}$, and $\overline{z} = \frac{M_{xy}}{M} = \frac{3}{4}$

69. $M = \frac{8\pi}{3}$; $M_{xy} = \int_0^{2\pi} \int_{\pi/3}^{\pi/2} \int_0^2 z\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \int_0^{2\pi} \int_{\pi/3}^{\pi/2} \int_0^2 \rho^3 \cos\phi \sin\phi\, d\rho\, d\phi\, d\theta = 4 \int_0^{2\pi} \int_{\pi/3}^{\pi/2} \cos\phi \sin\phi\, d\phi\, d\theta$
$= 4 \int_0^{2\pi} \left[\frac{\sin^2\phi}{2}\right]_{\pi/3}^{\pi/2} d\theta = 4 \int_0^{2\pi} \left(\frac{1}{2} - \frac{3}{8}\right) d\theta = \frac{1}{2} \int_0^{2\pi} d\theta = \pi \Rightarrow \overline{z} = \frac{M_{xy}}{M} = (\pi)\left(\frac{3}{8\pi}\right) = \frac{3}{8}$, and $\overline{x} = \overline{y} = 0$, by symmetry

70. $M = \int_0^{2\pi} \int_0^{\pi/4} \int_0^a \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \frac{a^3}{3} \int_0^{2\pi} \int_0^{\pi/4} \sin\phi\, d\phi\, d\theta = \frac{a^3}{3} \int_0^{2\pi} \frac{2 - \sqrt{2}}{2}\, d\theta = \frac{\pi a^3\left(2 - \sqrt{2}\right)}{3}$;
$M_{xy} = \int_0^{2\pi} \int_0^{\pi/4} \int_0^a \rho^3 \sin\phi \cos\phi\, d\rho\, d\phi\, d\theta = \frac{a^4}{4} \int_0^{2\pi} \int_0^{\pi/4} \sin\phi \cos\phi\, d\phi\, d\theta = \frac{a^4}{16} \int_0^{2\pi} d\theta = \frac{\pi a^4}{8}$

$\Rightarrow \bar{z} = \frac{M_{xy}}{M} = \left(\frac{\pi a^4}{8}\right)\left[\frac{3}{\pi a^3\left(2-\sqrt{2}\right)}\right] = \left(\frac{3a}{8}\right)\left(\frac{2+\sqrt{2}}{2}\right) = \frac{3\left(2+\sqrt{2}\right)a}{16}$, and $\bar{x} = \bar{y} = 0$, by symmetry

71. $M = \int_0^{2\pi}\int_0^4\int_0^{\sqrt{r}} dz\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^4 r^{3/2}\, dr\, d\theta = \frac{64}{5}\int_0^{2\pi} d\theta = \frac{128\pi}{5}$; $M_{xy} = \int_0^{2\pi}\int_0^4\int_0^{\sqrt{r}} z\, dz\, r\, dr\, d\theta$
$= \frac{1}{2}\int_0^{2\pi}\int_0^4 r^2\, dr\, d\theta = \frac{32}{3}\int_0^{2\pi} d\theta = \frac{64\pi}{3} \Rightarrow \bar{z} = \frac{M_{xy}}{M} = \frac{5}{6}$, and $\bar{x} = \bar{y} = 0$, by symmetry

72. $M = \int_{-\pi/3}^{\pi/3}\int_0^1\int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} dz\, r\, dr\, d\theta = \int_{-\pi/3}^{\pi/3}\int_0^1 2r\sqrt{1-r^2}\, dr\, d\theta = \int_{-\pi/3}^{\pi/3}\left[-\frac{2}{3}\left(1-r^2\right)^{3/2}\right]_0^1 d\theta$
$= \frac{2}{3}\int_{-\pi/3}^{\pi/3} d\theta = \left(\frac{2}{3}\right)\left(\frac{2\pi}{3}\right) = \frac{4\pi}{9}$; $M_{yz} = \int_{-\pi/3}^{\pi/3}\int_0^1\int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} r^2\cos\theta\, dz\, dr\, d\theta = 2\int_{-\pi/3}^{\pi/3}\int_0^1 r^2\sqrt{1-r^2}\cos\theta\, dr\, d\theta$
$= 2\int_{-\pi/3}^{\pi/3}\left[\frac{1}{8}\sin^{-1} r - \frac{1}{8} r\sqrt{1-r^2}\left(1-2r^2\right)\right]_0^1 \cos\theta\, d\theta = \frac{\pi}{8}\int_{-\pi/3}^{\pi/3}\cos\theta\, d\theta = \frac{\pi}{8}\left[\sin\theta\right]_{-\pi/3}^{\pi/3} = \left(\frac{\pi}{8}\right)\left(2\cdot\frac{\sqrt{3}}{2}\right) = \frac{\pi\sqrt{3}}{8}$
$\Rightarrow \bar{x} = \frac{M_{yz}}{M} = \frac{9\sqrt{3}}{32}$, and $\bar{y} = \bar{z} = 0$, by symmetry

73. We orient the cone with its vertex at the origin and axis along the z-axis $\Rightarrow \phi = \frac{\pi}{4}$. We use the the x-axis which is through the vertex and parallel to the base of the cone $\Rightarrow I_x = \int_0^{2\pi}\int_0^1\int_r^1\left(r^2\sin^2\theta + z^2\right) dz\, r\, dr\, d\theta$
$= \int_0^{2\pi}\int_0^1\left(r^3\sin^2\theta - r^4\sin^2\theta + \frac{r}{3} - \frac{r^4}{3}\right) dr\, d\theta = \int_0^{2\pi}\left(\frac{\sin^2\theta}{20} + \frac{1}{10}\right) d\theta = \left[\frac{\theta}{40} - \frac{\sin 2\theta}{80} + \frac{\theta}{10}\right]_0^{2\pi} = \frac{\pi}{20} + \frac{\pi}{5} = \frac{\pi}{4}$

74. $I_z = \int_0^{2\pi}\int_0^a\int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r^3\, dz\, dr\, d\theta = \int_0^{2\pi}\int_0^a 2r^3\sqrt{a^2-r^2}\, dr\, d\theta = 2\int_0^{2\pi}\left[\left(-\frac{r^2}{5} - \frac{2a^2}{15}\right)\left(a^2-r^2\right)^{3/2}\right]_0^a d\theta = 2\int_0^{2\pi}\frac{2}{15}a^5\, d\theta$
$= \frac{8\pi a^5}{15}$

75. $I_z = \int_0^{2\pi}\int_0^a\int_{\left(\frac{h}{a}\right)r}^h \left(x^2+y^2\right) dz\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^a\int_{\frac{hr}{a}}^h r^3\, dz\, dr\, d\theta = \int_0^{2\pi}\int_0^a\left(hr^3 - \frac{hr^4}{a}\right) dr\, d\theta = \int_0^{2\pi} h\left[\frac{r^4}{4} - \frac{r^5}{5a}\right]_0^a d\theta$
$= \int_0^{2\pi} h\left(\frac{a^4}{4} - \frac{a^5}{5a}\right) d\theta = \frac{ha^4}{20}\int_0^{2\pi} d\theta = \frac{\pi h a^4}{10}$

76. (a) $M = \int_0^{2\pi}\int_0^1\int_0^{r^2} z\, dz\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^1 \frac{1}{2} r^5\, dr\, d\theta = \frac{1}{12}\int_0^{2\pi} d\theta = \frac{\pi}{6}$; $M_{xy} = \int_0^{2\pi}\int_0^1\int_0^{r^2} z^2\, dz\, r\, dr\, d\theta$
$= \frac{1}{3}\int_0^{2\pi}\int_0^1 r^7\, dr\, d\theta = \frac{1}{24}\int_0^{2\pi} d\theta = \frac{\pi}{12} \Rightarrow \bar{z} = \frac{1}{2}$, and $\bar{x} = \bar{y} = 0$, by symmetry; $I_z = \int_0^{2\pi}\int_0^1\int_0^{r^2} zr^3\, dz\, dr\, d\theta$
$= \frac{1}{2}\int_0^{2\pi}\int_0^1 r^7\, dr\, d\theta = \frac{1}{16}\int_0^{2\pi} d\theta = \frac{\pi}{8}$

(b) $M = \int_0^{2\pi}\int_0^1\int_0^{r^2} r^2\, dz\, dr\, d\theta = \int_0^{2\pi}\int_0^1 r^4\, dr\, d\theta = \frac{1}{5}\int_0^{2\pi} d\theta = \frac{2\pi}{5}$; $M_{xy} = \int_0^{2\pi}\int_0^1\int_0^{r^2} zr^2\, dz\, dr\, d\theta$
$= \frac{1}{2}\int_0^{2\pi}\int_0^1 r^6\, dr\, d\theta = \frac{1}{14}\int_0^{2\pi} d\theta = \frac{\pi}{7} \Rightarrow \bar{z} = \frac{5}{14}$, and $\bar{x} = \bar{y} = 0$, by symmetry; $I_z = \int_0^{2\pi}\int_0^1\int_0^{r^2} r^4\, dz\, dr\, d\theta$
$= \int_0^{2\pi}\int_0^1 r^6\, dr\, d\theta = \frac{1}{7}\int_0^{2\pi} d\theta = \frac{2\pi}{7}$

77. (a) $M = \int_0^{2\pi}\int_0^1\int_r^1 z\, dz\, r\, dr\, d\theta = \frac{1}{2}\int_0^{2\pi}\int_0^1\left(r - r^3\right) dr\, d\theta = \frac{1}{8}\int_0^{2\pi} d\theta = \frac{\pi}{4}$; $M_{xy} = \int_0^{2\pi}\int_0^1\int_r^1 z^2\, dz\, r\, dr\, d\theta$
$= \frac{1}{3}\int_0^{2\pi}\int_0^1\left(r - r^4\right) dr\, d\theta = \frac{1}{10}\int_0^{2\pi} d\theta = \frac{\pi}{5} \Rightarrow \bar{z} = \frac{4}{5}$, and $\bar{x} = \bar{y} = 0$, by symmetry; $I_z = \int_0^{2\pi}\int_0^1\int_r^1 zr^3\, dz\, dr\, d\theta$
$= \frac{1}{2}\int_0^{2\pi}\int_0^1\left(r^3 - r^5\right) dr\, d\theta = \frac{1}{24}\int_0^{2\pi} d\theta = \frac{\pi}{12}$

(b) $M = \int_0^{2\pi}\int_0^1\int_r^1 z^2\, dz\, r\, dr\, d\theta = \frac{\pi}{5}$ from part (a); $M_{xy} = \int_0^{2\pi}\int_0^1\int_r^1 z^3\, dz\, r\, dr\, d\theta = \frac{1}{4}\int_0^{2\pi}\int_0^1\left(r - r^5\right) dr\, d\theta$
$= \frac{1}{12}\int_0^{2\pi} d\theta = \frac{\pi}{6} \Rightarrow \bar{z} = \frac{5}{6}$, and $\bar{x} = \bar{y} = 0$, by symmetry; $I_z = \int_0^{2\pi}\int_0^1\int_r^1 z^2 r^3\, dz\, dr\, d\theta = \frac{1}{3}\int_0^{2\pi}\int_0^1\left(r^3 - r^6\right) dr\, d\theta$
$= \frac{1}{28}\int_0^{2\pi} d\theta = \frac{\pi}{14}$

78. (a) $M = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^4 \sin\phi\, d\rho\, d\phi\, d\theta = \frac{a^5}{5}\int_0^{2\pi}\int_0^{\pi} \sin\phi\, d\phi\, d\theta = \frac{2a^5}{5}\int_0^{2\pi} d\theta = \frac{4\pi a^5}{5}$; $I_z = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^6 \sin^3\phi\, d\rho\, d\phi\, d\theta$
$= \frac{a^7}{7}\int_0^{2\pi}\int_0^{\pi} (1-\cos^2\phi)\sin\phi\, d\phi\, d\theta = \frac{a^7}{7}\int_0^{2\pi}\left[-\cos\phi + \frac{\cos^3\phi}{3}\right]_0^{\pi} d\theta = \frac{4a^7}{21}\int_0^{2\pi} d\theta = \frac{8a^7\pi}{21}$

(b) $M = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^3 \sin^2\phi\, d\rho\, d\phi\, d\theta = \frac{a^4}{4}\int_0^{2\pi}\int_0^{\pi} \frac{(1-\cos 2\phi)}{2}\, d\phi\, d\theta = \frac{\pi a^4}{8}\int_0^{2\pi} d\theta = \frac{\pi^2 a^4}{4}$;
$I_z = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^5 \sin^4\phi\, d\rho\, d\phi\, d\theta = \frac{a^6}{6}\int_0^{2\pi}\int_0^{\pi} \sin^4\phi\, d\phi\, d\theta$
$= \frac{a^6}{6}\int_0^{2\pi}\left(\left[\frac{-\sin^3\phi\cos\phi}{4}\right]_0^{\pi} + \frac{3}{4}\int_0^{\pi}\sin^2\phi\, d\phi\right) d\theta = \frac{a^6}{8}\int_0^{2\pi}\left[\frac{\phi}{2} - \frac{\sin 2\phi}{4}\right]_0^{\pi} d\theta = \frac{\pi a^6}{16}\int_0^{2\pi} d\theta = \frac{a^6\pi^2}{8}$

79. $M = \int_0^{2\pi}\int_0^{a}\int_0^{\frac{h}{a}\sqrt{a^2-r^2}} dz\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^{a} \frac{h}{a} r\sqrt{a^2-r^2}\, dr\, d\theta = \frac{h}{a}\int_0^{2\pi}\left[-\frac{1}{3}(a^2-r^2)^{3/2}\right]_0^{a} d\theta$
$= \frac{h}{a}\int_0^{2\pi}\frac{a^3}{3}\, d\theta = \frac{2ha^2\pi}{3}$; $M_{xy} = \int_0^{2\pi}\int_0^{a}\int_0^{\frac{h}{a}\sqrt{a^2-r^2}} z\, dz\, r\, dr\, d\theta = \frac{h^2}{2a^2}\int_0^{2\pi}\int_0^{a}(a^2r - r^3)\, dr\, d\theta$
$= \frac{h^2}{2a^2}\int_0^{2\pi}\left(\frac{a^4}{2} - \frac{a^4}{4}\right) d\theta = \frac{a^2h^2\pi}{4} \Rightarrow \bar{z} = \left(\frac{\pi a^2 h^2}{4}\right)\left(\frac{3}{2ha^2\pi}\right) = \frac{3}{8}h$, and $\bar{x} = \bar{y} = 0$, by symmetry

80. Let the base radius of the cone be a and the height h, and place the cone's axis of symmetry along the z-axis with the vertex at the origin. Then $M = \frac{\pi a^2 h}{3}$ and $M_{xy} = \int_0^{2\pi}\int_0^{a}\int_{\left(\frac{h}{a}\right)r}^{h} z\, dz\, r\, dr\, d\theta = \frac{1}{2}\int_0^{2\pi}\int_0^{a}\left(h^2 r - \frac{h^2}{a^2}r^3\right) dr\, d\theta$
$= \frac{h^2}{2}\int_0^{2\pi}\left[\frac{r^2}{2} - \frac{r^4}{4a^2}\right]_0^{a} d\theta = \frac{h^2}{2}\int_0^{2\pi}\left(\frac{a^2}{2} - \frac{a^2}{4}\right) d\theta = \frac{h^2a^2}{8}\int_0^{2\pi} d\theta = \frac{h^2a^2\pi}{4} \Rightarrow \bar{z} = \frac{M_{xy}}{M} = \left(\frac{h^2a^2\pi}{4}\right)\left(\frac{3}{\pi a^2 h}\right) = \frac{3}{4}h$, and
$\bar{x} = \bar{y} = 0$, by symmetry $\Rightarrow$ the centroid is one fourth of the way from the base to the vertex

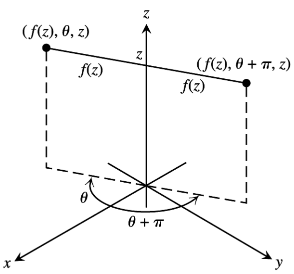
81. The density distribution function is linear so it has the form $\delta(\rho) = k\rho + C$, where ρ is the distance from the center of the planet. Now, $\delta(R) = 0 \Rightarrow kR + C = 0$, and $\delta(\rho) = k\rho - kR$. It remains to determine the constant k: $M = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} (k\rho - kR)\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi}\left[k\frac{\rho^4}{4} - kR\frac{\rho^3}{3}\right]_0^{R} \sin\phi\, d\phi\, d\theta$
$= \int_0^{2\pi}\int_0^{\pi} k\left(\frac{R^4}{4} - \frac{R^4}{3}\right)\sin\phi\, d\phi\, d\theta = \int_0^{2\pi} -\frac{k}{12}R^4\left[-\cos\phi\right]_0^{\pi} d\theta = \int_0^{2\pi} -\frac{k}{6}R^4\, d\theta = -\frac{k\pi R^4}{3} \Rightarrow k = -\frac{3M}{\pi R^4}$
$\Rightarrow \delta(\rho) = -\frac{3M}{\pi R^4}\rho + \frac{3M}{\pi R^4}R$. At the center of the planet $\rho = 0 \Rightarrow \delta(0) = \left(\frac{3M}{\pi R^4}\right)R = \frac{3M}{\pi R^3}$.

82. The mass of the plant's atmosphere to an altitude h above the surface of the planet is the triple integral
$M(h) = \int_0^{2\pi}\int_0^{\pi}\int_R^{h} \mu_0 e^{-c(\rho-R)}\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \int_R^{h}\int_0^{2\pi}\int_0^{\pi} \mu_0 e^{-c(\rho-R)}\rho^2 \sin\phi\, d\phi\, d\theta\, d\rho$
$= \int_R^{h}\int_0^{2\pi}\left[\mu_0 e^{-c(\rho-R)}\rho^2(-\cos\phi)\right]_0^{\pi} d\theta\, d\rho = 2\int_R^{h}\int_0^{2\pi} \mu_0 e^{cR} e^{-c\rho}\rho^2\, d\theta\, d\rho = 4\pi\mu_0 e^{cR}\int_R^{h} e^{-c\rho}\rho^2\, d\rho$
$= 4\pi\mu_0 e^{cR}\left[-\frac{\rho^2 e^{-c\rho}}{c} - \frac{2\rho e^{-c\rho}}{c^2} - \frac{2e^{-c\rho}}{c^3}\right]_R^{h}$ (by parts)
$= 4\pi\mu_0 e^{cR}\left(-\frac{h^2e^{-ch}}{c} - \frac{2he^{-ch}}{c^2} - \frac{2e^{-ch}}{c^3} + \frac{R^2e^{-cR}}{c} + \frac{2Re^{-cR}}{c^2} + \frac{2e^{-cR}}{c^3}\right)$.
The mass of the planet's atmosphere is therefore $M = \lim_{h\to\infty} M(h) = 4\pi\mu_0\left(\frac{R^2}{c} + \frac{2R}{c^2} + \frac{2}{c^3}\right)$.

83. (a) A plane perpendicular to the x-axis has the form x = a in rectangular coordinates $\Rightarrow r\cos\theta = a \Rightarrow r = \frac{a}{\cos\theta}$ $\Rightarrow r = a\sec\theta$, in cylindrical coordinates.
(b) A plane perpendicular to the y-axis has the form y = b in rectangular coordinates $\Rightarrow r\sin\theta = b \Rightarrow r = \frac{b}{\sin\theta}$ $\Rightarrow r = b\csc\theta$, in cylindrical coordinates.

84. $ax + by = c \Rightarrow a(r\cos\theta) + b(r\sin\theta) = c \Rightarrow r(a\cos\theta + b\sin\theta) = c \Rightarrow r = \frac{c}{a\cos\theta + b\sin\theta}$.

85. The equation $r = f(z)$ implies that the point $(r, \theta, z) = (f(z), \theta, z)$ will lie on the surface for all θ. In particular $(f(z), \theta + \pi, z)$ lies on the surface whenever $(f(z), \theta, z)$ does $\Rightarrow$ the surface is symmetric with respect to the z-axis.



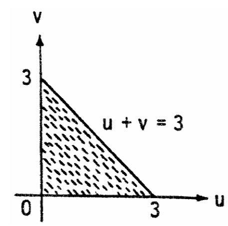
86. The equation $\rho = f(\phi)$ implies that the point $(\rho, \phi, \theta) = (f(\phi), \phi, \theta)$ lies on the surface for all θ. In particular, if $(f(\phi), \phi, \theta)$ lies on the surface, then $(f(\phi), \phi, \theta + \pi)$ lies on the surface, so the surface is symmetric wiith respect to the z-axis.

15.8 SUBSTITUTIONS IN MULTIPLE INTEGRALS

1. (a) $x - y = u$ and $2x + y = v \Rightarrow 3x = u + v$ and $y = x - u \Rightarrow x = \frac{1}{3}(u + v)$ and $y = \frac{1}{3}(-2u + v)$;

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{9} + \frac{2}{9} = \frac{1}{3}$$

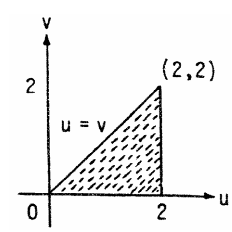
(b) The line segment $y = x$ from $(0,0)$ to $(1,1)$ is $x - y = 0 \Rightarrow u = 0$; the line segment $y = -2x$ from $(0,0)$ to $(1,-2)$ is $2x + y = 0 \Rightarrow v = 0$; the line segment $x = 1$ from $(1,1)$ to $(1,-2)$ is $(x - y) + (2x + y) = 3 \Rightarrow u + v = 3$. The transformed region is sketched at the right.



2. (a) $x + 2y = u$ and $x - y = v \Rightarrow 3y = u - v$ and $x = v + y \Rightarrow y = \frac{1}{3}(u - v)$ and $x = \frac{1}{3}(u + 2v)$;

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{vmatrix} = -\frac{1}{9} - \frac{2}{9} = -\frac{1}{3}$$

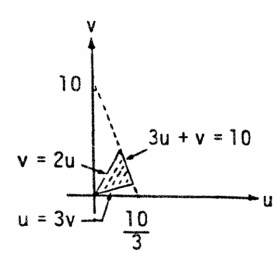
(b) The triangular region in the xy-plane has vertices $(0,0)$, $(2,0)$, and $\left(\frac{2}{3}, \frac{2}{3}\right)$. The line segment $y = x$ from $(0,0)$ to $\left(\frac{2}{3}, \frac{2}{3}\right)$ is $x - y = 0 \Rightarrow v = 0$; the line segment $y = 0$ from $(0,0)$ to $(2,0) \Rightarrow u = v$; the line segment $x + 2y = 2$ from $\left(\frac{2}{3}, \frac{2}{3}\right)$ to $(2,0) \Rightarrow u = 2$. The transformed region is sketched at the right.



3. (a) $3x + 2y = u$ and $x + 4y = v \Rightarrow -5x = -2u + v$ and $y = \frac{1}{2}(u - 3x) \Rightarrow x = \frac{1}{5}(2u - v)$ and $y = \frac{1}{10}(3v - u)$;

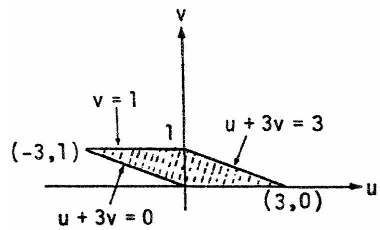
$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{10} & \frac{3}{10} \end{vmatrix} = \frac{6}{50} - \frac{1}{50} = \frac{1}{10}$$

(b) The x-axis $y = 0 \Rightarrow u = 3v$; the y-axis $x = 0 \Rightarrow v = 2u$; the line $x + y = 1 \Rightarrow \frac{1}{5}(2u - v) + \frac{1}{10}(3v - u) = 1 \Rightarrow 2(2u - v) + (3v - u) = 10 \Rightarrow 3u + v = 10$. The transformed region is sketched at the right.



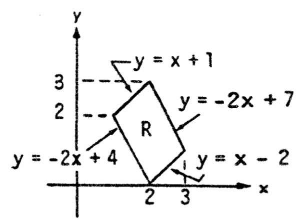
4. (a) $2x - 3y = u$ and $-x + y = v \Rightarrow -x = u + 3v$ and $y = v + x \Rightarrow x = -u - 3v$ and $y = -u - 2v$;
$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -1 & -3 \\ -1 & -2 \end{vmatrix} = 2 - 3 = -1$

(b) The line $x = -3 \Rightarrow -u - 3v = -3$ or $u + 3v = 3$; $x = 0 \Rightarrow u + 3v = 0$; $y = x \Rightarrow v = 0$; $y = x + 1 \Rightarrow v = 1$. The transformed region is the parallelogram sketched at the right.



5. $\int_0^4 \int_{y/2}^{(y/2)+1} \left(x - \frac{y}{2}\right) dx\,dy = \int_0^4 \left[\frac{x^2}{2} - \frac{xy}{2}\right]_{\frac{y}{2}}^{\frac{y}{2}+1} dy = \frac{1}{2}\int_0^4 \left[\left(\frac{y}{2}+1\right)^2 - \left(\frac{y}{2}\right)^2 - \left(\frac{y}{2}+1\right)y + \left(\frac{y}{2}\right)y\right] dy$
$= \frac{1}{2}\int_0^4 (y + 1 - y)\,dy = \frac{1}{2}\int_0^4 dy = \frac{1}{2}(4) = 2$

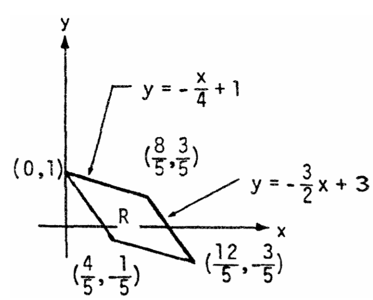
6. $\iint_R (2x^2 - xy - y^2)\,dx\,dy = \iint_R (x - y)(2x + y)\,dx\,dy$
$= \iint_G uv \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \frac{1}{3}\iint_G uv\,du\,dv$;
We find the boundaries of G from the boundaries of R, shown in the accompanying figure:



xy-equations for the boundary of R	Corresponding uv-equations for the boundary of G	Simplified uv-equations
$y = -2x + 4$	$\frac{1}{3}(-2u + v) = -\frac{2}{3}(u + v) + 4$	$v = 4$
$y = -2x + 7$	$\frac{1}{3}(-2u + v) = -\frac{2}{3}(u + v) + 7$	$v = 7$
$y = x - 2$	$\frac{1}{3}(-2u + v) = \frac{1}{3}(u + v) - 2$	$u = 2$
$y = x + 1$	$\frac{1}{3}(-2u + v) = \frac{1}{3}(u + v) + 1$	$u = -1$

$\Rightarrow \frac{1}{3}\iint_G uv\,du\,dv = \frac{1}{3}\int_{-1}^{2}\int_4^7 uv\,dv\,du = \frac{1}{3}\int_{-1}^{2} u\left[\frac{v^2}{2}\right]_4^7 du = \frac{11}{2}\int_{-1}^{2} u\,du = \left(\frac{11}{2}\right)\left[\frac{u^2}{2}\right]_{-1}^{2} = \left(\frac{11}{4}\right)(4 - 1) = \frac{33}{4}$

7. $\iint_R (3x^2 + 14xy + 8y^2)\,dx\,dy$
$= \iint_R (3x + 2y)(x + 4y)\,dx\,dy$
$= \iint_G uv \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \frac{1}{10}\iint_G uv\,du\,dv$;
We find the boundaries of G from the boundaries of R, shown in the accompanying figure:



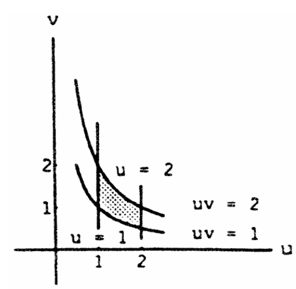
xy-equations for the boundary of R	Corresponding uv-equations for the boundary of G	Simplified uv-equations
$y = -\frac{3}{2}x + 1$	$\frac{1}{10}(3v - u) = -\frac{3}{10}(2u - v) + 1$	$u = 2$
$y = -\frac{3}{2}x + 3$	$\frac{1}{10}(3v - u) = -\frac{3}{10}(2u - v) + 3$	$u = 6$
$y = -\frac{1}{4}x$	$\frac{1}{10}(3v - u) = -\frac{1}{20}(2u - v)$	$v = 0$
$y = -\frac{1}{4}x + 1$	$\frac{1}{10}(3v - u) = -\frac{1}{20}(2u - v) + 1$	$v = 4$

$\Rightarrow \frac{1}{10}\iint_G uv\,du\,dv = \frac{1}{10}\int_2^6\int_0^4 uv\,dv\,du = \frac{1}{10}\int_2^6 u\left[\frac{v^2}{2}\right]_0^4 du = \frac{4}{5}\int_2^6 u\,du = \left(\frac{4}{5}\right)\left[\frac{u^2}{2}\right]_2^6 = \left(\frac{4}{5}\right)(18-2) = \frac{64}{5}$

8. $\iint_R 2(x-y)\,dx\,dy = \iint_G -2v\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \iint_G -2v\,du\,dv$; the region G is sketched in Exercise 4
$\Rightarrow \iint_G -2v\,du\,dv = \int_0^1\int_{-3v}^{3-3v} -2v\,du\,dv = \int_0^1 -2v(3-3v+3v)\,dv = \int_0^1 -6v\,dv = \left[-3v^2\right]_0^1 = -3$

9. $x = \frac{u}{v}$ and $y = uv \Rightarrow \frac{y}{x} = v^2$ and $xy = u^2$; $\frac{\partial(x,y)}{\partial(u,v)} = J(u,v) = \begin{vmatrix} v^{-1} & -uv^{-2} \\ v & u \end{vmatrix} = v^{-1}u + v^{-1}u = \frac{2u}{v}$;
$y = x \Rightarrow uv = \frac{u}{v} \Rightarrow v = 1$, and $y = 4x \Rightarrow v = 2$; $xy = 1 \Rightarrow u = 1$, and $xy = 9 \Rightarrow u = 3$; thus
$\iint_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy}\right) dx\,dy = \int_1^3\int_1^2 (v+u)\left(\frac{2u}{v}\right) dv\,du = \int_1^3\int_1^2 \left(2u + \frac{2u^2}{v}\right) dv\,du = \int_1^3 \left[2uv + 2u^2 \ln v\right]_1^2 du$
$= \int_1^3 (2u + 2u^2 \ln 2)\,du = \left[u^2 + \frac{2}{3}u^2 \ln 2\right]_1^3 = 8 + \frac{2}{3}(26)(\ln 2) = 8 + \frac{52}{3}(\ln 2)$

10. (a) $\frac{\partial(x,y)}{\partial(u,v)} = J(u,v) = \begin{vmatrix} 1 & 0 \\ v & u \end{vmatrix} = u$, and
the region G is sketched at the right

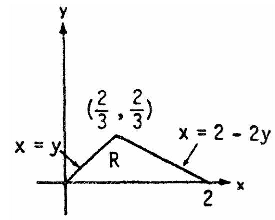


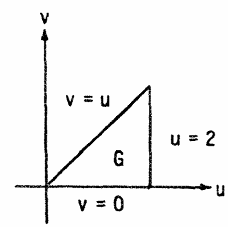
(b) $x = 1 \Rightarrow u = 1$, and $x = 2 \Rightarrow u = 2$; $y = 1 \Rightarrow uv = 1 \Rightarrow v = \frac{1}{u}$, and $y = 2 \Rightarrow uv = 2 \Rightarrow v = \frac{2}{u}$; thus,
$\int_1^2\int_1^2 \frac{y}{x}\,dy\,dx = \int_1^2\int_{1/u}^{2/u} \left(\frac{uv}{u}\right) u\,dv\,du = \int_1^2\int_{1/u}^{2/u} uv\,dv\,du = \int_1^2 u\left[\frac{v^2}{2}\right]_{1/u}^{2/u} du = \int_1^2 u\left(\frac{2}{u^2} - \frac{1}{2u^2}\right) du$
$= \frac{3}{2}\int_1^2 u\left(\frac{1}{u^2}\right) du = \frac{3}{2}\left[\ln u\right]_1^2 = \frac{3}{2}\ln 2$; $\int_1^2\int_1^2 \frac{y}{x}\,dy\,dx = \int_1^2 \left[\frac{1}{x}\cdot\frac{y^2}{2}\right]_1^2 dx = \frac{3}{2}\int_1^2 \frac{dx}{x} = \frac{3}{2}\left[\ln x\right]_1^2 = \frac{3}{2}\ln 2$

11. $x = ar\cos\theta$ and $y = ar\sin\theta \Rightarrow \frac{\partial(x,y)}{\partial(r,\theta)} = J(r,\theta) = \begin{vmatrix} a\cos\theta & -ar\sin\theta \\ b\sin\theta & br\cos\theta \end{vmatrix} = abr\cos^2\theta + abr\sin^2\theta = abr$;
$I_0 = \iint_R (x^2+y^2)\,dA = \int_0^{2\pi}\int_0^1 r^2(a^2\cos^2\theta + b^2\sin^2\theta)\,|J(r,\theta)|\,dr\,d\theta = \int_0^{2\pi}\int_0^1 abr^3(a^2\cos^2\theta + b^2\sin^2\theta)\,dr\,d\theta$
$= \frac{ab}{4}\int_0^{2\pi} (a^2\cos^2\theta + b^2\sin^2\theta)\,d\theta = \frac{ab}{4}\left[\frac{a^2\theta}{2} + \frac{a^2\sin 2\theta}{4} + \frac{b^2\theta}{2} - \frac{b^2\sin 2\theta}{4}\right]_0^{2\pi} = \frac{ab\pi(a^2+b^2)}{4}$

12. $\frac{\partial(x,y)}{\partial(u,v)} = J(u,v) = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab$; $A = \iint_R dy\,dx = \iint_G ab\,du\,dv = \int_{-1}^1\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} ab\,dv\,du$
$= 2ab\int_{-1}^1 \sqrt{1-u^2}\,du = 2ab\left[\frac{u}{2}\sqrt{1-u^2} + \frac{1}{2}\sin^{-1} u\right]_{-1}^1 = ab\left[\sin^{-1} 1 - \sin^{-1}(-1)\right] = ab\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = ab\pi$

13. The region of integration R in the xy-plane is sketched in the figure at the right. The boundaries of the image G are obtained as follows, with G sketched at the right:

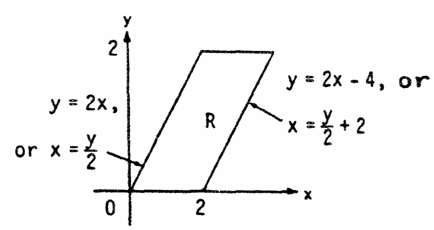




xy-equations for the boundary of R	Corresponding uv-equations for the boundary of G	Simplified uv-equations
$x = y$	$\frac{1}{3}(u + 2v) = \frac{1}{3}(u - v)$	$v = 0$
$x = 2 - 2y$	$\frac{1}{3}(u + 2v) = 2 - \frac{2}{3}(u - v)$	$u = 2$
$y = 0$	$0 = \frac{1}{3}(u - v)$	$v = u$

Also, from Exercise 2, $\frac{\partial(x,y)}{\partial(u,v)} = J(u,v) = -\frac{1}{3} \Rightarrow \int_0^{2/3} \int_y^{2-2y} (x + 2y)\, e^{(y-x)}\, dx\, dy = \int_0^2 \int_0^u ue^{-v} \left|-\frac{1}{3}\right| dv\, du$
$= \frac{1}{3} \int_0^2 u \left[-e^{-v}\right]_0^u du = \frac{1}{3} \int_0^2 u\,(1 - e^{-u})\, du = \frac{1}{3} \left[u\,(u + e^{-u}) - \frac{u^2}{2} + e^{-u}\right]_0^2 = \frac{1}{3}\left[2\,(2 + e^{-2}) - 2 + e^{-2} - 1\right]$
$= \frac{1}{3}(3e^{-2} + 1) \approx 0.4687$

14. $x = u + \frac{v}{2}$ and $y = v \Rightarrow 2x - y = (2u + v) - v = 2u$ and $\frac{\partial(x,y)}{\partial(u,v)} = J(u,v) = \begin{vmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{vmatrix} = 1$; next, $u = x - \frac{v}{2}$ $= x - \frac{y}{2}$ and $v = y$, so the boundaries of the region of integration R in the xy-plane are transformed to the boundaries of G:



xy-equations for the boundary of R	Corresponding uv-equations for the boundary of G	Simplified uv-equations
$x = \frac{y}{2}$	$u + \frac{v}{2} = \frac{v}{2}$	$u = 0$
$x = \frac{y}{2} + 2$	$u + \frac{v}{2} = \frac{v}{2} + 2$	$u = 2$
$y = 0$	$v = 0$	$v = 0$
$y = 2$	$v = 2$	$v = 2$

$\Rightarrow \int_0^2 \int_{y/2}^{(y/2)+2} y^3(2x - y)\, e^{(2x-y)^2}\, dx\, dy = \int_0^2 \int_0^2 v^3(2u)\, e^{4u^2}\, du\, dv = \int_0^2 v^3 \left[\frac{1}{4} e^{4u^2}\right]_0^2 dv = \frac{1}{4} \int_0^2 v^3 (e^{16} - 1)\, dv$
$= \frac{1}{4}(e^{16} - 1)\left[\frac{v^4}{4}\right]_0^2 = e^{16} - 1$

15. $x = \frac{u}{v}$ and $y = uv \Rightarrow \frac{y}{x} = v^2$ and $xy = u^2$; $\frac{\partial(x,y)}{\partial(u,v)} = J(u,v) = \begin{vmatrix} v^{-1} & -uv^{-2} \\ v & u \end{vmatrix} = v^{-1}u + v^{-1}u = \frac{2u}{v}$;
$y = x \Rightarrow uv = \frac{u}{v} \Rightarrow v = 1$, and $y = 4x \Rightarrow v = 2$; $xy = 1 \Rightarrow u = 1$, and $xy = 4 \Rightarrow u = 2$; thus
$\int_1^2 \int_{1/y}^{y} (x^2 + y^2)\, dx\, dy + \int_2^4 \int_{y/4}^{4/y} (x^2 + y^2)\, dx\, dy = \int_1^2 \int_1^2 \left(\frac{u^2}{v^2} + u^2v^2\right)\left(\frac{2u}{v}\right) du\, dv = \int_1^2 \int_1^2 \left(\frac{2u^3}{v^3} + 2u^3v\right) du\, dv$

$= \int_1^2 \left[\frac{u^4}{2v^3} + \frac{1}{2}u^4 v\right]_1^2 dv = \int_1^2 \left(\frac{15}{2v^3} + \frac{15v}{2}\right) dv = \left[-\frac{15}{4v^2} + \frac{15v^2}{4}\right]_1^2 = \frac{225}{16}$

16. $x = u^2 - v^2$ and $y = 2uv$; $\frac{\partial(x,y)}{\partial(u,v)} = J(u,v) = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4u^2 + 4v^2 = 4(u^2+v^2)$;

$y = 2\sqrt{1-x} \Rightarrow y^2 = 4(1-x) \Rightarrow (2uv)^2 = 4(1-(u^2-v^2)) \Rightarrow u = \pm 1$; $y = 0 \Rightarrow 2uv = 0 \Rightarrow u = 0$ or $v = 0$; $x = 0 \Rightarrow u^2 - v^2 = 0 \Rightarrow u = v$ or $u = -v$; This gives us four triangular regions, but only the one in the quadrant where both u, v are positive maps into the region R in the xy-plane.

$\int_0^1 \int_0^{2\sqrt{1-x}} \sqrt{x^2+y^2}\, dx\, dy = \int_0^1 \int_0^u \sqrt{(u^2-v^2)^2 + (2uv)^2} \cdot 4(u^2+v^2)\, dv\, du = 4\int_0^1 \int_0^u (u^2+v^2)^2\, dv\, du$

$= 4\int_1^2 \left[u^4 v + \frac{2}{3}u^2 v^3 + \frac{1}{5}v^5\right]_0^u du = \frac{112}{15}\int_1^2 u^5\, du = \frac{112}{15}\left[\frac{1}{6}u^6\right]_1^2 = \frac{56}{45}$

17. (a) $x = u\cos v$ and $y = u\sin v \Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \cos v & -u\sin v \\ \sin v & u\cos v \end{vmatrix} = u\cos^2 v + u\sin^2 v = u$

(b) $x = u\sin v$ and $y = u\cos v \Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \sin v & u\cos v \\ \cos v & -u\sin v \end{vmatrix} = -u\sin^2 v - u\cos^2 v = -u$

18. (a) $x = u\cos v,\ y = u\sin v,\ z = w \Rightarrow \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \cos v & -u\sin v & 0 \\ \sin v & u\cos v & 0 \\ 0 & 0 & 1 \end{vmatrix} = u\cos^2 v + u\sin^2 v = u$

(b) $x = 2u - 1,\ y = 3v - 4,\ z = \frac{1}{2}(w-4) \Rightarrow \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & \frac{1}{2} \end{vmatrix} = (2)(3)\left(\frac{1}{2}\right) = 3$

19. $\begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix}$

$= (\cos\phi)\begin{vmatrix} \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix} + (\rho\sin\phi)\begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix}$

$= (\rho^2\cos\phi)(\sin\phi\cos\phi\cos^2\theta + \sin\phi\cos\phi\sin^2\theta) + (\rho^2\sin\phi)(\sin^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta)$

$= \rho^2\sin\phi\cos^2\phi + \rho^2\sin^3\phi = (\rho^2\sin\phi)(\cos^2\phi + \sin^2\phi) = \rho^2\sin\phi$

20. Let $u = g(x) \Rightarrow J(x) = \frac{du}{dx} = g'(x) \Rightarrow \int_a^b f(u)\, du = \int_{g(a)}^{g(b)} f(g(x))g'(x)\, dx$ in accordance with Theorem 7 in Section 5.6. Note that $g'(x)$ represents the Jacobian of the transformation $u = g(x)$ or $x = g^{-1}(u)$.

21. $\int_0^3 \int_0^4 \int_{y/2}^{1+(y/2)} \left(\frac{2x-y}{2} + \frac{z}{3}\right) dx\, dy\, dz = \int_0^3 \int_0^4 \left[\frac{x^2}{2} - \frac{xy}{2} + \frac{xz}{3}\right]_{y/2}^{1+(y/2)} dy\, dz = \int_0^3 \int_0^4 \left[\frac{1}{2}(y+1) - \frac{y}{2} + \frac{z}{3}\right] dy\, dz$

$= \int_0^3 \left[\frac{(y+1)^2}{4} - \frac{y^2}{4} + \frac{yz}{3}\right]_0^4 dz = \int_0^3 \left(\frac{9}{4} + \frac{4z}{3} - \frac{1}{4}\right) dz = \int_0^3 \left(2 + \frac{4z}{3}\right) dz = \left[2z + \frac{2z^2}{3}\right]_0^3 = 12$

22. $J(u,v,w) = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$; the transformation takes the ellipsoid region $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1$ in xyz-space into the spherical region $u^2 + v^2 + w^2 \le 1$ in uvw-space (which has volume $V = \frac{4}{3}\pi$)

$\Rightarrow V = \iiint_R dx\, dy\, dz = \iiint_G abc\, du\, dv\, dw = \frac{4\pi abc}{3}$

23. $J(u,v,w) = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$; for R and G as in Exercise 22, $\iiint\limits_R |xyz|\,dx\,dy\,dz$

$= \iiint\limits_G a^2b^2c^2uvw\,dw\,dv\,du = 8a^2b^2c^2 \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 (\rho\sin\phi\cos\theta)(\rho\sin\phi\sin\theta)(\rho\cos\phi)(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta$

$= \frac{4a^2b^2c^2}{3}\int_0^{\pi/2}\int_0^{\pi/2} \sin\theta\cos\theta\sin^3\phi\cos\phi\,d\phi\,d\theta = \frac{a^2b^2c^2}{3}\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta = \frac{a^2b^2c^2}{6}$

24. $u = x$, $v = xy$, and $w = 3z \Rightarrow x = u$, $y = \frac{v}{u}$, and $z = \frac{1}{3}w \Rightarrow J(u,v,w) = \begin{vmatrix} 1 & 0 & 0 \\ -\frac{v}{u^2} & \frac{1}{u} & 0 \\ 0 & 0 & \frac{1}{3} \end{vmatrix} = \frac{1}{3u}$;

$\iiint\limits_D (x^2y + 3xyz)\,dx\,dy\,dz = \iiint\limits_G \left[u^2\left(\frac{v}{u}\right) + 3u\left(\frac{v}{u}\right)\left(\frac{w}{3}\right)\right]|J(u,v,w)|\,du\,dv\,dw = \frac{1}{3}\int_0^3\int_0^2\int_1^2 \left(v + \frac{vw}{u}\right)du\,dv\,dw$

$= \frac{1}{3}\int_0^3\int_0^2 (v + vw\ln 2)\,dv\,dw = \frac{1}{3}\int_0^3 (1 + w\ln 2)\left[\frac{v^2}{2}\right]_0^2 dw = \frac{2}{3}\int_0^3 (1 + w\ln 2)\,dw = \frac{2}{3}\left[w + \frac{w^2}{2}\ln 2\right]_0^3$

$= \frac{2}{3}\left(3 + \frac{9}{2}\ln 2\right) = 2 + 3\ln 2 = 2 + \ln 8$

25. The first moment about the xy-coordinate plane for the semi-ellipsoid, $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ using the transformation in Exercise 23 is, $M_{xy} = \iiint\limits_D z\,dz\,dy\,dx = \iiint\limits_G cw\,|J(u,v,w)|\,du\,dv\,dw$

$= abc^2\iiint\limits_G w\,du\,dv\,dw = (abc^2)\cdot(M_{xy}$ of the hemisphere $x^2 + y^2 + z^2 = 1,\ z \ge 0) = \frac{abc^2\pi}{4}$;

the mass of the semi-ellipsoid is $\frac{2abc\pi}{3} \Rightarrow \bar{z} = \left(\frac{abc^2\pi}{4}\right)\left(\frac{3}{2abc\pi}\right) = \frac{3}{8}c$

26. A solid of revolution is symmetric about the axis of revolution, therefore, the height of the solid is solely a function of r. That is, $y = f(x) = f(r)$. Using cylindrical coordinates with $x = r\cos\theta$, $y = y$ and $z = r\sin\theta$, we have

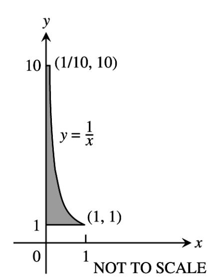
$V = \iiint\limits_G r\,dy\,d\theta\,dr = \int_a^b\int_0^{2\pi}\int_0^{f(r)} r\,dy\,d\theta\,dr = \int_a^b\int_0^{2\pi}[r\,y]_0^{f(r)}\,d\theta\,dr = \int_a^b\int_0^{2\pi} r\,f(r)\,d\theta\,dr = \int_a^b [r\theta f(r)]_0^{2\pi}\,dr$

$\int_a^b 2\pi r f(r)dr$. In the last integral, r is a dummy or stand-in variable and as such it can be replaced by any variable name. Choosing x instead of r we have $V = \int_a^b 2\pi x f(x)dx$, which is the same result obtained using the shell method.

CHAPTER 15 PRACTICE EXERCISES

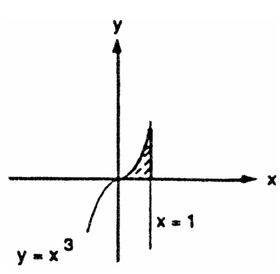
1. $\int_1^{10}\int_0^{1/y} ye^{xy}\,dx\,dy = \int_1^{10}[e^{xy}]_0^{1/y}\,dy$

$= \int_1^{10}(e - 1)\,dy = 9e - 9$

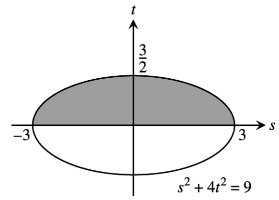


2. $\int_0^1\int_0^{x^3} e^{y/x}\,dy\,dx = \int_0^1 x\left[e^{y/x}\right]_0^{x^3} dx$

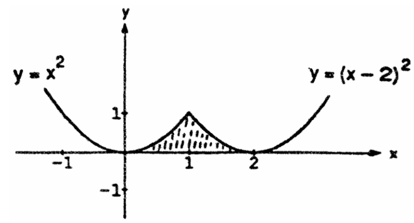
$= \int_0^1 \left(xe^{x^2} - x\right)dx = \left[\frac{1}{2}e^{x^2} - \frac{x^2}{2}\right]_0^1 = \frac{e-2}{2}$



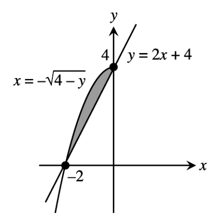
3. $\int_0^{3/2}\int_{-\sqrt{9-4t^2}}^{\sqrt{9-4t^2}} t\,ds\,dt = \int_0^{3/2} [ts]_{-\sqrt{9-4t^2}}^{\sqrt{9-4t^2}}\,dt$
$= \int_0^{3/2} 2t\sqrt{9-4t^2}\,dt = \left[-\frac{1}{6}\left(9-4t^2\right)^{3/2}\right]_0^{3/2}$
$= -\frac{1}{6}\left(0^{3/2} - 9^{3/2}\right) = \frac{27}{6} = \frac{9}{2}$



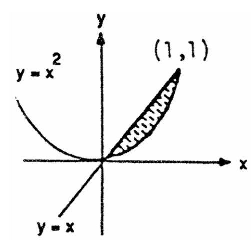
4. $\int_0^1\int_{\sqrt{y}}^{2-\sqrt{y}} xy\,dx\,dy = \int_0^1 y\left[\frac{x^2}{2}\right]_{\sqrt{y}}^{2-\sqrt{y}}\,dy$
$= \frac{1}{2}\int_0^1 y\left(4 - 4\sqrt{y} + y - y\right)dy$
$= \int_0^1 \left(2y - 2y^{3/2}\right)dy = \left[y^2 - \frac{4y^{5/2}}{5}\right]_0^1 = \frac{1}{5}$



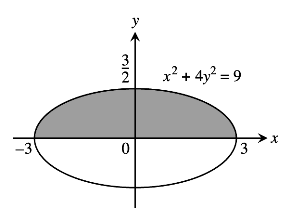
5. $\int_{-2}^0\int_{2x+4}^{4-x^2} dy\,dx = \int_{-2}^0 (-x^2 - 2x)\,dx$
$= \left[-\frac{x^3}{3} - x^2\right]_{-2}^0 = -\left(\frac{8}{3} - 4\right) = \frac{4}{3}$
$\int_0^4\int_{-\sqrt{4-y}}^{(y-4)/2} dx\,dy = \int_0^4 \left(\frac{y-4}{2} + \sqrt{4-y}\right)dy$
$= \left[\frac{y^2}{2} - 2y - \frac{2}{3}(4-y)^{3/2}\right]_0^4 = 4 - 8 + \frac{2}{3}\cdot 4^{3/2}$
$= -4 + \frac{16}{3} = \frac{4}{3}$



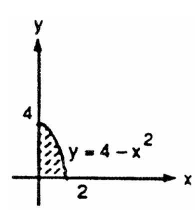
6. $\int_0^1\int_y^{\sqrt{y}} \sqrt{x}\,dx\,dy = \int_0^1 \left[\frac{2}{3}x^{3/2}\right]_y^{\sqrt{y}}\,dy$
$= \frac{2}{3}\int_0^1 \left(y^{3/4} - y^{3/2}\right)dy = \frac{2}{3}\left[\frac{4}{7}y^{7/4} - \frac{2}{5}y^{5/2}\right]_0^1$
$= \frac{2}{3}\left(\frac{4}{7} - \frac{2}{5}\right) = \frac{4}{35}$
$\int_0^1\int_{x^2}^{x} \sqrt{x}\,dy\,dx = \int_0^1 x^{1/2}(x - x^2)\,dx = \int_0^1 \left(x^{3/2} - x^{5/2}\right)dx$
$= \left[\frac{2}{5}x^{5/2} - \frac{2}{7}x^{7/2}\right]_0^1 = \frac{2}{5} - \frac{2}{7} = \frac{4}{35}$



7. $\int_{-3}^3\int_0^{(1/2)\sqrt{9-x^2}} y\,dy\,dx = \int_{-3}^3 \left[\frac{y^2}{2}\right]_0^{(1/2)\sqrt{9-x^2}}\,dx$
$= \int_{-3}^3 \frac{1}{8}(9 - x^2)\,dx = \left[\frac{9x}{8} - \frac{x^3}{24}\right]_{-3}^3$
$= \left(\frac{27}{8} - \frac{27}{24}\right) - \left(-\frac{27}{8} + \frac{27}{24}\right) = \frac{27}{6} = \frac{9}{2}$
$\int_0^{3/2}\int_{-\sqrt{9-4y^2}}^{\sqrt{9-4y^2}} y\,dx\,dy = \int_0^{3/2} 2y\sqrt{9-4y^2}\,dy$
$= -\frac{1}{4}\cdot\frac{2}{3}\left(9-4y^2\right)^{3/2}\Big|_0^{3/2} = \frac{1}{6}\cdot 9^{3/2} = \frac{27}{6} = \frac{9}{2}$



8. $\int_0^2\int_0^{4-x^2} 2x\,dy\,dx = \int_0^2 [2xy]_0^{4-x^2}\,dx$
$= \int_0^2 \left(2x(4 - x^2)\right)dx = \int_0^2 (8x - 2x^3)\,dx$
$= \left[4x^2 - \frac{x^4}{2}\right]_0^2 = 16 - \frac{16}{2} = 8$
$\int_0^4\int_0^{\sqrt{4-y}} 2x\,dx\,dy = \int_0^4 [x^2]_0^{\sqrt{4-y}}\,dy$
$= \int_0^4 (4 - y)\,dy = \left[4y - \frac{y^2}{2}\right]_0^4 = 16 - \frac{16}{2} = 8$



9. $\int_0^1 \int_{2y}^2 4\cos(x^2)\,dx\,dy = \int_0^2 \int_0^{x/2} 4\cos(x^2)\,dy\,dx = \int_0^2 2x\cos(x^2)\,dx = [\sin(x^2)]_0^2 = \sin 4$

10. $\int_0^2 \int_{y/2}^1 e^{x^2}\,dx\,dy = \int_0^1 \int_0^{2x} e^{x^2}\,dy\,dx = \int_0^1 2xe^{x^2}\,dx = [e^{x^2}]_0^1 = e - 1$

11. $\int_0^8 \int_{\sqrt[3]{x}}^2 \frac{1}{y^4+1}\,dy\,dx = \int_0^2 \int_0^{y^3} \frac{1}{y^4+1}\,dx\,dy = \frac{1}{4}\int_0^2 \frac{4y^3}{y^4+1}\,dy = \frac{\ln 17}{4}$

12. $\int_0^1 \int_{\sqrt[3]{y}}^1 \frac{2\pi\sin(\pi x^2)}{x^2}\,dx\,dy = \int_0^1 \int_0^{x^3} \frac{2\pi\sin(\pi x^2)}{x^2}\,dy\,dx = \int_0^1 2\pi x\sin(\pi x^2)\,dx = [-\cos(\pi x^2)]_0^1 = -(-1) - (-1) = 2$

13. $A = \int_{-2}^0 \int_{2x+4}^{4-x^2} dy\,dx = \int_{-2}^0 (-x^2 - 2x)\,dx = \frac{4}{3}$

14. $A = \int_1^4 \int_{2-y}^{\sqrt{y}} dx\,dy = \int_1^4 \left(\sqrt{y} - 2 + y\right) dy = \frac{37}{6}$

15. $V = \int_0^1 \int_x^{2-x} (x^2 + y^2)\,dy\,dx = \int_0^1 \left[x^2y + \frac{y^3}{3}\right]_x^{2-x} dx = \int_0^1 \left[2x^2 + \frac{(2-x)^3}{3} - \frac{7x^3}{3}\right] dx = \left[\frac{2x^3}{3} - \frac{(2-x)^4}{12} - \frac{7x^4}{12}\right]_0^1$
$= \left(\frac{2}{3} - \frac{1}{12} - \frac{7}{12}\right) + \frac{2^4}{12} = \frac{4}{3}$

16. $V = \int_{-3}^2 \int_x^{6-x^2} x^2\,dy\,dx = \int_{-3}^2 [x^2y]_x^{6-x^2}\,dx = \int_{-3}^2 (6x^2 - x^4 - x^3)\,dx = \frac{125}{4}$

17. average value $= \int_0^1 \int_0^1 xy\,dy\,dx = \int_0^1 \left[\frac{xy^2}{2}\right]_0^1 dx = \int_0^1 \frac{x}{2}\,dx = \frac{1}{4}$

18. average value $= \frac{1}{\left(\frac{\pi}{4}\right)} \int_0^1 \int_0^{\sqrt{1-x^2}} xy\,dy\,dx = \frac{4}{\pi}\int_0^1 \left[\frac{xy^2}{2}\right]_0^{\sqrt{1-x^2}} dx = \frac{2}{\pi}\int_0^1 (x - x^3)\,dx = \frac{1}{2\pi}$

19. $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{(1+x^2+y^2)^2}\,dy\,dx = \int_0^{2\pi} \int_0^1 \frac{2r}{(1+r^2)^2}\,dr\,d\theta = \int_0^{2\pi} \left[-\frac{1}{1+r^2}\right]_0^1 d\theta = \frac{1}{2}\int_0^{2\pi} d\theta = \pi$

20. $\int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln(x^2 + y^2 + 1)\,dx\,dy = \int_0^{2\pi} \int_0^1 r\ln(r^2+1)\,dr\,d\theta = \int_0^{2\pi} \int_1^2 \frac{1}{2}\ln u\,du\,d\theta = \frac{1}{2}\int_0^{2\pi} [u\ln u - u]_1^2\,d\theta$
$= \frac{1}{2}\int_0^{2\pi} (2\ln 2 - 1)\,d\theta = [\ln(4) - 1]\,\pi$

21. $(x^2 + y^2)^2 - (x^2 - y^2) = 0 \Rightarrow r^4 - r^2\cos 2\theta = 0 \Rightarrow r^2 = \cos 2\theta$ so the integral is $\int_{-\pi/4}^{\pi/4} \int_0^{\sqrt{\cos 2\theta}} \frac{r}{(1+r^2)^2}\,dr\,d\theta$
$= \int_{-\pi/4}^{\pi/4} \left[-\frac{1}{2(1+r^2)}\right]_0^{\sqrt{\cos 2\theta}} d\theta = \frac{1}{2}\int_{-\pi/4}^{\pi/4} \left(1 - \frac{1}{1+\cos 2\theta}\right) d\theta = \frac{1}{2}\int_{-\pi/4}^{\pi/4} \left(1 - \frac{1}{2\cos^2\theta}\right) d\theta$
$= \frac{1}{2}\int_{-\pi/4}^{\pi/4} \left(1 - \frac{\sec^2\theta}{2}\right) d\theta = \frac{1}{2}\left[\theta - \frac{\tan\theta}{2}\right]_{-\pi/4}^{\pi/4} = \frac{\pi-2}{4}$

22. (a) $\iint_R \frac{1}{(1+x^2+y^2)^2}\,dx\,dy = \int_0^{\pi/3} \int_0^{\sec\theta} \frac{r}{(1+r^2)^2}\,dr\,d\theta = \int_0^{\pi/3} \left[-\frac{1}{2(1+r^2)}\right]_0^{\sec\theta} d\theta$
$= \int_0^{\pi/3} \left[\frac{1}{2} - \frac{1}{2(1+\sec^2\theta)}\right] d\theta = \frac{1}{2}\int_0^{\pi/3} \frac{\sec^2\theta}{1+\sec^2\theta}\,d\theta; \begin{bmatrix} u = \tan\theta \\ du = \sec^2\theta\,d\theta \end{bmatrix} \to \frac{1}{2}\int_0^{\sqrt{3}} \frac{du}{2+u^2}$
$= \frac{1}{2}\left[\frac{1}{\sqrt{2}}\tan^{-1}\frac{u}{\sqrt{2}}\right]_0^{\sqrt{3}} = \frac{\sqrt{2}}{4}\tan^{-1}\sqrt{\frac{3}{2}}$

(b) $\iint_R \frac{1}{(1+x^2+y^2)^2}\,dx\,dy = \int_0^{\pi/2} \int_0^{\infty} \frac{r}{(1+r^2)^2}\,dr\,d\theta = \int_0^{\pi/2} \lim_{b\to\infty} \left[-\frac{1}{2(1+r^2)}\right]_0^b d\theta$
$= \int_0^{\pi/2} \lim_{b\to\infty} \left[\frac{1}{2} - \frac{1}{2(1+b^2)}\right] d\theta = \frac{1}{2}\int_0^{\pi/2} d\theta = \frac{\pi}{4}$

23. $\int_0^{\pi}\int_0^{\pi}\int_0^{\pi} \cos(x+y+z)\,dx\,dy\,dz = \int_0^{\pi}\int_0^{\pi} [\sin(z+y+\pi) - \sin(z+y)]\,dy\,dz$
$= \int_0^{\pi} [-\cos(z+2\pi) + \cos(z+\pi) - \cos z + \cos(z+\pi)]\,dz = 0$

24. $\int_{\ln 6}^{\ln 7}\int_0^{\ln 2}\int_{\ln 4}^{\ln 5} e^{(x+y+z)}\,dz\,dy\,dx = \int_{\ln 6}^{\ln 7}\int_0^{\ln 2} e^{(x+y)}\,dy\,dx = \int_{\ln 6}^{\ln 7} e^x\,dx = 1$

25. $\int_0^1\int_0^{x^2}\int_0^{x+y} (2x-y-z)\,dz\,dy\,dx = \int_0^1\int_0^{x^2}\left(\frac{3x^2}{2} - \frac{3y^2}{2}\right)dy\,dx = \int_0^1\left(\frac{3x^4}{2} - \frac{x^6}{2}\right)dx = \frac{8}{35}$

26. $\int_1^e\int_1^x\int_0^z \frac{2y}{z^3}\,dy\,dz\,dx = \int_1^e\int_1^x \frac{1}{z}\,dz\,dx = \int_1^e \ln x\,dx = [x\ln x - x]_1^e = 1$

27. $V = 2\int_0^{\pi/2}\int_{-\cos y}^0\int_0^{-2x} dz\,dx\,dy = 2\int_0^{\pi/2}\int_{-\cos y}^0 -2x\,dx\,dy = 2\int_0^{\pi/2}\cos^2 y\,dy = 2\left[\frac{y}{2} + \frac{\sin 2y}{4}\right]_0^{\pi/2} = \frac{\pi}{2}$

28. $V = 4\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^{4-x^2} dz\,dy\,dx = 4\int_0^2\int_0^{\sqrt{4-x^2}} (4-x^2)\,dy\,dx = 4\int_0^2 (4-x^2)^{3/2}\,dx$
$= \left[x(4-x^2)^{3/2} + 6x\sqrt{4-x^2} + 24\sin^{-1}\frac{x}{2}\right]_0^2 = 24\sin^{-1} 1 = 12\pi$

29. average $= \frac{1}{3}\int_0^1\int_0^3\int_0^1 30xz\sqrt{x^2+y}\,dz\,dy\,dx = \frac{1}{3}\int_0^1\int_0^3 15x\sqrt{x^2+y}\,dy\,dx = \frac{1}{3}\int_0^3\int_0^1 15x\sqrt{x^2+y}\,dx\,dy$
$= \frac{1}{3}\int_0^3\left[5(x^2+y)^{3/2}\right]_0^1 dy = \frac{1}{3}\int_0^3\left[5(1+y)^{3/2} - 5y^{3/2}\right]dy = \frac{1}{3}\left[2(1+y)^{5/2} - 2y^{5/2}\right]_0^3 = \frac{1}{3}\left[2(4)^{5/2} - 2(3)^{5/2} - 2\right]$
$= \frac{1}{3}\left[2\left(31 - 3^{5/2}\right)\right]$

30. average $= \frac{3}{4\pi a^3}\int_0^{2\pi}\int_0^{\pi}\int_0^a \rho^3\sin\phi\,d\rho\,d\phi\,d\theta = \frac{3a}{16\pi}\int_0^{2\pi}\int_0^{\pi}\sin\phi\,d\phi\,d\theta = \frac{3a}{8\pi}\int_0^{2\pi} d\theta = \frac{3a}{4}$

31. (a) $\int_{-\sqrt{2}}^{\sqrt{2}}\int_{-\sqrt{2-y^2}}^{\sqrt{2-y^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} 3\,dz\,dx\,dy$

(b) $\int_0^{2\pi}\int_0^{\pi/4}\int_0^2 3\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

(c) $\int_0^{2\pi}\int_0^{\sqrt{2}}\int_r^{\sqrt{4-r^2}} 3\,dz\,r\,dr\,d\theta = 3\int_0^{2\pi}\int_0^{\sqrt{2}}\left[r(4-r^2)^{1/2} - r^2\right]dr\,d\theta = 3\int_0^{2\pi}\left[-\frac{1}{3}(4-r^2)^{3/2} - \frac{r^3}{3}\right]_0^{\sqrt{2}} d\theta$
$= \int_0^{2\pi}\left(-2^{3/2} - 2^{3/2} + 4^{3/2}\right)d\theta = \left(8 - 4\sqrt{2}\right)\int_0^{2\pi} d\theta = 2\pi\left(8 - 4\sqrt{2}\right)$

32. (a) $\int_{-\pi/2}^{\pi/2}\int_0^1\int_{-r^2}^{r^2} 21(r\cos\theta)(r\sin\theta)^2\,dz\,r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\int_0^1\int_{-r^2}^{r^2} 21r^3\cos\theta\sin^2\theta\,dz\,r\,dr\,d\theta$

(b) $\int_{-\pi/2}^{\pi/2}\int_0^1\int_{-r^2}^{r^2} 21r^3\cos\theta\sin^2\theta\,dz\,r\,dr\,d\theta = 84\int_0^{\pi/2}\int_0^1 r^6\sin^2\theta\cos\theta\,dr\,d\theta = 12\int_0^{\pi/2}\sin^2\theta\cos\theta\,d\theta = 4$

33. (a) $\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sec\phi} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

(b) $\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sec\phi} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/4} (\sec\phi)(\sec\phi\tan\phi)\,d\phi\,d\theta = \frac{1}{3}\int_0^{2\pi}\left[\frac{1}{2}\tan^2\phi\right]_0^{\pi/4} d\theta = \frac{1}{6}\int_0^{2\pi} d\theta = \frac{\pi}{3}$

34. (a) $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{x^2+y^2}} (6+4y)\,dz\,dy\,dx$ (b) $\int_0^{\pi/2}\int_0^1\int_0^r (6+4r\sin\theta)\,dz\,r\,dr\,d\theta$

(c) $\int_0^{\pi/2}\int_{\pi/4}^{\pi/2}\int_0^{\csc\phi} (6+4\rho\sin\phi\sin\theta)(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta$

(d) $\int_0^{\pi/2}\int_0^1\int_0^r (6+4r\sin\theta)\,dz\,r\,dr\,d\theta = \int_0^{\pi/2}\int_0^1 (6r^2 + 4r^3\sin\theta)\,dr\,d\theta = \int_0^{\pi/2}[2r^3 + r^4\sin\theta]_0^1\,d\theta$
$= \int_0^{\pi/2}(2+\sin\theta)\,d\theta = [2\theta - \cos\theta]_0^{\pi/2} = \pi + 1$

35. $\int_0^1 \int_{\sqrt{1-x^2}}^{\sqrt{3-x^2}} \int_1^{\sqrt{4-x^2-y^2}} z^2yx\, dz\, dy\, dx + \int_1^{\sqrt{3}} \int_0^{\sqrt{3-x^2}} \int_1^{\sqrt{4-x^2-y^2}} z^2yx\, dz\, dy\, dx$

36. (a) Bounded on the top and bottom by the sphere $x^2 + y^2 + z^2 = 4$, on the right by the right circular cylinder $(x-1)^2 + y^2 = 1$, on the left by the plane $y = 0$

(b) $\int_0^{\pi/2} \int_0^{2\cos\theta} \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} dz\, r\, dr\, d\theta$

37. (a) $V = \int_0^{2\pi} \int_0^2 \int_2^{\sqrt{8-r^2}} dz\, r\, dr\, d\theta = \int_0^{2\pi} \int_0^2 \left(r\sqrt{8-r^2} - 2r\right) dr\, d\theta = \int_0^{2\pi} \left[-\frac{1}{3}(8-r^2)^{3/2} - r^2\right]_0^2 d\theta$

$= \int_0^{2\pi} \left[-\frac{1}{3}(4)^{3/2} - 4 + \frac{1}{3}(8)^{3/2}\right] d\theta = \int_0^{2\pi} \frac{4}{3}\left(-2 - 3 + 2\sqrt{8}\right) d\theta = \frac{4}{3}\left(4\sqrt{2} - 5\right) \int_0^{2\pi} d\theta = \frac{8\pi\left(4\sqrt{2}-5\right)}{3}$

(b) $V = \int_0^{2\pi} \int_0^{\pi/4} \int_{2\sec\phi}^{\sqrt{8}} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \frac{8}{3} \int_0^{2\pi} \int_0^{\pi/4} \left(2\sqrt{2}\sin\phi - \sec^3\phi \sin\phi\right) d\phi\, d\theta$

$= \frac{8}{3} \int_0^{2\pi} \int_0^{\pi/4} \left(2\sqrt{2}\sin\phi - \tan\phi \sec^2\phi\right) d\phi\, d\theta = \frac{8}{3} \int_0^{2\pi} \left[-2\sqrt{2}\cos\phi - \frac{1}{2}\tan^2\phi\right]_0^{\pi/4} d\theta$

$= \frac{8}{3} \int_0^{2\pi} \left(-2 - \frac{1}{2} + 2\sqrt{2}\right) d\theta = \frac{8}{3} \int_0^{2\pi} \left(\frac{-5+4\sqrt{2}}{2}\right) d\theta = \frac{8\pi\left(4\sqrt{2}-5\right)}{3}$

38. $I_z = \int_0^{2\pi} \int_0^{\pi/3} \int_0^2 (\rho\sin\phi)^2 (\rho^2 \sin\phi)\, d\rho\, d\phi\, d\theta = \int_0^{2\pi} \int_0^{\pi/3} \int_0^2 \rho^4 \sin^3\phi\, d\rho\, d\phi\, d\theta$

$= \frac{32}{5} \int_0^{2\pi} \int_0^{\pi/3} (\sin\phi - \cos^2\phi \sin\phi)\, d\phi\, d\theta = \frac{32}{5} \int_0^{2\pi} \left[-\cos\phi + \frac{\cos^3\phi}{3}\right]_0^{\pi/3} d\theta = \frac{8\pi}{3}$

39. With the centers of the spheres at the origin, $I_z = \int_0^{2\pi} \int_0^{\pi} \int_a^b \delta(\rho\sin\phi)^2 (\rho^2 \sin\phi)\, d\rho\, d\phi\, d\theta$

$= \frac{\delta(b^5 - a^5)}{5} \int_0^{2\pi} \int_0^{\pi} \sin^3\phi\, d\phi\, d\theta = \frac{\delta(b^5-a^5)}{5} \int_0^{2\pi} \int_0^{\pi} (\sin\phi - \cos^2\phi\sin\phi)\, d\phi\, d\theta$

$= \frac{\delta(b^5-a^5)}{5} \int_0^{2\pi} \left[-\cos\phi + \frac{\cos^3\phi}{3}\right]_0^{\pi} d\theta = \frac{4\delta(b^5-a^5)}{15} \int_0^{2\pi} d\theta = \frac{8\pi\delta(b^5-a^5)}{15}$

40. $I_z = \int_0^{2\pi} \int_0^{\pi} \int_0^{1-\cos\theta} (\rho\sin\phi)^2 (\rho^2\sin\phi)\, d\rho\, d\phi\, d\theta = \int_0^{2\pi} \int_0^{\pi} \int_0^{1-\cos\theta} \rho^4 \sin^3\phi\, d\rho\, d\phi\, d\theta$

$= \frac{1}{5} \int_0^{2\pi} \int_0^{\pi} (1-\cos\phi)^5 \sin^3\phi\, d\phi\, d\theta = \int_0^{2\pi} \int_0^{\pi} (1-\cos\phi)^6 (1+\cos\phi) \sin\phi\, d\phi\, d\theta;$

$\begin{bmatrix} u = 1 - \cos\phi \\ du = \sin\phi\, d\phi \end{bmatrix} \to \frac{1}{5} \int_0^{2\pi} \int_0^2 u^6(2-u)\, du\, d\theta = \frac{1}{5} \int_0^{2\pi} \left[\frac{2u^7}{7} - \frac{u^8}{8}\right]_0^2 d\theta = \frac{1}{5} \int_0^{2\pi} \left(\frac{1}{7} - \frac{1}{8}\right) 2^8\, d\theta$

$= \frac{1}{5} \int_0^{2\pi} \frac{2^3 \cdot 2^5}{56}\, d\theta = \frac{32}{35} \int_0^{2\pi} d\theta = \frac{64\pi}{35}$

41. $M = \int_1^2 \int_{2/x}^2 dy\, dx = \int_1^2 \left(2 - \frac{2}{x}\right) dx = 2 - \ln 4$; $M_y = \int_1^2 \int_{2/x}^2 x\, dy\, dx = \int_1^2 x\left(2 - \frac{2}{x}\right) dx = 1$;

$M_x = \int_1^2 \int_{2/x}^2 y\, dy\, dx = \int_1^2 \left(2 - \frac{2}{x^2}\right) dx = 1 \Rightarrow \bar{x} = \bar{y} = \frac{1}{2 - \ln 4}$

42. $M = \int_0^4 \int_{-2y}^{2y-y^2} dx\, dy = \int_0^4 (4y - y^2)\, dy = \frac{32}{3}$; $M_x = \int_0^4 \int_{-2y}^{2y-y^2} y\, dx\, dy = \int_0^4 (4y^2 - y^3)\, dy = \left[\frac{4y^3}{3} - \frac{y^4}{4}\right]_0^4 = \frac{64}{3}$;

$M_y = \int_0^4 \int_{-2y}^{2y-y^2} x\, dx\, dy = \int_0^4 \left[\frac{(2y-y^2)^2}{2} - 2y^2\right] dy = \left[\frac{y^5}{10} - \frac{y^4}{2}\right]_0^4 = -\frac{128}{5} \Rightarrow \bar{x} = \frac{M_y}{M} = -\frac{12}{5}$ and $\bar{y} = \frac{M_x}{M} = 2$

43. $I_o = \int_0^2 \int_{2x}^4 (x^2 + y^2)(3)\, dy\, dx = 3 \int_0^2 \left(4x^2 + \frac{64}{3} - \frac{14x^3}{3}\right) dx = 104$

44. (a) $I_o = \int_{-2}^2 \int_{-1}^1 (x^2 + y^2)\, dy\, dx = \int_{-2}^2 \left(2x^2 + \frac{2}{3}\right) dx = \frac{40}{3}$

(b) $I_x = \int_{-a}^{a}\int_{-b}^{b} y^2\, dy\, dx = \int_{-a}^{a} \frac{2b^3}{3}\, dx = \frac{4ab^3}{3}$; $I_y = \int_{-b}^{b}\int_{-a}^{a} x^2\, dx\, dy = \int_{-b}^{b} \frac{2a^3}{3}\, dy = \frac{4a^3b}{3} \Rightarrow I_o = I_x + I_y$
$= \frac{4ab^3}{3} + \frac{4a^3b}{3} = \frac{4ab\,(b^2+a^2)}{3}$

45. $M = \delta \int_0^3 \int_0^{2x/3} dy\, dx = \delta \int_0^3 \frac{2x}{3}\, dx = 3\delta$; $I_x = \delta \int_0^3 \int_0^{2x/3} y^2\, dy\, dx = \frac{8\delta}{81} \int_0^3 x^3\, dx = \left(\frac{8\delta}{81}\right)\left(\frac{3^4}{4}\right) = 2\delta$

46. $M = \int_0^1 \int_{x^2}^{x} (x+1)\, dy\, dx = \int_0^1 (x - x^3)\, dx = \frac{1}{4}$; $M_x = \int_0^1 \int_{x^2}^{x} y(x+1)\, dy\, dx = \frac{1}{2} \int_0^1 (x^3 - x^5 + x^2 - x^4)\, dx = \frac{13}{120}$;
$M_y = \int_0^1 \int_{x^2}^{x} x(x+1)\, dy\, dx = \int_0^1 (x^2 - x^4)\, dx = \frac{2}{15} \Rightarrow \overline{x} = \frac{8}{15}$ and $\overline{y} = \frac{13}{30}$; $I_x = \int_0^1 \int_{x^2}^{x} y^2(x+1)\, dy\, dx$
$= \frac{1}{3} \int_0^1 (x^4 - x^7 + x^3 - x^6)\, dx = \frac{17}{280} \Rightarrow R_x = \sqrt{\frac{I_x}{M}} = \sqrt{\frac{17}{70}}$; $I_y = \int_0^1 \int_{x^2}^{x} x^2(x+1)\, dy\, dx = \int_0^1 (x^3 - x^5)\, dx = \frac{1}{12}$

47. $M = \int_{-1}^{1}\int_{-1}^{1} \left(x^2 + y^2 + \frac{1}{3}\right) dy\, dx = \int_{-1}^{1} \left(2x^2 + \frac{4}{3}\right) dx = 4$; $M_x = \int_{-1}^{1}\int_{-1}^{1} y\left(x^2 + y^2 + \frac{1}{3}\right) dy\, dx = \int_{-1}^{1} 0\, dx = 0$;
$M_y = \int_{-1}^{1}\int_{-1}^{1} x\left(x^2 + y^2 + \frac{1}{3}\right) dy\, dx = \int_{-1}^{1} \left(2x^3 + \frac{4}{3}x\right) dx = 0$

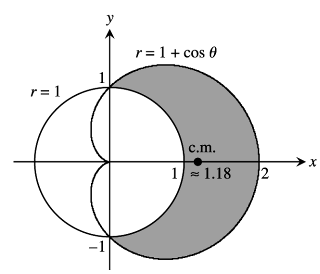
48. Place the ΔABC with its vertices at $A(0,0)$, $B(b,0)$ and $C(a,h)$. The line through the points A and C is $y = \frac{h}{a}x$; the line through the points C and B is $y = \frac{h}{a-b}(x-b)$. Thus, $M = \int_0^h \int_{ay/h}^{(a-b)y/h+b} \delta\, dx\, dy$
$= b\delta \int_0^h \left(1 - \frac{y}{h}\right) dy = \frac{\delta bh}{2}$; $I_x = \int_0^h \int_{ay/h}^{(a-b)y/h+b} y^2\delta\, dx\, dy = b\delta \int_0^h \left(y^2 - \frac{y^3}{h}\right) dy = \frac{\delta bh^3}{12}$

49. $M = \int_{-\pi/3}^{\pi/3} \int_0^3 r\, dr\, d\theta = \frac{9}{2} \int_{-\pi/3}^{\pi/3} d\theta = 3\pi$; $M_y = \int_{-\pi/3}^{\pi/3}\int_0^3 r^2 \cos\theta\, dr\, d\theta = 9 \int_{-\pi/3}^{\pi/3} \cos\theta\, d\theta = 9\sqrt{3} \Rightarrow \overline{x} = \frac{3\sqrt{3}}{\pi}$, and $\overline{y} = 0$ by symmetry

50. $M = \int_0^{\pi/2}\int_1^3 r\, dr\, d\theta = 4 \int_0^{\pi/2} d\theta = 2\pi$; $M_y = \int_0^{\pi/2}\int_1^3 r^2\cos\theta\, dr\, d\theta = \frac{26}{3}\int_0^{\pi/2} \cos\theta\, d\theta = \frac{26}{3} \Rightarrow \overline{x} = \frac{13}{3\pi}$, and $\overline{y} = \frac{13}{3\pi}$ by symmetry

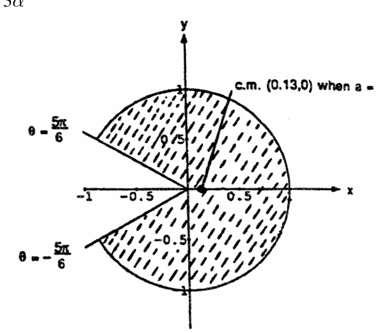
51. (a) $M = 2\int_0^{\pi/2} \int_1^{1+\cos\theta} r\, dr\, d\theta$
$= \int_0^{\pi/2} \left(2\cos\theta + \frac{1+\cos 2\theta}{2}\right) d\theta = \frac{8+\pi}{4}$;
$M_y = \int_{-\pi/2}^{\pi/2} \int_1^{1+\cos\theta} (r\cos\theta)\, r\, dr\, d\theta$
$= \int_{-\pi/2}^{\pi/2} \left(\cos^2\theta + \cos^3\theta + \frac{\cos^4\theta}{3}\right) d\theta$
$= \frac{32+15\pi}{24} \Rightarrow \overline{x} = \frac{15\pi+32}{6\pi+48}$, and
$\overline{y} = 0$ by symmetry

(b)

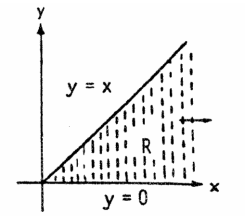


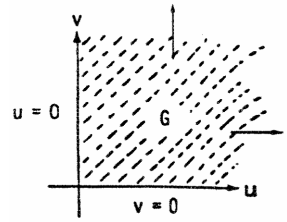
52. (a) $M = \int_{-\alpha}^{\alpha}\int_0^a r\, dr\, d\theta = \int_{-\alpha}^{\alpha} \frac{a^2}{2}\, d\theta = a^2\alpha$; $M_y = \int_{-\alpha}^{\alpha}\int_0^a (r\cos\theta)\, r\, dr\, d\theta = \int_{-\alpha}^{\alpha} \frac{a^3\cos\theta}{3}\, d\theta = \frac{2a^3\sin\alpha}{3}$
$\Rightarrow \overline{x} = \frac{2a\sin\alpha}{3\alpha}$, and $\overline{y} = 0$ by symmetry; $\lim_{\alpha\to\pi^-} \overline{x} = \lim_{\alpha\to\pi^-} \frac{2a\sin\alpha}{3\alpha} = 0$

(b) $\overline{x} = \frac{2a}{5\pi}$ and $\overline{y} = 0$



53. $x = u + y$ and $y = v \Rightarrow x = u + v$ and $y = v$
$\Rightarrow J(u, v) = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$; the boundary of the image G is obtained from the boundary of R as follows:





xy-equations for the boundary of R	Corresponding uv-equations for the boundary of G	Simplified uv-equations
$y = x$	$v = u + v$	$u = 0$
$y = 0$	$v = 0$	$v = 0$

$\Rightarrow \int_0^{\infty}\int_0^{x} e^{-sx} f(x-y, y)\, dy\, dx = \int_0^{\infty}\int_0^{\infty} e^{-s(u+v)} f(u, v)\, du\, dv$

54. If $s = \alpha x + \beta y$ and $t = \gamma x + \delta y$ where $(\alpha\delta - \beta\gamma)^2 = ac - b^2$, then $x = \frac{\delta s - \beta t}{\alpha\delta - \beta\gamma}$, $y = \frac{-\gamma s + \alpha t}{\alpha\delta - \beta\gamma}$,
and $J(s, t) = \frac{1}{(\alpha\delta - \beta\gamma)^2}\begin{vmatrix} \delta & -\beta \\ -\gamma & \alpha \end{vmatrix} = \frac{1}{\alpha\delta - \beta\gamma} \Rightarrow \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(s^2+t^2)} \frac{1}{\sqrt{ac - b^2}}\, ds\, dt$
$= \frac{1}{\sqrt{ac - b^2}} \int_0^{2\pi}\int_0^{\infty} re^{-r^2}\, dr\, d\theta = \frac{1}{2\sqrt{ac - b^2}} \int_0^{2\pi} d\theta = \frac{\pi}{\sqrt{ac - b^2}}$. Therefore, $\frac{\pi}{\sqrt{ac - b^2}} = 1 \Rightarrow ac - b^2 = \pi^2$.

CHAPTER 15 ADDITIONAL AND ADVANCED EXERCISES

1. (a) $V = \int_{-3}^{2}\int_{x}^{6-x^2} x^2\, dy\, dx$ (b) $V = \int_{-3}^{2}\int_{x}^{6-x^2}\int_0^{x^2} dz\, dy\, dx$
(c) $V = \int_{-3}^{2}\int_{x}^{6-x^2} x^2\, dy\, dx = \int_{-3}^{2}\int_{x}^{6-x^2} (6x^2 - x^4 - x^3)\, dx = \left[2x^3 - \frac{x^5}{5} - \frac{x^4}{4}\right]_{-3}^{2} = \frac{125}{4}$

2. Place the sphere's center at the origin with the surface of the water at $z = -3$. Then
$9 = 25 - x^2 - y^2 \Rightarrow x^2 + y^2 = 16$ is the projection of the volume of water onto the xy-plane
$\Rightarrow V = \int_0^{2\pi}\int_0^{4}\int_{-\sqrt{25-r^2}}^{-3} dz\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^{4}\left(r\sqrt{25 - r^2} - 3r\right) dr\, d\theta = \int_0^{2\pi}\left[-\frac{1}{3}(25 - r^2)^{3/2} - \frac{3}{2}r^2\right]_0^4 d\theta$
$= \int_0^{2\pi}\left[-\frac{1}{3}(9)^{3/2} - 24 + \frac{1}{3}(25)^{3/2}\right] d\theta = \int_0^{2\pi} \frac{26}{3}\, d\theta = \frac{52\pi}{3}$

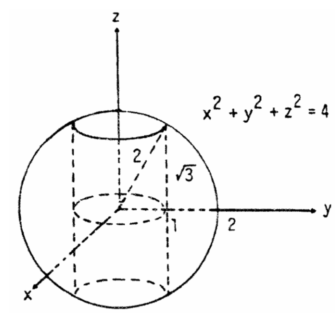
3. Using cylindrical coordinates, $V = \int_0^{2\pi}\int_0^{1}\int_0^{2 - r(\cos\theta + \sin\theta)} dz\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^{1}(2r - r^2\cos\theta - r^2\sin\theta)\, dr\, d\theta$
$= \int_0^{2\pi}\left(1 - \frac{1}{3}\cos\theta - \frac{1}{3}\sin\theta\right) d\theta = \left[\theta - \frac{1}{3}\sin\theta + \frac{1}{3}\cos\theta\right]_0^{2\pi} = 2\pi$

4. $V = 4\int_0^{\pi/2}\int_0^{1}\int_{r^2}^{\sqrt{2-r^2}} dz\, r\, dr\, d\theta = 4\int_0^{\pi/2}\int_0^{1}\left(r\sqrt{2 - r^2} - r^3\right) dr\, d\theta = 4\int_0^{\pi/2}\left[-\frac{1}{3}(2 - r^2)^{3/2} - \frac{r^4}{4}\right]_0^1 d\theta$
$= 4\int_0^{\pi/2}\left(-\frac{1}{3} - \frac{1}{4} + \frac{2\sqrt{2}}{3}\right) d\theta = \left(\frac{8\sqrt{2} - 7}{3}\right)\int_0^{\pi/2} d\theta = \frac{\pi\left(8\sqrt{2} - 7\right)}{6}$

5. The surfaces intersect when $3 - x^2 - y^2 = 2x^2 + 2y^2 \Rightarrow x^2 + y^2 = 1$. Thus the volume is
$V = 4\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{2x^2+2y^2}^{3-x^2-y^2} dz\,dy\,dx = 4\int_0^{\pi/2}\int_0^1\int_{2r^2}^{3-r^2} dz\,r\,dr\,d\theta = 4\int_0^{\pi/2}\int_0^1 (3r - 3r^3)\,dr\,d\theta = 3\int_0^{\pi/2} d\theta = \frac{3\pi}{2}$

6. $V = 8\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{2\sin\phi} \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta = \frac{64}{3}\int_0^{\pi/2}\int_0^{\pi/2} \sin^4\phi\,d\phi\,d\theta$
$= \frac{64}{3}\int_0^{\pi/2}\left[-\frac{\sin^3\phi\cos\phi}{4}\Big|_0^{\pi/2} + \frac{3}{4}\int_0^{\pi/2}\sin^2\phi\,d\phi\right] d\theta = 16\int_0^{\pi/2}\left[\frac{\phi}{2} - \frac{\sin 2\phi}{4}\right]_0^{\pi/2} d\theta = 4\pi\int_0^{\pi/2} d\theta = 2\pi^2$

7. (a) The radius of the hole is 1, and the radius of the sphere is 2.



(b) $V = 2\int_0^{2\pi}\int_0^{\sqrt{3}}\int_1^{\sqrt{4-z^2}} r\,dr\,dz\,d\theta = \int_0^{2\pi}\int_0^{\sqrt{3}} (3 - z^2)\,dz\,d\theta = 2\sqrt{3}\int_0^{2\pi} d\theta = 4\sqrt{3}\pi$

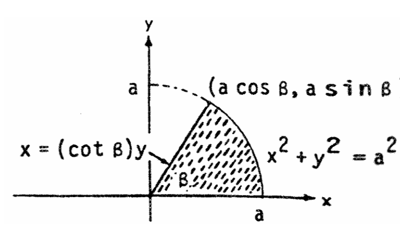
8. $V = \int_0^{\pi}\int_0^{3\sin\theta}\int_0^{\sqrt{9-r^2}} dz\,r\,dr\,d\theta = \int_0^{\pi}\int_0^{3\sin\theta} r\sqrt{9-r^2}\,dr\,d\theta = \int_0^{\pi}\left[-\frac{1}{3}(9-r^2)^{3/2}\right]_0^{3\sin\theta} d\theta$
$= \int_0^{\pi}\left[-\frac{1}{3}(9 - 9\sin^2\theta)^{3/2} + \frac{1}{3}(9)^{3/2}\right] d\theta = 9\int_0^{\pi}\left[1 - (1 - \sin^2\theta)^{3/2}\right] d\theta = 9\int_0^{\pi}(1 - \cos^3\theta)\,d\theta$
$= \int_0^{\pi}(1 - \cos\theta + \sin^2\theta\cos\theta)\,d\theta = 9\left[\theta - \sin\theta + \frac{\sin^3\theta}{3}\right]_0^{\pi} = 9\pi$

9. The surfaces intersect when $x^2 + y^2 = \frac{x^2+y^2+1}{2} \Rightarrow x^2 + y^2 = 1$. Thus the volume in cylindrical coordinates is
$V = 4\int_0^{\pi/2}\int_0^1\int_{r^2}^{(r^2+1)/2} dz\,r\,dr\,d\theta = 4\int_0^{\pi/2}\int_0^1\left(\frac{r}{2} - \frac{r^3}{2}\right) dr\,d\theta = 4\int_0^{\pi/2}\left[\frac{r^2}{4} - \frac{r^4}{8}\right]_0^1 d\theta = \frac{1}{2}\int_0^{\pi/2} d\theta = \frac{\pi}{4}$

10. $V = \int_0^{\pi/2}\int_1^2\int_0^{r^2\sin\theta\cos\theta} dz\,r\,dr\,d\theta = \int_0^{\pi/2}\int_1^2 r^3\sin\theta\cos\theta\,dr\,d\theta = \int_0^{\pi/2}\left[\frac{r^4}{4}\right]_1^2 \sin\theta\cos\theta\,d\theta$
$= \frac{15}{4}\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta = \frac{15}{4}\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2} = \frac{15}{8}$

11. $\int_0^{\infty}\frac{e^{-ax}-e^{-bx}}{x}\,dx = \int_0^{\infty}\int_a^b e^{-xy}\,dy\,dx = \int_a^b\int_0^{\infty} e^{-xy}\,dx\,dy = \int_a^b\left(\lim_{t\to\infty}\int_0^t e^{-xy}\,dx\right) dy$
$= \int_a^b \lim_{t\to\infty}\left[-\frac{e^{-xy}}{y}\right]_0^t dy = \int_a^b \lim_{t\to\infty}\left(\frac{1}{y} - \frac{e^{-yt}}{y}\right) dy = \int_a^b \frac{1}{y}\,dy = [\ln y]_a^b = \ln\left(\frac{b}{a}\right)$

12. (a) The region of integration is sketched at the right
$\Rightarrow \int_0^{a\sin\beta}\int_{y\cot\beta}^{\sqrt{a^2-y^2}} \ln(x^2+y^2)\,dx\,dy$
$= \int_0^{\beta}\int_0^{a} r\ln(r^2)\,dr\,d\theta;$
$\begin{bmatrix} u = r^2 \\ du = 2r\,dr \end{bmatrix} \to \frac{1}{2}\int_0^{\beta}\int_0^{a^2} \ln u\,du\,d\theta$
$= \frac{1}{2}\int_0^{\beta}[u\ln u - u]_0^{a^2}\,d\theta$
$= \frac{1}{2}\int_0^{\beta}\left[2a^2\ln a - a^2 - \lim_{t\to 0} t\ln t\right] d\theta = \frac{a^2}{2}\int_0^{\beta}(2\ln a - 1)\,d\theta = a^2\beta\left(\ln a - \frac{1}{2}\right)$



(b) $\int_0^{a\cos\beta}\int_0^{(\tan\beta)x} \ln(x^2+y^2)\,dy\,dx + \int_{a\cos\beta}^{a}\int_0^{\sqrt{a^2-x^2}} \ln(x^2+y^2)\,dy\,dx$

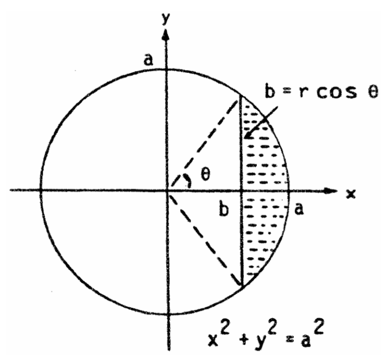
13. $\int_0^x \int_0^u e^{m(x-t)} f(t)\,dt\,du = \int_0^x \int_t^x e^{m(x-t)} f(t)\,du\,dt = \int_0^x (x-t)e^{m(x-t)} f(t)\,dt$; also
$\int_0^x \int_0^v \int_0^u e^{m(x-t)} f(t)\,dt\,du\,dv = \int_0^x \int_t^x \int_t^v e^{m(x-t)} f(t)\,du\,dv\,dt = \int_0^x \int_t^x (v-t)e^{m(x-t)} f(t)\,dv\,dt$
$= \int_0^x \left[\frac{1}{2}(v-t)^2 e^{m(x-t)} f(t)\right]_t^x dt = \int_0^x \frac{(x-t)^2}{2} e^{m(x-t)} f(t)\,dt$

14. $\int_0^1 f(x)\left(\int_0^x g(x-y)f(y)\,dy\right) dx = \int_0^1 \int_0^x g(x-y)f(x)f(y)\,dy\,dx$
$= \int_0^1 \int_y^1 g(x-y)f(x)f(y)\,dx\,dy = \int_0^1 f(y)\left(\int_y^1 g(x-y)f(x)\,dx\right) dy$;
$\int_0^1 \int_0^1 g(|x-y|)\,f(x)f(y)\,dx\,dy = \int_0^1 \int_0^x g(x-y)f(x)f(y)\,dy\,dx + \int_0^1 \int_x^1 g(y-x)f(x)f(y)\,dy\,dx$
$= \int_0^1 \int_y^1 g(x-y)f(x)f(y)\,dx\,dy + \int_0^1 \int_x^1 g(y-x)f(x)f(y)\,dy\,dx$
$= \int_0^1 \int_y^1 g(x-y)f(x)f(y)\,dx\,dy + \underbrace{\int_0^1 \int_y^1 g(x-y)f(y)f(x)\,dx\,dy}_{\text{simply interchange x and y variable names}}$
$= 2\int_0^1 \int_y^1 g(x-y)f(x)f(y)\,dx\,dy$, and the statement now follows.

15. $I_o(a) = \int_0^a \int_0^{x/a^2} (x^2+y^2)\,dy\,dx = \int_0^a \left[x^2 y + \frac{y^3}{3}\right]_0^{x/a^2} dx = \int_0^a \left(\frac{x^3}{a^2} + \frac{x^3}{3a^6}\right) dx = \left[\frac{x^4}{4a^2} + \frac{x^4}{12a^6}\right]_0^a$
$= \frac{a^2}{4} + \frac{1}{12}a^{-2}$; $I_o'(a) = \frac{1}{2}a - \frac{1}{6}a^{-3} = 0 \Rightarrow a^4 = \frac{1}{3} \Rightarrow a = \sqrt[4]{\frac{1}{3}} = \frac{1}{\sqrt[4]{3}}$. Since $I_o''(a) = \frac{1}{2} + \frac{1}{2}a^{-4} > 0$, the value of a does provide a minimum for the polar moment of inertia $I_o(a)$.

16. $I_o = \int_0^2 \int_{2x}^4 (x^2+y^2)(3)\,dy\,dx = 3\int_0^2 \left(4x^2 - \frac{14x^3}{3} + \frac{64}{3}\right) dx = 104$

17. $M = \int_{-\theta}^{\theta} \int_{b\sec\theta}^{a} r\,dr\,d\theta = \int_{-\theta}^{\theta} \left(\frac{a^2}{2} - \frac{b^2}{2}\sec^2\theta\right) d\theta$
$= a^2\theta - b^2\tan\theta = a^2\cos^{-1}\left(\frac{b}{a}\right) - b^2\left(\frac{\sqrt{a^2-b^2}}{b}\right)$
$= a^2\cos^{-1}\left(\frac{b}{a}\right) - b\sqrt{a^2-b^2}$; $I_o = \int_{-\theta}^{\theta} \int_{b\sec\theta}^{a} r^3\,dr\,d\theta$
$= \frac{1}{4}\int_{-\theta}^{\theta} (a^4 + b^4\sec^4\theta)\,d\theta$
$= \frac{1}{4}\int_{-\theta}^{\theta} [a^4 + b^4(1+\tan^2\theta)(\sec^2\theta)]\,d\theta$
$= \frac{1}{4}\left[a^4\theta - b^4\tan\theta - \frac{b^4\tan^3\theta}{3}\right]_{-\theta}^{\theta}$
$= \frac{a^4\theta}{2} - \frac{b^4\tan\theta}{2} - \frac{b^4\tan^3\theta}{6}$
$= \frac{1}{2}a^4\cos^{-1}\left(\frac{b}{a}\right) - \frac{1}{2}b^3\sqrt{a^2-b^2} - \frac{1}{6}b^3(a^2-b^2)^{3/2}$



18. $M = \int_{-2}^{2} \int_{1-(y^2/4)}^{2-(y^2/2)} dx\,dy = \int_{-2}^{2} \left(1 - \frac{y^2}{4}\right) dy = \left[y - \frac{y^3}{12}\right]_{-2}^{2} = \frac{8}{3}$; $M_y = \int_{-2}^{2} \int_{1-(y^2/4)}^{2-(y^2/2)} x\,dx\,dy$
$= \int_{-2}^{2} \left[\frac{x^2}{2}\right]_{1-(y^2/4)}^{2-(y^2/2)} dy = \int_{-2}^{2} \frac{3}{32}(4-y^2)\,dy = \frac{3}{32}\int_{-2}^{2} (16 - 8y^2 + y^4)\,dy = \frac{3}{16}\left[16y - \frac{8y^3}{3} + \frac{y^5}{5}\right]_0^2$
$= \frac{3}{16}\left(32 - \frac{64}{3} + \frac{32}{5}\right) = \left(\frac{3}{16}\right)\left(\frac{32\cdot 8}{15}\right) = \frac{48}{15} \Rightarrow \overline{x} = \frac{M_y}{M} = \left(\frac{48}{15}\right)\left(\frac{3}{8}\right) = \frac{6}{5}$, and $\overline{y} = 0$ by symmetry

19. $\int_0^a \int_0^b e^{\max(b^2x^2, a^2y^2)}\,dy\,dx = \int_0^a \int_0^{bx/a} e^{b^2x^2}\,dy\,dx + \int_0^b \int_0^{ay/b} e^{a^2y^2}\,dx\,dy$

$= \int_0^a \left(\frac{b}{a}x\right) e^{b^2x^2}\,dx + \int_0^b \left(\frac{a}{b}y\right) e^{a^2y^2}\,dy = \left[\frac{1}{2ab} e^{b^2x^2}\right]_0^a + \left[\frac{1}{2ba} e^{a^2y^2}\right]_0^b = \frac{1}{2ab}\left(e^{b^2a^2} - 1\right) + \frac{1}{2ab}\left(e^{a^2b^2} - 1\right)$
$= \frac{1}{ab}\left(e^{a^2b^2} - 1\right)$

20. $\int_{y_0}^{y_1}\int_{x_0}^{x_1} \frac{\partial^2 F(x,y)}{\partial x\,\partial y}\,dx\,dy = \int_{y_0}^{y_1}\left[\frac{\partial F(x,y)}{\partial y}\right]_{x_0}^{x_1} dy = \int_{y_0}^{y_1}\left[\frac{\partial F(x_1,y)}{\partial y} - \frac{\partial F(x_0,y)}{\partial y}\right] dx = [F(x_1,y) - F(x_0,y)]_{y_0}^{y_1}$
$= F(x_1,y_1) - F(x_0,y_1) - F(x_1,y_0) + F(x_0,y_0)$

21. (a) (i) Fubini's Theorem
(ii) Treating G(y) as a constant
(iii) Algebraic rearrangement
(iv) The definite integral is a constant number

(b) $\int_0^{\ln 2}\int_0^{\pi/2} e^x \cos y\,dy\,dx = \left(\int_0^{\ln 2} e^x\,dx\right)\left(\int_0^{\pi/2}\cos y\,dy\right) = \left(e^{\ln 2} - e^0\right)\left(\sin\frac{\pi}{2} - \sin 0\right) = (1)(1) = 1$

(c) $\int_1^2\int_{-1}^1 \frac{x}{y^2}\,dx\,dy = \left(\int_1^2 \frac{1}{y^2}\,dy\right)\left(\int_{-1}^1 x\,dx\right) = \left[-\frac{1}{y}\right]_1^2 \left[\frac{x^2}{2}\right]_{-1}^1 = \left(-\frac{1}{2} + 1\right)\left(\frac{1}{2} - \frac{1}{2}\right) = 0$

22. (a) $\nabla f = x\mathbf{i} + y\mathbf{j} \Rightarrow D_u f = u_1x + u_2y$; the area of the region of integration is $\frac{1}{2}$
$\Rightarrow$ average $= 2\int_0^1\int_0^{1-x} (u_1x + u_2y)\,dy\,dx = 2\int_0^1 \left[u_1x(1-x) + \frac{1}{2}u_2(1-x)^2\right] dx$
$= 2\left[u_1\left(\frac{x^2}{2} - \frac{x^3}{3}\right) - \left(\frac{1}{2}u_2\right)\frac{(1-x)^3}{3}\right]_0^1 = 2\left(\frac{1}{6}u_1 + \frac{1}{6}u_2\right) = \frac{1}{3}(u_1 + u_2)$

(b) average $= \frac{1}{\text{area}}\iint_R (u_1x + u_2y)\,dA = \frac{u_1}{\text{area}}\iint_R x\,dA + \frac{u_2}{\text{area}}\iint_R y\,dA = u_1\left(\frac{M_y}{M}\right) + u_2\left(\frac{M_x}{M}\right) = u_1\overline{x} + u_2\overline{y}$

23. (a) $I^2 = \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy = \int_0^{\pi/2}\int_0^\infty \left(e^{-r^2}\right) r\,dr\,d\theta = \int_0^{\pi/2}\left[\lim_{b\to\infty}\int_0^b re^{-r^2}\,dr\right] d\theta$
$= -\frac{1}{2}\int_0^{\pi/2} \lim_{b\to\infty}\left(e^{-b^2} - 1\right) d\theta = \frac{1}{2}\int_0^{\pi/2} d\theta = \frac{\pi}{4} \Rightarrow I = \frac{\sqrt{\pi}}{2}$

(b) $\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-1/2}e^{-t}\,dt = \int_0^\infty \left(y^2\right)^{-1/2} e^{-y^2}(2y)\,dy = 2\int_0^\infty e^{-y^2}\,dy = 2\left(\frac{\sqrt{\pi}}{2}\right) = \sqrt{\pi}$, where $y = \sqrt{t}$

24. $Q = \int_0^{2\pi}\int_0^R kr^2(1 - \sin\theta)\,dr\,d\theta = \frac{kR^3}{3}\int_0^{2\pi}(1 - \sin\theta)\,d\theta = \frac{kR^3}{3}[\theta + \cos\theta]_0^{2\pi} = \frac{2\pi kR^3}{3}$

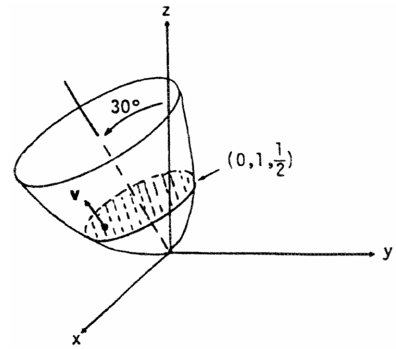
25. For a height h in the bowl the volume of water is $V = \int_{-\sqrt{h}}^{\sqrt{h}}\int_{-\sqrt{h-x^2}}^{\sqrt{h-x^2}}\int_{x^2+y^2}^{h} dz\,dy\,dx$
$= \int_{-\sqrt{h}}^{\sqrt{h}}\int_{-\sqrt{h-x^2}}^{\sqrt{h-x^2}} \left(h - x^2 - y^2\right) dy\,dx = \int_0^{2\pi}\int_0^{\sqrt{h}} \left(h - r^2\right) r\,dr\,d\theta = \int_0^{2\pi}\left[\frac{hr^2}{2} - \frac{r^4}{4}\right]_0^{\sqrt{h}} d\theta = \int_0^{2\pi} \frac{h^2}{4}\,d\theta = \frac{h^2\pi}{2}$.
Since the top of the bowl has area 10π, then we calibrate the bowl by comparing it to a right circular cylinder whose cross sectional area is 10π from $z = 0$ to $z = 10$. If such a cylinder contains $\frac{h^2\pi}{2}$ cubic inches of water to a depth w then we have $10\pi w = \frac{h^2\pi}{2} \Rightarrow w = \frac{h^2}{20}$. So for 1 inch of rain, $w = 1$ and $h = \sqrt{20}$; for 3 inches of rain, $w = 3$ and $h = \sqrt{60}$.

26. (a) An equation for the satellite dish in standard position is $z = \frac{1}{2}x^2 + \frac{1}{2}y^2$. Since the axis is tilted 30°, a unit vector $\mathbf{v} = 0\mathbf{i} + a\mathbf{j} + b\mathbf{k}$ normal to the plane of the water level satisfies $b = \mathbf{v} \cdot \mathbf{k} = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

$\Rightarrow\ a = -\sqrt{1-b^2} = -\frac{1}{2}\ \Rightarrow\ \mathbf{v} = -\frac{1}{2}\mathbf{j} + \frac{\sqrt{3}}{2}\mathbf{k}$

$\Rightarrow\ -\frac{1}{2}(y-1) + \frac{\sqrt{3}}{2}\left(z - \frac{1}{2}\right) = 0$

$\Rightarrow\ z = \frac{1}{\sqrt{3}}y + \left(\frac{1}{2} - \frac{1}{\sqrt{3}}\right)$



is an equation of the plane of the water level. Therefore the volume of water is $V = \iint_R \int_{\frac{1}{2}x^2+\frac{1}{2}y^2}^{\frac{1}{\sqrt{3}}y+\frac{1}{2}-\frac{1}{\sqrt{3}}} dz\, dy\, dx$, where R is the interior of the ellipse

$x^2 + y^2 - \frac{2}{\sqrt{3}}y - 1 + \frac{2}{\sqrt{3}} = 0$. When $x = 0$, then $y = \alpha$ or $y = \beta$, where $\alpha = \frac{\frac{2}{\sqrt{3}} + \sqrt{\frac{4}{3} - 4\left(\frac{2}{\sqrt{3}} - 1\right)}}{2}$

and $\beta = \frac{\frac{2}{\sqrt{3}} - \sqrt{\frac{4}{3} - 4\left(\frac{2}{\sqrt{3}} - 1\right)}}{2}\ \Rightarrow\ V = \int_\alpha^\beta \int_{-\left(\frac{2}{3}y+1-\frac{2}{\sqrt{3}}-y^2\right)^{1/2}}^{\left(\frac{2}{3}y+1-\frac{2}{\sqrt{3}}-y^2\right)^{1/2}} \int_{\frac{1}{2}x^2+\frac{1}{2}y^2}^{\frac{1}{\sqrt{3}}y+\frac{1}{2}-\frac{1}{\sqrt{3}}} 1\, dz\, dx\, dy$

(b) $x = 0\ \Rightarrow\ z = \frac{1}{2}y^2$ and $\frac{dz}{dy} = y$; $y = 1\ \Rightarrow\ \frac{dz}{dy} = 1\ \Rightarrow$ the tangent line has slope 1 or a 45° slant $\Rightarrow$ at 45° and thereafter, the dish will not hold water.

27. The cylinder is given by $x^2 + y^2 = 1$ from $z = 1$ to $\infty\ \Rightarrow\ \iiint_D z\left(r^2+z^2\right)^{-5/2} dV$

$= \int_0^{2\pi}\int_0^1\int_1^\infty \frac{z}{(r^2+z^2)^{5/2}}\, dz\, r\, dr\, d\theta = \lim_{a\to\infty} \int_0^{2\pi}\int_0^1\int_1^a \frac{rz}{(r^2+z^2)^{5/2}}\, dz\, dr\, d\theta$

$= \lim_{a\to\infty} \int_0^{2\pi}\int_0^1 \left[\left(-\frac{1}{3}\right)\frac{r}{(r^2+z^2)^{3/2}}\right]_1^a dr\, d\theta = \lim_{a\to\infty} \int_0^{2\pi}\int_0^1 \left[\left(-\frac{1}{3}\right)\frac{r}{(r^2+a^2)^{3/2}} + \left(\frac{1}{3}\right)\frac{r}{(r^2+1)^{3/2}}\right] dr\, d\theta$

$= \lim_{a\to\infty} \int_0^{2\pi} \left[\frac{1}{3}\left(r^2+a^2\right)^{-1/2} - \frac{1}{3}\left(r^2+1\right)^{-1/2}\right]_0^1 d\theta = \lim_{a\to\infty} \int_0^{2\pi} \left[\frac{1}{3}\left(1+a^2\right)^{-1/2} - \frac{1}{3}\left(2^{-1/2}\right) - \frac{1}{3}\left(a^2\right)^{-1/2} + \frac{1}{3}\right] d\theta$

$= \lim_{a\to\infty} 2\pi\left[\frac{1}{3}\left(1+a^2\right)^{-1/2} - \frac{1}{3}\left(\frac{\sqrt{2}}{2}\right) - \frac{1}{3}\left(\frac{1}{a}\right) + \frac{1}{3}\right] = 2\pi\left[\frac{1}{3} - \left(\frac{1}{3}\right)\frac{\sqrt{2}}{2}\right].$

28. Let's see?

The length of the "unit" line segment is: $L = 2\int_0^1 dx = 2.$

The area of the unit circle is: $A = 4\int_0^1\int_0^{\sqrt{1-x^2}} dy\, dx = \pi.$

The volume of the unit sphere is: $V = 8\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} dz\, dy\, dx = \frac{4}{3}\pi.$

Therefore, the hypervolume of the unit 4-sphere should be:

$$V_{\text{hyper}} = 16\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}}\int_0^{\sqrt{1-x^2-y^2-z^2}} dw\, dz\, dy\, dx.$$

Mathematica is able to handle this integral, but we'll use the brute force approach.

$V_{\text{hyper}} = 16\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}}\int_0^{\sqrt{1-x^2-y^2-z^2}} dw\, dz\, dy\, dx = 16\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \sqrt{1-x^2-y^2-z^2}\, dz\, dy\, dx$

$= 16\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \sqrt{1-x^2-y^2}\sqrt{1-\frac{z^2}{1-x^2-y^2}}\, dz\, dy\, dx = \left[\begin{array}{c}\frac{z}{\sqrt{1-x^2-y^2}} = \cos\theta \\ dz = -\sqrt{1-x^2-y^2}\sin\theta\, d\theta\end{array}\right]$

$= 16\int_0^1\int_0^{\sqrt{1-x^2}}\left(1-x^2-y^2\right)\int_{\pi/2}^0 -\sqrt{1-\cos^2\theta}\sin\theta\, d\theta\, dy\, dx = 16\int_0^1\int_0^{\sqrt{1-x^2}}\left(1-x^2-y^2\right)\int_{\pi/2}^0 -\sin^2\theta\, d\theta\, dy\, dx$

$= 16\int_0^1\int_0^{\sqrt{1-x^2}}\frac{\pi}{4}\left(1-x^2-y^2\right) dy\, dx = 4\pi\int_0^1\left(\sqrt{1-x^2} - x^2\sqrt{1-x^2} - \frac{1}{3}\left(1-x^2\right)^{3/2}\right) dx$

$= 4\pi\int_0^1\sqrt{1-x^2}\left[\left(1-x^2\right) - \frac{1-x^3}{3}\right] dx = \frac{8}{3}\pi\int_0^1\left(1-x^2\right)^{3/2} dx = \left[\begin{array}{c}x = \cos\theta \\ dx = -\sin\theta\, d\theta\end{array}\right] = -\frac{8}{3}\pi\int_{\pi/2}^0 \sin^4\theta\, d\theta$

$= -\frac{8}{3}\pi\int_{\pi/2}^0\left(\frac{1-\cos 2\theta}{2}\right)^2 d\theta = -\frac{2}{3}\pi\int_{\pi/2}^0\left(1 - 2\cos 2\theta + \cos^2 2\theta\right) d\theta = -\frac{2}{3}\pi\int_{\pi/2}^0\left(\frac{3}{2} - 2\cos 2\theta + \frac{\cos 4\theta}{2}\right) d\theta = \frac{\pi^2}{2}$

NOTES:

CHAPTER 16 INTEGRATION IN VECTOR FIELDS

16.1 LINE INTEGRALS

1. $\mathbf{r} = t\mathbf{i} + (1-t)\mathbf{j} \Rightarrow x = t$ and $y = 1 - t \Rightarrow y = 1 - x \Rightarrow$ (c)

2. $\mathbf{r} = \mathbf{i} + \mathbf{j} + t\mathbf{k} \Rightarrow x = 1, y = 1$, and $z = t \Rightarrow$ (e)

3. $\mathbf{r} = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j} \Rightarrow x = 2\cos t$ and $y = 2\sin t \Rightarrow x^2 + y^2 = 4 \Rightarrow$ (g)

4. $\mathbf{r} = t\mathbf{i} \Rightarrow x = t, y = 0$, and $z = 0 \Rightarrow$ (a)

5. $\mathbf{r} = t\mathbf{i} + t\mathbf{j} + t\mathbf{k} \Rightarrow x = t, y = t$, and $z = t \Rightarrow$ (d)

6. $\mathbf{r} = t\mathbf{j} + (2-2t)\mathbf{k} \Rightarrow y = t$ and $z = 2 - 2t \Rightarrow z = 2 - 2y \Rightarrow$ (b)

7. $\mathbf{r} = (t^2 - 1)\mathbf{j} + 2t\mathbf{k} \Rightarrow y = t^2 - 1$ and $z = 2t \Rightarrow y = \frac{z^2}{4} - 1 \Rightarrow$ (f)

8. $\mathbf{r} = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{k} \Rightarrow x = 2\cos t$ and $z = 2\sin t \Rightarrow x^2 + z^2 = 4 \Rightarrow$ (h)

9. $\mathbf{r}(t) = t\mathbf{i} + (1-t)\mathbf{j},\ 0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} - \mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{2}\,\mathbf{j}$; $x = t$ and $y = 1 - t \Rightarrow x + y = t + (1-t) = 1$
$\Rightarrow \int_C f(x,y,z)\,ds = \int_0^1 f(t, 1-t, 0)\left|\frac{d\mathbf{r}}{dt}\right| dt = \int_0^1 (1)\left(\sqrt{2}\right) dt = \left[\sqrt{2}\,t\right]_0^1 = \sqrt{2}$

10. $\mathbf{r}(t) = t\mathbf{i} + (1-t)\mathbf{j} + \mathbf{k},\ 0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} - \mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{2}$; $x = t, y = 1 - t$, and $z = 1 \Rightarrow x - y + z - 2$
$= t - (1-t) + 1 - 2 = 2t - 2 \Rightarrow \int_C f(x,y,z)\,ds = \int_0^1 (2t-2)\sqrt{2}\,dt = \sqrt{2}\left[t^2 - 2t\right]_0^1 = -\sqrt{2}$

11. $\mathbf{r}(t) = 2t\mathbf{i} + t\mathbf{j} + (2-2t)\mathbf{k},\ 0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{4+1+4} = 3$; $xy + y + z$
$= (2t)t + t + (2-2t) \Rightarrow \int_C f(x,y,z)\,ds = \int_0^1 (2t^2 - t + 2)\,3\,dt = 3\left[\frac{2}{3}t^3 - \frac{1}{2}t^2 + 2t\right]_0^1 = 3\left(\frac{2}{3} - \frac{1}{2} + 2\right) = \frac{13}{2}$

12. $\mathbf{r}(t) = (4\cos t)\mathbf{i} + (4\sin t)\mathbf{j} + 3t\mathbf{k},\ -2\pi \le t \le 2\pi \Rightarrow \frac{d\mathbf{r}}{dt} = (-4\sin t)\mathbf{i} + (4\cos t)\mathbf{j} + 3\mathbf{k}$
$\Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{16\sin^2 t + 16\cos^2 t + 9} = 5$; $\sqrt{x^2 + y^2} = \sqrt{16\cos^2 t + 16\sin^2 t} = 4 \Rightarrow \int_C f(x,y,z)\,ds = \int_{-2\pi}^{2\pi} (4)(5)\,dt$
$= \left[20t\right]_{-2\pi}^{2\pi} = 80\pi$

13. $\mathbf{r}(t) = (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) + t(-\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}) = (1-t)\mathbf{i} + (2-3t)\mathbf{j} + (3-2t)\mathbf{k},\ 0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = -\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$
$\Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{1+9+4} = \sqrt{14}$; $x + y + z = (1-t) + (2-3t) + (3-2t) = 6 - 6t \Rightarrow \int_C f(x,y,z)\,ds$
$= \int_0^1 (6-6t)\sqrt{14}\,dt = 6\sqrt{14}\left[t - \frac{t^2}{2}\right]_0^1 = \left(6\sqrt{14}\right)\left(\frac{1}{2}\right) = 3\sqrt{14}$

14. $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k},\ 1 \le t \le \infty \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{3}$; $\frac{\sqrt{3}}{x^2 + y^2 + z^2} = \frac{\sqrt{3}}{t^2 + t^2 + t^2} = \frac{\sqrt{3}}{3t^2}$
$\Rightarrow \int_C f(x,y,z)\,ds = \int_1^\infty \left(\frac{\sqrt{3}}{3t^2}\right)\sqrt{3}\,dt = \left[-\frac{1}{t}\right]_1^\infty = \lim_{b \to \infty}\left(-\frac{1}{b} + 1\right) = 1$

15. C_1: $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$, $0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{1+4t^2}$; $x + \sqrt{y} - z^2 = t + \sqrt{t^2} - 0 = t + |t| = 2t$ since $t \ge 0 \Rightarrow \int_{C_1} f(x,y,z)\,ds = \int_0^1 2t\sqrt{1+4t^2}\,dt = \left[\frac{1}{6}\left(1+4t^2\right)^{3/2}\right]_0^1 = \frac{1}{6}(5)^{3/2} - \frac{1}{6} = \frac{1}{6}\left(5\sqrt{5} - 1\right)$;
C_2: $\mathbf{r}(t) = \mathbf{i} + \mathbf{j} + t\mathbf{k}$, $0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{k} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = 1$; $x + \sqrt{y} - z^2 = 1 + \sqrt{1} - t^2 = 2 - t^2$
$\Rightarrow \int_{C_2} f(x,y,z)\,ds = \int_0^1 \left(2 - t^2\right)(1)\,dt = \left[2t - \frac{1}{3}t^3\right]_0^1 = 2 - \frac{1}{3} = \frac{5}{3}$; therefore $\int_C f(x,y,z)\,ds$
$= \int_{C_1} f(x,y,z)\,ds + \int_{C_2} f(x,y,z)\,ds = \frac{5}{6}\sqrt{5} + \frac{3}{2}$

16. C_1: $\mathbf{r}(t) = t\mathbf{k}$, $0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{k} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = 1$; $x + \sqrt{y} - z^2 = 0 + \sqrt{0} - t^2 = -t^2$
$\Rightarrow \int_{C_1} f(x,y,z)\,ds = \int_0^1 \left(-t^2\right)(1)\,dt = \left[-\frac{t^3}{3}\right]_0^1 = -\frac{1}{3}$;
C_2: $\mathbf{r}(t) = t\mathbf{j} + \mathbf{k}$, $0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = 1$; $x + \sqrt{y} - z^2 = 0 + \sqrt{t} - 1 = \sqrt{t} - 1$
$\Rightarrow \int_{C_2} f(x,y,z)\,ds = \int_0^1 \left(\sqrt{t} - 1\right)(1)\,dt = \left[\frac{2}{3}t^{3/2} - t\right]_0^1 = \frac{2}{3} - 1 = -\frac{1}{3}$;
C_3: $\mathbf{r}(t) = t\mathbf{i} + \mathbf{j} + \mathbf{k}$, $0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = 1$; $x + \sqrt{y} - z^2 = t + \sqrt{1} - 1 = t$
$\Rightarrow \int_{C_3} f(x,y,z)\,ds = \int_0^1 (t)(1)\,dt = \left[\frac{t^2}{2}\right]_0^1 = \frac{1}{2} \Rightarrow \int_C f(x,y,z)\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds + \int_{C_3} f\,ds = -\frac{1}{3} + \left(-\frac{1}{3}\right) + \frac{1}{2}$
$= -\frac{1}{6}$

17. $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}$, $0 < a \le t \le b \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{3}$; $\frac{x+y+z}{x^2+y^2+z^2} = \frac{t+t+t}{t^2+t^2+t^2} = \frac{1}{t}$
$\Rightarrow \int_C f(x,y,z)\,ds = \int_a^b \left(\frac{1}{t}\right)\sqrt{3}\,dt = \left[\sqrt{3}\ln|t|\right]_a^b = \sqrt{3}\ln\left(\frac{b}{a}\right)$, since $0 < a \le b$

18. $\mathbf{r}(t) = (a\cos t)\mathbf{j} + (a\sin t)\mathbf{k}$, $0 \le t \le 2\pi \Rightarrow \frac{d\mathbf{r}}{dt} = (-a\sin t)\mathbf{j} + (a\cos t)\mathbf{k} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{a^2\sin^2 t + a^2\cos^2 t} = |a|$;
$-\sqrt{x^2+z^2} = -\sqrt{0 + a^2\sin^2 t} = \begin{cases} -|a|\sin t, & 0 \le t \le \pi \\ |a|\sin t, & \pi \le t \le 2\pi \end{cases} \Rightarrow \int_C f(x,y,z)\,ds = \int_0^{\pi} -|a|^2\sin t\,dt + \int_{\pi}^{2\pi} |a|^2\sin t\,dt$
$= \left[a^2\cos t\right]_0^{\pi} - \left[a^2\cos t\right]_{\pi}^{2\pi} = \left[a^2(-1) - a^2\right] - \left[a^2 - a^2(-1)\right] = -4a^2$

19. (a) $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{2}t\mathbf{j}$, $0 \le t \le 4 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + \frac{1}{2}\mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \frac{\sqrt{5}}{2} \Rightarrow \int_C x\,ds = \int_0^4 t\,\frac{\sqrt{5}}{2}dt = \frac{\sqrt{5}}{2}\int_0^4 t\,dt = \left[\frac{\sqrt{5}}{4}t^2\right]_0^4 = 4\sqrt{5}$

(b) $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$, $0 \le t \le 2 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{1+4t^2} \Rightarrow \int_C x\,ds = \int_0^2 t\sqrt{1+4t^2}dt$
$= \left[\frac{1}{12}\left(1 + 4t^2\right)^{3/2}\right]_0^2 = \frac{17\sqrt{17}-1}{12}$

20. (a) $\mathbf{r}(t) = t\mathbf{i} + 4t\mathbf{j}$, $0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + 4\mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{17} \Rightarrow \int_C \sqrt{x+2y}\,ds = \int_0^1 \sqrt{t + 2(4t)}\,\sqrt{17}\,dt$
$= \sqrt{17}\int_0^1 \sqrt{9t}\,dt = 3\sqrt{17}\int_0^1 \sqrt{t}\,dt = \left[2\sqrt{17}\,t^{2/3}\right]_0^1 = 2\sqrt{17}$

(b) C_1: $\mathbf{r}(t) = t\mathbf{i}$, $0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = 1$; C_2: $\mathbf{r}(t) = \mathbf{i} + t\mathbf{j}$, $0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = 1$
$\int_C \sqrt{x+2y}\,ds = \int_{C_1} \sqrt{x+2y}\,ds + \int_{C_2} \sqrt{x+2y}\,ds = \int_0^1 \sqrt{t + 2(0)}\,dt + \int_0^2 \sqrt{1 + 2(t)}\,dt$
$= \int_0^1 \sqrt{t}\,dt + \int_0^2 \sqrt{1+2t}\,dt = \left[\frac{2}{3}t^{2/3}\right]_0^1 + \left[\frac{1}{3}(1+2t)^{2/3}\right]_0^2 = \frac{2}{3} + \left(\frac{5\sqrt{5}}{3} - \frac{1}{3}\right) = \frac{5\sqrt{5}+1}{3}$

21. $\mathbf{r}(t) = 4t\mathbf{i} - 3t\mathbf{j}$, $-1 \le t \le 2 \Rightarrow \frac{d\mathbf{r}}{dt} = 4\mathbf{i} - 3\mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = 5 \Rightarrow \int_C y\,e^{x^2}\,ds = \int_{-1}^2 (-3t)\,e^{(4t)^2} \cdot 5dt$
$= -15\int_{-1}^2 t\,e^{16t^2}dt = \left[-\frac{15}{32}e^{16t^2}\right]_{-1}^2 = -\frac{15}{32}e^{64} + \frac{15}{32}e^{16} = \frac{15}{32}\left(e^{16} - e^{64}\right)$

22. $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$, $0 \le t \le 2\pi \Rightarrow \frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{\sin^2 t + \cos^2 t} = 1 \Rightarrow \int_C (x - y + 3)\, ds$
$= \int_0^{2\pi} (\cos t - \sin t + 3) \cdot 1\, dt = [\sin t + \cos t + 3t]_0^{2\pi} = 6\pi$

23. $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j}$, $1 \le t \le 2 \Rightarrow \frac{d\mathbf{r}}{dt} = 2t\mathbf{i} + 3t^2\mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{(2t)^2 + (3t^2)^2} = t\sqrt{4 + 9t^2} \Rightarrow \int_C \frac{x^2}{y^{4/3}}\, ds$
$= \int_1^2 \frac{(t^2)^2}{(t^3)^{4/3}} \cdot t\sqrt{4 + 9t^2}\, dt = \int_1^2 t\sqrt{4 + 9t^2}\, dt = \left[\frac{1}{27}(4 + 9t^2)^{3/2}\right]_1^2 = \frac{80\sqrt{10} - 13\sqrt{13}}{27}$

24. $\mathbf{r}(t) = t^3\mathbf{i} + t^4\mathbf{j}$, $\frac{1}{2} \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = 3t^2\mathbf{i} + 4t^3\mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{(3t^2)^2 + (4t^3)^2} = t^2\sqrt{9 + 16t^2} \Rightarrow \int_C \frac{\sqrt{y}}{x}\, ds$
$= \int_{1/2}^1 \frac{\sqrt{t^4}}{t^3} \cdot t^2\sqrt{9 + 16t^2}\, dt = \int_{1/2}^1 t\sqrt{9 + 16t^2}\, dt = \left[\frac{1}{48}(9 + 16t^2)^{3/2}\right]_{1/2}^1 = \frac{125 - 13\sqrt{13}}{48}$

25. C_1: $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$, $0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{1 + 4t^2}$; C_2: $\mathbf{r}(t) = (1 - t)\mathbf{i} + (1 - t)\mathbf{j}$, $0 \le t \le 1$
$\Rightarrow \frac{d\mathbf{r}}{dt} = -\mathbf{i} - \mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{2} \Rightarrow \int_C \left(x + \sqrt{y}\right) ds = \int_{C_1} \left(x + \sqrt{y}\right) ds + \int_{C_2} \left(x + \sqrt{y}\right) ds$
$= \int_0^1 \left(t + \sqrt{t^2}\right)\sqrt{1 + 4t^2}\, dt + \int_0^1 \left((1 - t) + \sqrt{1 - t}\right)\sqrt{2}\, dt = \int_0^1 2t\sqrt{1 + 4t^2}\, dt + \int_0^1 \left(1 - t + \sqrt{1 - t}\right)\sqrt{2}\, dt$
$= \left[\frac{1}{6}(1 + 4t^2)^{3/2}\right]_0^1 + \sqrt{2}\left[t - \frac{1}{2}t^2 - \frac{2}{3}(1 - t)^{3/2}\right]_0^1 = \frac{5\sqrt{5} - 1}{6} + \frac{7\sqrt{2}}{6} = \frac{5\sqrt{5} + 7\sqrt{2} - 1}{6}$

26. C_1: $\mathbf{r}(t) = t\mathbf{i}$, $0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = 1$; C_2: $\mathbf{r}(t) = \mathbf{i} + t\mathbf{j}$, $0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = 1$;
C_3: $\mathbf{r}(t) = (1 - t)\mathbf{i} + \mathbf{j}$, $0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = -\mathbf{i} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = 1$; C_4: $\mathbf{r}(t) = (1 - t)\mathbf{j}$, $0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = -\mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = 1$;
$\Rightarrow \int_C \frac{1}{x^2 + y^2 + 1} ds = \int_{C_1} \frac{1}{x^2 + y^2 + 1} ds + \int_{C_2} \frac{1}{x^2 + y^2 + 1} ds + \int_{C_3} \frac{1}{x^2 + y^2 + 1} ds + \int_{C_4} \frac{1}{x^2 + y^2 + 1} ds$
$= \int_0^1 \frac{dt}{t^2 + 1} + \int_0^1 \frac{dt}{t^2 + 2} + \int_0^1 \frac{dt}{(1 - t)^2 + 2} + \int_0^1 \frac{dt}{(1 - t)^2 + 1}$
$= [\tan^{-1} t]_0^1 + \frac{1}{\sqrt{2}}\left[\tan^{-1}\left(\frac{t}{\sqrt{2}}\right)\right]_0^1 + \frac{1}{\sqrt{2}}\left[\tan^{-1}\left(\frac{t - 1}{\sqrt{2}}\right)\right]_0^1 + [-\tan^{-1}(1 - t)]_0^1 = \frac{\pi}{2} + \frac{2}{\sqrt{2}}\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$

27. $\mathbf{r}(x) = x\mathbf{i} + y\mathbf{j} = x\mathbf{i} + \frac{x^2}{2}\mathbf{j}$, $0 \le x \le 2 \Rightarrow \frac{d\mathbf{r}}{dx} = \mathbf{i} + x\mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dx}\right| = \sqrt{1 + x^2}$; $f(x, y) = f\left(x, \frac{x^2}{2}\right) = \frac{x^3}{\left(\frac{x^2}{2}\right)} = 2x \Rightarrow \int_C f\, ds$
$= \int_0^2 (2x)\sqrt{1 + x^2}\, dx = \left[\frac{2}{3}(1 + x^2)^{3/2}\right]_0^2 = \frac{2}{3}\left(5^{3/2} - 1\right) = \frac{10\sqrt{5} - 2}{3}$

28. $\mathbf{r}(t) = (1 - t)\mathbf{i} + \frac{1}{2}(1 - t)^2\mathbf{j}$, $0 \le t \le 1 \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{1 + (1 - t)^2}$; $f(x, y) = f\left((1 - t), \frac{1}{2}(1 - t)^2\right) = \frac{(1 - t) + \frac{1}{4}(1 - t)^4}{\sqrt{1 + (1 - t)^2}}$
$\Rightarrow \int_C f\, ds = \int_0^1 \frac{(1 - t) + \frac{1}{4}(1 - t)^4}{\sqrt{1 + (1 - t)^2}} \sqrt{1 + (1 - t)^2}\, dt = \int_0^1 \left((1 - t) + \frac{1}{4}(1 - t)^4\right) dt = \left[-\frac{1}{2}(1 - t)^2 - \frac{1}{20}(1 - t)^5\right]_0^1$
$= 0 - \left(-\frac{1}{2} - \frac{1}{20}\right) = \frac{11}{20}$

29. $\mathbf{r}(t) = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j}$, $0 \le t \le \frac{\pi}{2} \Rightarrow \frac{d\mathbf{r}}{dt} = (-2\sin t)\mathbf{i} + (2\cos t)\mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = 2$; $f(x, y) = f(2\cos t, 2\sin t)$
$= 2\cos t + 2\sin t \Rightarrow \int_C f\, ds = \int_0^{\pi/2} (2\cos t + 2\sin t)(2)\, dt = [4\sin t - 4\cos t]_0^{\pi/2} = 4 - (-4) = 8$

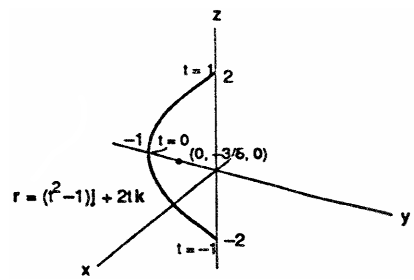
30. $\mathbf{r}(t) = (2\sin t)\mathbf{i} + (2\cos t)\mathbf{j}$, $0 \le t \le \frac{\pi}{4} \Rightarrow \frac{d\mathbf{r}}{dt} = (2\cos t)\mathbf{i} + (-2\sin t)\mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = 2$; $f(x, y) = f(2\sin t, 2\cos t)$
$= 4\sin^2 t - 2\cos t \Rightarrow \int_C f\, ds = \int_0^{\pi/4} (4\sin^2 t - 2\cos t)(2)\, dt = [4t - 2\sin 2t - 4\sin t]_0^{\pi/4} = \pi - 2\left(1 + \sqrt{2}\right)$

31. $y = x^2$, $0 \le x \le 2 \Rightarrow \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$, $0 \le t \le 2 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{1 + 4t^2} \Rightarrow A = \int_C f(x, y)\, ds$
$= \int_C \left(x + \sqrt{y}\right) ds = \int_0^2 \left(t + \sqrt{t^2}\right)\sqrt{1 + 4t^2}\, dt = \int_0^2 2t\sqrt{1 + 4t^2}\, dt = \left[\frac{1}{6}(1 + 4t^2)^{3/2}\right]_0^2 = \frac{17\sqrt{17} - 1}{6}$

32. $2x + 3y = 6,\ 0 \le x \le 6 \Rightarrow \mathbf{r}(t) = t\mathbf{i} + \left(2 - \frac{2}{3}t\right)\mathbf{j},\ 0 \le t \le 6 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} - \frac{2}{3}\mathbf{j} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \frac{\sqrt{13}}{3} \Rightarrow A = \int_C f(x, y)\, ds$
$= \int_C (4 + 3x + 2y)ds = \int_0^6 \left(4 + 3t + 2\left(2 - \frac{2}{3}t\right)\right)\frac{\sqrt{13}}{3}\, dt = \frac{\sqrt{13}}{3}\int_0^6 \left(8 + \frac{5}{3}t\right)dt = \frac{\sqrt{13}}{3}\left[8t + \frac{5}{6}t^2\right]_0^6 = 26\sqrt{13}$

33. $\mathbf{r}(t) = (t^2 - 1)\,\mathbf{j} + 2t\mathbf{k},\ 0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = 2t\mathbf{j} + 2\mathbf{k} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = 2\sqrt{t^2+1};\ M = \int_C \delta(x, y, z)\, ds = \int_0^1 \delta(t)\left(2\sqrt{t^2+1}\right) dt$
$= \int_0^1 \left(\frac{3}{2}t\right)\left(2\sqrt{t^2+1}\right) dt = \left[(t^2+1)^{3/2}\right]_0^1 = 2^{3/2} - 1 = 2\sqrt{2} - 1$

34. $\mathbf{r}(t) = (t^2 - 1)\,\mathbf{j} + 2t\mathbf{k},\ -1 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = 2t\mathbf{j} + 2\mathbf{k}$
$\Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = 2\sqrt{t^2+1};\ M = \int_C \delta(x, y, z)\, ds$
$= \int_{-1}^1 \left(15\sqrt{(t^2-1)+2}\right)\left(2\sqrt{t^2+1}\right) dt$
$= \int_{-1}^1 30\,(t^2+1)\, dt = \left[30\left(\frac{t^3}{3} + t\right)\right]_{-1}^1 = 60\left(\frac{1}{3} + 1\right) = 80;$
$M_{xz} = \int_C y\delta(x, y, z)\, ds = \int_{-1}^1 (t^2 - 1)\,[30\,(t^2+1)]\, dt$
$= \int_{-1}^1 30\,(t^4 - 1)\, dt = \left[30\left(\frac{t^5}{5} - t\right)\right]_{-1}^1 = 60\left(\frac{1}{5} - 1\right)$
$= -48 \Rightarrow \bar{y} = \frac{M_{xz}}{M} = -\frac{48}{80} = -\frac{3}{5};\ M_{yz} = \int_C x\delta(x, y, z)\, ds = \int_C 0\,\delta\, ds = 0 \Rightarrow \bar{x} = 0;\ \bar{z} = 0$ by symmetry (since δ is independent of z) $\Rightarrow (\bar{x}, \bar{y}, \bar{z}) = \left(0, -\frac{3}{5}, 0\right)$



35. $\mathbf{r}(t) = \sqrt{2}t\,\mathbf{i} + \sqrt{2}t\,\mathbf{j} + (4 - t^2)\,\mathbf{k},\ 0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j} - 2t\mathbf{k} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{2 + 2 + 4t^2} = 2\sqrt{1+t^2};$
(a) $M = \int_C \delta\, ds = \int_0^1 (3t)\left(2\sqrt{1+t^2}\right) dt = \left[2\,(1+t^2)^{3/2}\right]_0^1 = 2\left(2^{3/2} - 1\right) = 4\sqrt{2} - 2$
(b) $M = \int_C \delta\, ds = \int_0^1 (1)\left(2\sqrt{1+t^2}\right) dt = \left[t\sqrt{1+t^2} + \ln\left(t + \sqrt{1+t^2}\right)\right]_0^1 = \left[\sqrt{2} + \ln\left(1 + \sqrt{2}\right)\right] - (0 + \ln 1)$
$= \sqrt{2} + \ln\left(1 + \sqrt{2}\right)$

36. $\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} + \frac{2}{3}t^{3/2}\mathbf{k},\ 0 \le t \le 2 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2\mathbf{j} + t^{1/2}\mathbf{k} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{1 + 4 + t} = \sqrt{5 + t};$
$M = \int_C \delta\, ds = \int_0^2 \left(3\sqrt{5+t}\right)\left(\sqrt{5+t}\right) dt = \int_0^2 3(5+t)\, dt = \left[\frac{3}{2}(5+t)^2\right]_0^2 = \frac{3}{2}(7^2 - 5^2) = \frac{3}{2}(24) = 36;$
$M_{yz} = \int_C x\delta\, ds = \int_0^2 t[3(5+t)]\, dt = \int_0^2 (15t + 3t^2)\, dt = \left[\frac{15}{2}t^2 + t^3\right]_0^2 = 30 + 8 = 38;$
$M_{xz} = \int_C y\delta\, ds = \int_0^2 2t[3(5+t)]\, dt = 2\int_0^2 (15t + 3t^2)\, dt = 76;\ M_{xy} = \int_C z\delta\, ds = \int_0^2 \frac{2}{3}t^{3/2}[3(5+t)]\, dt$
$= \int_0^2 \left(10t^{3/2} + 2t^{5/2}\right) dt = \left[4t^{5/2} + \frac{4}{7}t^{7/2}\right]_0^2 = 4(2)^{5/2} + \frac{4}{7}(2)^{7/2} = 16\sqrt{2} + \frac{32}{7}\sqrt{2} = \frac{144}{7}\sqrt{2} \Rightarrow \bar{x} = \frac{M_{yz}}{M}$
$= \frac{38}{36} = \frac{19}{18},\ \bar{y} = \frac{M_{xz}}{M} = \frac{76}{36} = \frac{19}{9}$, and $\bar{z} = \frac{M_{xy}}{M} = \frac{144\sqrt{2}}{7 \cdot 36} = \frac{4}{7}\sqrt{2}$

37. Let $x = a\cos t$ and $y = a\sin t,\ 0 \le t \le 2\pi$. Then $\frac{dx}{dt} = -a\sin t,\ \frac{dy}{dt} = a\cos t,\ \frac{dz}{dt} = 0$
$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = a\, dt;\ I_z = \int_C (x^2 + y^2)\,\delta\, ds = \int_0^{2\pi} (a^2\sin^2 t + a^2\cos^2 t)\, a\delta\, dt$
$= \int_0^{2\pi} a^3\delta\, dt = 2\pi\delta a^3.$

38. $\mathbf{r}(t) = t\mathbf{j} + (2 - 2t)\mathbf{k},\ 0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{j} - 2\mathbf{k} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{5};\ M = \int_C \delta\, ds = \int_0^1 \delta\sqrt{5}\, dt = \delta\sqrt{5};$
$I_x = \int_C (y^2 + z^2)\,\delta\, ds = \int_0^1 [t^2 + (2 - 2t)^2]\,\delta\sqrt{5}\, dt = \int_0^1 (5t^2 - 8t + 4)\,\delta\sqrt{5}\, dt = \delta\sqrt{5}\left[\frac{5}{3}t^3 - 4t^2 + 4t\right]_0^1 = \frac{5}{3}\delta\sqrt{5};$

$I_y = \int_C (x^2+z^2)\,\delta\,ds = \int_0^1 [0^2+(2-2t)^2]\,\delta\sqrt{5}\,dt = \int_0^1 (4t^2-8t+4)\,\delta\sqrt{5}\,dt = \delta\sqrt{5}\left[\frac{4}{3}t^3-4t^2+4t\right]_0^1 = \frac{4}{3}\delta\sqrt{5}$;
$I_z = \int_C (x^2+y^2)\,\delta\,ds = \int_0^1 (0^2+t^2)\,\delta\sqrt{5}\,dt = \delta\sqrt{5}\left[\frac{t^3}{3}\right]_0^1 = \frac{1}{3}\delta\sqrt{5}$

39. $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k},\ 0 \le t \le 2\pi \Rightarrow \frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \mathbf{k} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$;
(a) $I_z = \int_C (x^2+y^2)\,\delta\,ds = \int_0^{2\pi} (\cos^2 t + \sin^2 t)\,\delta\sqrt{2}\,dt = 2\pi\delta\sqrt{2}$
(b) $I_z = \int_C (x^2+y^2)\,\delta\,ds = \int_0^{4\pi} \delta\sqrt{2}\,dt = 4\pi\delta\sqrt{2}$

40. $\mathbf{r}(t) = (t\cos t)\mathbf{i} + (t\sin t)\mathbf{j} + \frac{2\sqrt{2}}{3}t^{3/2}\mathbf{k},\ 0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = (\cos t - t\sin t)\mathbf{i} + (\sin t + t\cos t)\mathbf{j} + \sqrt{2t}\,\mathbf{k}$
$\Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{(t+1)^2} = t+1$ for $0 \le t \le 1$; $M = \int_C \delta\,ds = \int_0^1 (t+1)\,dt = \left[\frac{1}{2}(t+1)^2\right]_0^1 = \frac{1}{2}(2^2-1^2) = \frac{3}{2}$;
$M_{xy} = \int_C z\delta\,ds = \int_0^1 \left(\frac{2\sqrt{2}}{3}t^{3/2}\right)(t+1)\,dt = \frac{2\sqrt{2}}{3}\int_0^1 \left(t^{5/2}+t^{3/2}\right)dt = \frac{2\sqrt{2}}{3}\left[\frac{2}{7}t^{7/2}+\frac{2}{5}t^{5/2}\right]_0^1$
$= \frac{2\sqrt{2}}{3}\left(\frac{2}{7}+\frac{2}{5}\right) = \frac{2\sqrt{2}}{3}\left(\frac{24}{35}\right) = \frac{16\sqrt{2}}{35} \Rightarrow \overline{z} = \frac{M_{xy}}{M} = \left(\frac{16\sqrt{2}}{35}\right)\left(\frac{2}{3}\right) = \frac{32\sqrt{2}}{105}$; $I_z = \int_C (x^2+y^2)\,\delta\,ds$
$= \int_0^1 (t^2\cos^2 t + t^2\sin^2 t)(t+1)\,dt = \int_0^1 (t^3+t^2)\,dt = \left[\frac{t^4}{4}+\frac{t^3}{3}\right]_0^1 = \frac{1}{4}+\frac{1}{3} = \frac{7}{12}$

41. $\delta(x,y,z) = 2-z$ and $\mathbf{r}(t) = (\cos t)\mathbf{j} + (\sin t)\mathbf{k},\ 0 \le t \le \pi \Rightarrow M = 2\pi - 2$ as found in Example 3 of the text;
also $\left|\frac{d\mathbf{r}}{dt}\right| = 1$; $I_x = \int_C (y^2+z^2)\,\delta\,ds = \int_0^\pi (\cos^2 t + \sin^2 t)(2-\sin t)\,dt = \int_0^\pi (2-\sin t)\,dt = 2\pi - 2$

42. $\mathbf{r}(t) = t\mathbf{i} + \frac{2\sqrt{2}}{3}t^{3/2}\mathbf{j} + \frac{t^2}{2}\mathbf{k},\ 0 \le t \le 2 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + \sqrt{2}\,t^{1/2}\mathbf{j} + t\mathbf{k} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{1+2t+t^2} = \sqrt{(1+t)^2} = 1+t$ for
$0 \le t \le 2$; $M = \int_C \delta\,ds = \int_0^2 \left(\frac{1}{t+1}\right)(1+t)\,dt = \int_0^2 dt = 2$; $M_{yz} = \int_C x\delta\,ds = \int_0^2 t\left(\frac{1}{t+1}\right)(1+t)\,dt = \left[\frac{t^2}{2}\right]_0^2 = 2$;
$M_{xz} = \int_C y\delta\,ds = \int_0^2 \frac{2\sqrt{2}}{3}t^{3/2}\,dt = \left[\frac{4\sqrt{2}}{15}t^{5/2}\right]_0^2 = \frac{32}{15}$; $M_{xy} = \int_C z\delta\,ds = \int_0^2 \frac{t^2}{2}\,dt = \left[\frac{t^3}{6}\right]_0^2 = \frac{4}{3} \Rightarrow \overline{x} = \frac{M_{yz}}{M} = 1$,
$\overline{y} = \frac{M_{xz}}{M} = \frac{16}{15}$, and $\overline{z} = \frac{M_{xy}}{M} = \frac{2}{3}$; $I_x = \int_C (y^2+z^2)\,\delta\,ds = \int_0^2 \left(\frac{8}{9}t^3+\frac{1}{4}t^4\right)dt = \left[\frac{2}{9}t^4+\frac{t^5}{20}\right]_0^2 = \frac{32}{9}+\frac{32}{20} = \frac{232}{45}$;
$I_y = \int_C (x^2+z^2)\,\delta\,ds = \int_0^2 \left(t^2+\frac{1}{4}t^4\right)dt = \left[\frac{t^3}{3}+\frac{t^5}{20}\right]_0^2 = \frac{8}{3}+\frac{32}{20} = \frac{64}{15}$; $I_z = \int_C (x^2+y^2)\,\delta\,ds$
$= \int_0^2 \left(t^2+\frac{8}{9}t^3\right)dt = \left[\frac{t^3}{3}+\frac{2}{9}t^4\right]_0^2 = \frac{8}{3}+\frac{32}{9} = \frac{56}{9}$

43-46. Example CAS commands:

Maple:

```
f := (x,y,z) -> sqrt(1+30*x^2+10*y);
g := t -> t;
h := t -> t^2;
k := t -> 3*t^2;
a,b := 0,2;
ds := ( D(g)^2 + D(h)^2 + D(k)^2 )^(1/2):                    # (a)
'ds' = ds(t)*'dt';
F := f(g,h,k):                                               # (b)
'F(t)' = F(t);
Int( f, s=C..NULL ) = Int( simplify(F(t)*ds(t)), t=a..b );   # (c)
`` = value(rhs(%));
```

Mathematica: (functions and domains may vary)

```
Clear[x, y, z, r, t, f]
f[x_,y_,z_]:= Sqrt[1 + 30x^2 + 10y]
{a,b}= {0, 2};
x[t_]:= t
y[t_]:= t^2
z[t_]:= 3t^2
r[t_]:= {x[t], y[t], z[t]}
v[t_]:= D[r[t], t]
mag[vector_]:=Sqrt[vector.vector]
Integrate[f[x[t],y[t],z[t]] mag[v[t]], {t, a, b}]
N[%]
```

16.2 VECTOR FIELDS, WORK, CIRCULATION, AND FLUX

1. $f(x,y,z) = (x^2+y^2+z^2)^{-1/2} \Rightarrow \frac{\partial f}{\partial x} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2x) = -x(x^2+y^2+z^2)^{-3/2}$; similarly, $\frac{\partial f}{\partial y} = -y(x^2+y^2+z^2)^{-3/2}$ and $\frac{\partial f}{\partial z} = -z(x^2+y^2+z^2)^{-3/2} \Rightarrow \nabla f = \frac{-x\mathbf{i} - y\mathbf{j} - z\mathbf{k}}{(x^2+y^2+z^2)^{3/2}}$

2. $f(x,y,z) = \ln\sqrt{x^2+y^2+z^2} = \frac{1}{2}\ln(x^2+y^2+z^2) \Rightarrow \frac{\partial f}{\partial x} = \frac{1}{2}\left(\frac{1}{x^2+y^2+z^2}\right)(2x) = \frac{x}{x^2+y^2+z^2}$; similarly, $\frac{\partial f}{\partial y} = \frac{y}{x^2+y^2+z^2}$ and $\frac{\partial f}{\partial z} = \frac{z}{x^2+y^2+z^2} \Rightarrow \nabla f = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{x^2+y^2+z^2}$

3. $g(x,y,z) = e^z - \ln(x^2+y^2) \Rightarrow \frac{\partial g}{\partial x} = -\frac{2x}{x^2+y^2}$, $\frac{\partial g}{\partial y} = -\frac{2y}{x^2+y^2}$ and $\frac{\partial g}{\partial z} = e^z$ $\Rightarrow \nabla g = \left(\frac{-2x}{x^2+y^2}\right)\mathbf{i} - \left(\frac{2y}{x^2+y^2}\right)\mathbf{j} + e^z\mathbf{k}$

4. $g(x,y,z) = xy + yz + xz \Rightarrow \frac{\partial g}{\partial x} = y + z$, $\frac{\partial g}{\partial y} = x + z$, and $\frac{\partial g}{\partial z} = y + x \Rightarrow \nabla g = (y+z)\mathbf{i} + (x+z)\mathbf{j} + (x+y)\mathbf{k}$

5. $|\mathbf{F}|$ inversely proportional to the square of the distance from (x, y) to the origin $\Rightarrow \sqrt{(M(x,y))^2 + (N(x,y))^2} = \frac{k}{x^2+y^2}$, $k > 0$; $\mathbf{F}$ points toward the origin $\Rightarrow$ $\mathbf{F}$ is in the direction of $\mathbf{n} = \frac{-x}{\sqrt{x^2+y^2}}\mathbf{i} - \frac{y}{\sqrt{x^2+y^2}}\mathbf{j}$ $\Rightarrow \mathbf{F} = a\mathbf{n}$, for some constant $a > 0$. Then $M(x,y) = \frac{-ax}{\sqrt{x^2+y^2}}$ and $N(x,y) = \frac{-ay}{\sqrt{x^2+y^2}}$ $\Rightarrow \sqrt{(M(x,y))^2+(N(x,y))^2} = a \Rightarrow a = \frac{k}{x^2+y^2} \Rightarrow \mathbf{F} = \frac{-kx}{(x^2+y^2)^{3/2}}\mathbf{i} - \frac{ky}{(x^2+y^2)^{3/2}}\mathbf{j}$, for any constant $k > 0$

6. Given $x^2+y^2 = a^2+b^2$, let $x = \sqrt{a^2+b^2}\cos t$ and $y = -\sqrt{a^2+b^2}\sin t$. Then $\mathbf{r} = \left(\sqrt{a^2+b^2}\cos t\right)\mathbf{i} - \left(\sqrt{a^2+b^2}\sin t\right)\mathbf{j}$ traces the circle in a clockwise direction as t goes from 0 to 2π $\Rightarrow \mathbf{v} = \left(-\sqrt{a^2+b^2}\sin t\right)\mathbf{i} - \left(\sqrt{a^2+b^2}\cos t\right)\mathbf{j}$ is tangent to the circle in a clockwise direction. Thus, let $\mathbf{F} = \mathbf{v} \Rightarrow \mathbf{F} = y\mathbf{i} - x\mathbf{j}$ and $\mathbf{F}(0,0) = \mathbf{0}$.

7. Substitute the parametric representations for $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ representing each path into the vector field $\mathbf{F}$, and calculate $\int_C \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}$.
   (a) $\mathbf{F} = 3t\mathbf{i} + 2t\mathbf{j} + 4t\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i}+\mathbf{j}+\mathbf{k} \Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = 9t \Rightarrow \int_0^1 9t\,dt = \frac{9}{2}$
   (b) $\mathbf{F} = 3t^2\mathbf{i} + 2t\mathbf{j} + 4t^4\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 4t^3\mathbf{k} \Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = 7t^2 + 16t^7 \Rightarrow \int_0^1 (7t^2+16t^7)\,dt = \left[\frac{7}{3}t^3 + 2t^8\right]_0^1 = \frac{7}{3} + 2 = \frac{13}{3}$

(c) $\mathbf{r}_1 = t\mathbf{i} + t\mathbf{j}$ and $\mathbf{r}_2 = \mathbf{i} + \mathbf{j} + t\mathbf{k}$; $\mathbf{F}_1 = 3t\mathbf{i} + 2t\mathbf{j}$ and $\frac{d\mathbf{r}_1}{dt} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = 5t \Rightarrow \int_0^1 5t\,dt = \frac{5}{2}$;
$\mathbf{F}_2 = 3\mathbf{i} + 2\mathbf{j} + 4t\mathbf{k}$ and $\frac{d\mathbf{r}_2}{dt} = \mathbf{k} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = 4t \Rightarrow \int_0^1 4t\,dt = 2 \Rightarrow \frac{5}{2} + 2 = \frac{9}{2}$

8. Substitute the parametric representation for $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ representing each path into the vector field $\mathbf{F}$, and calculate $\int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}$.

(a) $\mathbf{F} = \left(\frac{1}{t^2+1}\right)\mathbf{j}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = \frac{1}{t^2+1} \Rightarrow \int_0^1 \frac{1}{t^2+1}\,dt = \left[\tan^{-1} t\right]_0^1 = \frac{\pi}{4}$

(b) $\mathbf{F} = \left(\frac{1}{t^2+1}\right)\mathbf{j}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 4t^3\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = \frac{2t}{t^2+1} \Rightarrow \int_0^1 \frac{2t}{t^2+1}\,dt = \left[\ln(t^2+1)\right]_0^1 = \ln 2$

(c) $\mathbf{r}_1 = t\mathbf{i} + t\mathbf{j}$ and $\mathbf{r}_2 = \mathbf{i} + \mathbf{j} + t\mathbf{k}$; $\mathbf{F}_1 = \left(\frac{1}{t^2+1}\right)\mathbf{j}$ and $\frac{d\mathbf{r}_1}{dt} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = \frac{1}{t^2+1}$; $\mathbf{F}_2 = \frac{1}{2}\mathbf{j}$ and $\frac{d\mathbf{r}_2}{dt} = \mathbf{k}$
$\Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = 0 \Rightarrow \int_0^1 \frac{1}{t^2+1}\,dt = \frac{\pi}{4}$

9. Substitute the parametric representation for $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ representing each path into the vector field $\mathbf{F}$, and calculate $\int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}$.

(a) $\mathbf{F} = \sqrt{t}\mathbf{i} - 2t\mathbf{j} + \sqrt{t}\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 2\sqrt{t} - 2t \Rightarrow \int_0^1 \left(2\sqrt{t} - 2t\right) dt = \left[\frac{4}{3}t^{3/2} - t^2\right]_0^1 = \frac{1}{3}$

(b) $\mathbf{F} = t^2\mathbf{i} - 2t\mathbf{j} + t\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 4t^3\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 4t^4 - 3t^2 \Rightarrow \int_0^1 \left(4t^4 - 3t^2\right) dt = \left[\frac{4}{5}t^5 - t^3\right]_0^1 = -\frac{1}{5}$

(c) $\mathbf{r}_1 = t\mathbf{i} + t\mathbf{j}$ and $\mathbf{r}_2 = \mathbf{i} + \mathbf{j} + t\mathbf{k}$; $\mathbf{F}_1 = -2t\mathbf{j} + \sqrt{t}\,\mathbf{k}$ and $\frac{d\mathbf{r}_1}{dt} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = -2t \Rightarrow \int_0^1 -2t\,dt = -1$;
$\mathbf{F}_2 = \sqrt{t}\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\frac{d\mathbf{r}_2}{dt} = \mathbf{k} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = 1 \Rightarrow \int_0^1 dt = 1 \Rightarrow -1 + 1 = 0$

10. Substitute the parametric representation for $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ representing each path into the vector field $\mathbf{F}$, and calculate $\int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}$.

(a) $\mathbf{F} = t^2\mathbf{i} + t^2\mathbf{j} + t^2\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 3t^2 \Rightarrow \int_0^1 3t^2\,dt = 1$

(b) $\mathbf{F} = t^3\mathbf{i} - t^6\mathbf{j} + t^5\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 4t^3\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = t^3 + 2t^7 + 4t^8 \Rightarrow \int_0^1 \left(t^3 + 2t^7 + 4t^8\right) dt$
$= \left[\frac{t^4}{4} + \frac{t^8}{4} + \frac{4}{9}t^9\right]_0^1 = \frac{17}{18}$

(c) $\mathbf{r}_1 = t\mathbf{i} + t\mathbf{j}$ and $\mathbf{r}_2 = \mathbf{i} + \mathbf{j} + t\mathbf{k}$; $\mathbf{F}_1 = t^2\mathbf{i}$ and $\frac{d\mathbf{r}_1}{dt} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = t^2 \Rightarrow \int_0^1 t^2\,dt = \frac{1}{3}$;
$\mathbf{F}_2 = \mathbf{i} + t\mathbf{j} + t\mathbf{k}$ and $\frac{d\mathbf{r}_2}{dt} = \mathbf{k} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = t \Rightarrow \int_0^1 t\,dt = \frac{1}{2} \Rightarrow \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$

11. Substitute the parametric representation for $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ representing each path into the vector field $\mathbf{F}$, and calculate $\int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}$.

(a) $\mathbf{F} = \left(3t^2 - 3t\right)\mathbf{i} + 3t\mathbf{j} + \mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 3t^2 + 1 \Rightarrow \int_0^1 \left(3t^2 + 1\right) dt = \left[t^3 + t\right]_0^1 = 2$

(b) $\mathbf{F} = \left(3t^2 - 3t\right)\mathbf{i} + 3t^4\mathbf{j} + \mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 4t^3\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 6t^5 + 4t^3 + 3t^2 - 3t$
$\Rightarrow \int_0^1 \left(6t^5 + 4t^3 + 3t^2 - 3t\right) dt = \left[t^6 + t^4 + t^3 - \frac{3}{2}t^2\right]_0^1 = \frac{3}{2}$

(c) $\mathbf{r}_1 = t\mathbf{i} + t\mathbf{j}$ and $\mathbf{r}_2 = \mathbf{i} + \mathbf{j} + t\mathbf{k}$; $\mathbf{F}_1 = \left(3t^2 - 3t\right)\mathbf{i} + \mathbf{k}$ and $\frac{d\mathbf{r}_1}{dt} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = 3t^2 - 3t$
$\Rightarrow \int_0^1 \left(3t^2 - 3t\right) dt = \left[t^3 - \frac{3}{2}t^2\right]_0^1 = -\frac{1}{2}$; $\mathbf{F}_2 = 3t\mathbf{j} + \mathbf{k}$ and $\frac{d\mathbf{r}_2}{dt} = \mathbf{k} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = 1 \Rightarrow \int_0^1 dt = 1$
$\Rightarrow -\frac{1}{2} + 1 = \frac{1}{2}$

12. Substitute the parametric representation for $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ representing each path into the vector field $\mathbf{F}$, and calculate $\int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}$.

(a) $\mathbf{F} = 2t\mathbf{i} + 2t\mathbf{j} + 2t\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 6t \Rightarrow \int_0^1 6t\,dt = \left[3t^2\right]_0^1 = 3$

(b) $\mathbf{F} = (t^2 + t^4)\,\mathbf{i} + (t^4 + t)\,\mathbf{j} + (t + t^2)\,\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 4t^3\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 6t^5 + 5t^4 + 3t^2$

$\Rightarrow \int_0^1 (6t^5 + 5t^4 + 3t^2)\,dt = \left[t^6 + t^5 + t^3\right]_0^1 = 3$

(c) $\mathbf{r}_1 = t\mathbf{i} + t\mathbf{j}$ and $\mathbf{r}_2 = \mathbf{i} + \mathbf{j} + t\mathbf{k}$; $\mathbf{F}_1 = t\mathbf{i} + t\mathbf{j} + 2t\mathbf{k}$ and $\frac{d\mathbf{r}_1}{dt} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = 2t \Rightarrow \int_0^1 2t\,dt = 1$;

$\mathbf{F}_2 = (1 + t)\mathbf{i} + (t + 1)\mathbf{j} + 2\mathbf{k}$ and $\frac{d\mathbf{r}_2}{dt} = \mathbf{k} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = 2 \Rightarrow \int_0^1 2\,dt = 2 \Rightarrow 1 + 2 = 3$

13. $x = t,\ y = 2t + 1,\ 0 \le t \le 3 \Rightarrow dx = dt \Rightarrow \int_C (x - y)\,dx = \int_0^3 (t - (2t + 1))\,dt = \int_0^3 (-t - 1)\,dt = \left[-\frac{1}{2}t^2 - t\right]_0^3 = -\frac{15}{2}$

14. $x = t,\ y = t^2,\ 1 \le t \le 2 \Rightarrow dy = 2t\,dt \Rightarrow \int_C \frac{x}{y}\,dy = \int_1^2 \frac{t}{t^2}(2t)\,dt = \int_1^2 2\,dt = [2t]_1^2 = 2$

15. C_1: $x = t,\ y = 0,\ 0 \le t \le 3 \Rightarrow dy = 0$; C_2: $x = 3,\ y = t,\ 0 \le t \le 3 \Rightarrow dy = dt \Rightarrow \int_C (x^2 + y^2)\,dy$

$= \int_{C_1} (x^2 + y^2)\,dx + \int_{C_2} (x^2 + y^2)\,dx = \int_0^3 (t^2 + 0^2) \cdot 0 + \int_0^3 (3^2 + t^2)\,dt = \int_0^3 (9 + t^2)dt = \left[9t + \frac{1}{3}t^3\right]_0^3 = 36$

16. C_1: $x = t,\ y = 3t,\ 0 \le t \le 1 \Rightarrow dx = dt$; C_2: $x = 1 - t,\ y = 3,\ 0 \le t \le 1 \Rightarrow dx = -dt$; C_3: $x = 0,\ y = 3 - t,\ 0 \le t \le 3$

$\Rightarrow dx = 0 \Rightarrow \int_C \sqrt{x + y}\,dx = \int_{C_1} \sqrt{x + y}\,dx + \int_{C_2} \sqrt{x + y}\,dx + \int_{C_3} \sqrt{x + y}\,dx$

$= \int_0^1 \sqrt{t + 3t}\,dt + \int_0^1 \sqrt{(1 - t) + 3}\,(-1)dt + \int_0^3 \sqrt{0 + (3 - t)} \cdot 0 = \int_0^1 2\sqrt{t}\,dt - \int_0^1 \sqrt{4 - t}\,dt$

$= \left[\frac{4}{3}t^{2/3}\right]_0^1 + \left[\frac{2}{3}(4 - t)^{2/3}\right]_0^1 = \frac{4}{3} + \left(2\sqrt{3} - \frac{16}{3}\right) = 2\sqrt{3} - 4$

17. $\mathbf{r}(t) = t\mathbf{i} - \mathbf{j} + t^2\mathbf{k},\ 0 \le t \le 1 \Rightarrow dx = dt,\ dy = 0,\ dz = 2t\,dt$

(a) $\int_C (x + y - z)\,dx = \int_0^1 (t - 1 - t^2)\,dt = \left[\frac{1}{2}t^2 - t - \frac{1}{3}t^3\right]_0^1 = -\frac{5}{6}$

(b) $\int_C (x + y - z)\,dy = \int_0^1 (t - 1 - t^2) \cdot 0 = 0$

(c) $\int_C (x + y - z)\,dz = \int_0^1 (t - 1 - t^2)\,2t\,dt = \int_0^1 (2t^2 - 2t - 2t^3)\,dt = = \left[\frac{2}{3}t^3 - t^2 - \frac{1}{2}t^4\right]_0^1 = -\frac{5}{6}$

18. $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} - (\cos t)\mathbf{k},\ 0 \le t \le \pi \Rightarrow dx = -\sin t\,dt,\ dy = \cos t\,dt,\ dz = \sin t\,dt$

(a) $\int_C xz\,dx = \int_0^\pi (\cos t)(-\cos t)(-\sin t)dt = \int_0^\pi \cos^2 t \sin t dt = \left[-\frac{1}{3}(\cos t)^3\right]_0^\pi = \frac{2}{3}$

(b) $\int_C xz\,dy = \int_0^\pi (\cos t)(-\cos t)(\cos t)dt = -\int_0^\pi \cos^3 t\,dt = -\int_0^\pi (1 - \sin^2 t)\cos t\,dt = \left[\frac{1}{3}(\sin t)^3 - \sin t\right]_0^\pi = 0$

(c) $\int_C xyz\,dz = \int_0^\pi (\cos t)(\sin t)(-\cos t)(\sin t)dt = -\int_0^\pi \cos^2 t \sin^2 t\,dt = -\frac{1}{4}\int_0^\pi \sin^2 2t\,dt = -\frac{1}{4}\int_0^\pi \frac{1 - \cos 4t}{2}\,dt$

$= -\frac{1}{8}\int_0^\pi (1 - \cos 4t)\,dt = \left[-\frac{1}{8}t + \frac{1}{32}\sin 4t\right]_0^\pi = -\frac{\pi}{8}$

19. $\mathbf{r} = t\mathbf{i} + t^2\mathbf{j} + t\mathbf{k},\ 0 \le t \le 1$, and $\mathbf{F} = xy\mathbf{i} + y\mathbf{j} - yz\mathbf{k} \Rightarrow \mathbf{F} = t^3\mathbf{i} + t^2\mathbf{j} - t^3\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + \mathbf{k}$

$\Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 2t^3 \Rightarrow \text{work} = \int_0^1 2t^3\,dt = \frac{1}{2}$

20. $\mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + \frac{t}{6}\mathbf{k},\ 0 \le t \le 2\pi$, and $\mathbf{F} = 2y\mathbf{i} + 3x\mathbf{j} + (x + y)\mathbf{k}$

$\Rightarrow \mathbf{F} = (2\sin t)\mathbf{i} + (3\cos t)\mathbf{j} + (\cos t + \sin t)\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \frac{1}{6}\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}$

$= 3\cos^2 t - 2\sin^2 t + \frac{1}{6}\cos t + \frac{1}{6}\sin t \Rightarrow \text{work} = \int_0^{2\pi} \left(3\cos^2 t - 2\sin^2 t + \frac{1}{6}\cos t + \frac{1}{6}\sin t\right) dt$

$= \left[\frac{3}{2}t + \frac{3}{4}\sin 2t - t + \frac{\sin 2t}{2} + \frac{1}{6}\sin t - \frac{1}{6}\cos t\right]_0^{2\pi} = \pi$

21. $\mathbf{r} = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} + t\mathbf{k}$, $0 \le t \le 2\pi$, and $\mathbf{F} = z\mathbf{i} + x\mathbf{j} + y\mathbf{k} \Rightarrow \mathbf{F} = t\mathbf{i} + (\sin t)\mathbf{j} + (\cos t)\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = t\cos t - \sin^2 t + \cos t \Rightarrow \text{work} = \int_0^{2\pi} (t\cos t - \sin^2 t + \cos t)\,dt$ $= \left[\cos t + t\sin t - \frac{t}{2} + \frac{\sin 2t}{4} + \sin t\right]_0^{2\pi} = -\pi$

22. $\mathbf{r} = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \frac{t}{6}\mathbf{k}$, $0 \le t \le 2\pi$, and $\mathbf{F} = 6z\mathbf{i} + y^2\mathbf{j} + 12x\mathbf{k} \Rightarrow \mathbf{F} = t\mathbf{i} + (\cos^2 t)\mathbf{j} + (12\sin t)\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j} + \frac{1}{6}\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = t\cos t - \sin t\cos^2 t + 2\sin t$ $\Rightarrow \text{work} = \int_0^{2\pi} (t\cos t - \sin t\cos^2 t + 2\sin t)\,dt = \left[\cos t + t\sin t + \frac{1}{3}\cos^3 t - 2\cos t\right]_0^{2\pi} = 0$

23. $x = t$ and $y = x^2 = t^2 \Rightarrow \mathbf{r} = t\mathbf{i} + t^2\mathbf{j}$, $-1 \le t \le 2$, and $\mathbf{F} = xy\mathbf{i} + (x+y)\mathbf{j} \Rightarrow \mathbf{F} = t^3\mathbf{i} + (t+t^2)\mathbf{j}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = t^3 + (2t^2 + 2t^3) = 3t^3 + 2t^2 \Rightarrow \int_C xy\,dx + (x+y)\,dy = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\,dt = \int_{-1}^{2} (3t^3 + 2t^2)\,dt$ $= \left[\frac{3}{4}t^4 + \frac{2}{3}t^3\right]_{-1}^{2} = \left(12 + \frac{16}{3}\right) - \left(\frac{3}{4} - \frac{2}{3}\right) = \frac{45}{4} + \frac{18}{3} = \frac{69}{4}$

24. Along $(0,0)$ to $(1,0)$: $\mathbf{r} = t\mathbf{i}$, $0 \le t \le 1$, and $\mathbf{F} = (x-y)\mathbf{i} + (x+y)\mathbf{j} \Rightarrow \mathbf{F} = t\mathbf{i} + t\mathbf{j}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = t$;
Along $(1,0)$ to $(0,1)$: $\mathbf{r} = (1-t)\mathbf{i} + t\mathbf{j}$, $0 \le t \le 1$, and $\mathbf{F} = (x-y)\mathbf{i} + (x+y)\mathbf{j} \Rightarrow \mathbf{F} = (1-2t)\mathbf{i} + \mathbf{j}$ and $\frac{d\mathbf{r}}{dt} = -\mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 2t$;
Along $(0,1)$ to $(0,0)$: $\mathbf{r} = (1-t)\mathbf{j}$, $0 \le t \le 1$, and $\mathbf{F} = (x-y)\mathbf{i} + (x+y)\mathbf{j} \Rightarrow \mathbf{F} = (t-1)\mathbf{i} + (1-t)\mathbf{j}$ and $\frac{d\mathbf{r}}{dt} = -\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = t - 1 \Rightarrow \int_C (x-y)\,dx + (x+y)\,dy = \int_0^1 t\,dt + \int_0^1 2t\,dt + \int_0^1 (t-1)\,dt = \int_0^1 (4t-1)\,dt$ $= [2t^2 - t]_0^1 = 2 - 1 = 1$

25. $\mathbf{r} = x\mathbf{i} + y\mathbf{j} = y^2\mathbf{i} + y\mathbf{j}$, $2 \ge y \ge -1$, and $\mathbf{F} = x^2\mathbf{i} - y\mathbf{j} = y^4\mathbf{i} - y\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dy} = 2y\mathbf{i} + \mathbf{j}$ and $\mathbf{F} \cdot \frac{d\mathbf{r}}{dy} = 2y^5 - y$ $\Rightarrow \int_C \mathbf{F} \cdot \mathbf{T}\,ds = \int_2^{-1} \mathbf{F} \cdot \frac{d\mathbf{r}}{dy}\,dy = \int_2^{-1} (2y^5 - y)\,dy = \left[\frac{1}{3}y^6 - \frac{1}{2}y^2\right]_2^{-1} = \left(\frac{1}{3} - \frac{1}{2}\right) - \left(\frac{64}{3} - \frac{4}{2}\right) = \frac{3}{2} - \frac{63}{3} = -\frac{39}{2}$

26. $\mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$, $0 \le t \le \frac{\pi}{2}$, and $\mathbf{F} = y\mathbf{i} - x\mathbf{j} \Rightarrow \mathbf{F} = (\sin t)\mathbf{i} - (\cos t)\mathbf{j}$ and $\frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$ $\Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -\sin^2 t - \cos^2 t = -1 \Rightarrow \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} (-1)\,dt = -\frac{\pi}{2}$

27. $\mathbf{r} = (\mathbf{i} + \mathbf{j}) + t(\mathbf{i} + 2\mathbf{j}) = (1+t)\mathbf{i} + (1+2t)\mathbf{j}$, $0 \le t \le 1$, and $\mathbf{F} = xy\mathbf{i} + (y-x)\mathbf{j} \Rightarrow \mathbf{F} = (1 + 3t + 2t^2)\mathbf{i} + t\mathbf{j}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 1 + 5t + 2t^2 \Rightarrow \text{work} = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\,dt = \int_0^1 (1 + 5t + 2t^2)\,dt = \left[t + \frac{5}{2}t^2 + \frac{2}{3}t^3\right]_0^1 = \frac{25}{6}$

28. $\mathbf{r} = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j}$, $0 \le t \le 2\pi$, and $\mathbf{F} = \nabla f = 2(x+y)\mathbf{i} + 2(x+y)\mathbf{j}$ $\Rightarrow \mathbf{F} = 4(\cos t + \sin t)\mathbf{i} + 4(\cos t + \sin t)\mathbf{j}$ and $\frac{d\mathbf{r}}{dt} = (-2\sin t)\mathbf{i} + (2\cos t)\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}$ $= -8(\sin t\cos t + \sin^2 t) + 8(\cos^2 t + \cos t\sin t) = 8(\cos^2 t - \sin^2 t) = 8\cos 2t \Rightarrow \text{work} = \int_C \nabla f \cdot d\mathbf{r}$ $= \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\,dt = \int_0^{2\pi} 8\cos 2t\,dt = [4\sin 2t]_0^{2\pi} = 0$

29. (a) $\mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$, $0 \le t \le 2\pi$, $\mathbf{F}_1 = x\mathbf{i} + y\mathbf{j}$, and $\mathbf{F}_2 = -y\mathbf{i} + x\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$, $\mathbf{F}_1 = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$, and $\mathbf{F}_2 = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}}{dt} = 0$ and $\mathbf{F}_2 \cdot \frac{d\mathbf{r}}{dt} = \sin^2 t + \cos^2 t = 1$ $\Rightarrow \text{Circ}_1 = \int_0^{2\pi} 0\,dt = 0$ and $\text{Circ}_2 = \int_0^{2\pi} dt = 2\pi$; $\mathbf{n} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \mathbf{n} = \cos^2 t + \sin^2 t = 1$ and $\mathbf{F}_2 \cdot \mathbf{n} = 0 \Rightarrow \text{Flux}_1 = \int_0^{2\pi} dt = 2\pi$ and $\text{Flux}_2 = \int_0^{2\pi} 0\,dt = 0$

(b) $\mathbf{r} = (\cos t)\mathbf{i} + (4\sin t)\mathbf{j}$, $0 \le t \le 2\pi \Rightarrow \frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (4\cos t)\mathbf{j}$, $\mathbf{F}_1 = (\cos t)\mathbf{i} + (4\sin t)\mathbf{j}$, and $\mathbf{F}_2 = (-4\sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}}{dt} = 15\sin t\cos t$ and $\mathbf{F}_2 \cdot \frac{d\mathbf{r}}{dt} = 4 \Rightarrow \text{Circ}_1 = \int_0^{2\pi} 15\sin t\cos t\,dt$ $= \left[\frac{15}{2}\sin^2 t\right]_0^{2\pi} = 0$ and $\text{Circ}_2 = \int_0^{2\pi} 4\,dt = 8\pi$; $\mathbf{n} = \left(\frac{4}{\sqrt{17}}\cos t\right)\mathbf{i} + \left(\frac{1}{\sqrt{17}}\sin t\right)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \mathbf{n}$

$= \frac{4}{\sqrt{17}}\cos^2 t + \frac{4}{\sqrt{17}}\sin^2 t$ and $\mathbf{F}_2 \cdot \mathbf{n} = -\frac{15}{\sqrt{17}}\sin t\cos t \Rightarrow \text{Flux}_1 = \int_0^{2\pi} (\mathbf{F}_1 \cdot \mathbf{n})\,|\mathbf{v}|\,dt = \int_0^{2\pi} \left(\frac{4}{\sqrt{17}}\right)\sqrt{17}\,dt$
$= 8\pi$ and $\text{Flux}_2 = \int_0^{2\pi} (\mathbf{F}_2 \cdot \mathbf{n})\,|\mathbf{v}|\,dt = \int_0^{2\pi}\left(-\frac{15}{\sqrt{17}}\sin t\cos t\right)\sqrt{17}\,dt = \left[-\frac{15}{2}\sin^2 t\right]_0^{2\pi} = 0$

30. $\mathbf{r} = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j}$, $0 \le t \le 2\pi$, $\mathbf{F}_1 = 2x\mathbf{i} - 3y\mathbf{j}$, and $\mathbf{F}_2 = 2x\mathbf{i} + (x-y)\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dt} = (-a\sin t)\mathbf{i} + (a\cos t)\mathbf{j}$,
$\mathbf{F}_1 = (2a\cos t)\mathbf{i} - (3a\sin t)\mathbf{j}$, and $\mathbf{F}_2 = (2a\cos t)\mathbf{i} + (a\cos t - a\sin t)\mathbf{j} \Rightarrow \mathbf{n}\,|\mathbf{v}| = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j}$,
$\mathbf{F}_1 \cdot \mathbf{n}\,|\mathbf{v}| = 2a^2\cos^2 t - 3a^2\sin^2 t$, and $\mathbf{F}_2 \cdot \mathbf{n}\,|\mathbf{v}| = 2a^2\cos^2 t + a^2\sin t\cos t - a^2\sin^2 t$
$\Rightarrow \text{Flux}_1 = \int_0^{2\pi}(2a^2\cos^2 t - 3a^2\sin^2 t)\,dt = 2a^2\left[\frac{t}{2} + \frac{\sin 2t}{4}\right]_0^{2\pi} - 3a^2\left[\frac{t}{2} - \frac{\sin 2t}{4}\right]_0^{2\pi} = -\pi a^2$, and
$\text{Flux}_2 = \int_0^{2\pi}(2a^2\cos^2 t - a^2\sin t\cos t - a^2\sin^2 t)\,dt = 2a^2\left[\frac{t}{2} + \frac{\sin 2t}{4}\right]_0^{2\pi} + \frac{a^2}{2}\left[\sin^2 t\right]_0^{2\pi} - a^2\left[\frac{t}{2} - \frac{\sin 2t}{4}\right]_0^{2\pi} = \pi a^2$

31. $\mathbf{F}_1 = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j}$, $\frac{d\mathbf{r}_1}{dt} = (-a\sin t)\mathbf{i} + (a\cos t)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = 0 \Rightarrow \text{Circ}_1 = 0$; $M_1 = a\cos t$,
$N_1 = a\sin t$, $dx = -a\sin t\,dt$, $dy = a\cos t\,dt \Rightarrow \text{Flux}_1 = \int_C M_1\,dy - N_1\,dx = \int_0^{\pi}(a^2\cos^2 t + a^2\sin^2 t)\,dt$
$= \int_0^{\pi} a^2\,dt = a^2\pi$;
$\mathbf{F}_2 = t\mathbf{i}$, $\frac{d\mathbf{r}_2}{dt} = \mathbf{i} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = t \Rightarrow \text{Circ}_2 = \int_{-a}^{a} t\,dt = 0$; $M_2 = t$, $N_2 = 0$, $dx = dt$, $dy = 0 \Rightarrow \text{Flux}_2$
$= \int_C M_2\,dy - N_2\,dx = \int_{-a}^{a} 0\,dt = 0$; therefore, $\text{Circ} = \text{Circ}_1 + \text{Circ}_2 = 0$ and $\text{Flux} = \text{Flux}_1 + \text{Flux}_2 = a^2\pi$

32. $\mathbf{F}_1 = (a^2\cos^2 t)\,\mathbf{i} + (a^2\sin^2 t)\,\mathbf{j}$, $\frac{d\mathbf{r}_1}{dt} = (-a\sin t)\mathbf{i} + (a\cos t)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = -a^3\sin t\cos^2 t + a^3\cos t\sin^2 t$
$\Rightarrow \text{Circ}_1 = \int_0^{\pi}(-a^3\sin t\cos^2 t + a^3\cos t\sin^2 t)\,dt = -\frac{2a^3}{3}$; $M_1 = a^2\cos^2 t$, $N_1 = a^2\sin^2 t$, $dy = a\cos t\,dt$,
$dx = -a\sin t\,dt \Rightarrow \text{Flux}_1 = \int_C M_1\,dy - N_1\,dx = \int_0^{\pi}(a^3\cos^3 t + a^3\sin^3 t)\,dt = \frac{4}{3}a^3$;
$\mathbf{F}_2 = t^2\mathbf{i}$, $\frac{d\mathbf{r}_2}{dt} = \mathbf{i} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = t^2 \Rightarrow \text{Circ}_2 = \int_{-a}^{a} t^2\,dt = \frac{2a^3}{3}$; $M_2 = t^2$, $N_2 = 0$, $dy = 0$, $dx = dt$
$\Rightarrow \text{Flux}_2 = \int_C M_2\,dy - N_2\,dx = 0$; therefore, $\text{Circ} = \text{Circ}_1 + \text{Circ}_2 = 0$ and $\text{Flux} = \text{Flux}_1 + \text{Flux}_2 = \frac{4}{3}a^3$

33. $\mathbf{F}_1 = (-a\sin t)\mathbf{i} + (a\cos t)\mathbf{j}$, $\frac{d\mathbf{r}_1}{dt} = (-a\sin t)\mathbf{i} + (a\cos t)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = a^2\sin^2 t + a^2\cos^2 t = a^2$
$\Rightarrow \text{Circ}_1 = \int_0^{\pi} a^2\,dt = a^2\pi$; $M_1 = -a\sin t$, $N_1 = a\cos t$, $dx = -a\sin t\,dt$, $dy = a\cos t\,dt$
$\Rightarrow \text{Flux}_1 = \int_C M_1\,dy - N_1\,dx = \int_0^{\pi}(-a^2\sin t\cos t + a^2\sin t\cos t)\,dt = 0$; $\mathbf{F}_2 = t\mathbf{j}$, $\frac{d\mathbf{r}_2}{dt} = \mathbf{i} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = 0$
$\Rightarrow \text{Circ}_2 = 0$; $M_2 = 0$, $N_2 = t$, $dx = dt$, $dy = 0 \Rightarrow \text{Flux}_2 = \int_C M_2\,dy - N_2\,dx = \int_{-a}^{a} -t\,dt = 0$; therefore,
$\text{Circ} = \text{Circ}_1 + \text{Circ}_2 = a^2\pi$ and $\text{Flux} = \text{Flux}_1 + \text{Flux}_2 = 0$

34. $\mathbf{F}_1 = (-a^2\sin^2 t)\,\mathbf{i} + (a^2\cos^2 t)\,\mathbf{j}$, $\frac{d\mathbf{r}_1}{dt} = (-a\sin t)\mathbf{i} + (a\cos t)\mathbf{j} \Rightarrow \mathbf{F}_1 \cdot \frac{d\mathbf{r}_1}{dt} = a^3\sin^3 t + a^3\cos^3 t$
$\Rightarrow \text{Circ}_1 = \int_0^{\pi}(a^3\sin^3 t + a^3\cos^3 t)\,dt = \frac{4}{3}a^3$; $M_1 = -a^2\sin^2 t$, $N_1 = a^2\cos^2 t$, $dy = a\cos t\,dt$, $dx = -a\sin t\,dt$
$\Rightarrow \text{Flux}_1 = \int_C M_1\,dy - N_1\,dx = \int_0^{\pi}(-a^3\cos t\sin^2 t + a^3\sin t\cos^2 t)\,dt = \frac{2}{3}a^3$; $\mathbf{F}_2 = t^2\mathbf{j}$, $\frac{d\mathbf{r}_2}{dt} = \mathbf{i} \Rightarrow \mathbf{F}_2 \cdot \frac{d\mathbf{r}_2}{dt} = 0$
$\Rightarrow \text{Circ}_2 = 0$; $M_2 = 0$, $N_2 = t^2$, $dy = 0$, $dx = dt \Rightarrow \text{Flux}_2 = \int_C M_2\,dy - N_2\,dx = \int_{-a}^{a} -t^2\,dt = -\frac{2}{3}a^3$; therefore,
$\text{Circ} = \text{Circ}_1 + \text{Circ}_2 = \frac{4}{3}a^3$ and $\text{Flux} = \text{Flux}_1 + \text{Flux}_2 = 0$

35. (a) $\mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$, $0 \le t \le \pi$, and $\mathbf{F} = (x+y)\mathbf{i} - (x^2+y^2)\,\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$ and
$\mathbf{F} = (\cos t + \sin t)\mathbf{i} - (\cos^2 t + \sin^2 t)\,\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -\sin t\cos t - \sin^2 t - \cos t \Rightarrow \int_C \mathbf{F} \cdot \mathbf{T}\,ds$
$= \int_0^{\pi}(-\sin t\cos t - \sin^2 t - \cos t)\,dt = \left[-\frac{1}{2}\sin^2 t - \frac{t}{2} + \frac{\sin 2t}{4} - \sin t\right]_0^{\pi} = -\frac{\pi}{2}$
(b) $\mathbf{r} = (1-2t)\mathbf{i}$, $0 \le t \le 1$, and $\mathbf{F} = (x+y)\mathbf{i} - (x^2+y^2)\,\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dt} = -2\mathbf{i}$ and $\mathbf{F} = (1-2t)\mathbf{i} - (1-2t)^2\,\mathbf{j} \Rightarrow$
$\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 4t - 2 \Rightarrow \int_C \mathbf{F} \cdot \mathbf{T}\,ds = \int_0^1 (4t-2)\,dt = \left[2t^2 - 2t\right]_0^1 = 0$

(c) $\mathbf{r}_1 = (1-t)\mathbf{i} - t\mathbf{j}$, $0 \le t \le 1$, and $\mathbf{F} = (x+y)\mathbf{i} - (x^2+y^2)\mathbf{j} \Rightarrow \frac{d\mathbf{r}_1}{dt} = -\mathbf{i} - \mathbf{j}$ and $\mathbf{F} = (1-2t)\mathbf{i} - (1-2t+2t^2)\mathbf{j}$
$\Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}_1}{dt} = (2t-1) + (1-2t+2t^2) = 2t^2 \Rightarrow \text{Flow}_1 = \int_{C_1} \mathbf{F} \cdot \frac{d\mathbf{r}_1}{dt} = \int_0^1 2t^2\, dt = \frac{2}{3}$; $\mathbf{r}_2 = -t\mathbf{i} + (t-1)\mathbf{j}$,
$0 \le t \le 1$, and $\mathbf{F} = (x+y)\mathbf{i} - (x^2+y^2)\mathbf{j} \Rightarrow \frac{d\mathbf{r}_2}{dt} = -\mathbf{i} + \mathbf{j}$ and $\mathbf{F} = -\mathbf{i} - (t^2+t^2-2t+1)\mathbf{j}$
$= -\mathbf{i} - (2t^2-2t+1)\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}_2}{dt} = 1 - (2t^2-2t+1) = 2t - 2t^2 \Rightarrow \text{Flow}_2 = \int_{C_2} \mathbf{F} \cdot \frac{d\mathbf{r}_2}{dt} = \int_0^1 (2t-2t^2)\, dt$
$= \left[t^2 - \frac{2}{3}t^3\right]_0^1 = \frac{1}{3} \Rightarrow \text{Flow} = \text{Flow}_1 + \text{Flow}_2 = \frac{2}{3} + \frac{1}{3} = 1$

36. From $(1,0)$ to $(0,1)$: $\mathbf{r}_1 = (1-t)\mathbf{i} + t\mathbf{j}$, $0 \le t \le 1$, and $\mathbf{F} = (x+y)\mathbf{i} - (x^2+y^2)\mathbf{j} \Rightarrow \frac{d\mathbf{r}_1}{dt} = -\mathbf{i} + \mathbf{j}$,
$\mathbf{F} = \mathbf{i} - (1-2t+2t^2)\mathbf{j}$, and $\mathbf{n}_1\,|\mathbf{v}_1| = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F} \cdot \mathbf{n}_1\,|\mathbf{v}_1| = 2t - 2t^2 \Rightarrow \text{Flux}_1 = \int_0^1 (2t-2t^2)\, dt$
$= \left[t^2 - \frac{2}{3}t^3\right]_0^1 = \frac{1}{3}$;
From $(0,1)$ to $(-1,0)$: $\mathbf{r}_2 = -t\mathbf{i} + (1-t)\mathbf{j}$, $0 \le t \le 1$, and $\mathbf{F} = (x+y)\mathbf{i} - (x^2+y^2)\mathbf{j} \Rightarrow \frac{d\mathbf{r}_2}{dt} = -\mathbf{i} - \mathbf{j}$,
$\mathbf{F} = (1-2t)\mathbf{i} - (1-2t+2t^2)\mathbf{j}$, and $\mathbf{n}_2\,|\mathbf{v}_2| = -\mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F} \cdot \mathbf{n}_2\,|\mathbf{v}_2| = (2t-1) + (-1+2t-2t^2) = -2 + 4t - 2t^2$
$\Rightarrow \text{Flux}_2 = \int_0^1 (-2+4t-2t^2)\, dt = \left[-2t + 2t^2 - \frac{2}{3}t^3\right]_0^1 = -\frac{2}{3}$;
From $(-1,0)$ to $(1,0)$: $\mathbf{r}_3 = (-1+2t)\mathbf{i}$, $0 \le t \le 1$, and $\mathbf{F} = (x+y)\mathbf{i} - (x^2+y^2)\mathbf{j} \Rightarrow \frac{d\mathbf{r}_3}{dt} = 2\mathbf{i}$,
$\mathbf{F} = (-1+2t)\mathbf{i} - (1-4t+4t^2)\mathbf{j}$, and $\mathbf{n}_3\,|\mathbf{v}_3| = -2\mathbf{j} \Rightarrow \mathbf{F} \cdot \mathbf{n}_3\,|\mathbf{v}_3| = 2(1-4t+4t^2)$
$\Rightarrow \text{Flux}_3 = 2\int_0^1 (1-4t+4t^2)\, dt = 2\left[t - 2t^2 + \frac{4}{3}t^3\right]_0^1 = \frac{2}{3} \Rightarrow \text{Flux} = \text{Flux}_1 + \text{Flux}_2 + \text{Flux}_3 = \frac{1}{3} - \frac{2}{3} + \frac{2}{3} = \frac{1}{3}$

37. (a) $y = 2x,\ 0 \le x \le 2 \Rightarrow \mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j},\ 0 \le t \le 2 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = \left((2t)^2\mathbf{i} + 2(t)(2t)\mathbf{j}\right) \cdot (\mathbf{i} + 2\mathbf{j})$
$= 4t^2 + 8t^2 = 12t^2 \Rightarrow \text{Flow} = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt = \int_0^2 12t^2\, dt = \left[4t^3\right]_0^2 = 32$

(b) $y = x^2,\ 0 \le x \le 2 \Rightarrow \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j},\ 0 \le t \le 2 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = \left((t^2)^2\mathbf{i} + 2(t)(t^2)\mathbf{j}\right) \cdot (\mathbf{i} + 2t\mathbf{j})$
$= t^4 + 4t^4 = 5t^4 \Rightarrow \text{Flow} = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt = \int_0^2 5t^4\, dt = \left[t^5\right]_0^2 = 32$

(c) answers will vary, one possible path is $y = \frac{1}{2}x^3,\ 0 \le x \le 2 \Rightarrow \mathbf{r}(t) = t\mathbf{i} + \frac{1}{2}t^3\mathbf{j},\ 0 \le t \le 2 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + 3t^2\mathbf{j}$
$\Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = \left(\left(\frac{1}{2}t^3\right)^2\mathbf{i} + 2(t)\left(\frac{1}{2}t^3\right)\mathbf{j}\right) \cdot (\mathbf{i} + 3t^2\mathbf{j}) = \frac{1}{4}t^6 + \frac{3}{2}t^6 = \frac{7}{4}t^6 \Rightarrow \text{Flow} = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt = \int_0^2 \frac{7}{4}t^6\, dt = \left[\frac{1}{4}t^7\right]_0^2$
$= 32$

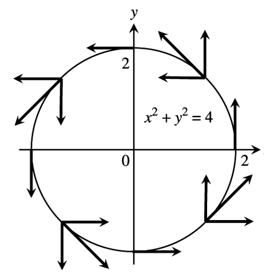
38. (a) C_1: $\mathbf{r}(t) = (1-t)\mathbf{i} + \mathbf{j},\ 0 \le t \le 2 \Rightarrow \frac{d\mathbf{r}}{dt} = -\mathbf{i} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = ((1)\mathbf{i} + ((1-t) + 2(1))\mathbf{j}) \cdot (-\mathbf{i}) = -1$;
C_2: $\mathbf{r}(t) = -\mathbf{i} + (1-t)\mathbf{j},\ 0 \le t \le 2 \Rightarrow \frac{d\mathbf{r}}{dt} = -\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = ((1-t)\mathbf{i} + ((-1) + 2(1-t))\mathbf{j}) \cdot (-\mathbf{j}) = 2t - 1$;
C_3: $\mathbf{r}(t) = (t-1)\mathbf{i} - \mathbf{j},\ 0 \le t \le 2 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = ((-1)\mathbf{i} + ((t-1) + 2(-1))\mathbf{j}) \cdot (\mathbf{i}) = -1$;
C_4: $\mathbf{r}(t) = \mathbf{i} + (t-1)\mathbf{j},\ 0 \le t \le 2 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = ((t-1)\mathbf{i} + ((1) + 2(t-1))\mathbf{j}) \cdot (\mathbf{j}) = 2t - 1$;
$\Rightarrow \text{Flow} = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt = \int_{C_1} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt + \int_{C_2} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt + \int_{C_3} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt + \int_{C_4} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt$
$= \int_0^2 (-1)\, dt + \int_0^2 (2t-1)\, dt + \int_0^2 (-1)\, dt + \int_0^2 (2t-1)\, dt = \left[-t\right]_0^2 + \left[t^2 - t\right]_0^2 + \left[-t\right]_0^2 + \left[t^2 - t\right]_0^2$
$= -2 + 2 - 2 + 2 = 0$

(b) $x^2 + y^2 = 4 \Rightarrow \mathbf{r}(t) = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j},\ 0 \le t \le 2\pi \Rightarrow \frac{d\mathbf{r}}{dt} = (-2\sin t)\mathbf{i} + (2\cos t)\mathbf{j}$
$\Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = ((2\sin t)\mathbf{i} + (2\cos t + 2(2\sin t))\mathbf{j}) \cdot ((-2\sin t)\mathbf{i} + (2\cos t)\mathbf{j}) = -4\sin^2 t + 4\cos^2 t + 8\sin t\cos t$
$= 4\cos 2t + 4\sin 2t \Rightarrow \text{Flow} = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt = \int_0^{2\pi} (4\cos 2t + 4\sin 2t)\, dt = \left[2\sin 2t - 2\cos 2t\right]_0^{2\pi} = 0$

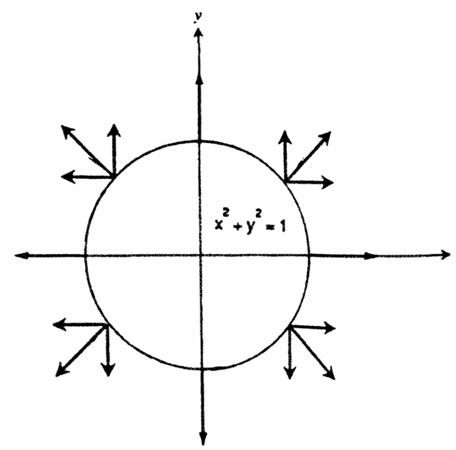
(c) answers will vary, one possible path is:
C_1: $\mathbf{r}(t) = t\mathbf{i},\ 0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = ((0)\mathbf{i} + (t + 2(1))\mathbf{j}) \cdot (\mathbf{i}) = 0$;
C_2: $\mathbf{r}(t) = (1-t)\mathbf{i} + t\mathbf{j},\ 0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = -\mathbf{i} + \mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = (t\mathbf{i} + ((1-t) + 2t)\mathbf{j}) \cdot (-\mathbf{i} + \mathbf{j}) = 1$;
C_3: $\mathbf{r}(t) = (1-t)\mathbf{j},\ 0 \le t \le 1 \Rightarrow \frac{d\mathbf{r}}{dt} = -\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = ((1-t)\mathbf{i} + (0 + 2(1-t))\mathbf{j}) \cdot (-\mathbf{j}) = 2t - 1$;

$\Rightarrow \text{Flow} = \int_C \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\,dt = \int_{C_1} \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\,dt + \int_{C_2} \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\,dt + \int_{C_3} \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\,dt = \int_0^1 (0)\,dt + \int_0^1 (1)\,dt + \int_0^1 (2t-1)\,dt$
$= 0 + [t]_0^1 + [t^2 - t]_0^1 = 1 + (-1) = 0$

39. $\mathbf{F} = -\frac{y}{\sqrt{x^2+y^2}}\mathbf{i} + \frac{x}{\sqrt{x^2+y^2}}\mathbf{j}$ on $x^2+y^2=4$;
at $(2,0)$, $\mathbf{F} = \mathbf{j}$; at $(0,2)$, $\mathbf{F} = -\mathbf{i}$; at $(-2,0)$,
$\mathbf{F} = -\mathbf{j}$; at $(0,-2)$, $\mathbf{F} = \mathbf{i}$; at $\left(\sqrt{2},\sqrt{2}\right)$, $\mathbf{F} = -\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$;
at $\left(\sqrt{2},-\sqrt{2}\right)$, $\mathbf{F} = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$; at $\left(-\sqrt{2},\sqrt{2}\right)$,
$\mathbf{F} = -\frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$; at $\left(-\sqrt{2},-\sqrt{2}\right)$, $\mathbf{F} = \frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$



40. $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ on $x^2+y^2=1$; at $(1,0)$, $\mathbf{F} = \mathbf{i}$;
at $(-1,0)$, $\mathbf{F} = -\mathbf{i}$; at $(0,1)$, $\mathbf{F} = \mathbf{j}$; at $(0,-1)$,
$\mathbf{F} = -\mathbf{j}$; at $\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$, $\mathbf{F} = \frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$;
at $\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)$, $\mathbf{F} = -\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$;
at $\left(\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$, $\mathbf{F} = \frac{1}{2}\mathbf{i} - \frac{\sqrt{3}}{2}\mathbf{j}$;
at $\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$, $\mathbf{F} = -\frac{1}{2}\mathbf{i} - \frac{\sqrt{3}}{2}\mathbf{j}$.



41. (a) $\mathbf{G} = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ is to have a magnitude $\sqrt{a^2+b^2}$ and to be tangent to $x^2+y^2 = a^2+b^2$ in a counterclockwise direction. Thus $x^2+y^2 = a^2+b^2 \Rightarrow 2x + 2yy' = 0 \Rightarrow y' = -\frac{x}{y}$ is the slope of the tangent line at any point on the circle $\Rightarrow y' = -\frac{a}{b}$ at (a,b). Let $\mathbf{v} = -b\mathbf{i} + a\mathbf{j} \Rightarrow |\mathbf{v}| = \sqrt{a^2+b^2}$, with $\mathbf{v}$ in a counterclockwise direction and tangent to the circle. Then let $P(x,y) = -y$ and $Q(x,y) = x$ $\Rightarrow \mathbf{G} = -y\mathbf{i} + x\mathbf{j} \Rightarrow$ for (a,b) on $x^2+y^2 = a^2+b^2$ we have $\mathbf{G} = -b\mathbf{i} + a\mathbf{j}$ and $|\mathbf{G}| = \sqrt{a^2+b^2}$.

(b) $\mathbf{G} = \left(\sqrt{x^2+y^2}\right)\mathbf{F} = \left(\sqrt{a^2+b^2}\right)\mathbf{F}$.

42. (a) From Exercise 41, part a, $-y\mathbf{i} + x\mathbf{j}$ is a vector tangent to the circle and pointing in a counterclockwise direction $\Rightarrow$ $y\mathbf{i} - x\mathbf{j}$ is a vector tangent to the circle pointing in a clockwise direction $\Rightarrow \mathbf{G} = \frac{y\mathbf{i} - x\mathbf{j}}{\sqrt{x^2+y^2}}$ is a unit vector tangent to the circle and pointing in a clockwise direction.

(b) $\mathbf{G} = -\mathbf{F}$

43. The slope of the line through (x,y) and the origin is $\frac{y}{x} \Rightarrow \mathbf{v} = x\mathbf{i} + y\mathbf{j}$ is a vector parallel to that line and pointing away from the origin $\Rightarrow \mathbf{F} = -\frac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2+y^2}}$ is the unit vector pointing toward the origin.

44. (a) From Exercise 43, $-\frac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2+y^2}}$ is a unit vector through (x,y) pointing toward the origin and we want $|\mathbf{F}|$ to have magnitude $\sqrt{x^2+y^2} \Rightarrow \mathbf{F} = \sqrt{x^2+y^2}\left(-\frac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2+y^2}}\right) = -x\mathbf{i} - y\mathbf{j}$.

(b) We want $|\mathbf{F}| = \frac{C}{\sqrt{x^2+y^2}}$ where $C \neq 0$ is a constant $\Rightarrow \mathbf{F} = \frac{C}{\sqrt{x^2+y^2}}\left(-\frac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2+y^2}}\right) = -C\left(\frac{x\mathbf{i} + y\mathbf{j}}{x^2+y^2}\right)$.

45. Yes. The work and area have the same numerical value because work $= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C y\mathbf{i} \cdot d\mathbf{r}$

$= \int_b^a [f(t)\mathbf{i}] \cdot \left[\mathbf{i} + \frac{df}{dt}\mathbf{j}\right] dt$ [On the path, y equals f(t)]

$= \int_a^b f(t)\,dt =$ Area under the curve [because $f(t) > 0$]

46. $\mathbf{r} = x\mathbf{i} + y\mathbf{j} = x\mathbf{i} + f(x)\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dx} = \mathbf{i} + f'(x)\mathbf{j}$; $\mathbf{F} = \frac{k}{\sqrt{x^2+y^2}}(x\mathbf{i} + y\mathbf{j})$ has constant magnitude k and points away from the origin $\Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dx} = \frac{kx}{\sqrt{x^2+y^2}} + \frac{k \cdot y \cdot f'(x)}{\sqrt{x^2+y^2}} = \frac{kx + k \cdot f(x) \cdot f'(x)}{\sqrt{x^2+[f(x)]^2}} = k\,\frac{d}{dx}\sqrt{x^2+[f(x)]^2}$, by the chain rule

$\Rightarrow \int_C \mathbf{F} \cdot \mathbf{T}\,ds = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dx}\,dx = \int_a^b k\,\frac{d}{dx}\sqrt{x^2+[f(x)]^2}\,dx = k\left[\sqrt{x^2+[f(x)]^2}\right]_a^b$

$= k\left(\sqrt{b^2+[f(b)]^2} - \sqrt{a^2+[f(a)]^2}\right)$, as claimed.

47. $\mathbf{F} = -4t^3\mathbf{i} + 8t^2\mathbf{j} + 2\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 12t^3 \Rightarrow \text{Flow} = \int_0^2 12t^3\,dt = \left[3t^4\right]_0^2 = 48$

48. $\mathbf{F} = 12t^2\mathbf{j} + 9t^2\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = 3\mathbf{j} + 4\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 72t^2 \Rightarrow \text{Flow} = \int_0^1 72t^2\,dt = \left[24t^3\right]_0^1 = 24$

49. $\mathbf{F} = (\cos t - \sin t)\mathbf{i} + (\cos t)\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -\sin t \cos t + 1$

$\Rightarrow \text{Flow} = \int_0^{\pi} (-\sin t \cos t + 1)\,dt = \left[\frac{1}{2}\cos^2 t + t\right]_0^{\pi} = \left(\frac{1}{2} + \pi\right) - \left(\frac{1}{2} + 0\right) = \pi$

50. $\mathbf{F} = (-2\sin t)\mathbf{i} - (2\cos t)\mathbf{j} + 2\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = (2\sin t)\mathbf{i} + (2\cos t)\mathbf{j} + 2\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -4\sin^2 t - 4\cos^2 t + 4 = 0$

$\Rightarrow \text{Flow} = 0$

51. C_1: $\mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$, $0 \le t \le \frac{\pi}{2} \Rightarrow \mathbf{F} = (2\cos t)\mathbf{i} + 2t\mathbf{j} + (2\sin t)\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \mathbf{k}$

$\Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -2\cos t \sin t + 2t\cos t + 2\sin t = -\sin 2t + 2t\cos t + 2\sin t$

$\Rightarrow \text{Flow}_1 = \int_0^{\pi/2} (-\sin 2t + 2t\cos t + 2\sin t)\,dt = \left[\frac{1}{2}\cos 2t + 2t\sin t + 2\cos t - 2\cos t\right]_0^{\pi/2} = -1 + \pi$;

C_2: $\mathbf{r} = \mathbf{j} + \frac{\pi}{2}(1-t)\mathbf{k}$, $0 \le t \le 1 \Rightarrow \mathbf{F} = \pi(1-t)\mathbf{j} + 2\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = -\frac{\pi}{2}\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -\pi$

$\Rightarrow \text{Flow}_2 = \int_0^1 -\pi\,dt = \left[-\pi t\right]_0^1 = -\pi$;

C_3: $\mathbf{r} = t\mathbf{i} + (1-t)\mathbf{j}$, $0 \le t \le 1 \Rightarrow \mathbf{F} = 2t\mathbf{i} + 2(1-t)\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} - \mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 2t$

$\Rightarrow \text{Flow}_3 = \int_0^1 2t\,dt = \left[t^2\right]_0^1 = 1 \Rightarrow \text{Circulation} = (-1+\pi) - \pi + 1 = 0$

52. $\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$, where $f(x,y,z) = \frac{1}{2}(x^2 + y^2 + x^2) \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = \frac{d}{dt}(f(\mathbf{r}(t)))$ by the chain rule $\Rightarrow$ Circulation $= \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\,dt = \int_a^b \frac{d}{dt}(f(\mathbf{r}(t)))\,dt = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$. Since C is an entire ellipse, $\mathbf{r}(b) = \mathbf{r}(a)$, thus the Circulation $= 0$.

53. Let $x = t$ be the parameter $\Rightarrow y = x^2 = t^2$ and $z = x = t \Rightarrow \mathbf{r} = t\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}$, $0 \le t \le 1$ from $(0,0,0)$ to $(1,1,1)$

$\Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + \mathbf{k}$ and $\mathbf{F} = xy\mathbf{i} + y\mathbf{j} - yz\mathbf{k} = t^3\mathbf{i} + t^2\mathbf{j} - t^3\mathbf{k} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = t^3 + 2t^3 - t^3 = 2t^3 \Rightarrow \text{Flow} = \int_0^1 2t^3\,dt$

$= \frac{1}{2}$

54. (a) $\mathbf{F} = \nabla(xy^2z^3) \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial z}{\partial z}\frac{dz}{dt} = \frac{df}{dt}$, where $f(x,y,z) = xy^2z^3 \Rightarrow \oint_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\,dt$

$= \int_a^b \frac{d}{dt}(f(\mathbf{r}(t)))\,dt = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) = 0$ since C is an entire ellipse.

(b) $\int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = \int_{(1,1,1)}^{(2,1,-1)} \frac{d}{dt}(xy^2z^3)\,dt = \left[xy^2z^3\right]_{(1,1,1)}^{(2,1,-1)} = (2)(1)^2(-1)^3 - (1)(1)^2(1)^3 = -2 - 1 = -3$

55-60. Example CAS commands:

Maple:

```
with( LinearAlgebra );#55
F := r -> < r[1]*r[2]^6 | 3*r[1]*(r[1]*r[2]^5+2) >;
r := t -> < 2*cos(t) | sin(t) >;
a,b := 0,2*Pi;
dr := map(diff,r(t),t);                                  # (a)
F(r(t));                                                 # (b)
q1 := simplify( F(r(t)) . dr ) assuming t::real;         # (c)
q2 := Int( q1, t=a..b );
value( q2 );
```

Mathematica: (functions and bounds will vary):

Exercises 55 and 56 use vectors in 2 dimensions

```
Clear[x, y, t, f, r, v]
f[x_, y_]:= {x y^6, 3x (x y^5 + 2)}
{a, b}={0, 2π};
x[t_]:= 2 Cos[t]
y[t_]:= Sin[t]
r[t_]:={x[t], y[t]}
v[t_]:= r'[t]
integrand= f[x[t], y[t]] . v[t] //Simplify
Integrate[integrand,{t, a, b}]
N[%]
```

If the integration takes too long or cannot be done, use NIntegrate to integrate numerically. This is suggested for exercises 57 - 60 that use vectors in 3 dimensions. Be certain to leave spaces between variables to be multiplied.

```
Clear[x, y, z, t, f, r, v]
f[x_, y_, z_]:= {y + y z Cos[x y z], x^2 + x z Cos[x y z], z + x y Cos[x y z]}
{a, b}={0, 2π};
x[t_]:= 2 Cos[t]
y[t_]:= 3 Sin[t]
z[t_]:= 1
r[t_]:={x[t], y[t], z[t]}
v[t_]:= r'[t]
integrand= f[x[t], y[t],z[t]] . v[t] //Simplify
NIntegrate[integrand,{t, a, b}]
```

16.3 PATH INDEPENDENCE, POTENTIAL FUNCTIONS, AND CONSERVATIVE FIELDS

1. $\frac{\partial P}{\partial y} = x = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = y = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = z = \frac{\partial M}{\partial y} \Rightarrow$ Conservative

2. $\frac{\partial P}{\partial y} = x \cos z = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = y \cos z = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = \sin z = \frac{\partial M}{\partial y} \Rightarrow$ Conservative

3. $\frac{\partial P}{\partial y} = -1 \neq 1 = \frac{\partial N}{\partial z} \Rightarrow$ Not Conservative

4. $\frac{\partial N}{\partial x} = 1 \neq -1 = \frac{\partial M}{\partial y} \Rightarrow$ Not Conservative

5. $\frac{\partial N}{\partial x} = 0 \neq 1 = \frac{\partial M}{\partial y} \Rightarrow$ Not Conservative

6. $\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = -e^x \sin y = \frac{\partial M}{\partial y} \Rightarrow$ Conservative

7. $\frac{\partial f}{\partial x} = 2x \Rightarrow f(x, y, z) = x^2 + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = 3y \Rightarrow g(y, z) = \frac{3y^2}{2} + h(z) \Rightarrow f(x, y, z) = x^2 + \frac{3y^2}{2} + h(z)$
$\Rightarrow \frac{\partial f}{\partial z} = h'(z) = 4z \Rightarrow h(z) = 2z^2 + C \Rightarrow f(x, y, z) = x^2 + \frac{3y^2}{2} + 2z^2 + C$

8. $\frac{\partial f}{\partial x} = y + z \Rightarrow f(x, y, z) = (y + z)x + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = x + \frac{\partial g}{\partial y} = x + z \Rightarrow \frac{\partial g}{\partial y} = z \Rightarrow g(y, z) = zy + h(z)$
$\Rightarrow f(x, y, z) = (y + z)x + zy + h(z) \Rightarrow \frac{\partial f}{\partial z} = x + y + h'(z) = x + y \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x, y, z)$
$= (y + z)x + zy + C$

9. $\frac{\partial f}{\partial x} = e^{y+2z} \Rightarrow f(x, y, z) = xe^{y+2z} + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = xe^{y+2z} + \frac{\partial g}{\partial y} = xe^{y+2z} \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow f(x, y, z)$
$= xe^{y+2z} + h(z) \Rightarrow \frac{\partial f}{\partial z} = 2xe^{y+2z} + h'(z) = 2xe^{y+2z} \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x, y, z) = xe^{y+2z} + C$

10. $\frac{\partial f}{\partial x} = y \sin z \Rightarrow f(x, y, z) = xy \sin z + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = x \sin z + \frac{\partial g}{\partial y} = x \sin z \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y, z) = h(z)$
$\Rightarrow f(x, y, z) = xy \sin z + h(z) \Rightarrow \frac{\partial f}{\partial z} = xy \cos z + h'(z) = xy \cos z \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x, y, z)$
$= xy \sin z + C$

11. $\frac{\partial f}{\partial z} = \frac{z}{y^2 + z^2} \Rightarrow f(x, y, z) = \frac{1}{2} \ln (y^2 + z^2) + g(x, y) \Rightarrow \frac{\partial f}{\partial x} = \frac{\partial g}{\partial x} = \ln x + \sec^2 (x + y) \Rightarrow g(x, y)$
$= (x \ln x - x) + \tan (x + y) + h(y) \Rightarrow f(x, y, z) = \frac{1}{2} \ln (y^2 + z^2) + (x \ln x - x) + \tan (x + y) + h(y)$
$\Rightarrow \frac{\partial f}{\partial y} = \frac{y}{y^2 + z^2} + \sec^2 (x + y) + h'(y) = \sec^2 (x + y) + \frac{y}{y^2 + z^2} \Rightarrow h'(y) = 0 \Rightarrow h(y) = C \Rightarrow f(x, y, z)$
$= \frac{1}{2} \ln (y^2 + z^2) + (x \ln x - x) + \tan (x + y) + C$

12. $\frac{\partial f}{\partial x} = \frac{y}{1 + x^2y^2} \Rightarrow f(x, y, z) = \tan^{-1} (xy) + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \frac{x}{1 + x^2y^2} + \frac{\partial g}{\partial y} = \frac{x}{1 + x^2y^2} + \frac{z}{\sqrt{1 - y^2z^2}}$
$\Rightarrow \frac{\partial g}{\partial y} = \frac{z}{\sqrt{1 - y^2z^2}} \Rightarrow g(y, z) = \sin^{-1} (yz) + h(z) \Rightarrow f(x, y, z) = \tan^{-1} (xy) + \sin^{-1} (yz) + h(z)$
$\Rightarrow \frac{\partial f}{\partial z} = \frac{y}{\sqrt{1 - y^2z^2}} + h'(z) = \frac{y}{\sqrt{1 - y^2z^2}} + \frac{1}{z} \Rightarrow h'(z) = \frac{1}{z} \Rightarrow h(z) = \ln |z| + C$
$\Rightarrow f(x, y, z) = \tan^{-1} (xy) + \sin^{-1} (yz) + \ln |z| + C$

13. Let $\mathbf{F}(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow \frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = 0 = \frac{\partial M}{\partial y} \Rightarrow M\,dx + N\,dy + P\,dz$ is exact; $\frac{\partial f}{\partial x} = 2x \Rightarrow f(x, y, z) = x^2 + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = 2y \Rightarrow g(y, z) = y^2 + h(z) \Rightarrow f(x, y, z) = x^2 + y^2 = h(z)$
$\Rightarrow \frac{\partial f}{\partial z} = h'(z) = 2z \Rightarrow h(z) = z^2 + C \Rightarrow f(x, y, z) = x^2 + y^2 + z^2 + C \Rightarrow \int_{(0,0,0)}^{(2,3,-6)} 2x\,dx + 2y\,dy + 2z\,dz$
$= f(2, 3, -6) - f(0, 0, 0) = 2^2 + 3^2 + (-6)^2 = 49$

14. Let $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} \Rightarrow \frac{\partial P}{\partial y} = x = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = y = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = z = \frac{\partial M}{\partial y} \Rightarrow M\,dx + N\,dy + P\,dz$ is exact; $\frac{\partial f}{\partial x} = yz \Rightarrow f(x, y, z) = xyz + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = xz + \frac{\partial g}{\partial y} = xz \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y, z) = h(z) \Rightarrow f(x, y, z)$
$= xyz + h(z) \Rightarrow \frac{\partial f}{\partial z} = xy + h'(z) = xy \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x, y, z) = xyz + C$
$\Rightarrow \int_{(1,1,2)}^{(3,5,0)} yz\,dx + xz\,dy + xy\,dz = f(3, 5, 0) - f(1, 1, 2) = 0 - 2 = -2$

15. Let $\mathbf{F}(x, y, z) = 2xy\mathbf{i} + (x^2 - z^2)\,\mathbf{j} - 2yz\mathbf{k} \Rightarrow \frac{\partial P}{\partial y} = -2z = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = 2x = \frac{\partial M}{\partial y}$
$\Rightarrow M\,dx + N\,dy + P\,dz$ is exact; $\frac{\partial f}{\partial x} = 2xy \Rightarrow f(x, y, z) = x^2y + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = x^2 + \frac{\partial g}{\partial y} = x^2 - z^2 \Rightarrow \frac{\partial g}{\partial y} = -z^2$
$\Rightarrow g(y, z) = -yz^2 + h(z) \Rightarrow f(x, y, z) = x^2y - yz^2 + h(z) \Rightarrow \frac{\partial f}{\partial z} = -2yz + h'(z) = -2yz \Rightarrow h'(z) = 0 \Rightarrow h(z) = C$
$\Rightarrow f(x, y, z) = x^2y - yz^2 + C \Rightarrow \int_{(0,0,0)}^{(1,2,3)} 2xy\,dx + (x^2 - z^2)\,dy - 2yz\,dz = f(1, 2, 3) - f(0, 0, 0) = 2 - 2(3)^2 = -16$

16. Let $\mathbf{F}(x, y, z) = 2x\mathbf{i} - y^2\mathbf{j} - \left(\frac{4}{1 + z^2}\right)\mathbf{k} \Rightarrow \frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = 0 = \frac{\partial M}{\partial y}$
$\Rightarrow M\,dx + N\,dy + P\,dz$ is exact; $\frac{\partial f}{\partial x} = 2x \Rightarrow f(x, y, z) = x^2 + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = -y^2 \Rightarrow g(y, z) = -\frac{y^3}{3} + h(z)$

$\Rightarrow f(x,y,z) = x^2 - \frac{y^3}{3} + h(z) \Rightarrow \frac{\partial f}{\partial z} = h'(z) = -\frac{4}{1+z^2} \Rightarrow h(z) = -4\tan^{-1} z + C \Rightarrow f(x,y,z)$
$= x^2 - \frac{y^3}{3} - 4\tan^{-1} z + C \Rightarrow \int_{(0,0,0)}^{(3,3,1)} 2x\,dx - y^2\,dy - \frac{4}{1+z^2}\,dz = f(3,3,1) - f(0,0,0)$
$= \left(9 - \frac{27}{3} - 4 \cdot \frac{\pi}{4}\right) - (0 - 0 - 0) = -\pi$

17. Let $\mathbf{F}(x,y,z) = (\sin y \cos x)\mathbf{i} + (\cos y \sin x)\mathbf{j} + \mathbf{k} \Rightarrow \frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = \cos y \cos x = \frac{\partial M}{\partial y}$
$\Rightarrow M\,dx + N\,dy + P\,dz$ is exact; $\frac{\partial f}{\partial x} = \sin y \cos x \Rightarrow f(x,y,z) = \sin y \sin x + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = \cos y \sin x + \frac{\partial g}{\partial y}$
$= \cos y \sin x \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y,z) = h(z) \Rightarrow f(x,y,z) = \sin y \sin x + h(z) \Rightarrow \frac{\partial f}{\partial z} = h'(z) = 1 \Rightarrow h(z) = z + C$
$\Rightarrow f(x,y,z) = \sin y \sin x + z + C \Rightarrow \int_{(1,0,0)}^{(0,1,1)} \sin y \cos x\,dx + \cos y \sin x\,dy + dz = f(0,1,1) - f(1,0,0)$
$= (0+1) - (0+0) = 1$

18. Let $\mathbf{F}(x,y,z) = (2\cos y)\mathbf{i} + \left(\frac{1}{y} - 2x \sin y\right)\mathbf{j} + \left(\frac{1}{z}\right)\mathbf{k} \Rightarrow \frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = -2\sin y = \frac{\partial M}{\partial y}$
$\Rightarrow M\,dx + N\,dy + P\,dz$ is exact; $\frac{\partial f}{\partial x} = 2\cos y \Rightarrow f(x,y,z) = 2x\cos y + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = -2x\sin y + \frac{\partial g}{\partial y}$
$= \frac{1}{y} - 2x\sin y \Rightarrow \frac{\partial g}{\partial y} = \frac{1}{y} \Rightarrow g(y,z) = \ln|y| + h(z) \Rightarrow f(x,y,z) = 2x\cos y + \ln|y| + h(z) \Rightarrow \frac{\partial f}{\partial z} = h'(z) = \frac{1}{z}$
$\Rightarrow h(z) = \ln|z| + C \Rightarrow f(x,y,z) = 2x\cos y + \ln|y| + \ln|z| + C$
$\Rightarrow \int_{(0,2,1)}^{(1,\pi/2,2)} 2\cos y\,dx + \left(\frac{1}{y} - 2x\sin y\right)dy + \frac{1}{z}\,dz = f\left(1, \frac{\pi}{2}, 2\right) - f(0,2,1)$
$= \left(2 \cdot 0 + \ln\frac{\pi}{2} + \ln 2\right) - (0 \cdot \cos 2 + \ln 2 + \ln 1) = \ln\frac{\pi}{2}$

19. Let $\mathbf{F}(x,y,z) = 3x^2\mathbf{i} + \left(\frac{z^2}{y}\right)\mathbf{j} + (2z\ln y)\mathbf{k} \Rightarrow \frac{\partial P}{\partial y} = \frac{2z}{y} = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = 0 = \frac{\partial M}{\partial y}$
$\Rightarrow M\,dx + N\,dy + P\,dz$ is exact; $\frac{\partial f}{\partial x} = 3x^2 \Rightarrow f(x,y,z) = x^3 + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = \frac{z^2}{y} \Rightarrow g(y,z) = z^2\ln y + h(z)$
$\Rightarrow f(x,y,z) = x^3 + z^2\ln y + h(z) \Rightarrow \frac{\partial f}{\partial z} = 2z\ln y + h'(z) = 2z\ln y \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x,y,z)$
$= x^3 + z^2\ln y + C \Rightarrow \int_{(1,1,1)}^{(1,2,3)} 3x^2\,dx + \frac{z^2}{y}\,dy + 2z\ln y\,dz = f(1,2,3) - f(1,1,1)$
$= (1 + 9\ln 2 + C) - (1 + 0 + C) = 9\ln 2$

20. Let $\mathbf{F}(x,y,z) = (2x\ln y - yz)\mathbf{i} + \left(\frac{x^2}{y} - xz\right)\mathbf{j} - (xy)\mathbf{k} \Rightarrow \frac{\partial P}{\partial y} = -x = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = -y = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = \frac{2x}{y} - z = \frac{\partial M}{\partial y}$
$\Rightarrow M\,dx + N\,dy + P\,dz$ is exact; $\frac{\partial f}{\partial x} = 2x\ln y - yz \Rightarrow f(x,y,z) = x^2\ln y - xyz + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = \frac{x^2}{y} - xz + \frac{\partial g}{\partial y}$
$= \frac{x^2}{y} - xz \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y,z) = h(z) \Rightarrow f(x,y,z) = x^2\ln y - xyz + h(z) \Rightarrow \frac{\partial f}{\partial z} = -xy + h'(z) = -xy \Rightarrow h'(z) = 0$
$\Rightarrow h(z) = C \Rightarrow f(x,y,z) = x^2\ln y - xyz + C \Rightarrow \int_{(1,2,1)}^{(2,1,1)} (2x\ln y - yz)\,dx + \left(\frac{x^2}{y} - xz\right)dy - xy\,dz$
$= f(2,1,1) - f(1,2,1) = (4\ln 1 - 2 + C) - (\ln 2 - 2 + C) = -\ln 2$

21. Let $\mathbf{F}(x,y,z) = \left(\frac{1}{y}\right)\mathbf{i} + \left(\frac{1}{z} - \frac{x}{y^2}\right)\mathbf{j} - \left(\frac{y}{z^2}\right)\mathbf{k} \Rightarrow \frac{\partial P}{\partial y} = -\frac{1}{z^2} = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = -\frac{1}{y^2} = \frac{\partial M}{\partial y}$
$\Rightarrow M\,dx + N\,dy + P\,dz$ is exact; $\frac{\partial f}{\partial x} = \frac{1}{y} \Rightarrow f(x,y,z) = \frac{x}{y} + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = -\frac{x}{y^2} + \frac{\partial g}{\partial y} = \frac{1}{z} - \frac{x}{y^2}$
$\Rightarrow \frac{\partial g}{\partial y} = \frac{1}{z} \Rightarrow g(y,z) = \frac{y}{z} + h(z) \Rightarrow f(x,y,z) = \frac{x}{y} + \frac{y}{z} + h(z) \Rightarrow \frac{\partial f}{\partial z} = -\frac{y}{z^2} + h'(z) = -\frac{y}{z^2} \Rightarrow h'(z) = 0 \Rightarrow h(z) = C$
$\Rightarrow f(x,y,z) = \frac{x}{y} + \frac{y}{z} + C \Rightarrow \int_{(1,1,1)}^{(2,2,2)} \frac{1}{y}\,dx + \left(\frac{1}{z} - \frac{x}{y^2}\right)dy - \frac{y}{z^2}\,dz = f(2,2,2) - f(1,1,1) = \left(\frac{2}{2} + \frac{2}{2} + C\right) - \left(\frac{1}{1} + \frac{1}{1} + C\right)$
$= 0$

22. Let $\mathbf{F}(x,y,z) = \frac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{x^2+y^2+z^2}$ $\left(\text{and let } \rho^2 = x^2 + y^2 + z^2 \Rightarrow \frac{\partial \rho}{\partial x} = \frac{x}{\rho}, \frac{\partial \rho}{\partial y} = \frac{y}{\rho}, \frac{\partial \rho}{\partial z} = \frac{z}{\rho}\right)$
$\Rightarrow \frac{\partial P}{\partial y} = -\frac{4yz}{\rho^4} = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = -\frac{4xz}{\rho^4} = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = -\frac{4xy}{\rho^4} = \frac{\partial M}{\partial y} \Rightarrow M\,dx + N\,dy + P\,dz$ is exact;
$\frac{\partial f}{\partial x} = \frac{2x}{x^2+y^2+z^2} \Rightarrow f(x,y,z) = \ln(x^2+y^2+z^2) + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = \frac{2y}{x^2+y^2+z^2} + \frac{\partial g}{\partial y} = \frac{2y}{x^2+y^2+z^2}$

$\Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y,z) = h(z) \Rightarrow f(x,y,z) = \ln(x^2+y^2+z^2) + h(z) \Rightarrow \frac{\partial f}{\partial z} = \frac{2z}{x^2+y^2+z^2} + h'(z)$
$= \frac{2z}{x^2+y^2+z^2} \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x,y,z) = \ln(x^2+y^2+z^2) + C$
$\Rightarrow \int_{(-1,-1,-1)}^{(2,2,2)} \frac{2x\,dx + 2y\,dy + 2z\,dz}{x^2+y^2+z^2} = f(2,2,2) - f(-1,-1,-1) = \ln 12 - \ln 3 = \ln 4$

23. $\mathbf{r} = (\mathbf{i}+\mathbf{j}+\mathbf{k}) + t(\mathbf{i}+2\mathbf{j}-2\mathbf{k}) = (1+t)\mathbf{i} + (1+2t)\mathbf{j} + (1-2t)\mathbf{k},\ 0 \le t \le 1 \Rightarrow dx = dt,\ dy = 2\,dt,\ dz = -2\,dt$
$\Rightarrow \int_{(1,1,1)}^{(2,3,-1)} y\,dx + x\,dy + 4\,dz = \int_0^1 (2t+1)\,dt + (t+1)(2\,dt) + 4(-2)\,dt = \int_0^1 (4t-5)\,dt = [2t^2-5t]_0^1 = -3$

24. $\mathbf{r} = t(3\mathbf{j}+4\mathbf{k}),\ 0 \le t \le 1 \Rightarrow dx = 0,\ dy = 3\,dt,\ dz = 4\,dt \Rightarrow \int_{(0,0,0)}^{(0,3,4)} x^2\,dx + yz\,dy + \left(\frac{y^2}{2}\right)dz$
$= \int_0^1 (12t^2)(3\,dt) + \left(\frac{9t^2}{2}\right)(4\,dt) = \int_0^1 54t^2\,dt = [18t^2]_0^1 = 18$

25. $\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z},\ \frac{\partial M}{\partial z} = 2z = \frac{\partial P}{\partial x},\ \frac{\partial N}{\partial x} = 0 = \frac{\partial M}{\partial y} \Rightarrow M\,dx + N\,dy + P\,dz$ is exact $\Rightarrow$ **F** is conservative
$\Rightarrow$ path independence

26. $\frac{\partial P}{\partial y} = -\frac{yz}{\left(\sqrt{x^2+y^2+z^2}\right)^3} = \frac{\partial N}{\partial z},\ \frac{\partial M}{\partial z} = -\frac{xz}{\left(\sqrt{x^2+y^2+z^2}\right)^3} = \frac{\partial P}{\partial x},\ \frac{\partial N}{\partial x} = -\frac{xy}{\left(\sqrt{x^2+y^2+z^2}\right)^3} = \frac{\partial M}{\partial y}$
$\Rightarrow M\,dx + N\,dy + P\,dz$ is exact $\Rightarrow$ **F** is conservative $\Rightarrow$ path independence

27. $\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z},\ \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x},\ \frac{\partial N}{\partial x} = -\frac{2x}{y^2} = \frac{\partial M}{\partial y} \Rightarrow$ **F** is conservative $\Rightarrow$ there exists an f so that $\mathbf{F} = \nabla f$;
$\frac{\partial f}{\partial x} = \frac{2x}{y} \Rightarrow f(x,y) = \frac{x^2}{y} + g(y) \Rightarrow \frac{\partial f}{\partial y} = -\frac{x^2}{y^2} + g'(y) = \frac{1-x^2}{y^2} \Rightarrow g'(y) = \frac{1}{y^2} \Rightarrow g(y) = -\frac{1}{y} + C$
$\Rightarrow f(x,y) = \frac{x^2}{y} - \frac{1}{y} + C \Rightarrow \mathbf{F} = \nabla\left(\frac{x^2-1}{y}\right)$

28. $\frac{\partial P}{\partial y} = \cos z = \frac{\partial N}{\partial z},\ \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x},\ \frac{\partial N}{\partial x} = \frac{e^x}{y} = \frac{\partial M}{\partial y} \Rightarrow$ **F** is conservative $\Rightarrow$ there exists an f so that $\mathbf{F} = \nabla f$;
$\frac{\partial f}{\partial x} = e^x \ln y \Rightarrow f(x,y,z) = e^x \ln y + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = \frac{e^x}{y} + \frac{\partial g}{\partial y} = \frac{e^x}{y} + \sin z \Rightarrow \frac{\partial g}{\partial y} = \sin z \Rightarrow g(y,z)$
$= y \sin z + h(z) \Rightarrow f(x,y,z) = e^x \ln y + y \sin z + h(z) \Rightarrow \frac{\partial f}{\partial z} = y \cos z + h'(z) = y \cos z \Rightarrow h'(z) = 0$
$\Rightarrow h(z) = C \Rightarrow f(x,y,z) = e^x \ln y + y \sin z + C \Rightarrow \mathbf{F} = \nabla(e^x \ln y + y \sin z)$

29. $\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z},\ \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x},\ \frac{\partial N}{\partial x} = 1 = \frac{\partial M}{\partial y} \Rightarrow$ **F** is conservative $\Rightarrow$ there exists an f so that $\mathbf{F} = \nabla f$;
$\frac{\partial f}{\partial x} = x^2 + y \Rightarrow f(x,y,z) = \frac{1}{3}x^3 + xy + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = x + \frac{\partial g}{\partial y} = y^2 + x \Rightarrow \frac{\partial g}{\partial y} = y^2 \Rightarrow g(y,z) = \frac{1}{3}y^3 + h(z)$
$\Rightarrow f(x,y,z) = \frac{1}{3}x^3 + xy + \frac{1}{3}y^3 + h(z) \Rightarrow \frac{\partial f}{\partial z} = h'(z) = ze^z \Rightarrow h(z) = ze^z - e^z + C \Rightarrow f(x,y,z)$
$= \frac{1}{3}x^3 + xy + \frac{1}{3}y^3 + ze^z - e^z + C \Rightarrow \mathbf{F} = \nabla\left(\frac{1}{3}x^3 + xy + \frac{1}{3}y^3 + ze^z - e^z\right)$

(a) work $= \int_A^B \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\,dt = \int_A^B \mathbf{F} \cdot d\mathbf{r} = \left[\frac{1}{3}x^3 + xy + \frac{1}{3}y^3 + ze^z - e^z\right]_{(1,0,0)}^{(1,0,1)} = \left(\frac{1}{3} + 0 + 0 + e - e\right) - \left(\frac{1}{3} + 0 + 0 - 1\right)$
$= 1$

(b) work $= \int_A^B \mathbf{F} \cdot d\mathbf{r} = \left[\frac{1}{3}x^3 + xy + \frac{1}{3}y^3 + ze^z - e^z\right]_{(1,0,0)}^{(1,0,1)} = 1$

(c) work $= \int_A^B \mathbf{F} \cdot d\mathbf{r} = \left[\frac{1}{3}x^3 + xy + \frac{1}{3}y^3 + ze^z - e^z\right]_{(1,0,0)}^{(1,0,1)} = 1$

Note: Since **F** is conservative, $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ is independent of the path from $(1,0,0)$ to $(1,0,1)$.

30. $\frac{\partial P}{\partial y} = xe^{yz} + xyze^{yz} + \cos y = \frac{\partial N}{\partial z},\ \frac{\partial M}{\partial z} = ye^{yz} = \frac{\partial P}{\partial x},\ \frac{\partial N}{\partial x} = ze^{yz} = \frac{\partial M}{\partial y} \Rightarrow$ **F** is conservative $\Rightarrow$ there exists an f so
that $\mathbf{F} = \nabla f$; $\frac{\partial f}{\partial x} = e^{yz} \Rightarrow f(x,y,z) = xe^{yz} + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = xze^{yz} + \frac{\partial g}{\partial y} = xze^{yz} + z\cos y \Rightarrow \frac{\partial g}{\partial y} = z\cos y$
$\Rightarrow g(y,z) = z\sin y + h(z) \Rightarrow f(x,y,z) = xe^{yz} + z\sin y + h(z) \Rightarrow \frac{\partial f}{\partial z} = xye^{yz} + \sin y + h'(z) = xye^{yz} + \sin y$
$\Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x,y,z) = xe^{yz} + z\sin y + C \Rightarrow \mathbf{F} = \nabla(xe^{yz} + z\sin y)$

(a) work $= \int_A^B \mathbf{F}\cdot d\mathbf{r} = \left[xe^{yz} + z\sin y\right]_{(1,0,1)}^{(1,\pi/2,0)} = (1+0)-(1+0) = 0$

(b) work $= \int_A^B \mathbf{F}\cdot d\mathbf{r} = \left[xe^{yz} + z\sin y\right]_{(1,0,1)}^{(1,\pi/2,0)} = 0$

(c) work $= \int_A^B \mathbf{F}\cdot d\mathbf{r} = \left[xe^{yz} + z\sin y\right]_{(1,0,1)}^{(1,\pi/2,0)} = 0$

Note: Since $\mathbf{F}$ is conservative, $\int_A^B \mathbf{F}\cdot d\mathbf{r}$ is independent of the path from $(1,0,1)$ to $\left(1,\frac{\pi}{2},0\right)$.

31. (a) $\mathbf{F} = \nabla(x^3y^2) \Rightarrow \mathbf{F} = 3x^2y^2\mathbf{i} + 2x^3y\mathbf{j}$; let C_1 be the path from $(-1,1)$ to $(0,0)$ $\Rightarrow x = t-1$ and $y = -t+1,\ 0 \le t \le 1 \Rightarrow \mathbf{F} = 3(t-1)^2(-t+1)^2\mathbf{i} + 2(t-1)^3(-t+1)\mathbf{j} = 3(t-1)^4\mathbf{i} - 2(t-1)^4\mathbf{j}$ and $\mathbf{r}_1 = (t-1)\mathbf{i} + (-t+1)\mathbf{j} \Rightarrow d\mathbf{r}_1 = dt\,\mathbf{i} - dt\,\mathbf{j} \Rightarrow \int_{C_1}\mathbf{F}\cdot d\mathbf{r}_1 = \int_0^1 \left[3(t-1)^4 + 2(t-1)^4\right] dt$ $= \int_0^1 5(t-1)^4\,dt = \left[(t-1)^5\right]_0^1 = 1$; let C_2 be the path from $(0,0)$ to $(1,1)$ $\Rightarrow x = t$ and $y = t$, $0 \le t \le 1 \Rightarrow \mathbf{F} = 3t^4\mathbf{i} + 2t^4\mathbf{j}$ and $\mathbf{r}_2 = t\mathbf{i} + t\mathbf{j} \Rightarrow d\mathbf{r}_2 = dt\,\mathbf{i} + dt\,\mathbf{j} \Rightarrow \int_{C_2}\mathbf{F}\cdot d\mathbf{r}_2 = \int_0^1 (3t^4 + 2t^4)\,dt$ $= \int_0^1 5t^4\,dt = 1 \Rightarrow \int_C \mathbf{F}\cdot d\mathbf{r} = \int_{C_1}\mathbf{F}\cdot d\mathbf{r}_1 + \int_{C_2}\mathbf{F}\cdot d\mathbf{r}_2 = 2$

(b) Since $f(x,y) = x^3y^2$ is a potential function for $\mathbf{F}$, $\int_{(-1,1)}^{(1,1)} \mathbf{F}\cdot d\mathbf{r} = f(1,1) - f(-1,1) = 2$

32. $\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z},\ \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x},\ \frac{\partial N}{\partial x} = -2x\sin y = \frac{\partial M}{\partial y} \Rightarrow \mathbf{F}$ is conservative $\Rightarrow$ there exists an f so that $\mathbf{F} = \nabla f$; $\frac{\partial f}{\partial x} = 2x\cos y \Rightarrow f(x,y,z) = x^2\cos y + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = -x^2\sin y + \frac{\partial g}{\partial y} = -x^2\sin y \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y,z) = h(z)$ $\Rightarrow f(x,y,z) = x^2\cos y + h(z) \Rightarrow \frac{\partial f}{\partial z} = h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x,y,z) = x^2\cos y + C \Rightarrow \mathbf{F} = \nabla(x^2\cos y)$

(a) $\int_C 2x\cos y\,dx - x^2\sin y\,dy = \left[x^2\cos y\right]_{(1,0)}^{(0,1)} = 0 - 1 = -1$

(b) $\int_C 2x\cos y\,dx - x^2\sin y\,dy = \left[x^2\cos y\right]_{(-1,\pi)}^{(1,0)} = 1 - (-1) = 2$

(c) $\int_C 2x\cos y\,dx - x^2\sin y\,dy = \left[x^2\cos y\right]_{(-1,0)}^{(1,0)} = 1 - 1 = 0$

(d) $\int_C 2x\cos y\,dx - x^2\sin y\,dy = \left[x^2\cos y\right]_{(1,0)}^{(1,0)} = 1 - 1 = 0$

33. (a) If the differential form is exact, then $\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z} \Rightarrow 2ay = cy$ for all y $\Rightarrow 2a = c$, $\frac{\partial M}{\partial z} = \frac{\partial P}{\partial x} \Rightarrow 2cx = 2cx$ for all x, and $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} \Rightarrow by = 2ay$ for all y $\Rightarrow b = 2a$ and $c = 2a$

(b) $\mathbf{F} = \nabla f \Rightarrow$ the differential form with $a = 1$ in part (a) is exact $\Rightarrow b = 2$ and $c = 2$

34. $\mathbf{F} = \nabla f \Rightarrow g(x,y,z) = \int_{(0,0,0)}^{(x,y,z)} \mathbf{F}\cdot d\mathbf{r} = \int_{(0,0,0)}^{(x,y,z)} \nabla f\cdot d\mathbf{r} = f(x,y,z) - f(0,0,0) \Rightarrow \frac{\partial g}{\partial x} = \frac{\partial f}{\partial x} - 0,\ \frac{\partial g}{\partial y} = \frac{\partial f}{\partial y} - 0$, and $\frac{\partial g}{\partial z} = \frac{\partial f}{\partial z} - 0 \Rightarrow \nabla g = \nabla f = \mathbf{F}$, as claimed

35. The path will not matter; the work along any path will be the same because the field is conservative.

36. The field is not conservative, for otherwise the work would be the same along C_1 and C_2.

37. Let the coordinates of points A and B be (x_A, y_A, z_A) and (x_B, y_B, z_B), respectively. The force $\mathbf{F} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is conservative because all the partial derivatives of M, N, and P are zero. Therefore, the potential function is $f(x,y,z) = ax + by + cz + C$, and the work done by the force in moving a particle along any path from A to B is $f(B) - f(A) = f(x_B, y_B, z_B) - f(x_A, y_A, z_A) = (ax_B + by_B + cz_B + C) - (ax_A + by_A + cz_A + C)$ $= a(x_B - x_A) + b(y_B - y_A) + c(z_B - z_A) = \mathbf{F}\cdot\overrightarrow{BA}$

38. (a) Let $-GmM = C \Rightarrow \mathbf{F} = C\left[\frac{x}{(x^2+y^2+z^2)^{3/2}}\mathbf{i} + \frac{y}{(x^2+y^2+z^2)^{3/2}}\mathbf{j} + \frac{z}{(x^2+y^2+z^2)^{3/2}}\mathbf{k}\right]$ $\Rightarrow \frac{\partial P}{\partial y} = \frac{-3yzC}{(x^2+y^2+z^2)^{5/2}} = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = \frac{-3xzC}{(x^2+y^2+z^2)^{5/2}} = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = \frac{-3xyC}{(x^2+y^2+z^2)^{5/2}} = \frac{\partial M}{\partial y} \Rightarrow \mathbf{F} = \nabla f$ for some f; $\frac{\partial f}{\partial x} = \frac{xC}{(x^2+y^2+z^2)^{3/2}} \Rightarrow f(x,y,z) = -\frac{C}{(x^2+y^2+z^2)^{1/2}} + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = \frac{yC}{(x^2+y^2+z^2)^{3/2}} + \frac{\partial g}{\partial y}$ $= \frac{yC}{(x^2+y^2+z^2)^{3/2}} \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y,z) = h(z) \Rightarrow \frac{\partial f}{\partial z} = \frac{zC}{(x^2+y^2+z^2)^{3/2}} + h'(z) = \frac{zC}{(x^2+y^2+z^2)^{3/2}}$ $\Rightarrow h(z) = C_1 \Rightarrow f(x,y,z) = -\frac{C}{(x^2+y^2+z^2)^{1/2}} + C_1$. Let $C_1 = 0 \Rightarrow f(x,y,z) = \frac{GmM}{(x^2+y^2+z^2)^{1/2}}$ is a potential function for $\mathbf{F}$.

(b) If s is the distance of (x, y, z) from the origin, then $s = \sqrt{x^2+y^2+z^2}$. The work done by the gravitational field $\mathbf{F}$ is work $= \int_{P_1}^{P_2} \mathbf{F}\cdot d\mathbf{r} = \left[\frac{GmM}{\sqrt{x^2+y^2+z^2}}\right]_{P_1}^{P_2} = \frac{GmM}{s_2} - \frac{GmM}{s_1} = GmM\left(\frac{1}{s_2} - \frac{1}{s_1}\right)$, as claimed.

16.4 GREEN'S THEOREM IN THE PLANE

1. $M = -y = -a\sin t$, $N = x = a\cos t$, $dx = -a\sin t\,dt$, $dy = a\cos t\,dt \Rightarrow \frac{\partial M}{\partial x} = 0, \frac{\partial M}{\partial y} = -1, \frac{\partial N}{\partial x} = 1$, and $\frac{\partial N}{\partial y} = 0$;

Equation (3): $\oint_C M\,dy - N\,dx = \int_0^{2\pi}[(-a\sin t)(a\cos t) - (a\cos t)(-a\sin t)]\,dt = \int_0^{2\pi} 0\,dt = 0$; $\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)dx\,dy = \iint_R 0\,dx\,dy = 0$, Flux

Equation (4): $\oint_C M\,dx + N\,dy = \int_0^{2\pi}[(-a\sin t)(-a\sin t) - (a\cos t)(a\cos t)]\,dt = \int_0^{2\pi} a^2\,dt = 2\pi a^2$; $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy = \int_{-a}^{a}\int_{-c}^{\sqrt{a^2-x^2}} 2\,dy\,dx = \int_{-a}^{a} 4\sqrt{a^2-x^2}\,dx = 4\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_{-a}^{a}$ $= 2a^2\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = 2a^2\pi$, Circulation

2. $M = y = a\sin t$, $N = 0$, $dx = -a\sin t\,dt$, $dy = a\cos t\,dt \Rightarrow \frac{\partial M}{\partial x} = 0, \frac{\partial M}{\partial y} = 1, \frac{\partial N}{\partial x} = 0$, and $\frac{\partial N}{\partial y} = 0$;

Equation (3): $\oint_C M\,dy - N\,dx = \int_0^{2\pi} a^2\sin t\cos t\,dt = a^2\left[\frac{1}{2}\sin^2 t\right]_0^{2\pi} = 0$; $\iint_R 0\,dx\,dy = 0$, Flux

Equation (4): $\oint_C M\,dx + N\,dy = \int_0^{2\pi}(-a^2\sin^2 t)\,dt = -a^2\left[\frac{t}{2} - \frac{\sin 2t}{4}\right]_0^{2\pi} = -\pi a^2$; $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy$ $= \iint_R -1\,dx\,dy = \int_0^{2\pi}\int_0^a -r\,dr\,d\theta = \int_0^{2\pi} -\frac{a^2}{2}\,d\theta = -\pi a^2$, Circulation

3. $M = 2x = 2a\cos t$, $N = -3y = -3a\sin t$, $dx = -a\sin t\,dt$, $dy = a\cos t\,dt \Rightarrow \frac{\partial M}{\partial x} = 2, \frac{\partial M}{\partial y} = 0, \frac{\partial N}{\partial x} = 0$, and $\frac{\partial N}{\partial y} = -3$;

Equation (3): $\oint_C M\,dy - N\,dx = \int_0^{2\pi}[(2a\cos t)(a\cos t) + (3a\sin t)(-a\sin t)]\,dt$ $= \int_0^{2\pi}(2a^2\cos^2 t - 3a^2\sin^2 t)\,dt = 2a^2\left[\frac{t}{2} + \frac{\sin 2t}{4}\right]_0^{2\pi} - 3a^2\left[\frac{t}{2} - \frac{\sin 2t}{4}\right]_0^{2\pi} = 2\pi a^2 - 3\pi a^2 = -\pi a^2$; $\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) = \iint_R -1\,dx\,dy = \int_0^{2\pi}\int_0^a -r\,dr\,d\theta = \int_0^{2\pi} -\frac{a^2}{2}\,d\theta = -\pi a^2$, Flux

Equation (4): $\oint_C M\,dx + N\,dy = \int_0^{2\pi}[(2a\cos t)(-a\sin t) + (-3a\sin t)(a\cos t)]\,dt$ $= \int_0^{2\pi}(-2a^2\sin t\cos t - 3a^2\sin t\cos t)\,dt = -5a^2\left[\frac{1}{2}\sin^2 t\right]_0^{2\pi} = 0$; $\iint_R 0\,dx\,dy = 0$, Circulation

4. $M = -x^2y = -a^3\cos^2 t$, $N = xy^2 = a^3\cos t\sin^2 t$, $dx = -a\sin t\,dt$, $dy = a\cos t\,dt$ $\Rightarrow \frac{\partial M}{\partial x} = -2xy, \frac{\partial M}{\partial y} = -x^2, \frac{\partial N}{\partial x} = y^2$, and $\frac{\partial N}{\partial y} = 2xy$;

Equation (3): $\oint_C M\,dy - N\,dx = \int_0^{2\pi}(-a^4\cos^3 t\sin t + a^4\cos t\sin^3 t) = \left[\frac{a^4}{4}\cos^4 t + \frac{a^4}{4}\sin^4 t\right]_0^{2\pi} = 0$;

$\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy = \iint_R (-2xy + 2xy)\,dx\,dy = 0$, Flux

Equation (4): $\oint_C M\,dx + N\,dy = \int_0^{2\pi} (a^4 \cos^2 t \sin^2 t + a^4 \cos^2 t \sin^2 t)\,dt = \int_0^{2\pi} (2a^4 \cos^2 t \sin^2 t)\,dt$
$= \int_0^{2\pi} \frac{1}{2} a^4 \sin^2 2t\,dt = \frac{a^4}{4} \int_0^{4\pi} \sin^2 u\,du = \frac{a^4}{4}\left[\frac{u}{2} - \frac{\sin 2u}{4}\right]_0^{4\pi} = \frac{\pi a^4}{2}$; $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R (y^2 + x^2)\,dx\,dy$
$= \int_0^{2\pi} \int_0^a r^2 \cdot r\,dr\,d\theta = \int_0^{2\pi} \frac{a^4}{4}\,d\theta = \frac{\pi a^4}{2}$, Circulation

5. $M = x - y,\ N = y - x \Rightarrow \frac{\partial M}{\partial x} = 1,\ \frac{\partial M}{\partial y} = -1,\ \frac{\partial N}{\partial x} = -1,\ \frac{\partial N}{\partial y} = 1 \Rightarrow$ Flux $= \iint_R 2\,dx\,dy = \int_0^1 \int_0^1 2\,dx\,dy = 2$;
Circ $= \iint_R [-1 - (-1)]\,dx\,dy = 0$

6. $M = x^2 + 4y,\ N = x + y^2 \Rightarrow \frac{\partial M}{\partial x} = 2x,\ \frac{\partial M}{\partial y} = 4,\ \frac{\partial N}{\partial x} = 1,\ \frac{\partial N}{\partial y} = 2y \Rightarrow$ Flux $= \iint_R (2x + 2y)\,dx\,dy$
$= \int_0^1 \int_0^1 (2x + 2y)\,dx\,dy = \int_0^1 \left[x^2 + 2xy\right]_0^1 dy = \int_0^1 (1 + 2y)\,dy = \left[y + y^2\right]_0^1 = 2$; Circ $= \iint_R (1 - 4)\,dx\,dy$
$= \int_0^1 \int_0^1 -3\,dx\,dy = -3$

7. $M = y^2 - x^2,\ N = x^2 + y^2 \Rightarrow \frac{\partial M}{\partial x} = -2x,\ \frac{\partial M}{\partial y} = 2y,\ \frac{\partial N}{\partial x} = 2x,\ \frac{\partial N}{\partial y} = 2y \Rightarrow$ Flux $= \iint_R (-2x + 2y)\,dx\,dy$
$= \int_0^3 \int_0^x (-2x + 2y)\,dy\,dx = \int_0^3 (-2x^2 + x^2)\,dx = \left[-\frac{1}{3}x^3\right]_0^3 = -9$; Circ $= \iint_R (2x - 2y)\,dx\,dy$
$= \int_0^3 \int_0^x (2x - 2y)\,dy\,dx = \int_0^3 x^2\,dx = 9$

8. $M = x + y,\ N = -(x^2 + y^2) \Rightarrow \frac{\partial M}{\partial x} = 1,\ \frac{\partial M}{\partial y} = 1,\ \frac{\partial N}{\partial x} = -2x,\ \frac{\partial N}{\partial y} = -2y \Rightarrow$ Flux $= \iint_R (1 - 2y)\,dx\,dy$
$= \int_0^1 \int_0^x (1 - 2y)\,dy\,dx = \int_0^1 (x - x^2)\,dx = \frac{1}{6}$; Circ $= \iint_R (-2x - 1)\,dx\,dy = \int_0^1 \int_0^x (-2x - 1)\,dy\,dx$
$= \int_0^1 (-2x^2 - x)\,dx = -\frac{7}{6}$

9. $M = xy + y^2,\ N = x - y \Rightarrow \frac{\partial M}{\partial x} = y,\ \frac{\partial M}{\partial y} = x + 2y,\ \frac{\partial N}{\partial x} = 1,\ \frac{\partial N}{\partial y} = -1 \Rightarrow$ Flux $= \iint_R (y + (-1))\,dy\,dx$
$= \int_0^1 \int_{x^2}^{\sqrt{x}} (y - 1)\,dy\,dx = \int_0^1 \left(\frac{1}{2}x - \sqrt{x} - \frac{1}{2}x^4 + x^2\right) dx = -\frac{11}{60}$; Circ $= \iint_R (1 - (x + 2y))\,dy\,dx$
$= \int_0^1 \int_{x^2}^{\sqrt{x}} (1 - x - 2y)\,dy\,dx = \int_0^1 \left(\sqrt{x} - x^{3/2} - x - x^2 + x^3 + x^4\right) dx = -\frac{7}{60}$

10. $M = x + 3y,\ N = 2x - y \Rightarrow \frac{\partial M}{\partial x} = 1,\ \frac{\partial M}{\partial y} = 3,\ \frac{\partial N}{\partial x} = 2,\ \frac{\partial N}{\partial y} = -1 \Rightarrow$ Flux $= \iint_R (1 + (-1))\,dy\,dx = 0$
Circ $= \iint_R (2 - 3)\,dy\,dx = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{(2-x^2)/2}}^{\sqrt{(2-x^2)/2}} (-1)\,dy\,dx = -\frac{2}{\sqrt{2}} \int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{2 - x^2}\,dx = -\pi\sqrt{2}$

11. $M = x^3y^2,\ N = \frac{1}{2}x^4y \Rightarrow \frac{\partial M}{\partial x} = 3x^2y^2,\ \frac{\partial M}{\partial y} = 2x^3y,\ \frac{\partial N}{\partial x} = 2x^3y,\ \frac{\partial N}{\partial y} = \frac{1}{2}x^4 \Rightarrow$ Flux $= \iint_R \left(3x^2y^2 + \frac{1}{2}x^4\right) dy\,dx$
$= \int_0^2 \int_{x^2 - x}^{x} \left(3x^2y^2 + \frac{1}{2}x^4\right) dy\,dx = \int_0^2 \left(3x^5 - \frac{7}{2}x^6 + 3x^7 - x^8\right) dx = \frac{64}{9}$; Circ $= \iint_R (2x^3y - 2x^3y)\,dy\,dx = 0$

12. $M = \frac{x}{1+y^2}$, $N = \tan^{-1} y \Rightarrow \frac{\partial M}{\partial x} = \frac{1}{1+y^2}$, $\frac{\partial M}{\partial y} = \frac{-2xy}{(1+y^2)^2}$, $\frac{\partial N}{\partial x} = 0$, $\frac{\partial N}{\partial y} = \frac{1}{1+y^2} \Rightarrow \text{Flux} = \iint_R \left(\frac{1}{1+y^2} + \frac{1}{1+y^2}\right) dx\,dy$
$= \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{2}{1+y^2}\,dx\,dy = \int_{-1}^{1} \frac{4\sqrt{1-y^2}}{1+y^2}\,dx = 4\pi\sqrt{2} - 4\pi$; $\text{Circ} = \iint_R \left(0 - \left(\frac{-2xy}{(1+y^2)^2}\right)\right) dy\,dx$
$= \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \left(\frac{2xy}{(1+y^2)^2}\right) dy\,dx = \int_{-1}^{1} (0)\,dx = 0$

13. $M = x + e^x \sin y$, $N = x + e^x \cos y \Rightarrow \frac{\partial M}{\partial x} = 1 + e^x \sin y$, $\frac{\partial M}{\partial y} = e^x \cos y$, $\frac{\partial N}{\partial x} = 1 + e^x \cos y$, $\frac{\partial N}{\partial y} = -e^x \sin y$
$\Rightarrow \text{Flux} = \iint_R dx\,dy = \int_{-\pi/4}^{\pi/4}\int_0^{\sqrt{\cos 2\theta}} r\,dr\,d\theta = \int_{-\pi/4}^{\pi/4} \left(\frac{1}{2}\cos 2\theta\right) d\theta = \left[\frac{1}{4}\sin 2\theta\right]_{-\pi/4}^{\pi/4} = \frac{1}{2}$;
$\text{Circ} = \iint_R (1 + e^x \cos y - e^x \cos y)\,dx\,dy = \iint_R dx\,dy = \int_{-\pi/4}^{\pi/4}\int_0^{\sqrt{\cos 2\theta}} r\,dr\,d\theta = \int_{-\pi/4}^{\pi/4} \left(\frac{1}{2}\cos 2\theta\right) d\theta = \frac{1}{2}$

14. $M = \tan^{-1} \frac{y}{x}$, $N = \ln(x^2 + y^2) \Rightarrow \frac{\partial M}{\partial x} = \frac{-y}{x^2+y^2}$, $\frac{\partial M}{\partial y} = \frac{x}{x^2+y^2}$, $\frac{\partial N}{\partial x} = \frac{2x}{x^2+y^2}$, $\frac{\partial N}{\partial y} = \frac{2y}{x^2+y^2}$
$\Rightarrow \text{Flux} = \iint_R \left(\frac{-y}{x^2+y^2} + \frac{2y}{x^2+y^2}\right) dx\,dy = \int_0^{\pi}\int_1^2 \left(\frac{r\sin\theta}{r^2}\right) r\,dr\,d\theta = \int_0^{\pi} \sin\theta\,d\theta = 2$;
$\text{Circ} = \iint_R \left(\frac{2x}{x^2+y^2} - \frac{x}{x^2+y^2}\right) dx\,dy = \int_0^{\pi}\int_1^2 \left(\frac{r\cos\theta}{r^2}\right) r\,dr\,d\theta = \int_0^{\pi} \cos\theta\,d\theta = 0$

15. $M = xy$, $N = y^2 \Rightarrow \frac{\partial M}{\partial x} = y$, $\frac{\partial M}{\partial y} = x$, $\frac{\partial N}{\partial x} = 0$, $\frac{\partial N}{\partial y} = 2y \Rightarrow \text{Flux} = \iint_R (y + 2y)\,dy\,dx = \int_0^1\int_{x^2}^{x} 3y\,dy\,dx$
$= \int_0^1 \left(\frac{3x^2}{2} - \frac{3x^4}{2}\right) dx = \frac{1}{5}$; $\text{Circ} = \iint_R -x\,dy\,dx = \int_0^1\int_{x^2}^{x} -x\,dy\,dx = \int_0^1 (-x^2 + x^3)\,dx = -\frac{1}{12}$

16. $M = -\sin y$, $N = x\cos y \Rightarrow \frac{\partial M}{\partial x} = 0$, $\frac{\partial M}{\partial y} = -\cos y$, $\frac{\partial N}{\partial x} = \cos y$, $\frac{\partial N}{\partial y} = -x\sin y$
$\Rightarrow \text{Flux} = \iint_R (-x\sin y)\,dx\,dy = \int_0^{\pi/2}\int_0^{\pi/2} (-x\sin y)\,dx\,dy = \int_0^{\pi/2} \left(-\frac{\pi^2}{8}\sin y\right) dy = -\frac{\pi^2}{8}$;
$\text{Circ} = \iint_R [\cos y - (-\cos y)]\,dx\,dy = \int_0^{\pi/2}\int_0^{\pi/2} 2\cos y\,dx\,dy = \int_0^{\pi/2} \pi\cos y\,dy = [\pi \sin y]_0^{\pi/2} = \pi$

17. $M = 3xy - \frac{x}{1+y^2}$, $N = e^x + \tan^{-1} y \Rightarrow \frac{\partial M}{\partial x} = 3y - \frac{1}{1+y^2}$, $\frac{\partial N}{\partial y} = \frac{1}{1+y^2}$
$\Rightarrow \text{Flux} = \iint_R \left(3y - \frac{1}{1+y^2} + \frac{1}{1+y^2}\right) dx\,dy = \iint_R 3y\,dx\,dy = \int_0^{2\pi}\int_0^{a(1+\cos\theta)} (3r\sin\theta)\,r\,dr\,d\theta$
$= \int_0^{2\pi} a^3(1+\cos\theta)^3(\sin\theta)\,d\theta = \left[-\frac{a^3}{4}(1+\cos\theta)^4\right]_0^{2\pi} = -4a^3 - (-4a^3) = 0$

18. $M = y + e^x \ln y$, $N = \frac{e^x}{y} \Rightarrow \frac{\partial M}{\partial y} = 1 + \frac{e^x}{y}$, $\frac{\partial N}{\partial x} = \frac{e^x}{y} \Rightarrow \text{Circ} = \iint_R \left[\frac{e^x}{y} - \left(1 + \frac{e^x}{y}\right)\right] dx\,dy = \iint_R (-1)\,dx\,dy$
$= \int_{-1}^{1}\int_{x^4+1}^{3-x^2} -dy\,dx = -\int_{-1}^{1} [(3 - x^2) - (x^4 + 1)]\,dx = \int_{-1}^{1} (x^4 + x^2 - 2)\,dx = -\frac{44}{15}$

19. $M = 2xy^3$, $N = 4x^2y^2 \Rightarrow \frac{\partial M}{\partial y} = 6xy^2$, $\frac{\partial N}{\partial x} = 8xy^2 \Rightarrow \text{work} = \oint_C 2xy^3\,dx + 4x^2y^2\,dy = \iint_R (8xy^2 - 6xy^2)\,dx\,dy$
$= \int_0^1\int_0^{x^3} 2xy^2\,dy\,dx = \int_0^1 \frac{2}{3} x^{10}\,dx = \frac{2}{33}$

20. $M = 4x - 2y$, $N = 2x - 4y \Rightarrow \frac{\partial M}{\partial y} = -2$, $\frac{\partial N}{\partial x} = 2 \Rightarrow \text{work} = \oint_C (4x - 2y)\,dx + (2x - 4y)\,dy$
$= \iint_R [2 - (-2)]\,dx\,dy = 4\iint_R dx\,dy = 4(\text{Area of the circle}) = 4(\pi \cdot 4) = 16\pi$

21. $M = y^2, N = x^2 \Rightarrow \frac{\partial M}{\partial y} = 2y, \frac{\partial N}{\partial x} = 2x \Rightarrow \oint_C y^2\,dx + x^2\,dy = \iint_R (2x - 2y)\,dy\,dx$
$= \int_0^1 \int_0^{1-x} (2x - 2y)\,dy\,dx = \int_0^1 (-3x^2 + 4x - 1)\,dx = \left[-x^3 + 2x^2 - x\right]_0^1 = -1 + 2 - 1 = 0$

22. $M = 3y, N = 2x \Rightarrow \frac{\partial M}{\partial y} = 3, \frac{\partial N}{\partial x} = 2 \Rightarrow \oint_C 3y\,dx + 2x\,dy = \iint_R (2 - 3)\,dx\,dy = \int_0^{\pi} \int_0^{\sin x} (-1)dy\,dx$
$= -\int_0^{\pi} \sin x\,dx = -2$

23. $M = 6y + x, N = y + 2x \Rightarrow \frac{\partial M}{\partial y} = 6, \frac{\partial N}{\partial x} = 2 \Rightarrow \oint_C (6y + x)\,dx + (y + 2x)\,dy = \iint_R (2 - 6)\,dy\,dx$
$= -4(\text{Area of the circle}) = -16\pi$

24. $M = 2x + y^2, N = 2xy + 3y \Rightarrow \frac{\partial M}{\partial y} = 2y, \frac{\partial N}{\partial x} = 2y \Rightarrow \oint_C (2x + y^2)\,dx + (2xy + 3y)\,dy = \iint_R (2y - 2y)\,dx\,dy = 0$

25. $M = x = a\cos t, N = y = a\sin t \Rightarrow dx = -a\sin t\,dt, dy = a\cos t\,dt \Rightarrow \text{Area} = \frac{1}{2}\oint_C x\,dy - y\,dx$
$= \frac{1}{2}\int_0^{2\pi} (a^2\cos^2 t + a^2\sin^2 t)\,dt = \frac{1}{2}\int_0^{2\pi} a^2\,dt = \pi a^2$

26. $M = x = a\cos t, N = y = b\sin t \Rightarrow dx = -a\sin t\,dt, dy = b\cos t\,dt \Rightarrow \text{Area} = \frac{1}{2}\oint_C x\,dy - y\,dx$
$= \frac{1}{2}\int_0^{2\pi} (ab\cos^2 t + ab\sin^2 t)\,dt = \frac{1}{2}\int_0^{2\pi} ab\,dt = \pi ab$

27. $M = x = \cos^3 t, N = y = \sin^3 t \Rightarrow dx = -3\cos^2 t\sin t\,dt, dy = 3\sin^2 t\cos t\,dt \Rightarrow \text{Area} = \frac{1}{2}\oint_C x\,dy - y\,dx$
$= \frac{1}{2}\int_0^{2\pi} (3\sin^2 t\cos^2 t)(\cos^2 t + \sin^2 t)\,dt = \frac{1}{2}\int_0^{2\pi} (3\sin^2 t\cos^2 t)\,dt = \frac{3}{8}\int_0^{2\pi} \sin^2 2t\,dt = \frac{3}{16}\int_0^{4\pi} \sin^2 u\,du$
$= \frac{3}{16}\left[\frac{u}{2} - \frac{\sin 2u}{4}\right]_0^{4\pi} = \frac{3}{8}\pi$

28. C_1: $M = x = t, N = y = 0 \Rightarrow dx = dt, dy = 0$; C_2: $M = x = (2\pi - t) - \sin(2\pi - t) = 2\pi - t + \sin t, N = y$
$= 1 - \cos(2\pi - t) = 1 - \cos t \Rightarrow dx = (\cos t - 1)\,dt,\ dy = \sin t\,dt$
$\Rightarrow \text{Area} = \frac{1}{2}\oint_C x\,dy - y\,dx = \frac{1}{2}\oint_{C_1} x\,dy - y\,dx + \frac{1}{2}\oint_{C_2} x\,dy - y\,dx$
$= \frac{1}{2}\int_0^{2\pi} (0)dt + \frac{1}{2}\int_0^{2\pi} [(2\pi - t + \sin t)(\sin t) - (1 - \cos t)(\cos t - 1)]\,dt = -\frac{1}{2}\int_0^{2\pi} (2\cos t + t\sin t - 2 - 2\pi\sin t)\,dt$
$= -\frac{1}{2}\left[3\sin t - t\cos t - 2t - 2\pi\cos t\right]_0^{2\pi} = 3\pi$

29. (a) $M = f(x), N = g(y) \Rightarrow \frac{\partial M}{\partial y} = 0, \frac{\partial N}{\partial x} = 0 \Rightarrow \oint_C f(x)\,dx + g(y)\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R 0\,dx\,dy = 0$

(b) $M = ky, N = hx \Rightarrow \frac{\partial M}{\partial y} = k, \frac{\partial N}{\partial x} = h \Rightarrow \oint_C ky\,dx + hx\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$
$= \iint_R (h - k)\,dx\,dy = (h - k)(\text{Area of the region})$

30. $M = xy^2, N = x^2y + 2x \Rightarrow \frac{\partial M}{\partial y} = 2xy, \frac{\partial N}{\partial x} = 2xy + 2 \Rightarrow \oint_C xy^2\,dx + (x^2y + 2x)\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$
$= \iint_R (2xy + 2 - 2xy)\,dx\,dy = 2\iint_R dx\,dy = 2$ times the area of the square

31. The integral is 0 for any simple closed plane curve C. The reasoning: By the tangential form of Green's Theorem, with $M = 4x^3y$ and $N = x^4$, $\oint_C 4x^3y\,dx + x^4\,dy = \iint_R \left[\frac{\partial}{\partial x}(x^4) - \frac{\partial}{\partial y}(4x^3y)\right] dx\,dy$
$= \iint_R \underbrace{(4x^3 - 4x^3)}_{0}\,dx\,dy = 0.$

32. The integral is 0 for any simple closed curve C. The reasoning: By the normal form of Green's theorem, with $M = x^3$ and $N = -y^3$, $\oint_C -y^3\,dy + x^3\,dx = \iint_R \left[\underbrace{\frac{\partial}{\partial x}(-y^3)}_{0} - \underbrace{\frac{\partial}{\partial y}(x^3)}_{0}\right] dx\,dy = 0.$

33. Let $M = x$ and $N = 0 \Rightarrow \frac{\partial M}{\partial x} = 1$ and $\frac{\partial N}{\partial y} = 0 \Rightarrow \oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy \Rightarrow \oint_C x\,dy$
$= \iint_R (1+0)\,dx\,dy \Rightarrow$ Area of $R = \iint_R dx\,dy = \oint_C x\,dy$; similarly, $M = y$ and $N = 0 \Rightarrow \frac{\partial M}{\partial y} = 1$ and $\frac{\partial N}{\partial x} = 0 \Rightarrow \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} + \frac{\partial M}{\partial y}\right) dy\,dx \Rightarrow \oint_C y\,dx = \iint_R (0-1)\,dy\,dx \Rightarrow -\oint_C y\,dx$
$= \iint_R dx\,dy =$ Area of R

34. $\int_a^b f(x)\,dx =$ Area of $R = -\oint_C y\,dx$, from Exercise 33

35. Let $\delta(x,y) = 1 \Rightarrow \overline{x} = \frac{M_y}{M} = \frac{\iint_R x\,\delta(x,y)\,dA}{\iint_R \delta(x,y)\,dA} = \frac{\iint_R x\,dA}{\iint_R dA} = \frac{\iint_R x\,dA}{A} \Rightarrow A\overline{x} = \iint_R x\,dA = \iint_R (x+0)\,dx\,dy$
$= \oint_C \frac{x^2}{2}\,dy$, $A\overline{x} = \iint_R x\,dA = \iint_R (0+x)\,dx\,dy = -\oint_C xy\,dx$, and $A\overline{x} = \iint_R x\,dA = \iint_R \left(\frac{2}{3}x + \frac{1}{3}x\right) dx\,dy$
$= \oint_C \frac{1}{3}x^2\,dy - \frac{1}{3}xy\,dx \Rightarrow \frac{1}{2}\oint_C x^2\,dy = -\oint_C xy\,dx = \frac{1}{3}\oint_C x^2\,dy - xy\,dx = A\overline{x}$

36. If $\delta(x,y) = 1$, then $I_y = \iint_R x^2\,\delta(x,y)\,dA = \iint_R x^2\,dA = \iint_R (x^2+0)\,dy\,dx = \frac{1}{3}\oint_C x^3\,dy$,
$\iint_R x^2\,dA = \iint_R (0+x^2)\,dy\,dx = -\oint_C x^2y\,dx$, and $\iint_R x^2\,dA = \iint_R \left(\frac{3}{4}x^2 + \frac{1}{4}x^2\right) dy\,dx$
$= \oint_C \frac{1}{4}x^3\,dy - \frac{1}{4}x^2y\,dx = \frac{1}{4}\oint_C x^3\,dy - x^2y\,dx \Rightarrow \frac{1}{3}\oint_C x^3\,dy = -\oint_C x^2y\,dx = \frac{1}{4}\oint_C x^3\,dy - x^2y\,dx = I_y$

37. $M = \frac{\partial f}{\partial y}$, $N = -\frac{\partial f}{\partial x} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y^2}$, $\frac{\partial N}{\partial x} = -\frac{\partial^2 f}{\partial x^2} \Rightarrow \oint_C \frac{\partial f}{\partial y}\,dx - \frac{\partial f}{\partial x}\,dy = \iint_R \left(-\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2}\right) dx\,dy = 0$ for such curves C

38. $M = \frac{1}{4}x^2y + \frac{1}{3}y^3$, $N = x \Rightarrow \frac{\partial M}{\partial y} = \frac{1}{4}x^2 + y^2$, $\frac{\partial N}{\partial x} = 1 \Rightarrow \text{Curl} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1 - \left(\frac{1}{4}x^2 + y^2\right) > 0$ in the interior of the ellipse $\frac{1}{4}x^2 + y^2 = 1 \Rightarrow$ work $= \int_C \mathbf{F}\cdot d\mathbf{r} = \iint_R \left(1 - \frac{1}{4}x^2 - y^2\right) dx\,dy$ will be maximized on the region $R = \{(x,y)\,|\,\text{curl}\,\mathbf{F}\} \geq 0$ or over the region enclosed by $1 = \frac{1}{4}x^2 + y^2$

39. (a) $\nabla f = \left(\frac{2x}{x^2+y^2}\right)\mathbf{i} + \left(\frac{2y}{x^2+y^2}\right)\mathbf{j} \Rightarrow M = \frac{2x}{x^2+y^2}$, $N = \frac{2y}{x^2+y^2}$; since M, N are discontinuous at $(0,0)$, we compute $\int_C \nabla f\cdot\mathbf{n}\,ds$ directly since Green's Theorem does not apply. Let $x = a\cos t$, $y = a\sin t \Rightarrow dx = -a\sin t\,dt$, $dy = a\cos t\,dt$, $M = \frac{2}{a}\cos t$, $N = \frac{2}{a}\sin t$, $0 \leq t \leq 2\pi$, so $\int_C \nabla f\cdot\mathbf{n}\,ds = \int_C M\,dy - N\,dx$
$= \int_0^{2\pi}\left[\left(\frac{2}{a}\cos t\right)(a\cos t) - \left(\frac{2}{a}\sin t\right)(-a\sin t)\right]dt = \int_0^{2\pi} 2(\cos^2 t + \sin^2 t)\,dt = 4\pi$. Note that this holds for any

a > 0, so $\int_C \nabla f \cdot \mathbf{n}\, ds = 4\pi$ for any circle C centered at (0, 0) traversed counterclockwise and $\int_C \nabla f \cdot \mathbf{n}\, ds = -4\pi$ if C is traversed clockwise.

(b) If K does not enclose the point (0, 0) we may apply Green's Theorem: $\int_C \nabla f \cdot \mathbf{n}\, ds = \int_C M\, dy - N\, dx$ $= \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\, dy = \iint_R \left(\frac{2(y^2-x^2)}{(x^2+y^2)^2} + \frac{2(x^2-y^2)}{(x^2+y^2)^2}\right) dx\, dy = \iint_R 0\, dx\, dy = 0$. If K does enclose the point (0, 0) we proceed as follows:

Choose a small enough so that the circle C centered at (0, 0) of radius a lies entirely within K. Green's Theorem applies to the region R that lies between K and C. Thus, as before, $0 = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\, dy$ $= \int_K M\, dy - N\, dx + \int_C M\, dy - N\, dx$ where K is traversed counterclockwise and C is traversed clockwise. Hence by part (a) $0 = \left[\int_K M\, dy - N\, dx\right] - 4\pi \Rightarrow 4\pi = \int_K M\, dy - N\, dx = \int_K \nabla f \cdot \mathbf{n}\, ds$. We have shown:

$$\int_K \nabla f \cdot \mathbf{n}\, ds = \begin{cases} 0 & \text{if } (0,0) \text{ lies inside K} \\ 4\pi & \text{if } (0,0) \text{ lies outside K} \end{cases}$$

40. Assume a particle has a closed trajectory in R and let C_1 be the path $\Rightarrow$ C_1 encloses a simply connected region $R_1 \Rightarrow C_1$ is a simple closed curve. Then the flux over R_1 is $\oint_{C_1} \mathbf{F} \cdot \mathbf{n}\, ds = 0$, since the velocity vectors **F** are tangent to C_1. But $0 = \oint_{C_1} \mathbf{F} \cdot \mathbf{n}\, ds = \oint_{C_1} M\, dy - N\, dx = \iint_{R_1} \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\, dy \Rightarrow M_x + N_y = 0$, which is a contradiction. Therefore, C_1 cannot be a closed trajectory.

41. $\int_{g_1(y)}^{g_2(y)} \frac{\partial N}{\partial x}\, dx\, dy = N(g_2(y), y) - N(g_1(y), y) \Rightarrow \int_c^d \int_{g_1(y)}^{g_2(y)} \left(\frac{\partial N}{\partial x}\, dx\right) dy = \int_c^d [N(g_2(y), y) - N(g_1(y), y)]\, dy$
$= \int_c^d N(g_2(y), y)\, dy - \int_c^d N(g_1(y), y)\, dy = \int_c^d N(g_2(y), y)\, dy + \int_d^c N(g_1(y), y)\, dy = \int_{C_2} N\, dy + \int_{C_1} N\, dy$
$= \oint_C dy \Rightarrow \oint_C N\, dy = \iint_R \frac{\partial N}{\partial x}\, dx\, dy$

42. The curl of a conservative two-dimensional field is zero. The reasoning: A two-dimensional field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ can be considered to be the restriction to the xy-plane of a three-dimensional field whose k component is zero, and whose **i** and **j** components are independent of z. For such a field to be conservative, we must have $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$ by the component test in Section 16.3 $\Rightarrow$ curl $\mathbf{F} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$.

43-46. Example CAS commands:

<u>Maple</u>:

```
with( plots );#43
M := (x,y) -> 2*x-y;
N := (x,y) -> x+3*y;
C := x^2 + 4*y^2 = 4;
implicitplot( C, x=-2..2, y=-2..2, scaling=constrained, title="#43(a) (Section 16.4)" );
curlF_k := D[1](N) - D[2](M):                    # (b)
'curlF_k' = curlF_k(x,y);
top,bot := solve( C, y );                        # (c)
left,right := -2, 2;
q1 := Int( Int( curlF_k(x,y), y=bot..top ), x=left..right );
value( q1 );
```

<u>Mathematica</u>: (functions and bounds will vary)

The **ImplicitPlot** command will be useful for 43 and 44, but is not needed for 43 and 44. In 44, the equation of the line from (0, 4) to (2, 0) must be determined first.

```
Clear[x, y, f]
<<Graphics`ImplicitPlot`
f[x_, y_]:= {2x − y, x + 3y}
curve= x^2 + 4y^2 ==4
ImplicitPlot[curve, {x, −3, 3},{y, −2, 2}, AspectRatio → Automatic, AxesLabel → {x, y}];
ybounds= Solve[curve, y]
{y1, y2}=y/.ybounds;
integrand:=D[f[x,y][[2]], x] − D[f[x,y][[1]], y]//Simplify
Integrate[integrand, {x, −2, 2}, {y, y1, y2}]
N[%]
```

Bounds for y are determined differently in 45 and 46. In 46, note equation of the line from (0, 4) to (2, 0).

```
Clear[x, y, f]
f[x_, y_]:= {x Exp[y], 4x^2 Log[y]}
ybound = 4 − 2x
Plot[{0, ybound}, {x, 0,2. 1}, AspectRatio → Automatic, AxesLabel → {x, y}];
integrand:=D[f[x, y][[2]], x] − D[f[x, y][[1]], y]//Simplify
Integrate[integrand, {x, 0, 2}, {y, 0, ybound}]
N[%]
```

16.5 SURFACES AND AREA

1. In cylindrical coordinates, let $x = r\cos\theta$, $y = r\sin\theta$, $z = \left(\sqrt{x^2+y^2}\right)^2 = r^2$. Then $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r^2\mathbf{k}$, $0 \le r \le 2$, $0 \le \theta \le 2\pi$.

2. In cylindrical coordinates, let $x = r\cos\theta$, $y = r\sin\theta$, $z = 9 - x^2 - y^2 = 9 - r^2$. Then $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + (9-r^2)\,\mathbf{k}$; $z \ge 0 \Rightarrow 9 - r^2 \ge 0 \Rightarrow r^2 \le 9 \Rightarrow -3 \le r \le 3$, $0 \le \theta \le 2\pi$. But $-3 \le r \le 0$ gives the same points as $0 \le r \le 3$, so let $0 \le r \le 3$.

3. In cylindrical coordinates, let $x = r\cos\theta$, $y = r\sin\theta$, $z = \frac{\sqrt{x^2+y^2}}{2} \Rightarrow z = \frac{r}{2}$. Then $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + \left(\frac{r}{2}\right)\mathbf{k}$. For $0 \le z \le 3$, $0 \le \frac{r}{2} \le 3 \Rightarrow 0 \le r \le 6$; to get only the first octant, let $0 \le \theta \le \frac{\pi}{2}$.

4. In cylindrical coordinates, let $x = r\cos\theta$, $y = r\sin\theta$, $z = 2\sqrt{x^2+y^2} \Rightarrow z = 2r$. Then $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + 2r\mathbf{k}$. For $2 \le z \le 4$, $2 \le 2r \le 4 \Rightarrow 1 \le r \le 2$, and let $0 \le \theta \le 2\pi$.

5. In cylindrical coordinates, let $x = r\cos\theta$, $y = r\sin\theta$; since $x^2 + y^2 = r^2 \Rightarrow z^2 = 9 - (x^2+y^2) = 9 - r^2$ $\Rightarrow z = \sqrt{9-r^2}$, $z \ge 0$. Then $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + \sqrt{9-r^2}\mathbf{k}$. Let $0 \le \theta \le 2\pi$. For the domain of r: $z = \sqrt{x^2+y^2}$ and $x^2 + y^2 + z^2 = 9 \Rightarrow x^2 + y^2 + \left(\sqrt{x^2+y^2}\right)^2 = 9 \Rightarrow 2(x^2+y^2) = 9 \Rightarrow 2r^2 = 9$ $\Rightarrow r = \frac{3}{\sqrt{2}} \Rightarrow 0 \le r \le \frac{3}{\sqrt{2}}$.

6. In cylindrical coordinates, $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + \sqrt{4-r^2}\,\mathbf{k}$ (see Exercise 5 above with $x^2 + y^2 + z^2 = 4$, instead of $x^2 + y^2 + z^2 = 9$). For the first octant, let $0 \le \theta \le \frac{\pi}{2}$. For the domain of r: $z = \sqrt{x^2+y^2}$ and $x^2 + y^2 + z^2 = 4 \Rightarrow x^2 + y^2 + \left(\sqrt{x^2+y^2}\right)^2 = 4 \Rightarrow 2(x^2+y^2) = 4 \Rightarrow 2r^2 = 4 \Rightarrow r = \sqrt{2}$. Thus, let $\sqrt{2} \le r \le 2$ (to get the portion of the sphere between the cone and the xy-plane).

7. In spherical coordinates, $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, $\rho = \sqrt{x^2+y^2+z^2} \Rightarrow \rho^2 = 3 \Rightarrow \rho = \sqrt{3}$ $\Rightarrow z = \sqrt{3}\cos\phi$ for the sphere; $z = \frac{\sqrt{3}}{2} = \sqrt{3}\cos\phi \Rightarrow \cos\phi = \frac{1}{2} \Rightarrow \phi = \frac{\pi}{3}$; $z = -\frac{\sqrt{3}}{2} \Rightarrow -\frac{\sqrt{3}}{2} = \sqrt{3}\cos\phi$ $\Rightarrow \cos\phi = -\frac{1}{2} \Rightarrow \phi = \frac{2\pi}{3}$. Then $\mathbf{r}(\phi,\theta) = \left(\sqrt{3}\sin\phi\cos\theta\right)\mathbf{i} + \left(\sqrt{3}\sin\phi\sin\theta\right)\mathbf{j} + \left(\sqrt{3}\cos\phi\right)\mathbf{k}$, $\frac{\pi}{3} \le \phi \le \frac{2\pi}{3}$ and $0 \le \theta \le 2\pi$.

8. In spherical coordinates, $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, $\rho = \sqrt{x^2+y^2+z^2} \Rightarrow \rho^2 = 8 \Rightarrow \rho = \sqrt{8} = 2\sqrt{2}$ $\Rightarrow x = 2\sqrt{2}\sin\phi\cos\theta$, $y = 2\sqrt{2}\sin\phi\sin\theta$, and $z = 2\sqrt{2}\cos\phi$. Thus let $\mathbf{r}(\phi,\theta) = \left(2\sqrt{2}\sin\phi\cos\theta\right)\mathbf{i} + \left(2\sqrt{2}\sin\phi\sin\theta\right)\mathbf{j} + \left(2\sqrt{2}\cos\phi\right)\mathbf{k}$; $z = -2 \Rightarrow -2 = 2\sqrt{2}\cos\phi$ $\Rightarrow \cos\phi = -\frac{1}{\sqrt{2}} \Rightarrow \phi = \frac{3\pi}{4}$; $z = 2\sqrt{2} \Rightarrow 2\sqrt{2} = 2\sqrt{2}\cos\phi \Rightarrow \cos\phi = 1 \Rightarrow \phi = 0$. Thus $0 \le \phi \le \frac{3\pi}{4}$ and $0 \le \theta \le 2\pi$.

9. Since $z = 4 - y^2$, we can let $\mathbf{r}$ be a function of x and y $\Rightarrow \mathbf{r}(x,y) = x\mathbf{i} + y\mathbf{j} + (4-y^2)\mathbf{k}$. Then $z = 0$ $\Rightarrow 0 = 4 - y^2 \Rightarrow y = \pm 2$. Thus, let $-2 \le y \le 2$ and $0 \le x \le 2$.

10. Since $y = x^2$, we can let $\mathbf{r}$ be a function of x and z $\Rightarrow \mathbf{r}(x,z) = x\mathbf{i} + x^2\mathbf{j} + z\mathbf{k}$. Then $y = 2$ $\Rightarrow x^2 = 2 \Rightarrow x = \pm\sqrt{2}$. Thus, let $-\sqrt{2} \le x \le \sqrt{2}$ and $0 \le z \le 3$.

11. When $x = 0$, let $y^2 + z^2 = 9$ be the circular section in the yz-plane. Use polar coordinates in the yz-plane $\Rightarrow y = 3\cos\theta$ and $z = 3\sin\theta$. Thus let $x = u$ and $\theta = v \Rightarrow \mathbf{r}(u,v) = u\mathbf{i} + (3\cos v)\mathbf{j} + (3\sin v)\mathbf{k}$ where $0 \le u \le 3$, and $0 \le v \le 2\pi$.

12. When $y = 0$, let $x^2 + z^2 = 4$ be the circular section in the xz-plane. Use polar coordinates in the xz-plane $\Rightarrow x = 2\cos\theta$ and $z = 2\sin\theta$. Thus let $y = u$ and $\theta = v \Rightarrow \mathbf{r}(u,v) = (2\cos v)\mathbf{i} + u\mathbf{j} + (3\sin v)\mathbf{k}$ where $-2 \le u \le 2$, and $0 \le v \le \pi$ (since we want the portion <u>above</u> the xy-plane).

13. (a) $x + y + z = 1 \Rightarrow z = 1 - x - y$. In cylindrical coordinates, let $x = r\cos\theta$ and $y = r\sin\theta$ $\Rightarrow z = 1 - r\cos\theta - r\sin\theta \Rightarrow \mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + (1 - r\cos\theta - r\sin\theta)\mathbf{k}$, $0 \le \theta \le 2\pi$ and $0 \le r \le 3$.

(b) In a fashion similar to cylindrical coordinates, but working in the yz-plane instead of the xy-plane, let $y = u\cos v$, $z = u\sin v$ where $u = \sqrt{y^2+z^2}$ and v is the angle formed by (x,y,z), $(x,0,0)$, and $(x,y,0)$ with $(x,0,0)$ as vertex. Since $x + y + z = 1 \Rightarrow x = 1 - y - z \Rightarrow x = 1 - u\cos v - u\sin v$, then $\mathbf{r}$ is a function of u and v $\Rightarrow \mathbf{r}(u,v) = (1 - u\cos v - u\sin v)\mathbf{i} + (u\cos v)\mathbf{j} + (u\sin v)\mathbf{k}$, $0 \le u \le 3$ and $0 \le v \le 2\pi$.

14. (a) In a fashion similar to cylindrical coordinates, but working in the xz-plane instead of the xy-plane, let $x = u\cos v$, $z = u\sin v$ where $u = \sqrt{x^2+z^2}$ and v is the angle formed by (x,y,z), $(y,0,0)$, and $(x,y,0)$ with vertex $(y,0,0)$. Since $x - y + 2z = 2 \Rightarrow y = x + 2z - 2$, then $\mathbf{r}(u,v)$ $= (u\cos v)\mathbf{i} + (u\cos v + 2u\sin v - 2)\mathbf{j} + (u\sin v)\mathbf{k}$, $0 \le u \le \sqrt{3}$ and $0 \le v \le 2\pi$.

(b) In a fashion similar to cylindrical coordinates, but working in the yz-plane instead of the xy-plane, let $y = u\cos v$, $z = u\sin v$ where $u = \sqrt{y^2+z^2}$ and v is the angle formed by (x,y,z), $(x,0,0)$, and $(x,y,0)$ with vertex $(x,0,0)$. Since $x - y + 2z = 2 \Rightarrow x = y - 2z + 2$, then $\mathbf{r}(u,v)$ $= (u\cos v - 2u\sin v + 2)\mathbf{i} + (u\cos v)\mathbf{j} + (u\sin v)\mathbf{k}$, $0 \le u \le \sqrt{2}$ and $0 \le v \le 2\pi$.

15. Let $x = w\cos v$ and $z = w\sin v$. Then $(x-2)^2 + z^2 = 4 \Rightarrow x^2 - 4x + z^2 = 0 \Rightarrow w^2\cos^2 v - 4w\cos v + w^2\sin^2 v$ $= 0 \Rightarrow w^2 - 4w\cos v = 0 \Rightarrow w = 0$ or $w - 4\cos v = 0 \Rightarrow w = 0$ or $w = 4\cos v$. Now $w = 0 \Rightarrow x = 0$ and $y = 0$, which is a line not a cylinder. Therefore, let $w = 4\cos v \Rightarrow x = (4\cos v)(\cos v) = 4\cos^2 v$ and $z = 4\cos v\sin v$. Finally, let $y = u$. Then $\mathbf{r}(u,v) = (4\cos^2 v)\,\mathbf{i} + u\mathbf{j} + (4\cos v\sin v)\mathbf{k}$, $-\frac{\pi}{2} \le v \le \frac{\pi}{2}$ and $0 \le u \le 3$.

16. Let $y = w\cos v$ and $z = w\sin v$. Then $y^2 + (z-5)^2 = 25 \Rightarrow y^2 + z^2 - 10z = 0$
$\Rightarrow w^2\cos^2 v + w^2\sin^2 v - 10w\sin v = 0 \Rightarrow w^2 - 10w\sin v = 0 \Rightarrow w(w - 10\sin v) = 0 \Rightarrow w = 0$ or $w = 10\sin v$. Now $w = 0 \Rightarrow y = 0$ and $z = 0$, which is a line not a cylinder. Therefore, let $w = 10\sin v$ $\Rightarrow y = 10\sin v\cos v$ and $z = 10\sin^2 v$. Finally, let $x = u$. Then $\mathbf{r}(u,v) = u\mathbf{i} + (10\sin v\cos v)\mathbf{j} + (10\sin^2 v)\,\mathbf{k}$, $0 \le u \le 10$ and $0 \le v \le \pi$.

17. Let $x = r\cos\theta$ and $y = r\sin\theta$. Then $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + \left(\frac{2 - r\sin\theta}{2}\right)\mathbf{k}$, $0 \le r \le 1$ and $0 \le \theta \le 2\pi$
$\Rightarrow \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} - \left(\frac{\sin\theta}{2}\right)\mathbf{k}$ and $\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j} - \left(\frac{r\cos\theta}{2}\right)\mathbf{k}$

$$\Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & -\frac{\sin\theta}{2} \\ -r\sin\theta & r\cos\theta & -\frac{r\cos\theta}{2} \end{vmatrix}$$

$$= \left(\frac{-r\sin\theta\cos\theta}{2} + \frac{(\sin\theta)(r\cos\theta)}{2}\right)\mathbf{i} + \left(\frac{r\sin^2\theta}{2} + \frac{r\cos^2\theta}{2}\right)\mathbf{j} + (r\cos^2\theta + r\sin^2\theta)\,\mathbf{k} = \frac{r}{2}\,\mathbf{j} + r\mathbf{k}$$

$$\Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{\frac{r^2}{4} + r^2} = \frac{\sqrt{5}\,r}{2} \Rightarrow A = \int_0^{2\pi}\int_0^1 \frac{\sqrt{5}\,r}{2}\,dr\,d\theta = \int_0^{2\pi}\left[\frac{\sqrt{5}\,r^2}{4}\right]_0^1 d\theta = \int_0^{2\pi} d\theta = \frac{\pi\sqrt{5}}{2}$$

18. Let $x = r\cos\theta$ and $y = r\sin\theta \Rightarrow z = -x = -r\cos\theta$, $0 \le r \le 2$ and $0 \le \theta \le 2\pi$. Then $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} - (r\cos\theta)\mathbf{k} \Rightarrow \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} - (\cos\theta)\mathbf{k}$ and $\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j} + (r\sin\theta)\mathbf{k}$

$$\Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & -\cos\theta \\ -r\sin\theta & r\cos\theta & r\sin\theta \end{vmatrix}$$

$$= (r\sin^2\theta + r\cos^2\theta)\,\mathbf{i} + (r\sin\theta\cos\theta - r\sin\theta\cos\theta)\mathbf{j} + (r\cos^2\theta + r\sin^2\theta)\,\mathbf{k} = r\mathbf{i} + r\mathbf{k}$$

$$\Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{r^2 + r^2} = r\sqrt{2} \Rightarrow A = \int_0^{2\pi}\int_0^2 r\sqrt{2}\,dr\,d\theta = \int_0^{2\pi}\left[\frac{r^2\sqrt{2}}{2}\right]_0^2 d\theta = \int_0^{2\pi} 2\sqrt{2}\,d\theta = 4\pi\sqrt{2}$$

19. Let $x = r\cos\theta$ and $y = r\sin\theta \Rightarrow z = 2\sqrt{x^2 + y^2} = 2r$, $1 \le r \le 3$ and $0 \le \theta \le 2\pi$. Then $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + 2r\mathbf{k} \Rightarrow \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + 2\mathbf{k}$ and $\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j}$

$$\Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 2 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (-2r\cos\theta)\mathbf{i} - (2r\sin\theta)\mathbf{j} + (r\cos^2\theta + r\sin^2\theta)\,\mathbf{k}$$

$$= (-2r\cos\theta)\mathbf{i} - (2r\sin\theta)\mathbf{j} + r\mathbf{k} \Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{4r^2\cos^2\theta + 4r^2\sin^2\theta + r^2} = \sqrt{5r^2} = r\sqrt{5}$$

$$\Rightarrow A = \int_0^{2\pi}\int_1^3 r\sqrt{5}\,dr\,d\theta = \int_0^{2\pi}\left[\frac{r^2\sqrt{5}}{2}\right]_1^3 d\theta = \int_0^{2\pi} 4\sqrt{5}\,d\theta = 8\pi\sqrt{5}$$

20. Let $x = r\cos\theta$ and $y = r\sin\theta \Rightarrow z = \frac{\sqrt{x^2 + y^2}}{3} = \frac{r}{3}$, $3 \le r \le 4$ and $0 \le \theta \le 2\pi$. Then $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + \left(\frac{r}{3}\right)\mathbf{k} \Rightarrow \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + \left(\frac{1}{3}\right)\mathbf{k}$ and $\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j}$

$$\Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & \frac{1}{3} \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \left(-\frac{1}{3}r\cos\theta\right)\mathbf{i} - \left(\frac{1}{3}r\sin\theta\right)\mathbf{j} + (r\cos^2\theta + r\sin^2\theta)\,\mathbf{k}$$

$$= \left(-\frac{1}{3}r\cos\theta\right)\mathbf{i} - \left(\frac{1}{3}r\sin\theta\right)\mathbf{j} + r\mathbf{k} \Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{\frac{1}{9}r^2\cos^2\theta + \frac{1}{9}r^2\sin^2\theta + r^2} = \sqrt{\frac{10r^2}{9}} = \frac{r\sqrt{10}}{3}$$

$$\Rightarrow A = \int_0^{2\pi}\int_3^4 \frac{r\sqrt{10}}{3}\,dr\,d\theta = \int_0^{2\pi}\left[\frac{r^2\sqrt{10}}{6}\right]_3^4 d\theta = \int_0^{2\pi}\frac{7\sqrt{10}}{6}\,d\theta = \frac{7\pi\sqrt{10}}{3}$$

21. Let $x = r\cos\theta$ and $y = r\sin\theta \Rightarrow r^2 = x^2 + y^2 = 1$, $1 \le z \le 4$ and $0 \le \theta \le 2\pi$. Then $\mathbf{r}(z,\theta) = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + z\mathbf{k} \Rightarrow \mathbf{r}_z = \mathbf{k}$ and $\mathbf{r}_\theta = (-\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j}$

$$\Rightarrow \mathbf{r}_\theta \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (\cos\theta)\mathbf{i} + (\sin\theta)\,\mathbf{j} \Rightarrow |\mathbf{r}_\theta \times \mathbf{r}_z| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

$$\Rightarrow A = \int_0^{2\pi}\int_1^4 1\,dr\,d\theta = \int_0^{2\pi} 3\,d\theta = 6\pi$$

22. Let $x = u\cos v$ and $z = u\sin v \Rightarrow u^2 = x^2 + z^2 = 10, -1 \le y \le 1, 0 \le v \le 2\pi$. Then $\mathbf{r}(y, v) = (u\cos v)\mathbf{i} + y\mathbf{j} + (u\sin v)\mathbf{k} = \left(\sqrt{10}\cos v\right)\mathbf{i} + y\mathbf{j} + \left(\sqrt{10}\sin v\right)\mathbf{k}$

$$\Rightarrow \mathbf{r}_v = \left(-\sqrt{10}\sin v\right)\mathbf{i} + \left(\sqrt{10}\cos v\right)\mathbf{k} \text{ and } \mathbf{r}_y = \mathbf{j} \Rightarrow \mathbf{r}_v \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sqrt{10}\sin v & 0 & \sqrt{10}\cos v \\ 0 & 1 & 0 \end{vmatrix}$$

$$= \left(-\sqrt{10}\cos v\right)\mathbf{i} - \left(\sqrt{10}\sin v\right)\mathbf{k} \Rightarrow |\mathbf{r}_v \times \mathbf{r}_y| = \sqrt{10} \Rightarrow A = \int_0^{2\pi}\int_{-1}^{1}\sqrt{10}\,du\,dv = \int_0^{2\pi}\left[\sqrt{10}u\right]_{-1}^{1} dv$$

$$= \int_0^{2\pi} 2\sqrt{10}\,dv = 4\pi\sqrt{10}$$

23. $z = 2 - x^2 - y^2$ and $z = \sqrt{x^2 + y^2} \Rightarrow z = 2 - z^2 \Rightarrow z^2 + z - 2 = 0 \Rightarrow z = -2$ or $z = 1$. Since $z = \sqrt{x^2 + y^2} \ge 0$, we get $z = 1$ where the cone intersects the paraboloid. When $x = 0$ and $y = 0$, $z = 2 \Rightarrow$ the vertex of the paraboloid is $(0, 0, 2)$. Therefore, z ranges from 1 to 2 on the "cap" $\Rightarrow$ r ranges from 1 (when $x^2 + y^2 = 1$) to 0 (when $x = 0$ and $y = 0$ at the vertex). Let $x = r\cos\theta$, $y = r\sin\theta$, and $z = 2 - r^2$. Then $\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + (2 - r^2)\,\mathbf{k}$, $0 \le r \le 1$, $0 \le \theta \le 2\pi \Rightarrow \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} - 2r\mathbf{k}$ and

$$\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j} \Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & -2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

$$= (2r^2\cos\theta)\,\mathbf{i} + (2r^2\sin\theta)\,\mathbf{j} + r\mathbf{k} \Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{4r^4\cos^2\theta + 4r^4\sin^2\theta + r^2} = r\sqrt{4r^2 + 1}$$

$$\Rightarrow A = \int_0^{2\pi}\int_0^1 r\sqrt{4r^2 + 1}\,dr\,d\theta = \int_0^{2\pi}\left[\tfrac{1}{12}\left(4r^2 + 1\right)^{3/2}\right]_0^1 d\theta = \int_0^{2\pi}\left(\tfrac{5\sqrt{5} - 1}{12}\right)d\theta = \tfrac{\pi}{6}\left(5\sqrt{5} - 1\right)$$

24. Let $x = r\cos\theta$, $y = r\sin\theta$ and $z = x^2 + y^2 = r^2$. Then $\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r^2\mathbf{k}$, $1 \le r \le 2$, $0 \le \theta \le 2\pi \Rightarrow \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + 2r\mathbf{k}$ and $\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j}$

$$\Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (-2r^2\cos\theta)\,\mathbf{i} - (2r^2\sin\theta)\,\mathbf{j} + r\mathbf{k} \Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta|$$

$$= \sqrt{4r^4\cos^2\theta + 4r^4\sin^2\theta + r^2} = r\sqrt{4r^2 + 1} \Rightarrow A = \int_0^{2\pi}\int_1^2 r\sqrt{4r^2 + 1}\,dr\,d\theta = \int_0^{2\pi}\left[\tfrac{1}{12}\left(4r^2 + 1\right)^{3/2}\right]_1^2 d\theta$$

$$= \int_0^{2\pi}\left(\tfrac{17\sqrt{17} - 5\sqrt{5}}{12}\right)d\theta = \tfrac{\pi}{6}\left(17\sqrt{17} - 5\sqrt{5}\right)$$

25. Let $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, and $z = \rho\cos\phi \Rightarrow \rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{2}$ on the sphere. Next, $x^2 + y^2 + z^2 = 2$ and $z = \sqrt{x^2 + y^2} \Rightarrow z^2 + z^2 = 2 \Rightarrow z^2 = 1 \Rightarrow z = 1$ since $z \ge 0 \Rightarrow \phi = \frac{\pi}{4}$. For the lower portion of the sphere cut by the cone, we get $\phi = \pi$. Then

$$\mathbf{r}(\phi, \theta) = \left(\sqrt{2}\sin\phi\cos\theta\right)\mathbf{i} + \left(\sqrt{2}\sin\phi\sin\theta\right)\mathbf{j} + \left(\sqrt{2}\cos\phi\right)\mathbf{k},\ \tfrac{\pi}{4} \le \phi \le \pi,\ 0 \le \theta \le 2\pi$$

$$\Rightarrow \mathbf{r}_\phi = \left(\sqrt{2}\cos\phi\cos\theta\right)\mathbf{i} + \left(\sqrt{2}\cos\phi\sin\theta\right)\mathbf{j} - \left(\sqrt{2}\sin\phi\right)\mathbf{k} \text{ and } \mathbf{r}_\theta = \left(-\sqrt{2}\sin\phi\sin\theta\right)\mathbf{i} + \left(\sqrt{2}\sin\phi\cos\theta\right)\mathbf{j}$$

$$\Rightarrow \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sqrt{2}\cos\phi\cos\theta & \sqrt{2}\cos\phi\sin\theta & -\sqrt{2}\sin\phi \\ -\sqrt{2}\sin\phi\sin\theta & \sqrt{2}\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= (2\sin^2\phi\cos\theta)\,\mathbf{i} + (2\sin^2\phi\sin\theta)\,\mathbf{j} + (2\sin\phi\cos\phi)\mathbf{k}$$

$\Rightarrow |\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{4\sin^4\phi\cos^2\theta + 4\sin^4\phi\sin^2\theta + 4\sin^2\phi\cos^2\phi} = \sqrt{4\sin^2\phi} = 2|\sin\phi| = 2\sin\phi$

$\Rightarrow A = \int_0^{2\pi}\int_{\pi/4}^{\pi} 2\sin\phi\, d\phi\, d\theta = \int_0^{2\pi}\left(2+\sqrt{2}\right)d\theta = \left(4+2\sqrt{2}\right)\pi$

26. Let $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, and $z = \rho\cos\phi \Rightarrow \rho = \sqrt{x^2+y^2+z^2} = 2$ on the sphere. Next, $z = -1 \Rightarrow -1 = 2\cos\phi \Rightarrow \cos\phi = -\frac{1}{2} \Rightarrow \phi = \frac{2\pi}{3}$; $z = \sqrt{3} \Rightarrow \sqrt{3} = 2\cos\phi \Rightarrow \cos\phi = \frac{\sqrt{3}}{2} \Rightarrow \phi = \frac{\pi}{6}$. Then $\mathbf{r}(\phi,\theta) = (2\sin\phi\cos\theta)\mathbf{i} + (2\sin\phi\sin\theta)\mathbf{j} + (2\cos\phi)\mathbf{k}$, $\frac{\pi}{6} \le \phi \le \frac{2\pi}{3}$, $0 \le \theta \le 2\pi$

$\Rightarrow \mathbf{r}_\phi = (2\cos\phi\cos\theta)\mathbf{i} + (2\cos\phi\sin\theta)\mathbf{j} - (2\sin\phi)\mathbf{k}$ and

$\mathbf{r}_\theta = (-2\sin\phi\sin\theta)\mathbf{i} + (2\sin\phi\cos\theta)\mathbf{j}$

$$\Rightarrow \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2\cos\phi\cos\theta & 2\cos\phi\sin\theta & -2\sin\phi \\ -2\sin\phi\sin\theta & 2\sin\phi\cos\theta & 0 \end{vmatrix}$$

$= (4\sin^2\phi\cos\theta)\mathbf{i} + (4\sin^2\phi\sin\theta)\mathbf{j} + (4\sin\phi\cos\phi)\mathbf{k}$

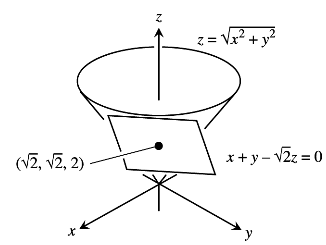
$\Rightarrow |\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{16\sin^4\phi\cos^2\theta + 16\sin^4\phi\sin^2\theta + 16\sin^2\phi\cos^2\phi} = \sqrt{16\sin^2\phi} = 4|\sin\phi| = 4\sin\phi$

$\Rightarrow A = \int_0^{2\pi}\int_{\pi/6}^{2\pi/3} 4\sin\phi\, d\phi\, d\theta = \int_0^{2\pi}\left(2+2\sqrt{3}\right)d\theta = \left(4+4\sqrt{3}\right)\pi$

27. The parametrization $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}$ at $P_0 = \left(\sqrt{2},\sqrt{2},2\right) \Rightarrow \theta = \frac{\pi}{4}$, $r = 2$,

$\mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + \mathbf{k} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j} + \mathbf{k}$ and

$\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j} = -\sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}$



$$\Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sqrt{2}/2 & \sqrt{2}/2 & 1 \\ -\sqrt{2} & \sqrt{2} & 0 \end{vmatrix}$$

$= -\sqrt{2}\mathbf{i} - \sqrt{2}\mathbf{j} + 2\mathbf{k} \Rightarrow$ the tangent plane is

$0 = \left(-\sqrt{2}\mathbf{i} - \sqrt{2}\mathbf{j} + 2\mathbf{k}\right)\cdot\left[\left(x-\sqrt{2}\right)\mathbf{i} + \left(y-\sqrt{2}\right)\mathbf{j} + (z-2)\mathbf{k}\right] \Rightarrow \sqrt{2}x + \sqrt{2}y - 2z = 0$, or $x + y - \sqrt{2}z = 0$.

The parametrization $\mathbf{r}(r,\theta) \Rightarrow x = r\cos\theta$, $y = r\sin\theta$ and $z = r \Rightarrow x^2 + y^2 = r^2 = z^2 \Rightarrow$ the surface is $z = \sqrt{x^2+y^2}$.

28. The parametrization $\mathbf{r}(\phi,\theta)$

$= (4\sin\phi\cos\theta)\mathbf{i} + (4\sin\phi\sin\theta)\mathbf{j} + (4\cos\phi)\mathbf{k}$

at $P_0 = \left(\sqrt{2},\sqrt{2},2\sqrt{3}\right) \Rightarrow \rho = 4$ and $z = 2\sqrt{3}$

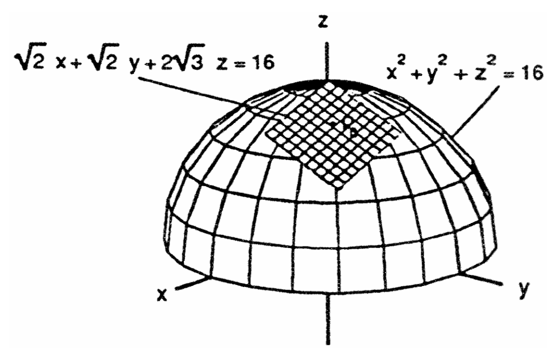
$= 4\cos\phi \Rightarrow \phi = \frac{\pi}{6}$; also $x = \sqrt{2}$ and $y = \sqrt{2}$

$\Rightarrow \theta = \frac{\pi}{4}$. Then $\mathbf{r}_\phi$

$= (4\cos\phi\cos\theta)\mathbf{i} + (4\cos\phi\sin\theta)\mathbf{j} - (4\sin\phi)\mathbf{k}$

$= \sqrt{6}\mathbf{i} + \sqrt{6}\mathbf{j} - 2\mathbf{k}$ and

$\mathbf{r}_\theta = (-4\sin\phi\sin\theta)\mathbf{i} + (4\sin\phi\cos\theta)\mathbf{j}$



$$= -\sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j} \text{ at } P_0 \Rightarrow \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sqrt{6} & \sqrt{6} & -2 \\ -\sqrt{2} & \sqrt{2} & 0 \end{vmatrix}$$

$= 2\sqrt{2}\mathbf{i} + 2\sqrt{2}\mathbf{j} + 4\sqrt{3}\mathbf{k} \Rightarrow$ the tangent plane is

$\left(2\sqrt{2}\mathbf{i} + 2\sqrt{2}\mathbf{j} + 4\sqrt{3}\mathbf{k}\right)\cdot\left[\left(x-\sqrt{2}\right)\mathbf{i} + \left(y-\sqrt{2}\right)\mathbf{j} + \left(z-2\sqrt{3}\right)\mathbf{k}\right] = 0 \Rightarrow \sqrt{2}x + \sqrt{2}y + 2\sqrt{3}z = 16$,

or $x + y + \sqrt{6}z = 8\sqrt{2}$. The parametrization $\Rightarrow x = 4\sin\phi\cos\theta$, $y = 4\sin\phi\sin\theta$, $z = 4\cos\phi$

$\Rightarrow$ the surface is $x^2 + y^2 + z^2 = 16$, $z \ge 0$.

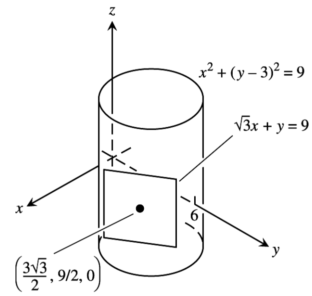
29. The parametrization $\mathbf{r}(\theta, z) = (3 \sin 2\theta)\mathbf{i} + (6 \sin^2 \theta)\mathbf{j} + z\mathbf{k}$ at $P_0 = \left(\frac{3\sqrt{3}}{2}, \frac{9}{2}, 0\right) \Rightarrow \theta = \frac{\pi}{3}$ and $z = 0$. Then $\mathbf{r}_\theta = (6 \cos 2\theta)\mathbf{i} + (12 \sin\theta \cos\theta)\mathbf{j}$ $= -3\mathbf{i} + 3\sqrt{3}\mathbf{j}$ and $\mathbf{r}_z = \mathbf{k}$ at P_0

$$\Rightarrow \mathbf{r}_\theta \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 3\sqrt{3} & 0 \\ 0 & 0 & 1 \end{vmatrix} = 3\sqrt{3}\mathbf{i} + 3\mathbf{j}$$

$\Rightarrow$ the tangent plane is

$$\left(3\sqrt{3}\mathbf{i} + 3\mathbf{j}\right) \cdot \left[\left(x - \frac{3\sqrt{3}}{2}\right)\mathbf{i} + \left(y - \frac{9}{2}\right)\mathbf{j} + (z - 0)\mathbf{k}\right] = 0$$

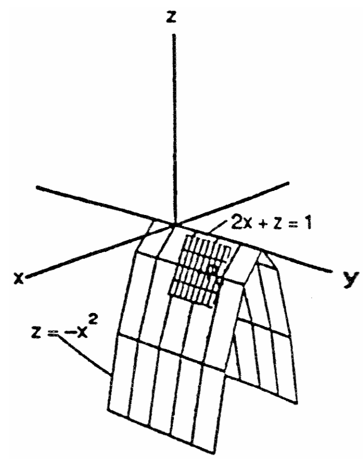
$\Rightarrow \sqrt{3}x + y = 9$. The parametrization $\Rightarrow x = 3 \sin 2\theta$ and $y = 6 \sin^2\theta \Rightarrow x^2 + y^2 = 9 \sin^2 2\theta + (6 \sin^2\theta)^2$ $= 9(4 \sin^2\theta \cos^2\theta) + 36 \sin^4\theta = 6(6 \sin^2\theta) = 6y \Rightarrow x^2 + y^2 - 6y + 9 = 9 \Rightarrow x^2 + (y-3)^2 = 9$



30. The parametrization $\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} - x^2\mathbf{k}$ at $P_0 = (1, 2, -1) \Rightarrow \mathbf{r}_x = \mathbf{i} - 2x\mathbf{k} = \mathbf{i} - 2\mathbf{k}$ and $\mathbf{r}_y = \mathbf{j}$ at P_0

$$\Rightarrow \mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -2 \\ 0 & 1 & 0 \end{vmatrix} = 2\mathbf{i} + \mathbf{k} \Rightarrow \text{the tangent plane}$$

is $(2\mathbf{i} + \mathbf{k}) \cdot [(x-1)\mathbf{i} + (y-2)\mathbf{j} + (z+1)\mathbf{k}] = 0$ $\Rightarrow 2x + z = 1$. The parametrization $\Rightarrow x = x$, $y = y$ and $z = -x^2 \Rightarrow$ the surface is $z = -x^2$



31. (a) An arbitrary point on the circle C is $(x, z) = (R + r \cos u, r \sin u) \Rightarrow (x, y, z)$ is on the torus with $x = (R + r \cos u)\cos v$, $y = (R + r \cos u)\sin v$, and $z = r \sin u$, $0 \le u \le 2\pi$, $0 \le v \le 2\pi$

(b) $\mathbf{r}_u = (-r \sin u \cos v)\mathbf{i} - (r \sin u \sin v)\mathbf{j} + (r \cos u)\mathbf{k}$ and $\mathbf{r}_v = (-(R + r \cos u)\sin v)\mathbf{i} + ((R + r \cos u)\cos v)\mathbf{j}$

$$\Rightarrow \mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r \sin u \cos v & -r \sin u \sin v & r \cos u \\ -(R + r \cos u)\sin v & (R + r \cos u)\cos v & 0 \end{vmatrix}$$

$= -(R + r \cos u)(r \cos v \cos u)\mathbf{i} - (R + r \cos u)(r \sin v \cos u)\mathbf{j} + (-r \sin u)(R + r \cos u)\mathbf{k}$

$\Rightarrow |\mathbf{r}_u \times \mathbf{r}_v|^2 = (R + r \cos u)^2 (r^2 \cos^2 v \cos^2 u + r^2 \sin^2 v \cos^2 u + r^2 \sin^2 u) \Rightarrow |\mathbf{r}_u \times \mathbf{r}_v| = r(R + r \cos u)$

$\Rightarrow A = \int_0^{2\pi}\int_0^{2\pi} (rR + r^2 \cos u)\, du\, dv = \int_0^{2\pi} 2\pi rR\, dv = 4\pi^2 rR$

32. (a) The point (x, y, z) is on the surface for fixed $x = f(u)$ when $y = g(u) \sin\left(\frac{\pi}{2} - v\right)$ and $z = g(u) \cos\left(\frac{\pi}{2} - v\right)$ $\Rightarrow x = f(u)$, $y = g(u) \cos v$, and $z = g(u) \sin v \Rightarrow \mathbf{r}(u, v) = f(u)\mathbf{i} + (g(u) \cos v)\mathbf{j} + (g(u) \sin v)\mathbf{k}$, $0 \le v \le 2\pi$, $a \le u \le b$

(b) Let $u = y$ and $x = u^2 \Rightarrow f(u) = u^2$ and $g(u) = u \Rightarrow \mathbf{r}(u, v) = u^2\mathbf{i} + (u \cos v)\mathbf{j} + (u \sin v)\mathbf{k}$, $0 \le v \le 2\pi$, $0 \le u$

33. (a) Let $w^2 + \frac{z^2}{c^2} = 1$ where $w = \cos\phi$ and $\frac{z}{c} = \sin\phi \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2\phi \Rightarrow \frac{x}{a} = \cos\phi \cos\theta$ and $\frac{y}{b} = \cos\phi \sin\theta$ $\Rightarrow x = a \cos\theta \cos\phi$, $y = b \sin\theta \cos\phi$, and $z = c \sin\phi$ $\Rightarrow \mathbf{r}(\theta, \phi) = (a \cos\theta \cos\phi)\mathbf{i} + (b \sin\theta \cos\phi)\mathbf{j} + (c \sin\phi)\mathbf{k}$

(b) $\mathbf{r}_\theta = (-a\sin\theta\cos\phi)\mathbf{i} + (b\cos\theta\cos\phi)\mathbf{j}$ and $\mathbf{r}_\phi = (-a\cos\theta\sin\phi)\mathbf{i} - (b\sin\theta\sin\phi)\mathbf{j} + (c\cos\phi)\mathbf{k}$

$$\Rightarrow \mathbf{r}_\theta \times \mathbf{r}_\phi = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin\theta\cos\phi & b\cos\theta\cos\phi & 0 \\ -a\cos\theta\sin\phi & -b\sin\theta\sin\phi & c\cos\phi \end{vmatrix}$$

$= (bc\cos\theta\cos^2\phi)\,\mathbf{i} + (ac\sin\theta\cos^2\phi)\,\mathbf{j} + (ab\sin\phi\cos\phi)\mathbf{k}$

$\Rightarrow |\mathbf{r}_\theta \times \mathbf{r}_\phi|^2 = b^2c^2\cos^2\theta\cos^4\phi + a^2c^2\sin^2\theta\cos^4\phi + a^2b^2\sin^2\phi\cos^2\phi$, and the result follows.

$A \Rightarrow \int_0^{2\pi}\int_0^{\pi} |\mathbf{r}_\theta \times \mathbf{r}_\phi|\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi} \left[a^2b^2\sin^2\phi\cos^2\phi + b^2c^2\cos^2\theta\cos^4\phi + a^2c^2\sin^2\theta\cos^4\phi\right]^{1/2} d\phi\, d\theta$

34. (a) $\mathbf{r}(\theta, u) = (\cosh u\cos\theta)\mathbf{i} + (\cosh u\sin\theta)\mathbf{j} + (\sinh u)\mathbf{k}$

(b) $\mathbf{r}(\theta, u) = (a\cosh u\cos\theta)\mathbf{i} + (b\cosh u\sin\theta)\mathbf{j} + (c\sinh u)\mathbf{k}$

35. $\mathbf{r}(\theta, u) = (5\cosh u\cos\theta)\mathbf{i} + (5\cosh u\sin\theta)\mathbf{j} + (5\sinh u)\mathbf{k} \Rightarrow \mathbf{r}_\theta = (-5\cosh u\sin\theta)\mathbf{i} + (5\cosh u\cos\theta)\mathbf{j}$ and $\mathbf{r}_u = (5\sinh u\cos\theta)\mathbf{i} + (5\sinh u\sin\theta)\mathbf{j} + (5\cosh u)\mathbf{k}$

$$\Rightarrow r_\theta \times \mathbf{r}_u = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -5\cosh u\sin\theta & 5\cosh u\cos\theta & 0 \\ 5\sinh u\cos\theta & 5\sinh u\sin\theta & 5\cosh u \end{vmatrix}$$

$= (25\cosh^2 u\cos\theta)\,\mathbf{i} + (25\cosh^2 u\sin\theta)\,\mathbf{j} - (25\cosh u\sinh u)\mathbf{k}$. At the point $(x_0, y_0, 0)$, where $x_0^2 + y_0^2 = 25$ we have $5\sinh u = 0 \Rightarrow u = 0$ and $x_0 = 25\cos\theta$, $y_0 = 25\sin\theta \Rightarrow$ the tangent plane is $5(x_0\mathbf{i} + y_0\mathbf{j}) \cdot [(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + z\mathbf{k}] = 0 \Rightarrow x_0x - x_0^2 + y_0y - y_0^2 = 0 \Rightarrow x_0x + y_0y = 25$

36. Let $\frac{z^2}{c^2} - w^2 = 1$ where $\frac{z}{c} = \cosh u$ and $w = \sinh u \Rightarrow w^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2} \Rightarrow \frac{x}{a} = w\cos\theta$ and $\frac{y}{b} = w\sin\theta$

$\Rightarrow x = a\sinh u\cos\theta$, $y = b\sinh u\sin\theta$, and $z = c\cosh u$

$\Rightarrow \mathbf{r}(\theta, u) = (a\sinh u\cos\theta)\mathbf{i} + (b\sinh u\sin\theta)\mathbf{j} + (c\cosh u)\mathbf{k}$, $0 \le \theta \le 2\pi$, $-\infty < u < \infty$

37. $\mathbf{p} = \mathbf{k}$, $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k} \Rightarrow |\nabla f| = \sqrt{(2x)^2 + (2y)^2 + (-1)^2} = \sqrt{4x^2 + 4y^2 + 1}$ and $|\nabla f \cdot \mathbf{p}| = 1$; $z = 2 \Rightarrow x^2 + y^2 = 2$; thus $S = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}\, dA = \iint_R \sqrt{4x^2 + 4y^2 + 1}\, dx\, dy$

$= \iint_R \sqrt{4r^2\cos^2\theta + 4r^2\sin^2\theta + 1}\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^{\sqrt{2}} \sqrt{4r^2 + 1}\, r\, dr\, d\theta = \int_0^{2\pi} \left[\frac{1}{12}(4r^2 + 1)^{3/2}\right]_0^{\sqrt{2}} d\theta$

$= \int_0^{2\pi} \frac{13}{6}\, d\theta = \frac{13}{3}\pi$

38. $\mathbf{p} = \mathbf{k}$, $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k} \Rightarrow |\nabla f| = \sqrt{4x^2 + 4y^2 + 1}$ and $|\nabla f \cdot \mathbf{p}| = 1$; $2 \le x^2 + y^2 \le 6$

$\Rightarrow S = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}\, dA = \iint_R \sqrt{4x^2 + 4y^2 + 1}\, dx\, dy = \iint_R \sqrt{4r^2 + 1}\, r\, dr\, d\theta = \int_0^{2\pi}\int_{\sqrt{2}}^{\sqrt{6}} \sqrt{4r^2 + 1}\, r\, dr\, d\theta$

$= \int_0^{2\pi} \left[\frac{1}{12}(4r^2 + 1)^{3/2}\right]_{\sqrt{2}}^{\sqrt{6}} d\theta = \int_0^{2\pi} \frac{49}{6}\, d\theta = \frac{49}{3}\pi$

39. $\mathbf{p} = \mathbf{k}$, $\nabla f = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \Rightarrow |\nabla f| = 3$ and $|\nabla f \cdot \mathbf{p}| = 2$; $x = y^2$ and $x = 2 - y^2$ intersect at $(1, 1)$ and $(1, -1)$

$\Rightarrow S = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}\, dA = \iint_R \frac{3}{2}\, dx\, dy = \int_{-1}^{1}\int_{y^2}^{2-y^2} \frac{3}{2}\, dx\, dy = \int_{-1}^{1} (3 - 3y^2)\, dy = 4$

40. $\mathbf{p} = \mathbf{k}$, $\nabla f = 2x\mathbf{i} - 2\mathbf{k} \Rightarrow |\nabla f| = \sqrt{4x^2 + 4} = 2\sqrt{x^2 + 1}$ and $|\nabla f \cdot \mathbf{p}| = 2 \Rightarrow S = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}\, dA$

$= \iint_R \frac{2\sqrt{x^2+1}}{2}\, dx\, dy = \int_0^{\sqrt{3}}\int_0^{x} \sqrt{x^2 + 1}\, dy\, dx = \int_0^{\sqrt{3}} x\sqrt{x^2 + 1}\, dx = \left[\frac{1}{3}(x^2 + 1)^{3/2}\right]_0^{\sqrt{3}} = \frac{1}{3}(4)^{3/2} - \frac{1}{3} = \frac{7}{3}$

41. $\mathbf{p} = \mathbf{k}$, $\nabla f = 2x\mathbf{i} - 2\mathbf{j} - 2\mathbf{k} \Rightarrow |\nabla f| = \sqrt{(2x)^2 + (-2)^2 + (-2)^2} = \sqrt{4x^2 + 8} = 2\sqrt{x^2+2}$ and $|\nabla f \cdot \mathbf{p}| = 2$
$\Rightarrow S = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}\, dA = \iint_R \frac{2\sqrt{x^2+2}}{2}\, dx\, dy = \int_0^2 \int_0^{3x} \sqrt{x^2+2}\, dy\, dx = \int_0^2 3x\sqrt{x^2+2}\, dx = \left[(x^2+2)^{3/2}\right]_0^2$
$= 6\sqrt{6} - 2\sqrt{2}$

42. $\mathbf{p} = \mathbf{k}$, $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow |\nabla f| = \sqrt{4x^2 + 4y^2 + 4z^2} = \sqrt{8} = 2\sqrt{2}$ and $|\nabla f \cdot \mathbf{p}| = 2z$; $x^2 + y^2 + z^2 = 2$ and $z = \sqrt{x^2 + y^2} \Rightarrow x^2 + y^2 = 1$; thus, $S = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}\, dA = \iint_R \frac{2\sqrt{2}}{2z}\, dA = \sqrt{2} \iint_R \frac{1}{z}\, dA$
$= \sqrt{2} \iint_R \frac{1}{\sqrt{2 - (x^2+y^2)}}\, dA = \sqrt{2} \int_0^{2\pi} \int_0^1 \frac{r\, dr\, d\theta}{\sqrt{2 - r^2}} = \sqrt{2} \int_0^{2\pi} \left(-1 + \sqrt{2}\right) d\theta = 2\pi\left(2 - \sqrt{2}\right)$

43. $\mathbf{p} = \mathbf{k}$, $\nabla f = c\mathbf{i} - \mathbf{k} \Rightarrow |\nabla f| = \sqrt{c^2+1}$ and $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow S = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}\, dA = \iint_R \sqrt{c^2+1}\, dx\, dy$
$= \int_0^{2\pi} \int_0^1 \sqrt{c^2+1}\, r\, dr\, d\theta = \int_0^{2\pi} \frac{\sqrt{c^2+1}}{2}\, d\theta = \pi\sqrt{c^2+1}$

44. $\mathbf{p} = \mathbf{k}$, $\nabla f = 2x\mathbf{i} + 2z\mathbf{j} \Rightarrow |\nabla f| = \sqrt{(2x)^2 + (2z)^2} = 2$ and $|\nabla f \cdot \mathbf{p}| = 2z$ for the upper surface, $z \ge 0$
$\Rightarrow S = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}\, dA = \iint_R \frac{2}{2z}\, dA = \iint_R \frac{1}{\sqrt{1-x^2}}\, dy\, dx = 2\int_{-1/2}^{1/2}\int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\, dy\, dx = \int_{-1/2}^{1/2} \frac{1}{\sqrt{1-x^2}}\, dx$
$= \left[\sin^{-1} x\right]_{-1/2}^{1/2} = \frac{\pi}{6} - \left(-\frac{\pi}{6}\right) = \frac{\pi}{3}$

45. $\mathbf{p} = \mathbf{i}$, $\nabla f = \mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow |\nabla f| = \sqrt{1^2 + (2y)^2 + (2z)^2} = \sqrt{1 + 4y^2 + 4z^2}$ and $|\nabla f \cdot \mathbf{p}| = 1$; $1 \le y^2 + z^2 \le 4$
$\Rightarrow S = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}\, dA = \iint_R \sqrt{1 + 4y^2 + 4z^2}\, dy\, dz = \int_0^{2\pi} \int_1^2 \sqrt{1 + 4r^2\cos^2\theta + 4r^2\sin^2\theta}\, r\, dr\, d\theta$
$= \int_0^{2\pi} \int_1^2 \sqrt{1+4r^2}\, r\, dr\, d\theta = \int_0^{2\pi} \left[\frac{1}{12}\left(1+4r^2\right)^{3/2}\right]_1^2 d\theta = \int_0^{2\pi} \frac{1}{12}\left(17\sqrt{17} - 5\sqrt{5}\right) d\theta = \frac{\pi}{6}\left(17\sqrt{17} - 5\sqrt{5}\right)$

46. $\mathbf{p} = \mathbf{j}$, $\nabla f = 2x\mathbf{i} + \mathbf{j} + 2z\mathbf{k} \Rightarrow |\nabla f| = \sqrt{4x^2 + 4z^2 + 1}$ and $|\nabla f \cdot \mathbf{p}| = 1$; $y = 0$ and $x^2 + y + z^2 = 2 \Rightarrow x^2 + z^2 = 2$; thus, $S = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}\, dA = \iint_R \sqrt{4x^2 + 4z^2 + 1}\, dx\, dz = \int_0^{2\pi} \int_0^{\sqrt{2}} \sqrt{4r^2+1}\, r\, dr\, d\theta = \int_0^{2\pi} \frac{13}{6}\, d\theta = \frac{13}{3}\pi$

47. $\mathbf{p} = \mathbf{k}$, $\nabla f = \left(2x - \frac{2}{x}\right)\mathbf{i} + \sqrt{15}\,\mathbf{j} - \mathbf{k} \Rightarrow |\nabla f| = \sqrt{\left(2x - \frac{2}{x}\right)^2 + \left(\sqrt{15}\right)^2 + (-1)^2} = \sqrt{4x^2 + 8 + \frac{4}{x^2}} = \sqrt{\left(2x + \frac{2}{x}\right)^2}$
$= 2x + \frac{2}{x}$, on $1 \le x \le 2$ and $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow S = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}\, dA = \iint_R \left(2x + 2x^{-1}\right) dx\, dy$
$= \int_0^1 \int_1^2 \left(2x + 2x^{-1}\right) dx\, dy = \int_0^1 \left[x^2 + 2\ln x\right]_1^2 dy = \int_0^1 (3 + 2\ln 2)\, dy = 3 + 2\ln 2$

48. $\mathbf{p} = \mathbf{k}$, $\nabla f = 3\sqrt{x}\,\mathbf{i} + 3\sqrt{y}\,\mathbf{j} - 3\mathbf{k} \Rightarrow |\nabla f| = \sqrt{9x + 9y + 9} = 3\sqrt{x+y+1}$ and $|\nabla f \cdot \mathbf{p}| = 3$
$\Rightarrow S = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}\, dA = \iint_R \sqrt{x+y+1}\, dx\, dy = \int_0^1 \int_0^1 \sqrt{x+y+1}\, dx\, dy = \int_0^1 \left[\frac{2}{3}(x+y+1)^{3/2}\right]_0^1 dy$
$= \int_0^1 \left[\frac{2}{3}(y+2)^{3/2} - \frac{2}{3}(y+1)^{3/2}\right] dy = \left[\frac{4}{15}(y+2)^{5/2} - \frac{4}{15}(y+1)^{5/2}\right]_0^1 = \frac{4}{15}\left[(3)^{5/2} - (2)^{5/2} - (2)^{5/2} + 1\right]$
$= \frac{4}{15}\left(9\sqrt{3} - 8\sqrt{2} + 1\right)$

49. $f_x(x,y) = 2x$, $f_y(x,y) = 2y \Rightarrow \sqrt{f_x^2 + f_y^2 + 1} = \sqrt{4x^2 + 4y^2 + 1} \Rightarrow \text{Area} = \iint_R \sqrt{4x^2 + 4y^2 + 1}\, dx\, dy$
$= \int_0^{2\pi} \int_0^{\sqrt{3}} \sqrt{4r^2+1}\, r\, dr\, d\theta = \frac{\pi}{6}\left(13\sqrt{13} - 1\right)$

50. $f_y(y,z) = -2y,\ f_z(y,z) = -2z \Rightarrow \sqrt{f_y^2 + f_z^2 + 1} = \sqrt{4y^2 + 4z^2 + 1} \Rightarrow \text{Area} = \iint\limits_R \sqrt{4y^2 + 4z^2 + 1}\, dy\, dz$
$= \int_0^{2\pi}\int_0^1 \sqrt{4r^2 + 1}\, r\, dr\, d\theta = \frac{\pi}{6}\left(5\sqrt{5} - 1\right)$

51. $f_x(x,y) = \frac{x}{\sqrt{x^2+y^2}},\ f_y(x,y) = \frac{y}{\sqrt{x^2+y^2}} \Rightarrow \sqrt{f_x^2 + f_y^2 + 1} = \sqrt{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + 1} = \sqrt{2} \Rightarrow \text{Area} = \iint\limits_{R_{xy}} \sqrt{2}\, dx\, dy$
$= \sqrt{2}(\text{Area between the ellipse and the circle}) = \sqrt{2}(6\pi - \pi) = 5\pi\sqrt{2}$

52. Over R_{xy}: $z = 2 - \frac{2}{3}x - 2y \Rightarrow f_x(x,y) = -\frac{2}{3},\ f_y(x,y) = -2 \Rightarrow \sqrt{f_x^2 + f_y^2 + 1} = \sqrt{\frac{4}{9} + 4 + 1} = \frac{7}{3}$
$\Rightarrow \text{Area} = \iint\limits_{R_{xy}} \frac{7}{3}\, dA = \frac{7}{3}(\text{Area of the shadow triangle in the xy-plane}) = \left(\frac{7}{3}\right)\left(\frac{3}{2}\right) = \frac{7}{2}.$

Over R_{xz}: $y = 1 - \frac{1}{3}x - \frac{1}{2}z \Rightarrow f_x(x,z) = -\frac{1}{3},\ f_z(x,z) = -\frac{1}{2} \Rightarrow \sqrt{f_x^2 + f_z^2 + 1} = \sqrt{\frac{1}{9} + \frac{1}{4} + 1} = \frac{7}{6}$
$\Rightarrow \text{Area} = \iint\limits_{R_{xz}} \frac{7}{6}\, dA = \frac{7}{6}(\text{Area of the shadow triangle in the xz-plane}) = \left(\frac{7}{6}\right)(3) = \frac{7}{2}.$

Over R_{yz}: $x = 3 - 3y - \frac{3}{2}z \Rightarrow f_y(y,z) = -3,\ f_z(y,z) = -\frac{3}{2} \Rightarrow \sqrt{f_y^2 + f_z^2 + 1} = \sqrt{9 + \frac{9}{4} + 1} = \frac{7}{2}$
$\Rightarrow \text{Area} = \iint\limits_{R_{yz}} \frac{7}{2}\, dA = \frac{7}{2}(\text{Area of the shadow triangle in the yz-plane}) = \left(\frac{7}{2}\right)(1) = \frac{7}{2}.$

53. $y = \frac{2}{3}z^{3/2} \Rightarrow f_x(x,z) = 0,\ f_z(x,z) = z^{1/2} \Rightarrow \sqrt{f_x^2 + f_z^2 + 1} = \sqrt{z+1};\ y = \frac{16}{3} \Rightarrow \frac{16}{3} = \frac{2}{3}z^{3/2} \Rightarrow z = 4$
$\Rightarrow \text{Area} = \int_0^4\int_0^1 \sqrt{z+1}\, dx\, dz = \int_0^4 \sqrt{z+1}\, dz = \frac{2}{3}\left(5\sqrt{5} - 1\right)$

54. $y = 4 - z \Rightarrow f_x(x,z) = 0,\ f_z(x,z) = -1 \Rightarrow \sqrt{f_x^2 + f_z^2 + 1} = \sqrt{2} \Rightarrow \text{Area} = \iint\limits_{R_{xz}} \sqrt{2}\, dA = \int_0^2\int_0^{4-z^2} \sqrt{2}\, dx\, dz$
$= \sqrt{2}\int_0^2 (4 - z^2)\, dz = \frac{16\sqrt{2}}{3}$

55. $\mathbf{r}(x,y) = x\,\mathbf{i} + y\,\mathbf{j} + f(x,y)\,\mathbf{k} \Rightarrow \mathbf{r}_x(x,y) = \mathbf{i} + f_x(x,y)\,\mathbf{k},\ \mathbf{r}_y(x,y) = \mathbf{j} + f_y(x,y)\mathbf{k}$
$\Rightarrow \mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & f_x(x,y) \\ 0 & 1 & f_y(x,y) \end{vmatrix} = -f_x(x,y)\,\mathbf{i} - f_y(x,y)\,\mathbf{j} + \mathbf{k}$
$\Rightarrow |\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{(-f_x(x,y))^2 + (-f_y(x,y))^2 + 1^2} = \sqrt{f_x(x,y)^2 + f_y(x,y)^2 + 1}$
$\Rightarrow d\sigma = \sqrt{f_x(x,y)^2 + f_y(x,y)^2 + 1}\, dA$

56. S is obtained by rotating $y = f(x)$, $a \le x \le b$ about the x-axis where $f(x) \ge 0$

(a) Let (x, y, z) be a point on S. Consider the cross section when $x = x^*$, the cross section is a circle with radius $r = f(x^*)$. The set of parametric equations for this circle are given by $y(\theta) = r\cos\theta = f(x^*)\cos\theta$ and $z(\theta) = r\sin\theta = f(x^*)\sin\theta$ where $0 \le \theta \le 2\pi$. Since x can take on any value between a and b we have $x(x,\theta) = x$, $y(x,\theta) = f(x)\cos\theta$, $z(x,\theta) = f(x)\sin\theta$ where $a \le x \le b$ and $0 \le \theta \le 2\pi$. Thus $\mathbf{r}(x,\theta) = x\,\mathbf{i} + f(x)\cos\theta\,\mathbf{j} + f(x)\sin\theta\,\mathbf{k}$

(b) $\mathbf{r}_x(x,\theta) = \mathbf{i} + f'(x)\cos\theta\,\mathbf{j} + f'(x)\sin\theta\,\mathbf{k}$ and $\mathbf{r}_\theta(x,\theta) = -f(x)\sin\theta\,\mathbf{j} + f(x)\cos\theta\,\mathbf{k}$
$\Rightarrow \mathbf{r}_x \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & f'(x)\cos\theta & f'(x)\sin\theta \\ 0 & -f(x)\sin\theta & f(x)\cos\theta \end{vmatrix} = f(x)\cdot f'(x)\,\mathbf{i} - f(x)\cos\theta\,\mathbf{j} - f(x)\sin\theta\,\mathbf{k}$
$\Rightarrow |\mathbf{r}_x \times \mathbf{r}_\theta| = \sqrt{(f(x)\cdot f'(x))^2 + (-f(x)\cos\theta)^2 + (-f(x)\sin\theta)^2} = f(x)\sqrt{1 + (f'(x))^2}$
$A = \int_a^b\int_0^{2\pi} f(x)\sqrt{1 + (f'(x))^2}\, d\theta\, dx = \int_a^b \left[\left(f(x)\sqrt{1 + (f'(x))^2}\right)\theta\right]_0^{2\pi} dx = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\, dx$

16.6 SURFACE INTEGRALS

1. Let the parametrization be $\mathbf{r}(x,z) = x\mathbf{i} + x^2\mathbf{j} + z\mathbf{k} \Rightarrow \mathbf{r}_x = \mathbf{i} + 2x\mathbf{j}$ and $\mathbf{r}_z = \mathbf{k} \Rightarrow \mathbf{r}_x \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2x & 0 \\ 0 & 0 & 1 \end{vmatrix}$

$= 2x\mathbf{i} + \mathbf{j} \Rightarrow |\mathbf{r}_x \times \mathbf{r}_z| = \sqrt{4x^2+1} \Rightarrow \iint_S G(x,y,z)\, d\sigma = \int_0^3 \int_0^2 x\sqrt{4x^2+1}\, dx\, dz = \int_0^3 \left[\frac{1}{12}\left(4x^2+1\right)^{3/2}\right]_0^2 dz$

$= \int_0^3 \frac{1}{12}\left(17\sqrt{17}-1\right) dz = \frac{17\sqrt{17}-1}{4}$

2. Let the parametrization be $\mathbf{r}(x,y) = x\mathbf{i} + y\mathbf{j} + \sqrt{4-y^2}\,\mathbf{k}$, $-2 \le y \le 2 \Rightarrow \mathbf{r}_x = \mathbf{i}$ and $\mathbf{r}_y = \mathbf{j} - \frac{y}{\sqrt{4-y^2}}\mathbf{k}$

$\Rightarrow \mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & -\frac{y}{\sqrt{4-y^2}} \end{vmatrix} = \frac{y}{\sqrt{4-y^2}}\mathbf{j} + \mathbf{k} \Rightarrow |\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{\frac{y^2}{4-y^2}+1} = \frac{2}{\sqrt{4-y^2}}$

$\Rightarrow \iint_S G(x,y,z)\, d\sigma = \int_1^4 \int_{-2}^2 \sqrt{4-y^2}\left(\frac{2}{\sqrt{4-y^2}}\right) dy\, dx = 24$

3. Let the parametrization be $\mathbf{r}(\phi,\theta) = (\sin\phi\cos\theta)\mathbf{i} + (\sin\phi\sin\theta)\mathbf{j} + (\cos\phi)\mathbf{k}$ (spherical coordinates with $\rho = 1$ on the sphere), $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi \Rightarrow \mathbf{r}_\phi = (\cos\phi\cos\theta)\mathbf{i} + (\cos\phi\sin\theta)\mathbf{j} - (\sin\phi)\mathbf{k}$ and

$\mathbf{r}_\theta = (-\sin\phi\sin\theta)\mathbf{i} + (\sin\phi\cos\theta)\mathbf{j} \Rightarrow \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\phi\sin\theta & \sin\phi\cos\theta & 0 \end{vmatrix}$

$= (\sin^2\phi\cos\theta)\,\mathbf{i} + (\sin^2\phi\sin\theta)\,\mathbf{j} + (\sin\phi\cos\phi)\mathbf{k} \Rightarrow |\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{\sin^4\phi\cos^2\theta + \sin^4\phi\sin^2\theta + \sin^2\phi\cos^2\phi}$

$= \sin\phi$; $x = \sin\phi\cos\theta \Rightarrow G(x,y,z) = \cos^2\theta\sin^2\phi \Rightarrow \iint_S G(x,y,z)\, d\sigma = \int_0^{2\pi}\int_0^\pi (\cos^2\theta\sin^2\phi)(\sin\phi)\, d\phi\, d\theta$

$= \int_0^{2\pi}\int_0^\pi (\cos^2\theta)(1-\cos^2\phi)(\sin\phi)\, d\phi\, d\theta;\ \begin{bmatrix} u = \cos\phi \\ du = -\sin\phi\, d\phi \end{bmatrix} \to \int_0^{2\pi}\int_1^{-1} (\cos^2\theta)(u^2-1)\, du\, d\theta$

$= \int_0^{2\pi} (\cos^2\theta)\left[\frac{u^3}{3} - u\right]_1^{-1} d\theta = \frac{4}{3}\int_0^{2\pi} \cos^2\theta\, d\theta = \frac{4}{3}\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{2\pi} = \frac{4\pi}{3}$

4. Let the parametrization be $\mathbf{r}(\phi,\theta) = (a\sin\phi\cos\theta)\mathbf{i} + (a\sin\phi\sin\theta)\mathbf{j} + (a\cos\phi)\mathbf{k}$ (spherical coordinates with $\rho = a$, $a \ge 0$, on the sphere), $0 \le \phi \le \frac{\pi}{2}$ (since $z \ge 0$), $0 \le \theta \le 2\pi \Rightarrow \mathbf{r}_\phi = (a\cos\phi\cos\theta)\mathbf{i} + (a\cos\phi\sin\theta)\mathbf{j} - (a\sin\phi)\mathbf{k}$ and

$\mathbf{r}_\theta = (-a\sin\phi\sin\theta)\mathbf{i} + (a\sin\phi\cos\theta)\mathbf{j} \Rightarrow \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix}$

$= (a^2\sin^2\phi\cos\theta)\,\mathbf{i} + (a^2\sin^2\phi\sin\theta)\,\mathbf{j} + (a^2\sin\phi\cos\phi)\mathbf{k}$

$\Rightarrow |\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{a^4\sin^4\phi\cos^2\theta + a^4\sin^4\phi\sin^2\theta + a^4\sin^2\phi\cos^2\phi} = a^2\sin\phi$; $z = a\cos\phi$

$\Rightarrow G(x,y,z) = a^2\cos^2\phi \Rightarrow \iint_S G(x,y,z)\, d\sigma = \int_0^{2\pi}\int_0^{\pi/2} (a^2\cos^2\phi)(a^2\sin\phi)\, d\phi\, d\theta = \frac{2}{3}\pi a^4$

5. Let the parametrization be $\mathbf{r}(x,y) = x\mathbf{i} + y\mathbf{j} + (4-x-y)\mathbf{k} \Rightarrow \mathbf{r}_x = \mathbf{i} - \mathbf{k}$ and $\mathbf{r}_y = \mathbf{j} - \mathbf{k}$

$\Rightarrow \mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow |\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{3} \Rightarrow \iint_S F(x,y,z)\, d\sigma = \int_0^1\int_0^1 (4-x-y)\sqrt{3}\, dy\, dx$

$= \int_0^1 \sqrt{3}\left[4y - xy - \frac{y^2}{2}\right]_0^1 dx = \int_0^1 \sqrt{3}\left(\frac{7}{2} - x\right) dx = \sqrt{3}\left[\frac{7}{2}x - \frac{x^2}{2}\right]_0^1 = 3\sqrt{3}$

6. Let the parametrization be $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}$, $0 \le r \le 1$ (since $0 \le z \le 1$) and $0 \le \theta \le 2\pi$ $\Rightarrow \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + \mathbf{k}$ and $\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j} \Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$ $= (-r\cos\theta)\mathbf{i} - (r\sin\theta)\mathbf{j} + r\mathbf{k} \Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{(-r\cos\theta)^2 + (-r\sin\theta)^2 + r^2} = r\sqrt{2}$; $z = r$ and $x = r\cos\theta$ $\Rightarrow F(x,y,z) = r - r\cos\theta \Rightarrow \iint_S F(x,y,z)\,d\sigma = \int_0^{2\pi}\int_0^1 (r - r\cos\theta)\left(r\sqrt{2}\right) dr\,d\theta = \sqrt{2}\int_0^{2\pi}\int_0^1 (1-\cos\theta)\,r^2\,dr\,d\theta$ $= \frac{2\pi\sqrt{2}}{3}$

7. Let the parametrization be $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + (1-r^2)\,\mathbf{k}$, $0 \le r \le 1$ (since $0 \le z \le 1$) and $0 \le \theta \le 2\pi$ $\Rightarrow \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} - 2r\mathbf{k}$ and $\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j} \Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & -2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$ $= (2r^2\cos\theta)\,\mathbf{i} + (2r^2\sin\theta)\,\mathbf{j} + r\mathbf{k} \Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{(2r^2\cos\theta)^2 + (2r^2\sin\theta) + r^2} = r\sqrt{1+4r^2}$; $z = 1 - r^2$ and $x = r\cos\theta \Rightarrow H(x,y,z) = (r^2\cos^2\theta)\sqrt{1+4r^2} \Rightarrow \iint_S H(x,y,z)\,d\sigma$ $= \int_0^{2\pi}\int_0^1 (r^2\cos^2\theta)\left(\sqrt{1+4r^2}\right)\left(r\sqrt{1+4r^2}\right) dr\,d\theta = \int_0^{2\pi}\int_0^1 r^3\,(1+4r^2)\cos^2\theta\,dr\,d\theta = \frac{11\pi}{12}$

8. Let the parametrization be $\mathbf{r}(\phi,\theta) = (2\sin\phi\cos\theta)\mathbf{i} + (2\sin\phi\sin\theta)\mathbf{j} + (2\cos\phi)\mathbf{k}$ (spherical coordinates with $\rho = 2$ on the sphere), $0 \le \phi \le \frac{\pi}{4}$; $x^2+y^2+z^2 = 4$ and $z = \sqrt{x^2+y^2} \Rightarrow z^2+z^2 = 4 \Rightarrow z^2 = 2 \Rightarrow z = \sqrt{2}$ (since $z \ge 0$) $\Rightarrow 2\cos\phi = \sqrt{2} \Rightarrow \cos\phi = \frac{\sqrt{2}}{2} \Rightarrow \phi = \frac{\pi}{4}$, $0 \le \theta \le 2\pi$; $\mathbf{r}_\phi = (2\cos\phi\cos\theta)\mathbf{i} + (2\cos\phi\sin\theta)\mathbf{j} - (2\sin\phi)\mathbf{k}$ and $r_\theta = (-2\sin\phi\sin\theta)\mathbf{i} + (2\sin\phi\cos\theta)\mathbf{j} \Rightarrow \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2\cos\phi\cos\theta & 2\cos\phi\sin\theta & -2\sin\phi \\ -2\sin\phi\sin\theta & 2\sin\phi\cos\theta & 0 \end{vmatrix}$ $= (4\sin^2\phi\cos\theta)\,\mathbf{i} + (4\sin^2\phi\sin\theta)\,\mathbf{j} + (4\sin\phi\cos\phi)\mathbf{k}$ $\Rightarrow |\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{16\sin^4\phi\cos^2\theta + 16\sin^4\phi\sin^2\theta + 16\sin^2\phi\cos^2\phi} = 4\sin\phi$; $y = 2\sin\phi\sin\theta$ and $z = 2\cos\phi \Rightarrow H(x,y,z) = 4\cos\phi\sin\phi\sin\theta \Rightarrow \iint_S H(x,y,z)\,d\sigma = \int_0^{2\pi}\int_0^{\pi/4} (4\cos\phi\sin\phi\sin\theta)(4\sin\phi)\,d\phi\,d\theta$ $= \int_0^{2\pi}\int_0^{\pi/4} 16\sin^2\phi\cos\phi\sin\theta\,d\phi\,d\theta = 0$

9. The bottom face S of the cube is in the xy-plane $\Rightarrow z = 0 \Rightarrow G(x,y,0) = x + y$ and $f(x,y,z) = z = 0 \Rightarrow \mathbf{p} = \mathbf{k}$ and $\nabla f = \mathbf{k} \Rightarrow |\nabla f| = 1$ and $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dx\,dy \Rightarrow \iint_S G\,d\sigma = \iint_R (x+y)\,dx\,dy$ $= \int_0^a\int_0^a (x+y)\,dx\,dy = \int_0^a \left(\frac{a^2}{2} + ay\right) dy = a^3$. Because of symmetry, we also get a^3 over the face of the cube in the xz-plane and a^3 over the face of the cube in the yz-plane. Next, on the top of the cube, $G(x,y,z)$ $= G(x,y,a) = x + y + a$ and $f(x,y,z) = z = a \Rightarrow \mathbf{p} = \mathbf{k}$ and $\nabla f = \mathbf{k} \Rightarrow |\nabla f| = 1$ and $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dx\,dy$ $\iint_S G\,d\sigma = \iint_R (x+y+a)\,dx\,dy = \int_0^a\int_0^a (x+y+a)\,dx\,dy = \int_0^a\int_0^a (x+y)\,dx\,dy + \int_0^a\int_0^a a\,dx\,dy = 2a^3$. Because of symmetry, the integral is also $2a^3$ over each of the other two faces. Therefore, $\iint_{cube} (x+y+z)\,d\sigma = 3\,(a^3 + 2a^3) = 9a^3$.

10. On the face S in the xz-plane, we have $y = 0 \Rightarrow f(x,y,z) = y = 0$ and $G(x,y,z) = G(x,0,z) = z \Rightarrow \mathbf{p} = \mathbf{j}$ and $\nabla f = \mathbf{j}$ $\Rightarrow |\nabla f| = 1$ and $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dx\,dz \Rightarrow \iint_S G\,d\sigma = \iint_S (y+z)\,d\sigma = \int_0^1\int_0^2 z\,dx\,dz = \int_0^1 2z\,dz = 1.$
On the face in the xy-plane, we have $z = 0 \Rightarrow f(x,y,z) = z = 0$ and $G(x,y,z) = G(x,y,0) = y \Rightarrow \mathbf{p} = \mathbf{k}$ and $\nabla f = \mathbf{k}$

$\Rightarrow |\nabla f| = 1$ and $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dx\,dy \Rightarrow \iint_S G\,d\sigma = \iint_S y\,d\sigma = \int_0^1 \int_0^2 y\,dx\,dy = 1.$

On the triangular face in the plane $x = 2$ we have $f(x,y,z) = x = 2$ and $G(x,y,z) = G(2,y,z) = y + z \Rightarrow \mathbf{p} = \mathbf{i}$ and $\nabla f = \mathbf{i} \Rightarrow |\nabla f| = 1$ and $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dz\,dy \Rightarrow \iint_S G\,d\sigma = \iint_S (y+z)\,d\sigma = \int_0^1 \int_0^{1-y} (y+z)\,dz\,dy$
$= \int_0^1 \frac{1}{2}(1 - y^2)\,dy = \frac{1}{3}.$

On the triangular face in the yz-plane, we have $x = 0 \Rightarrow f(x,y,z) = x = 0$ and $G(x,y,z) = G(0,y,z) = y + z$
$\Rightarrow \mathbf{p} = \mathbf{i}$ and $\nabla f = \mathbf{i} \Rightarrow |\nabla f| = 1$ and $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dz\,dy \Rightarrow \iint_S G\,d\sigma = \iint_S (y+z)\,d\sigma$
$= \int_0^1 \int_0^{1-y} (y+z)\,dz\,dy = \frac{1}{3}.$

Finally, on the sloped face, we have $y + z = 1 \Rightarrow f(x,y,z) = y + z = 1$ and $G(x,y,z) = y + z = 1 \Rightarrow \mathbf{p} = \mathbf{k}$ and $\nabla f = \mathbf{j} + \mathbf{k} \Rightarrow |\nabla f| = \sqrt{2}$ and $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = \sqrt{2}\,dx\,dy \Rightarrow \iint_S G\,d\sigma = \iint_S (y+z)\,d\sigma$
$= \int_0^1 \int_0^2 \sqrt{2}\,dx\,dy = 2\sqrt{2}.$ Therefore, $\iint_{\text{wedge}} G(x,y,z)\,d\sigma = 1 + 1 + \frac{1}{3} + \frac{1}{3} + 2\sqrt{2} = \frac{8}{3} + 2\sqrt{2}$

11. On the faces in the coordinate planes, $G(x,y,z) = 0 \Rightarrow$ the integral over these faces is 0.
On the face $x = a$, we have $f(x,y,z) = x = a$ and $G(x,y,z) = G(a,y,z) = ayz \Rightarrow \mathbf{p} = \mathbf{i}$ and $\nabla f = \mathbf{i} \Rightarrow |\nabla f| = 1$ and $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dy\,dz \Rightarrow \iint_S G\,d\sigma = \iint_S ayz\,d\sigma = \int_0^c \int_0^b ayz\,dy\,dz = \frac{ab^2c^2}{4}.$
On the face $y = b$, we have $f(x,y,z) = y = b$ and $G(x,y,z) = G(x,b,z) = bxz \Rightarrow \mathbf{p} = \mathbf{j}$ and $\nabla f = \mathbf{j} \Rightarrow |\nabla f| = 1$ and $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dx\,dz \Rightarrow \iint_S G\,d\sigma = \iint_S bxz\,d\sigma = \int_0^c \int_0^a bxz\,dx\,dz = \frac{a^2bc^2}{4}.$
On the face $z = c$, we have $f(x,y,z) = z = c$ and $G(x,y,z) = G(x,y,c) = cxy \Rightarrow \mathbf{p} = \mathbf{k}$ and $\nabla f = \mathbf{k} \Rightarrow |\nabla f| = 1$ and $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dy\,dx \Rightarrow \iint_S G\,d\sigma = \iint_S cxy\,d\sigma = \int_0^b \int_0^a cxy\,dx\,dy = \frac{a^2b^2c}{4}.$ Therefore,
$\iint_S G(x,y,z)\,d\sigma = \frac{abc(ab+ac+bc)}{4}.$

12. On the face $x = a$, we have $f(x,y,z) = x = a$ and $G(x,y,z) = G(a,y,z) = ayz \Rightarrow \mathbf{p} = \mathbf{i}$ and $\nabla f = \mathbf{i} \Rightarrow |\nabla f| = 1$ and $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dz\,dy \Rightarrow \iint_S G\,d\sigma = \iint_S ayz\,d\sigma = \int_{-b}^b \int_{-c}^c ayz\,dz\,dy = 0.$ Because of the symmetry of G on all the other faces, all the integrals are 0, and $\iint_S G(x,y,z)\,d\sigma = 0.$

13. $f(x,y,z) = 2x + 2y + z = 2 \Rightarrow \nabla f = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $G(x,y,z) = x + y + (2 - 2x - 2y) = 2 - x - y \Rightarrow \mathbf{p} = \mathbf{k}$, $|\nabla f| = 3$ and $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = 3\,dy\,dx$; $z = 0 \Rightarrow 2x + 2y = 2 \Rightarrow y = 1 - x \Rightarrow \iint_S G\,d\sigma = \iint_S (2 - x - y)\,d\sigma$
$= 3\int_0^1 \int_0^{1-x} (2 - x - y)\,dy\,dx = 3\int_0^1 \left[(2-x)(1-x) - \frac{1}{2}(1-x)^2\right] dx = 3\int_0^1 \left(\frac{3}{2} - 2x + \frac{x^2}{2}\right) dx = 2$

14. $f(x,y,z) = y^2 + 4z = 16 \Rightarrow \nabla f = 2y\mathbf{j} + 4\mathbf{k} \Rightarrow |\nabla f| = \sqrt{4y^2 + 16} = 2\sqrt{y^2 + 4}$ and $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 4$
$\Rightarrow d\sigma = \frac{2\sqrt{y^2+4}}{4}\,dx\,dy \Rightarrow \iint_S G\,d\sigma = \int_{-4}^4 \int_0^1 \left(x\sqrt{y^2+4}\right)\left(\frac{\sqrt{y^2+4}}{2}\right) dx\,dy = \int_{-4}^4 \int_0^1 \frac{x(y^2+4)}{2}\,dx\,dy$
$= \int_{-4}^4 \frac{1}{4}(y^2 + 4)\,dy = \frac{1}{2}\left[\frac{y^3}{3} + 4y\right]_0^4 = \frac{1}{2}\left(\frac{64}{3} + 16\right) = \frac{56}{3}$

15. $f(x, y, z) = x + y^2 - z = 0 \Rightarrow \nabla f = \mathbf{i} + 2y\mathbf{j} - \mathbf{k} \Rightarrow |\nabla f| = \sqrt{4y^2 + 2} = \sqrt{2}\sqrt{2y^2 + 1}$ and $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 1$
$\Rightarrow d\sigma = \frac{\sqrt{2}\sqrt{2y^2+1}}{1}\, dx\, dy \Rightarrow \iint_S G\, d\sigma = \int_0^1 \int_0^y (x + y^2 - x)\sqrt{2}\sqrt{2y^2 + 1}\, dx\, dy = \sqrt{2}\int_0^1 \int_0^y y^2 \sqrt{2y^2 + 1}\, dx\, dy$
$= \sqrt{2}\int_0^1 y^3 \sqrt{2y^2 + 1}\, dy = \frac{6\sqrt{6} + \sqrt{2}}{30}$

16. $f(x, y, z) = x^2 + y - z = 0 \Rightarrow \nabla f = 2x\mathbf{i} + \mathbf{j} - \mathbf{k} \Rightarrow |\nabla f| = \sqrt{4x^2 + 2} = \sqrt{2}\sqrt{2x^2 + 1}$ and $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 1$
$\Rightarrow d\sigma = \frac{\sqrt{2}\sqrt{2x^2+1}}{1}\, dx\, dy \Rightarrow \iint_S G\, d\sigma = \int_{-1}^1 \int_0^1 x\sqrt{2}\sqrt{2x^2 + 1}\, dx\, dy = \sqrt{2}\int_{-1}^1 \int_0^1 x\sqrt{2x^2 + 1}\, dx\, dy$
$= \frac{3\sqrt{6} - \sqrt{2}}{6}\int_0^1 dy = \frac{3\sqrt{6} - \sqrt{2}}{3}$

17. $f(x, y, z) = 2x + y + z = 2 \Rightarrow \nabla f = 2\mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow |\nabla f| = \sqrt{6}$ and $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = \frac{\sqrt{6}}{1}\, dy\, dx$
$\Rightarrow \iint_S G\, d\sigma = \int_0^1 \int_{1-x}^{2-2x} x\, y(2 - 2x - y)\sqrt{6}\, dy\, dx = \sqrt{6}\int_0^1 \int_{1-x}^{2-2x} (2x\, y - 2x^2y - x\, y^2)\, dy\, dx$
$= \sqrt{6}\int_0^1 \left(\frac{2}{3}x - 2x^2 + 2x^3 - \frac{2}{3}x^4\right) dx = \frac{\sqrt{6}}{30}$

18. $f(x, y, z) = x + y = 1 \Rightarrow \nabla f = \mathbf{i} + \mathbf{j} \Rightarrow |\nabla f| = \sqrt{2}$ and $\mathbf{p} = \mathbf{j} \Rightarrow |\nabla f \cdot \mathbf{p}| = 1$
$\Rightarrow d\sigma = \frac{\sqrt{2}}{1}\, dz\, dx \Rightarrow \iint_S G\, d\sigma = \int_0^1 \int_0^1 (x - (1 - x) - z)\sqrt{2}\, dz\, dx = \sqrt{2}\int_0^1 \int_0^1 (2x - z - 1)\, dz\, dx$
$= \sqrt{2}\int_0^1 \left(2x - \frac{3}{2}\right) dx = -\frac{\sqrt{2}}{2}$

19. Let the parametrization be $\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + (4 - y^2)\mathbf{k}$, $0 \le x \le 1$, $-2 \le y \le 2$; $z = 0 \Rightarrow 0 = 4 - y^2$
$\Rightarrow y = \pm 2$; $\mathbf{r}_x = \mathbf{i}$ and $\mathbf{r}_y = \mathbf{j} - 2y\mathbf{k} \Rightarrow \mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & -2y \end{vmatrix} = 2y\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F} \cdot \mathbf{n}\, d\sigma$
$= \mathbf{F} \cdot \frac{\mathbf{r}_x \times \mathbf{r}_y}{|\mathbf{r}_x \times \mathbf{r}_y|}|\mathbf{r}_x \times \mathbf{r}_y|\, dy\, dx = (2xy - 3z)\, dy\, dx = [2xy - 3(4 - y^2)]\, dy\, dx \Rightarrow \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$
$= \int_0^1 \int_{-2}^2 (2xy + 3y^2 - 12)\, dy\, dx = \int_0^1 [xy^2 + y^3 - 12y]_{-2}^2\, dx = \int_0^1 -32\, dx = -32$

20. Let the parametrization be $\mathbf{r}(x, y) = x\mathbf{i} + x^2\mathbf{j} + z\mathbf{k}$, $-1 \le x \le 1$, $0 \le z \le 2 \Rightarrow \mathbf{r}_x = \mathbf{i} + 2x\mathbf{j}$ and $\mathbf{r}_z = \mathbf{k}$
$\Rightarrow \mathbf{r}_x \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2x & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2x\mathbf{i} - \mathbf{j} \Rightarrow \mathbf{F} \cdot \mathbf{n}\, d\sigma = \mathbf{F} \cdot \frac{\mathbf{r}_x \times \mathbf{r}_z}{|\mathbf{r}_x \times \mathbf{r}_z|}|\mathbf{r}_x \times \mathbf{r}_z|\, dz\, dx = -x^2\, dz\, dx$
$\Rightarrow \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \int_{-1}^1 \int_0^2 -x^2\, dz\, dx = -\frac{4}{3}$

21. Let the parametrization be $\mathbf{r}(\phi, \theta) = (a \sin\phi \cos\theta)\mathbf{i} + (a \sin\phi \sin\theta)\mathbf{j} + (a \cos\phi)\mathbf{k}$ (spherical coordinates with $\rho = a$, $a \ge 0$, on the sphere), $0 \le \phi \le \frac{\pi}{2}$ (for the first octant), $0 \le \theta \le \frac{\pi}{2}$ (for the first octant)
$\Rightarrow \mathbf{r}_\phi = (a \cos\phi \cos\theta)\mathbf{i} + (a \cos\phi \sin\theta)\mathbf{j} - (a \sin\phi)\mathbf{k}$ and $\mathbf{r}_\theta = (-a \sin\phi \sin\theta)\mathbf{i} + (a \sin\phi \cos\theta)\mathbf{j}$
$\Rightarrow \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix}$
$= (a^2 \sin^2\phi \cos\theta)\mathbf{i} + (a^2 \sin^2\phi \sin\theta)\mathbf{j} + (a^2 \sin\phi \cos\phi)\mathbf{k} \Rightarrow \mathbf{F} \cdot \mathbf{n}\, d\sigma = \mathbf{F} \cdot \frac{\mathbf{r}_\phi \times \mathbf{r}_\theta}{|\mathbf{r}_\phi \times \mathbf{r}_\theta|}|\mathbf{r}_\phi \times \mathbf{r}_\theta|\, d\theta\, d\phi$
$= a^3 \cos^2\phi \sin\phi\, d\theta\, d\phi$ since $\mathbf{F} = z\mathbf{k} = (a \cos\phi)\mathbf{k} \Rightarrow \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \int_0^{\pi/2} \int_0^{\pi/2} a^3 \cos^2\phi \sin\phi\, d\phi\, d\theta = \frac{\pi a^3}{6}$

22. Let the parametrization be $\mathbf{r}(\phi, \theta) = (a \sin\phi \cos\theta)\mathbf{i} + (a \sin\phi \sin\theta)\mathbf{j} + (a \cos\phi)\mathbf{k}$ (spherical coordinates with $\rho = a$, $a \ge 0$, on the sphere), $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$

$\Rightarrow \mathbf{r}_\phi = (a \cos\phi \cos\theta)\mathbf{i} + (a \cos\phi \sin\theta)\mathbf{j} - (a \sin\phi)\mathbf{k}$ and $\mathbf{r}_\theta = (-a \sin\phi \sin\theta)\mathbf{i} + (a \sin\phi \cos\theta)\mathbf{j}$

$$\Rightarrow \mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix}$$

$= (a^2 \sin^2\phi \cos\theta)\,\mathbf{i} + (a^2 \sin^2\phi \sin\theta)\,\mathbf{j} + (a^2 \sin\phi \cos\phi)\,\mathbf{k} \Rightarrow \mathbf{F} \cdot \mathbf{n}\, d\sigma = \mathbf{F} \cdot \frac{\mathbf{r}_\phi \times \mathbf{r}_\theta}{|\mathbf{r}_\phi \times \mathbf{r}_\theta|}\, |\mathbf{r}_\phi \times \mathbf{r}_\theta|\, d\theta\, d\phi$

$= (a^3 \sin^3\phi \cos^2\phi + a^3 \sin^3\phi \sin^2\theta + a^3 \sin\phi \cos^2\phi)\, d\theta\, d\phi = a^3 \sin\phi\, d\theta\, d\phi$ since $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$= (a \sin\phi \cos\theta)\mathbf{i} + (a \sin\phi \sin\theta)\mathbf{j} + (a \cos\phi)\mathbf{k} \Rightarrow \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \int_0^{2\pi}\int_0^{\pi} a^3 \sin\phi\, d\phi\, d\theta = 4\pi a^3$

23. Let the parametrization be $\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + (2a - x - y)\mathbf{k}$, $0 \le x \le a$, $0 \le y \le a \Rightarrow \mathbf{r}_x = \mathbf{i} - \mathbf{k}$ and $\mathbf{r}_y = \mathbf{j} - \mathbf{k}$

$$\Rightarrow \mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F} \cdot \mathbf{n}\, d\sigma = \mathbf{F} \cdot \frac{\mathbf{r}_x \times \mathbf{r}_y}{|\mathbf{r}_x \times \mathbf{r}_y|}\, |\mathbf{r}_x \times \mathbf{r}_y|\, dy\, dx$$

$= [2xy + 2y(2a - x - y) + 2x(2a - x - y)]\, dy\, dx$ since $\mathbf{F} = 2xy\mathbf{i} + 2yz\mathbf{j} + 2xz\mathbf{k}$

$= 2xy\mathbf{i} + 2y(2a - x - y)\mathbf{j} + 2x(2a - x - y)\mathbf{k} \Rightarrow \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$

$= \int_0^a\int_0^a [2xy + 2y(2a - x - y) + 2x(2a - x - y)]\, dy\, dx = \int_0^a\int_0^a (4ay - 2y^2 + 4ax - 2x^2 - 2xy)\, dy\, dx$

$= \int_0^a \left(\frac{4}{3}a^3 + 3a^2x - 2ax^2\right) dx = \left(\frac{4}{3} + \frac{3}{2} - \frac{2}{3}\right) a^4 = \frac{13a^4}{6}$

24. Let the parametrization be $\mathbf{r}(\theta, z) = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + z\mathbf{k}$, $0 \le z \le a$, $0 \le \theta \le 2\pi$ (where $r = \sqrt{x^2 + y^2} = 1$ on the cylinder) $\Rightarrow \mathbf{r}_\theta = (-\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j}$ and $\mathbf{r}_z = \mathbf{k} \Rightarrow \mathbf{r}_\theta \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}$

$\Rightarrow \mathbf{F} \cdot \mathbf{n}\, d\sigma = \mathbf{F} \cdot \frac{\mathbf{r}_\theta \times \mathbf{r}_z}{|\mathbf{r}_\theta \times \mathbf{r}_z|}\, |\mathbf{r}_\theta \times \mathbf{r}_z|\, dz\, d\theta = (\cos^2\theta + \sin^2\theta)\, dz\, d\theta = dz\, d\theta$, since $\mathbf{F} = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + z\mathbf{k}$

$\Rightarrow \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \int_0^{2\pi}\int_0^a 1\, dz\, d\theta = 2\pi a$

25. Let the parametrization be $\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}$, $0 \le r \le 1$ (since $0 \le z \le 1$) and $0 \le \theta \le 2\pi$

$\Rightarrow \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + \mathbf{k}$ and $\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j} \Rightarrow \mathbf{r}_\theta \times \mathbf{r}_r = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & 1 \end{vmatrix}$

$= (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} - r\mathbf{k} \Rightarrow \mathbf{F} \cdot \mathbf{n}\, d\sigma = \mathbf{F} \cdot \frac{\mathbf{r}_\theta \times \mathbf{r}_r}{|\mathbf{r}_\theta \times \mathbf{r}_r|}\, |\mathbf{r}_\theta \times \mathbf{r}_r|\, d\theta\, dr = (r^3 \sin\theta \cos^2\theta + r^2)\, d\theta\, dr$ since

$\mathbf{F} = (r^2 \sin\theta \cos\theta)\,\mathbf{i} - r\mathbf{k} \Rightarrow \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \int_0^{2\pi}\int_0^1 (r^3 \sin\theta \cos^2\theta + r^2)\, dr\, d\theta = \int_0^{2\pi} \left(\frac{1}{4}\sin\theta \cos^2\theta + \frac{1}{3}\right) d\theta$

$= \left[-\frac{1}{12}\cos^3\theta + \frac{\theta}{3}\right]_0^{2\pi} = \frac{2\pi}{3}$

26. Let the parametrization be $\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + 2r\mathbf{k}$, $0 \le r \le 1$ (since $0 \le z \le 2$) and $0 \le \theta \le 2\pi$

$\Rightarrow \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + 2\mathbf{k}$ and $\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j} \Rightarrow \mathbf{r}_\theta \times \mathbf{r}_r = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & 2 \end{vmatrix}$

$= (2r\cos\theta)\mathbf{i} + (2r\sin\theta)\mathbf{j} - r\mathbf{k} \Rightarrow \mathbf{F} \cdot \mathbf{n}\, d\sigma = \mathbf{F} \cdot \frac{\mathbf{r}_\theta \times \mathbf{r}_r}{|\mathbf{r}_\theta \times \mathbf{r}_r|}\, |\mathbf{r}_\theta \times \mathbf{r}_r|\, d\theta\, dr$

$= (2r^3 \sin^2\theta \cos\theta + 4r^3 \cos\theta \sin\theta + r)\, d\theta\, dr$ since

$\mathbf{F} = (r^2 \sin^2\theta)\,\mathbf{i} + (2r^2 \cos\theta)\,\mathbf{j} - \mathbf{k} \Rightarrow \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \int_0^{2\pi}\int_0^1 (2r^3 \sin^2\theta \cos\theta + 4r^3 \cos\theta \sin\theta + r)\, dr\, d\theta$

$= \int_0^{2\pi} \left(\frac{1}{2}\sin^2\theta \cos\theta + \cos\theta \sin\theta + \frac{1}{2}\right) d\theta = \left[\frac{1}{6}\sin^3\theta + \frac{1}{2}\sin^2\theta + \frac{1}{2}\theta\right]_0^{2\pi} = \pi$

27. Let the parametrization be $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}$, $1 \le r \le 2$ (since $1 \le z \le 2$) and $0 \le \theta \le 2\pi$

$\Rightarrow\ \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + \mathbf{k}$ and $\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j}\ \Rightarrow\ \mathbf{r}_\theta \times \mathbf{r}_r = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & 1 \end{vmatrix}$

$= (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} - r\mathbf{k}\ \Rightarrow\ \mathbf{F}\cdot\mathbf{n}\,d\sigma = \mathbf{F}\cdot\frac{\mathbf{r}_\theta\times\mathbf{r}_r}{|\mathbf{r}_\theta\times\mathbf{r}_r|}\,|\mathbf{r}_\theta\times\mathbf{r}_r|\,d\theta\,dr = (-r^2\cos^2\theta - r^2\sin^2\theta - r^3)\,d\theta\,dr$

$= (-r^2 - r^3)\,d\theta\,dr$ since $\mathbf{F} = (-r\cos\theta)\mathbf{i} - (r\sin\theta)\mathbf{j} + r^2\mathbf{k}\ \Rightarrow \iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \int_0^{2\pi}\int_1^2 (-r^2 - r^3)\,dr\,d\theta = -\frac{73\pi}{6}$

28. Let the parametrization be $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r^2\mathbf{k}$, $0 \le r \le 1$ (since $0 \le z \le 1$) and $0 \le \theta \le 2\pi$

$\Rightarrow\ \mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + 2r\mathbf{k}$ and $\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j}\ \Rightarrow\ \mathbf{r}_\theta \times \mathbf{r}_r = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & 2r \end{vmatrix}$

$= (2r^2\cos\theta)\,\mathbf{i} + (2r^2\sin\theta)\,\mathbf{j} - r\mathbf{k}\ \Rightarrow\ \mathbf{F}\cdot\mathbf{n}\,d\sigma = \mathbf{F}\cdot\frac{\mathbf{r}_\theta\times\mathbf{r}_r}{|\mathbf{r}_\theta\times\mathbf{r}_r|}\,|\mathbf{r}_\theta\times\mathbf{r}_r|\,d\theta\,dr = (8r^3\cos^2\theta + 8r^3\sin^2\theta - 2r)\,d\theta\,dr$

$= (8r^3 - 2r)\,d\theta\,dr$ since $\mathbf{F} = (4r\cos\theta)\mathbf{i} + (4r\sin\theta)\mathbf{j} + 2\mathbf{k}\ \Rightarrow \iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \int_0^{2\pi}\int_0^1 (8r^3 - 2r)\,dr\,d\theta = 2\pi$

29. $g(x,y,z) = z$, $\mathbf{p} = \mathbf{k}\ \Rightarrow\ \nabla g = \mathbf{k}\ \Rightarrow\ |\nabla g| = 1$ and $|\nabla g\cdot\mathbf{p}| = 1\ \Rightarrow$ Flux $= \iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint_R (\mathbf{F}\cdot\mathbf{k})\,dA$

$= \int_0^2\int_0^3 3\,dy\,dx = 18$

30. $g(x,y,z) = y$, $\mathbf{p} = -\mathbf{j}\ \Rightarrow\ \nabla g = \mathbf{j}\ \Rightarrow\ |\nabla g| = 1$ and $|\nabla g\cdot\mathbf{p}| = 1\ \Rightarrow$ Flux $= \iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint_R (\mathbf{F}\cdot -\mathbf{j})\,dA$

$= \int_{-1}^2\int_2^7 2\,dz\,dx = \int_{-1}^2 2(7-2)\,dx = 10(2+1) = 30$

31. $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}\ \Rightarrow\ |\nabla g| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2a$; $\mathbf{n} = \frac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{2\sqrt{x^2+y^2+z^2}} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a}\ \Rightarrow\ \mathbf{F}\cdot\mathbf{n} = \frac{z^2}{a}$;

$|\nabla g\cdot\mathbf{k}| = 2z\ \Rightarrow\ d\sigma = \frac{2a}{2z}\,dA\ \Rightarrow$ Flux $= \iint_R \left(\frac{z^2}{a}\right)\left(\frac{a}{z}\right)dA = \iint_R z\,dA = \iint_R \sqrt{a^2 - (x^2+y^2)}\,dx\,dy$

$= \int_0^{\pi/2}\int_0^a \sqrt{a^2 - r^2}\,r\,dr\,d\theta = \frac{\pi a^3}{6}$

32. $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}\ \Rightarrow\ |\nabla g| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2a$; $\mathbf{n} = \frac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{2\sqrt{x^2+y^2+z^2}} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a}\ \Rightarrow\ \mathbf{F}\cdot\mathbf{n} = \frac{-xy}{a} + \frac{xy}{a}$

$= 0$; $|\nabla g\cdot\mathbf{k}| = 2z\ \Rightarrow\ d\sigma = \frac{2a}{2z}\,dA\ \Rightarrow$ Flux $= \iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint_S 0\,d\sigma = 0$

33. From Exercise 31, $\mathbf{n} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a}$ and $d\sigma = \frac{a}{z}\,dA\ \Rightarrow\ \mathbf{F}\cdot\mathbf{n} = \frac{xy}{a} - \frac{xy}{a} + \frac{z}{a} = \frac{z}{a}\ \Rightarrow$ Flux $= \iint_R \left(\frac{z}{a}\right)\left(\frac{a}{z}\right)dA$

$= \iint_R 1\,dA = \frac{\pi a^2}{4}$

34. From Exercise 31, $\mathbf{n} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a}$ and $d\sigma = \frac{a}{z}\,dA\ \Rightarrow\ \mathbf{F}\cdot\mathbf{n} = \frac{zx^2}{a} + \frac{zy^2}{a} + \frac{z^3}{a} = z\left(\frac{x^2+y^2+z^2}{a}\right) = az$

$\Rightarrow$ Flux $= \iint_R (za)\left(\frac{a}{z}\right)dx\,dy = \iint_R a^2\,dx\,dy = a^2(\text{Area of R}) = \frac{1}{4}\pi a^4$

35. From Exercise 31, $\mathbf{n} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a}$ and $d\sigma = \frac{a}{z}\,dA\ \Rightarrow\ \mathbf{F}\cdot\mathbf{n} = \frac{x^2}{a} + \frac{y^2}{a} + \frac{z^2}{a} = a\ \Rightarrow$ Flux

$= \iint_R a\left(\frac{a}{z}\right)dA = \iint_R \frac{a^2}{z}\,dA = \iint_R \frac{a^2}{\sqrt{a^2 - (x^2+y^2)}}\,dA = \int_0^{\pi/2}\int_0^a \frac{a^2}{\sqrt{a^2 - r^2}}\,r\,dr\,d\theta$

$= \int_0^{\pi/2} a^2\left[-\sqrt{a^2 - r^2}\right]_0^a d\theta = \frac{\pi a^3}{2}$

36. From Exercise 31, $\mathbf{n} = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{a}$ and $d\sigma = \frac{a}{z}\,dA \Rightarrow \mathbf{F}\cdot\mathbf{n} = \frac{\left(\frac{x^2}{a}\right)+\left(\frac{y^2}{a}\right)+\left(\frac{z^2}{a}\right)}{\sqrt{x^2+y^2+z^2}} = \frac{\left(\frac{a^2}{a}\right)}{a} = 1$

$\Rightarrow \text{Flux} = \iint_R \frac{a}{z}\,dx\,dy = \iint_R \frac{a}{\sqrt{a^2-(x^2+y^2)}}\,dx\,dy = \int_0^{\pi/2}\int_0^a \frac{a}{\sqrt{a^2-r^2}}\,r\,dr\,d\theta = \frac{\pi a^2}{2}$

37. $g(x,y,z) = y^2+z = 4 \Rightarrow \nabla g = 2y\mathbf{j}+\mathbf{k} \Rightarrow |\nabla g| = \sqrt{4y^2+1} \Rightarrow \mathbf{n} = \frac{2y\mathbf{j}+\mathbf{k}}{\sqrt{4y^2+1}}$

$\Rightarrow \mathbf{F}\cdot\mathbf{n} = \frac{2xy-3z}{\sqrt{4y^2+1}}$; $\mathbf{p}=\mathbf{k} \Rightarrow |\nabla g\cdot\mathbf{p}| = 1 \Rightarrow d\sigma = \sqrt{4y^2+1}\,dA \Rightarrow$ Flux

$= \iint_R \left(\frac{2xy-3z}{\sqrt{4y^2+1}}\right)\sqrt{4y^2+1}\,dA = \iint_R (2xy-3z)\,dA$; $z=0$ and $z=4-y^2 \Rightarrow y^2=4$

$\Rightarrow \text{Flux} = \iint_R [2xy-3(4-y^2)]\,dA = \int_0^1\int_{-2}^2 (2xy-12+3y^2)\,dy\,dx = \int_0^1 \left[xy^2-12y+y^3\right]_{-2}^{2}\,dx$

$= \int_0^1 -32\,dx = -32$

38. $g(x,y,z) = x^2+y^2-z = 0 \Rightarrow \nabla g = 2x\mathbf{i}+2y\mathbf{j}-\mathbf{k} \Rightarrow |\nabla g| = \sqrt{4x^2+4y^2+1} = \sqrt{4(x^2+y^2)+1}$

$\Rightarrow \mathbf{n} = \frac{2x\mathbf{i}+2y\mathbf{j}-\mathbf{k}}{\sqrt{4(x^2+y^2)+1}} \Rightarrow \mathbf{F}\cdot\mathbf{n} = \frac{8x^2+8y^2-2}{\sqrt{4(x^2+y^2)+1}}$; $\mathbf{p}=\mathbf{k} \Rightarrow |\nabla g\cdot\mathbf{p}| = 1 \Rightarrow d\sigma = \sqrt{4(x^2+y^2)+1}\,dA$

$\Rightarrow \text{Flux} = \iint_R \left(\frac{8x^2+8y^2-2}{\sqrt{4(x^2+y^2)+1}}\right)\sqrt{4(x^2+y^2)+1}\,dA = \iint_R (8x^2+8y^2-2)\,dA$; $z=1$ and $x^2+y^2=z$

$\Rightarrow x^2+y^2 = 1 \Rightarrow \text{Flux} = \int_0^{2\pi}\int_0^1 (8r^2-2)\,r\,dr\,d\theta = 2\pi$

39. $g(x,y,z) = y-e^x = 0 \Rightarrow \nabla g = -e^x\mathbf{i}+\mathbf{j} \Rightarrow |\nabla g| = \sqrt{e^{2x}+1} \Rightarrow \mathbf{n} = \frac{e^x\mathbf{i}-\mathbf{j}}{\sqrt{e^{2x}+1}} \Rightarrow \mathbf{F}\cdot\mathbf{n} = \frac{-2e^x-2y}{\sqrt{e^{2x}+1}}$; $\mathbf{p}=\mathbf{i}$

$\Rightarrow |\nabla g\cdot\mathbf{p}| = e^x \Rightarrow d\sigma = \frac{\sqrt{e^{2x}+1}}{e^x}\,dA \Rightarrow \text{Flux} = \iint_R \left(\frac{-2e^x-2y}{\sqrt{e^{2x}+1}}\right)\left(\frac{\sqrt{e^{2x}+1}}{e^x}\right)dA = \iint_R \frac{-2e^x-2e^x}{e^x}\,dA$

$= \iint_R -4\,dA = \int_0^1\int_1^2 -4\,dy\,dz = -4$

40. $g(x,y,z) = y-\ln x = 0 \Rightarrow \nabla g = -\frac{1}{x}\mathbf{i}+\mathbf{j} \Rightarrow |\nabla g| = \sqrt{\frac{1}{x^2}+1} = \frac{\sqrt{1+x^2}}{x}$ since $1 \le x \le e$

$\Rightarrow \mathbf{n} = \frac{\left(-\frac{1}{x}\mathbf{i}+\mathbf{j}\right)}{\left(\frac{\sqrt{1+x^2}}{x}\right)} = \frac{-\mathbf{i}+x\mathbf{j}}{\sqrt{1+x^2}} \Rightarrow \mathbf{F}\cdot\mathbf{n} = \frac{2xy}{\sqrt{1+x^2}}$; $\mathbf{p}=\mathbf{j} \Rightarrow |\nabla g\cdot\mathbf{p}| = 1 \Rightarrow d\sigma = \frac{\sqrt{1+x^2}}{x}\,dA$

$\Rightarrow \text{Flux} = \iint_R \left(\frac{2xy}{\sqrt{1+x^2}}\right)\left(\frac{\sqrt{1+x^2}}{x}\right)dA = \int_0^1\int_1^e 2y\,dx\,dz = \int_1^e\int_0^1 2\ln x\,dz\,dx = \int_1^e 2\ln x\,dx$

$= 2\left[x\ln x - x\right]_1^e = 2(e-e)-2(0-1) = 2$

41. On the face $z=a$: $g(x,y,z) = z \Rightarrow \nabla g = \mathbf{k} \Rightarrow |\nabla g| = 1$; $\mathbf{n}=\mathbf{k} \Rightarrow \mathbf{F}\cdot\mathbf{n} = 2xz = 2ax$ since $z=a$;

$d\sigma = dx\,dy \Rightarrow \text{Flux} = \iint_R 2ax\,dx\,dy = \int_0^a\int_0^a 2ax\,dx\,dy = a^4$.

On the face $z=0$: $g(x,y,z) = z \Rightarrow \nabla g = \mathbf{k} \Rightarrow |\nabla g| = 1$; $\mathbf{n}=-\mathbf{k} \Rightarrow \mathbf{F}\cdot\mathbf{n} = -2xz = 0$ since $z=0$;

$d\sigma = dx\,dy \Rightarrow \text{Flux} = \iint_R 0\,dx\,dy = 0$.

On the face $x=a$: $g(x,y,z) = x \Rightarrow \nabla g = \mathbf{i} \Rightarrow |\nabla g| = 1$; $\mathbf{n}=\mathbf{i} \Rightarrow \mathbf{F}\cdot\mathbf{n} = 2xy = 2ay$ since $x=a$;

$d\sigma = dy\,dz \Rightarrow \text{Flux} = \int_0^a\int_0^a 2ay\,dy\,dz = a^4$.

On the face $x=0$: $g(x,y,z) = x \Rightarrow \nabla g = \mathbf{i} \Rightarrow |\nabla g| = 1$; $\mathbf{n}=-\mathbf{i} \Rightarrow \mathbf{F}\cdot\mathbf{n} = -2xy = 0$ since $x=0$

$\Rightarrow \text{Flux} = 0$.

On the face $y=a$: $g(x,y,z) = y \Rightarrow \nabla g = \mathbf{j} \Rightarrow |\nabla g| = 1$; $\mathbf{n}=\mathbf{j} \Rightarrow \mathbf{F}\cdot\mathbf{n} = 2yz = 2az$ since $y=a$;

$d\sigma = dz\,dx \Rightarrow \text{Flux} = \int_0^a\int_0^a 2az\,dz\,dx = a^4$.

On the face $y=0$: $g(x,y,z) = y \Rightarrow \nabla g = \mathbf{j} \Rightarrow |\nabla g| = 1$; $\mathbf{n}=-\mathbf{j} \Rightarrow \mathbf{F}\cdot\mathbf{n} = -2yz = 0$ since $y=0$

$\Rightarrow \text{Flux} = 0$. Therefore, Total Flux $= 3a^4$.

42. Across the cap: $g(x,y,z) = x^2 + y^2 + z^2 = 25 \Rightarrow \nabla g = 2x\,\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow |\nabla g| = \sqrt{4x^2 + 4y^2 + 4z^2} = 10$
$\Rightarrow \mathbf{n} = \frac{\nabla g}{|\nabla g|} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{5} \Rightarrow \mathbf{F} \cdot \mathbf{n} = \frac{x^2 z}{5} + \frac{y^2 z}{5} + \frac{z}{5}$; $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla g \cdot \mathbf{p}| = 2z$ since $z \ge 0 \Rightarrow d\sigma = \frac{10}{2z}\, dA$
$\Rightarrow \text{Flux}_{\text{cap}} = \iint\limits_{\text{cap}} \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint\limits_{R} \left(\frac{x^2 z}{5} + \frac{y^2 z}{5} + \frac{z}{5}\right)\left(\frac{5}{z}\right) dA = \iint\limits_{R} (x^2 + y^2 + 1)\, dx\, dy = \int_0^{2\pi}\int_0^4 (r^2 + 1)\, r\, dr\, d\theta$
$= \int_0^{2\pi} 72\, d\theta = 144\pi.$
Across the bottom: $g(x,y,z) = z = 3 \Rightarrow \nabla g = \mathbf{k} \Rightarrow |\nabla g| = 1 \Rightarrow \mathbf{n} = -\mathbf{k} \Rightarrow \mathbf{F} \cdot \mathbf{n} = -1$; $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla g \cdot \mathbf{p}| = 1$
$\Rightarrow d\sigma = dA \Rightarrow \text{Flux}_{\text{bottom}} = \iint\limits_{\text{bottom}} \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint\limits_{R} -1\, dA = -1$(Area of the circular region) $= -16\pi$. Therefore,
$\text{Flux} = \text{Flux}_{\text{cap}} + \text{Flux}_{\text{bottom}} = 128\pi$

43. $\nabla f = 2x\,\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow |\nabla f| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2a$; $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 2z$ since $z \ge 0 \Rightarrow d\sigma = \frac{2a}{2z} dA$
$= \frac{a}{z}\, dA$; $M = \iint\limits_{S} \delta\, d\sigma = \frac{\delta}{8}$ (surface area of sphere) $= \frac{\delta\pi a^2}{2}$; $M_{xy} = \iint\limits_{S} z\delta\, d\sigma = \delta \iint\limits_{R} z\left(\frac{a}{z}\right) dA = a\delta \iint\limits_{R} dA$
$= a\delta \int_0^{\pi/2}\int_0^a r\, dr\, d\theta = \frac{\delta\pi a^3}{4} \Rightarrow \bar{z} = \frac{M_{xy}}{M} = \left(\frac{\delta\pi a^3}{4}\right)\left(\frac{2}{\delta\pi a^2}\right) = \frac{a}{2}$. Because of symmetry, $\bar{x} = \bar{y} = \frac{a}{2} \Rightarrow$ the centroid is
$\left(\frac{a}{2}, \frac{a}{2}, \frac{a}{2}\right)$.

44. $\nabla f = 2y\,\mathbf{j} + 2z\mathbf{k} \Rightarrow |\nabla f| = \sqrt{4y^2 + 4z^2} = \sqrt{4(y^2 + z^2)} = 6$; $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{k}| = 2z$ since $z \ge 0 \Rightarrow d\sigma = \frac{6}{2z}\, dA$
$= \frac{3}{z}\, dA$; $M = \iint\limits_{S} 1\, d\sigma = \int_{-3}^{3}\int_0^3 \frac{3}{z}\, dx\, dy = \int_{-3}^{3}\int_0^3 \frac{3}{\sqrt{9-y^2}}\, dx\, dy = 9\pi$; $M_{xy} = \iint\limits_{S} z\, d\sigma = \int_{-3}^{3}\int_0^3 z\left(\frac{3}{z}\right) dx\, dy = 54$;
$M_{xz} = \iint\limits_{S} y\, d\sigma = \int_{-3}^{3}\int_0^3 y\left(\frac{3}{z}\right) dx\, dy = \int_{-3}^{3}\int_0^3 \frac{3y}{\sqrt{9-y^2}}\, dx\, dy = 0$; $M_{yz} = \iint\limits_{S} x\, d\sigma = \int_{-3}^{3}\int_0^3 \frac{3x}{\sqrt{9-y^2}}\, dx\, dy = \frac{27}{2}\pi$.
Therefore, $\bar{x} = \frac{\left(\frac{27}{2}\pi\right)}{9\pi} = \frac{3}{2}$, $\bar{y} = 0$, and $\bar{z} = \frac{54}{9\pi} = \frac{6}{\pi}$

45. Because of symmetry, $\bar{x} = \bar{y} = 0$; $M = \iint\limits_{S} \delta\, d\sigma = \delta \iint\limits_{S} d\sigma = (\text{Area of } S)\delta = 3\pi\sqrt{2}\,\delta$; $\nabla f = 2x\,\mathbf{i} + 2y\mathbf{j} - 2z\mathbf{k}$
$\Rightarrow |\nabla f| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2\sqrt{x^2 + y^2 + z^2}$; $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 2z \Rightarrow d\sigma = \frac{2\sqrt{x^2+y^2+z^2}}{2z}\, dA$
$= \frac{\sqrt{x^2+y^2+(x^2+y^2)}}{z}\, dA = \frac{\sqrt{2}\sqrt{x^2+y^2}}{z}\, dA \Rightarrow M_{xy} = \delta \iint\limits_{R} z\left(\frac{\sqrt{2}\sqrt{x^2+y^2}}{z}\right) dA = \delta \iint\limits_{R} \sqrt{2}\sqrt{x^2 + y^2}\, dA$
$= \delta \int_0^{2\pi}\int_1^2 \sqrt{2}\, r^2\, dr\, d\theta = \frac{14\pi\sqrt{2}}{3}\delta \Rightarrow \bar{z} = \frac{\left(\frac{14\pi\sqrt{2}}{3}\delta\right)}{3\pi\sqrt{2}\,\delta} = \frac{14}{9} \Rightarrow (\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, \frac{14}{9}\right)$. Next, $I_z = \iint\limits_{S} (x^2 + y^2)\,\delta\, d\sigma$
$= \iint\limits_{R} (x^2 + y^2)\left(\frac{\sqrt{2}\sqrt{x^2+y^2}}{z}\right)\delta\, dA = \delta\sqrt{2}\iint\limits_{R} (x^2 + y^2)\, dA = \delta\sqrt{2}\int_0^{2\pi}\int_1^2 r^3\, dr\, d\theta = \frac{15\pi\sqrt{2}}{2}\delta \Rightarrow R_z = \sqrt{\frac{I_z}{M}} = \frac{\sqrt{10}}{2}$

46. $f(x,y,z) = 4x^2 + 4y^2 - z^2 = 0 \Rightarrow \nabla f = 8x\mathbf{i} + 8y\mathbf{j} - 2z\mathbf{k} \Rightarrow |\nabla f| = \sqrt{64x^2 + 64y^2 + 4z^2}$
$= 2\sqrt{16x^2 + 16y^2 + z^2} = 2\sqrt{4z^2 + z^2} = 2\sqrt{5}\,z$ since $z \ge 0$; $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 2z \Rightarrow d\sigma = \frac{2\sqrt{5}z}{2z}\, dA = \sqrt{5}\, dA$
$\Rightarrow I_z = \iint\limits_{S} (x^2 + y^2)\,\delta\, d\sigma = \delta\sqrt{5}\iint\limits_{R} (x^2 + y^2)\, dx\, dy = \delta\sqrt{5}\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^3\, dr\, d\theta = \frac{3\sqrt{5}\pi\delta}{2}$

47. (a) Let the diameter lie on the z-axis and let $f(x,y,z) = x^2 + y^2 + z^2 = a^2$, $z \ge 0$ be the upper hemisphere
$\Rightarrow \nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow |\nabla f| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2a$, $a > 0$; $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 2z$ since $z \ge 0$
$\Rightarrow d\sigma = \frac{a}{z}\, dA \Rightarrow I_z = \iint\limits_{S} \delta(x^2 + y^2)\left(\frac{a}{z}\right) d\sigma = a\delta \iint\limits_{R} \frac{x^2+y^2}{\sqrt{a^2-(x^2+y^2)}}\, dA = a\delta \int_0^{2\pi}\int_0^a \frac{r^2}{\sqrt{a^2-r^2}}\, r\, dr\, d\theta$
$= a\delta \int_0^{2\pi}\left[-r^2\sqrt{a^2 - r^2} - \frac{2}{3}(a^2 - r^2)^{3/2}\right]_0^a d\theta = a\delta \int_0^{2\pi} \frac{2}{3}a^3\, d\theta = \frac{4\pi}{3}a^4\delta \Rightarrow$ the moment of inertia is $\frac{8\pi}{3}a^4\delta$ for the whole sphere

(b) $I_L = I_{c.m.} + mh^2$, where m is the mass of the body and h is the distance between the parallel lines; now, $I_{c.m.} = \frac{8\pi}{3} a^4\delta$ (from part a) and $\frac{m}{2} = \iint_S \delta\, d\sigma = \delta \iint_R \left(\frac{a}{z}\right) dA = a\delta \iint_R \frac{1}{\sqrt{a^2-(x^2+y^2)}}\, dy\, dx$

$= a\delta \int_0^{2\pi}\int_0^a \frac{1}{\sqrt{a^2-r^2}}\, r\, dr\, d\theta = a\delta \int_0^{2\pi} \left[-\sqrt{a^2-r^2}\right]_0^a d\theta = a\delta \int_0^{2\pi} a\, d\theta = 2\pi a^2\delta$ and $h = a$

$\Rightarrow I_L = \frac{8\pi}{3} a^4\delta + 4\pi a^2\delta a^2 = \frac{20\pi}{3} a^4\delta$

48. Let $z = \frac{h}{a}\sqrt{x^2+y^2}$ be the cone from $z = 0$ to $z = h$, $h > 0$. Because of symmetry, $\overline{x} = 0$ and $\overline{y} = 0$; $z = \frac{h}{a}\sqrt{x^2+y^2} \Rightarrow f(x,y,z) = \frac{h^2}{a^2}(x^2+y^2) - z^2 = 0 \Rightarrow \nabla f = \frac{2xh^2}{a^2}\mathbf{i} + \frac{2yh^2}{a^2}\mathbf{j} - 2z\mathbf{k}$

$\Rightarrow |\nabla f| = \sqrt{\frac{4x^2h^4}{a^4} + \frac{4y^2h^4}{a^4} + 4z^2} = 2\sqrt{\frac{h^4}{a^4}(x^2+y^2) + \frac{h^2}{a^2}(x^2+y^2)} = 2\sqrt{\left(\frac{h^2}{a^2}\right)(x^2+y^2)\left(\frac{h^2}{a^2}+1\right)}$

$= 2\sqrt{z^2\left(\frac{h^2+a^2}{a^2}\right)} = \left(\frac{2z}{a}\right)\sqrt{h^2+a^2}$ since $z \ge 0$; $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 2z \Rightarrow d\sigma = \frac{\left(\frac{2z}{a}\right)\sqrt{h^2+a^2}}{2z}\, dA$

$= \frac{\sqrt{h^2+a^2}}{a}\, dA$; $M = \iint_S d\sigma = \iint_R \frac{\sqrt{h^2+a^2}}{a}\, dA = \frac{\sqrt{h^2+a^2}}{a}(\pi a^2) = \pi a\sqrt{h^2+a^2}$;

$M_{xy} = \iint_S z\, d\sigma = \iint_R z\left(\frac{\sqrt{h^2+a^2}}{a}\right) dA = \frac{\sqrt{h^2+a^2}}{a}\iint_R \frac{h}{a}\sqrt{x^2+y^2}\, dx\, dy = \frac{h\sqrt{h^2+a^2}}{a^2}\int_0^{2\pi}\int_0^a r^2\, dr\, d\theta$

$= \frac{2\pi ah\sqrt{h^2+a^2}}{3} \Rightarrow \overline{z} = \frac{M_{xy}}{M} = \frac{2h}{3} \Rightarrow$ the centroid is $\left(0, 0, \frac{2h}{3}\right)$

16.7 STOKES' THEOREM

1. $\text{curl}\, \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 2x & z^2 \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + (2-0)\mathbf{k} = 2\mathbf{k}$ and $\mathbf{n} = \mathbf{k} \Rightarrow \text{curl}\, \mathbf{F}\cdot\mathbf{n} = 2 \Rightarrow d\sigma = dx\, dy$

$\Rightarrow \oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_R 2\, dA = 2(\text{Area of the ellipse}) = 4\pi$

2. $\text{curl}\, \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y & 3x & -z^2 \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + (3-2)\mathbf{k} = \mathbf{k}$ and $\mathbf{n} = \mathbf{k} \Rightarrow \text{curl}\, \mathbf{F}\cdot\mathbf{n} = 1 \Rightarrow d\sigma = dx\, dy$

$\Rightarrow \oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_R dx\, dy = \text{Area of circle} = 9\pi$

3. $\text{curl}\, \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & xz & x^2 \end{vmatrix} = -x\mathbf{i} - 2x\mathbf{j} + (z-1)\mathbf{k}$ and $\mathbf{n} = \frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}} \Rightarrow \text{curl}\, \mathbf{F}\cdot\mathbf{n}$

$= \frac{1}{\sqrt{3}}(-x - 2x + z - 1) \Rightarrow d\sigma = \frac{\sqrt{3}}{1}\, dA \Rightarrow \oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_R \frac{1}{\sqrt{3}}(-3x + z - 1)\sqrt{3}\, dA$

$= \int_0^1\int_0^{1-x} [-3x + (1-x-y) - 1]\, dy\, dx = \int_0^1\int_0^{1-x} (-4x - y)\, dy\, dx = \int_0^1 -\left[4x(1-x) + \frac{1}{2}(1-x)^2\right] dx$

$= -\int_0^1 \left(\frac{1}{2} + 3x - \frac{7}{2}x^2\right) dx = -\frac{5}{6}$

4. $\text{curl}\, \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2+z^2 & x^2+z^2 & x^2+y^2 \end{vmatrix} = (2y-2z)\mathbf{i} + (2z-2x)\mathbf{j} + (2x-2y)\mathbf{k}$ and $\mathbf{n} = \frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}}$

$\Rightarrow \text{curl}\, \mathbf{F}\cdot\mathbf{n} = \frac{1}{\sqrt{3}}(2y - 2z + 2z - 2x + 2x - 2y) = 0 \Rightarrow \oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S 0\, d\sigma = 0$

5. curl $\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2+z^2 & x^2+y^2 & x^2+y^2 \end{vmatrix} = 2y\mathbf{i} + (2z-2x)\mathbf{j} + (2x-2y)\mathbf{k}$ and $\mathbf{n} = \mathbf{k}$

$\Rightarrow$ curl $\mathbf{F}\cdot\mathbf{n} = 2x - 2y \Rightarrow d\sigma = dx\,dy \Rightarrow \oint_C \mathbf{F}\cdot d\mathbf{r} = \int_{-1}^{1}\int_{-1}^{1}(2x-2y)\,dx\,dy = \int_{-1}^{1}\left[x^2-2xy\right]_{-1}^{1} dy$

$= \int_{-1}^{1} -4y\,dy = 0$

6. curl $\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y^3 & 1 & z \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} - 3x^2y^2\mathbf{k}$ and $\mathbf{n} = \frac{2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}}{2\sqrt{x^2+y^2+z^2}} = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{4}$

$\Rightarrow$ curl $\mathbf{F}\cdot\mathbf{n} = -\frac{3}{4}x^2y^2z$; $d\sigma = \frac{4}{z}\,dA$ (Section 16.6, Example 6, with $a = 4$) $\Rightarrow \oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_R \left(-\frac{3}{4}x^2y^2z\right)\left(\frac{4}{z}\right) dA$

$= -3\int_0^{2\pi}\int_0^2 (r^2\cos^2\theta)(r^2\sin^2\theta)\,r\,dr\,d\theta = -3\int_0^{2\pi}\left[\frac{r^6}{6}\right]_0^2 (\cos\theta\sin\theta)^2\,d\theta = -32\int_0^{2\pi}\frac{1}{4}\sin^2 2\theta\,d\theta = -4\int_0^{4\pi}\sin^2 u\,du$

$= -4\left[\frac{u}{2} - \frac{\sin 2u}{4}\right]_0^{4\pi} = -8\pi$

7. $x = 3\cos t$ and $y = 2\sin t \Rightarrow \mathbf{F} = (2\sin t)\mathbf{i} + (9\cos^2 t)\mathbf{j} + (9\cos^2 t + 16\sin^4 t)\sin e^{\sqrt{(6\sin t\cos t)(0)}}\mathbf{k}$ at the base of the shell; $\mathbf{r} = (3\cos t)\mathbf{i} + (2\sin t)\mathbf{j} \Rightarrow d\mathbf{r} = (-3\sin t)\mathbf{i} + (2\cos t)\mathbf{j} \Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = -6\sin^2 t + 18\cos^3 t$

$\Rightarrow \iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma = \int_0^{2\pi}(-6\sin^2 t + 18\cos^3 t)\,dt = \left[-3t + \frac{3}{2}\sin 2t + 6(\sin t)(\cos^2 t + 2)\right]_0^{2\pi} = -6\pi$

8. curl $\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -z + \frac{1}{2+x} & \tan^{-1} y & x + \frac{1}{4+z} \end{vmatrix} = -2\mathbf{j}$; $f(x,y,z) = 4x^2 + y + z^2 \Rightarrow \nabla f = 8x\mathbf{i} + \mathbf{j} + 2z\mathbf{k}$

$\Rightarrow \mathbf{n} = \frac{\nabla f}{|\nabla f|}$ and $\mathbf{p} = \mathbf{j} \Rightarrow |\nabla f\cdot\mathbf{p}| = 1 \Rightarrow d\sigma = \frac{|\nabla f|}{|\nabla f\cdot\mathbf{p}|}\,dA = |\nabla f|\,dA$; $\nabla\times\mathbf{F}\cdot\mathbf{n} = \frac{1}{|\nabla f|}(-2\mathbf{j}\cdot\nabla f) = \frac{-2}{|\nabla f|}$

$\Rightarrow \nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma = -2\,dA \Rightarrow \iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint_R -2\,dA = -2(\text{Area of R}) = -2(\pi\cdot 1\cdot 2) = -4\pi$, where R is the elliptic region in the xz-plane enclosed by $4x^2 + z^2 = 4$.

9. Flux of $\nabla\times\mathbf{F} = \iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma = \oint_C \mathbf{F}\cdot d\mathbf{r}$, so let C be parametrized by $\mathbf{r} = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j}$, $0 \le t \le 2\pi \Rightarrow \frac{d\mathbf{r}}{dt} = (-a\sin t)\mathbf{i} + (a\cos t)\mathbf{j} \Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = ay\sin t + ax\cos t = a^2\sin^2 t + a^2\cos^2 t = a^2$

$\Rightarrow$ Flux of $\nabla\times\mathbf{F} = \oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} a^2\,dt = 2\pi a^2$

10. $\nabla\times(y\mathbf{i}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & 0 & 0 \end{vmatrix} = -\mathbf{k}$; $\mathbf{n} = \frac{\nabla f}{|\nabla f|} = \frac{2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}}{2\sqrt{x^2+y^2+z^2}} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$\Rightarrow \nabla\times(y\mathbf{i})\cdot\mathbf{n} = -z$; $d\sigma = \frac{1}{z}\,dA$ (Section 16.6, Example 6, with $a = 1$) $\Rightarrow \iint_S \nabla\times(y\mathbf{i})\cdot\mathbf{n}\,d\sigma$

$= \iint_R (-z)\left(\frac{1}{z}\,dA\right) = -\iint_R dA = -\pi$, where R is the disk $x^2 + y^2 \le 1$ in the xy-plane.

11. Let S_1 and S_2 be oriented surfaces that span C and that induce the same positive direction on C. Then
$\iint_{S_1} \nabla\times\mathbf{F}\cdot\mathbf{n}_1\,d\sigma_1 = \oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_{S_2} \nabla\times\mathbf{F}\cdot\mathbf{n}_2\,d\sigma_2$

12. $\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_{S_1} \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma + \iint_{S_2} \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$, and since S_1 and S_2 are joined by the simple closed curve C, each of the above integrals will be equal to a circulation integral on C. But for one surface the circulation will be counterclockwise, and for the other surface the circulation will be clockwise. Since the integrands are the same, the sum will be 0 $\Rightarrow \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = 0$.

13. $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z & 3x & 5y \end{vmatrix} = 5\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$; $\mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} - 2r\mathbf{k}$ and $\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j}$

$\Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & -2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (2r^2\cos\theta)\,\mathbf{i} + (2r^2\sin\theta)\,\mathbf{j} + r\mathbf{k}$; $\mathbf{n} = \frac{\mathbf{r}_r \times \mathbf{r}_\theta}{|\mathbf{r}_r \times \mathbf{r}_\theta|}$ and $d\sigma = |\mathbf{r}_r \times \mathbf{r}_\theta|\, dr\, d\theta$

$\Rightarrow \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = (\nabla \times \mathbf{F}) \cdot (\mathbf{r}_r \times \mathbf{r}_\theta)\, dr\, d\theta = (10r^2\cos\theta + 4r^2\sin\theta + 3r)\, dr\, d\theta \Rightarrow \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$

$= \int_0^{2\pi}\int_0^2 (10r^2\cos\theta + 4r^2\sin\theta + 3r)\, dr\, d\theta = \int_0^{2\pi} \left[\frac{10}{3} r^3\cos\theta + \frac{4}{3} r^3 \sin\theta + \frac{3}{2} r^2\right]_0^2 d\theta$

$= \int_0^{2\pi} \left(\frac{80}{3}\cos\theta + \frac{32}{3}\sin\theta + 6\right) d\theta = 6(2\pi) = 12\pi$

14. $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y - z & z - x & x + z \end{vmatrix} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$; $\mathbf{r}_r \times \mathbf{r}_\theta = (2r^2\cos\theta)\,\mathbf{i} + (2r^2\sin\theta)\,\mathbf{j} + r\mathbf{k}$ and

$\nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = (\nabla \times \mathbf{F}) \cdot (\mathbf{r}_r \times \mathbf{r}_\theta)\, dr\, d\theta$ (see Exercise 13 above) $\Rightarrow \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$

$= \int_0^{2\pi}\int_0^3 (-2r^2\cos\theta - 4r^2\sin\theta - 2r)\, dr\, d\theta = \int_0^{2\pi} \left[-\frac{2}{3} r^3\cos\theta - \frac{4}{3} r^3\sin\theta - r^2\right]_0^3 d\theta$

$= \int_0^{2\pi} (-18\cos\theta - 36\sin\theta - 9)\, d\theta = -9(2\pi) = -18\pi$

15. $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & 2y^3 z & 3z \end{vmatrix} = -2y^3\mathbf{i} + 0\mathbf{j} - x^2\mathbf{k}$; $\mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$

$= (-r\cos\theta)\mathbf{i} - (r\sin\theta)\mathbf{j} + r\mathbf{k}$ and $\nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = (\nabla \times \mathbf{F}) \cdot (\mathbf{r}_r \times \mathbf{r}_\theta)\, dr\, d\theta$ (see Exercise 13 above)

$\Rightarrow \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_R (2ry^3\cos\theta - rx^2)\, dr\, d\theta = \int_0^{2\pi}\int_0^1 (2r^4\sin^3\theta\cos\theta - r^3\cos^2\theta)\, dr\, d\theta$

$= \int_0^{2\pi} \left(\frac{2}{5}\sin^3\theta\cos\theta - \frac{1}{4}\cos^2\theta\right) d\theta = \left[\frac{1}{10}\sin^4\theta - \frac{1}{4}\left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right)\right]_0^{2\pi} = -\frac{\pi}{4}$

16. $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x - y & y - z & z - x \end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k}$; $\mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & -1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$

$= (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}$ and $\nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = (\nabla \times \mathbf{F}) \cdot (\mathbf{r}_r \times \mathbf{r}_\theta)\, dr\, d\theta$ (see Exercise 13 above)

$\Rightarrow \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = \int_0^{2\pi}\int_0^5 (r\cos\theta + r\sin\theta + r)\, dr\, d\theta = \int_0^{2\pi} \left[(\cos\theta + \sin\theta + 1)\frac{r^2}{2}\right]_0^5 d\theta = \left(\frac{25}{2}\right)(2\pi) = 25\pi$

17. $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y & 5 - 2x & z^2 - 2 \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} - 5\mathbf{k}$; $\mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sqrt{3}\cos\phi\cos\theta & \sqrt{3}\cos\phi\sin\theta & -\sqrt{3}\sin\phi \\ -\sqrt{3}\sin\phi\sin\theta & \sqrt{3}\sin\phi\cos\theta & 0 \end{vmatrix}$

$= (3\sin^2\phi\cos\theta)\,\mathbf{i} + (3\sin^2\phi\sin\theta)\,\mathbf{j} + (3\sin\phi\cos\phi)\mathbf{k}$; $\nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = (\nabla \times \mathbf{F}) \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta)\, d\phi\, d\theta$ (see Exercise 13 above) $\Rightarrow \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = \int_0^{2\pi}\int_0^{\pi/2} -15\cos\phi\sin\phi\, d\phi\, d\theta = \int_0^{2\pi} \left[\frac{15}{2}\cos^2\phi\right]_0^{\pi/2} d\theta = \int_0^{2\pi} -\frac{15}{2}\, d\theta = -15\pi$

18. $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & z^2 & x \end{vmatrix} = -2z\mathbf{i} - \mathbf{j} - 2y\mathbf{k}$; $\mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2\cos\phi\cos\theta & 2\cos\phi\sin\theta & -2\sin\phi \\ -2\sin\phi\sin\theta & 2\sin\phi\cos\theta & 0 \end{vmatrix}$

$= (4\sin^2\phi\cos\theta)\,\mathbf{i} + (4\sin^2\phi\sin\theta)\,\mathbf{j} + (4\sin\phi\cos\phi)\mathbf{k}$; $\nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = (\nabla \times \mathbf{F}) \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta)\, d\phi\, d\theta$ (see Exercise 13 above) $\Rightarrow \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_R (-8z\sin^2\phi\cos\theta - 4\sin^2\phi\sin\theta - 8y\sin\phi\cos\theta)\, d\phi\, d\theta$

$= \int_0^{2\pi}\int_0^{\pi/2} (-16\sin^2\phi\cos\phi\cos\theta - 4\sin^2\phi\sin\theta - 16\sin^2\phi\sin\theta\cos\theta)\, d\phi\, d\theta$

$= \int_0^{2\pi} \left[-\frac{16}{3}\sin^3\phi\cos\theta - 4\left(\frac{\phi}{2} - \frac{\sin 2\phi}{4}\right)(\sin\theta) - 16\left(\frac{\phi}{2} - \frac{\sin 2\phi}{4}\right)(\sin\theta\cos\theta)\right]_0^{\pi/2} d\theta$

$= \int_0^{2\pi} \left(-\frac{16}{3}\cos\theta - \pi\sin\theta - 4\pi\sin\theta\cos\theta\right) d\theta = \left[-\frac{16}{3}\sin\theta + \pi\cos\theta - 2\pi\sin^2\theta\right]_0^{2\pi} = 0$

19. (a) $\mathbf{F} = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow \text{curl}\,\mathbf{F} = \mathbf{0} \Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_S 0\, d\sigma = 0$

(b) Let $f(x, y, z) = x^2y^2z^3 \Rightarrow \nabla \times \mathbf{F} = \nabla \times \nabla f = \mathbf{0} \Rightarrow \text{curl}\,\mathbf{F} = \mathbf{0} \Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_S 0\, d\sigma$ $= 0$

(c) $\mathbf{F} = \nabla \times (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = \mathbf{0} \Rightarrow \nabla \times \mathbf{F} = \mathbf{0} \Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_S 0\, d\sigma = 0$

(d) $\mathbf{F} = \nabla f \Rightarrow \nabla \times \mathbf{F} = \nabla \times \nabla f = \mathbf{0} \Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_S 0\, d\sigma = 0$

20. $\mathbf{F} = \nabla f = -\frac{1}{2}(x^2 + y^2 + z^2)^{-3/2}(2x)\mathbf{i} - \frac{1}{2}(x^2 + y^2 + z^2)^{-3/2}(2y)\mathbf{j} - \frac{1}{2}(x^2 + y^2 + z^2)^{-3/2}(2z)\mathbf{k}$

$= -x(x^2 + y^2 + z^2)^{-3/2}\mathbf{i} - y(x^2 + y^2 + z^2)^{-3/2}\mathbf{j} - z(x^2 + y^2 + z^2)^{-3/2}\mathbf{k}$

(a) $\mathbf{r} = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j}$, $0 \le t \le 2\pi \Rightarrow \frac{d\mathbf{r}}{dt} = (-a\sin t)\mathbf{i} + (a\cos t)\mathbf{j}$

$\Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -x(x^2 + y^2 + z^2)^{-3/2}(-a\sin t) - y(x^2 + y^2 + z^2)^{-3/2}(a\cos t)$

$= \left(-\frac{a\cos t}{a^3}\right)(-a\sin t) - \left(\frac{a\sin t}{a^3}\right)(a\cos t) = 0 \Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = 0$

(b) $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_S \nabla \times \nabla f \cdot \mathbf{n}\, d\sigma = \iint_S \mathbf{0} \cdot \mathbf{n}\, d\sigma = \iint_S 0\, d\sigma = 0$

21. Let $\mathbf{F} = 2y\mathbf{i} + 3z\mathbf{j} - x\mathbf{k} \Rightarrow \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y & 3z & -x \end{vmatrix} = -3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$; $\mathbf{n} = \frac{2\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{3}$

$\Rightarrow \nabla \times \mathbf{F} \cdot \mathbf{n} = -2 \Rightarrow \oint_C 2y\, dx + 3z\, dy - x\, dz = \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_S -2\, d\sigma$

$= -2\iint_S d\sigma$, where $\iint_S d\sigma$ is the area of the region enclosed by C on the plane S: $2x + 2y + z = 2$

22. $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0$

23. Suppose $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ exists such that $\nabla \times \mathbf{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}$ $= x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Then $\frac{\partial}{\partial x}\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right) = \frac{\partial}{\partial x}(x) \Rightarrow \frac{\partial^2 P}{\partial x \partial y} - \frac{\partial^2 N}{\partial x \partial z} = 1$. Likewise, $\frac{\partial}{\partial y}\left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right) = \frac{\partial}{\partial y}(y)$ $\Rightarrow \frac{\partial^2 M}{\partial y \partial z} - \frac{\partial^2 P}{\partial y \partial x} = 1$ and $\frac{\partial}{\partial z}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{\partial}{\partial z}(z) \Rightarrow \frac{\partial^2 N}{\partial z \partial x} - \frac{\partial^2 M}{\partial z \partial y} = 1$. Summing the calculated equations $\Rightarrow \left(\frac{\partial^2 P}{\partial x \partial y} - \frac{\partial^2 P}{\partial y \partial x}\right) + \left(\frac{\partial^2 N}{\partial z \partial x} - \frac{\partial^2 N}{\partial x \partial z}\right) + \left(\frac{\partial^2 M}{\partial y \partial z} - \frac{\partial^2 M}{\partial z \partial y}\right) = 3$ or $0 = 3$ (assuming the second mixed partials are equal). This result is a contradiction, so there is no field $\mathbf{F}$ such that curl $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

24. Yes: If $\nabla \times \mathbf{F} = \mathbf{0}$, then the circulation of $\mathbf{F}$ around the boundary C of any oriented surface S in the domain of $\mathbf{F}$ is zero. The reason is this: By Stokes's theorem, circulation $= \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_S \mathbf{0} \cdot \mathbf{n}\, d\sigma = 0.$

25. $r = \sqrt{x^2+y^2} \Rightarrow r^4 = (x^2+y^2)^2 \Rightarrow \mathbf{F} = \nabla(r^4) = 4x(x^2+y^2)\mathbf{i} + 4y(x^2+y^2)\mathbf{j} = M\mathbf{i} + N\mathbf{j}$

$\Rightarrow \oint_C \nabla(r^4) \cdot \mathbf{n}\, ds = \oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \oint_C M\, dy - N\, dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\, dy$

$= \iint_R [4(x^2+y^2) + 8x^2 + 4(x^2+y^2) + 8y^2]\, dA = \iint_R 16(x^2+y^2)\, dA = 16 \iint_R x^2\, dA + 16 \iint_R y^2\, dA$

$= 16I_y + 16I_x.$

26. $\frac{\partial P}{\partial y} = 0,\ \frac{\partial N}{\partial z} = 0,\ \frac{\partial M}{\partial z} = 0,\ \frac{\partial P}{\partial x} = 0,\ \frac{\partial N}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2},\ \frac{\partial M}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2} \Rightarrow \text{curl}\, \mathbf{F} = \left[\frac{y^2-x^2}{(x^2+y^2)^2} - \frac{y^2-x^2}{(x^2+y^2)^2}\right]\mathbf{k} = \mathbf{0}.$

However, $x^2+y^2 = 1 \Rightarrow \mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} \Rightarrow \frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$

$\Rightarrow \mathbf{F} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = \sin^2 t + \cos^2 t = 1 \Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_0^{2\pi} 1\, dt = 2\pi$ which is not zero.

16.8 THE DIVERGENCE THEOREM AND A UNIFIED THEORY

1. $\mathbf{F} = \frac{-y\mathbf{i}+x\mathbf{j}}{\sqrt{x^2+y^2}} \Rightarrow \text{div}\, \mathbf{F} = \frac{xy-xy}{(x^2+y^2)^{3/2}} = 0$

2. $\mathbf{F} = x\mathbf{i} + y\mathbf{j} \Rightarrow \text{div}\, \mathbf{F} = 1 + 1 = 2$

3. $\mathbf{F} = -\frac{GM(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})}{(x^2+y^2+z^2)^{3/2}} \Rightarrow \text{div}\, \mathbf{F} = -GM\left[\frac{(x^2+y^2+z^2)^{3/2} - 3x^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3}\right]$

$- GM\left[\frac{(x^2+y^2+z^2)^{3/2} - 3y^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3}\right] - GM\left[\frac{(x^2+y^2+z^2)^{3/2} - 3z^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3}\right]$

$= -GM\left[\frac{3(x^2+y^2+z^2)^2 - 3(x^2+y^2+z^2)(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{7/2}}\right] = 0$

4. $z = a^2 - r^2$ in cylindrical coordinates $\Rightarrow z = a^2 - (x^2+y^2) \Rightarrow \mathbf{v} = (a^2 - x^2 - y^2)\mathbf{k} \Rightarrow \text{div}\, \mathbf{v} = 0$

5. $\frac{\partial}{\partial x}(y-x) = -1,\ \frac{\partial}{\partial y}(z-y) = -1,\ \frac{\partial}{\partial z}(y-x) = 0 \Rightarrow \nabla \cdot \mathbf{F} = -2 \Rightarrow \text{Flux} = \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 -2\, dx\, dy\, dz = -2(2^3) = -16$

6. $\frac{\partial}{\partial x}(x^2) = 2x,\ \frac{\partial}{\partial y}(y^2) = 2y,\ \frac{\partial}{\partial x}(z^2) = 2z \Rightarrow \nabla \cdot \mathbf{F} = 2x + 2y + 2z$

(a) Flux $= \int_0^1 \int_0^1 \int_0^1 (2x+2y+2z)\, dx\, dy\, dz = \int_0^1 \int_0^1 [x^2 + 2x(y+z)]_0^1\, dy\, dz = \int_0^1 \int_0^1 (1+2y+2z)\, dy\, dz$

$= \int_0^1 [y(1+2z) + y^2]_0^1\, dz = \int_0^1 (2+2z)\, dz = [2z + z^2]_0^1 = 3$

(b) Flux $= \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 (2x+2y+2z)\, dx\, dy\, dz = \int_{-1}^1 \int_{-1}^1 [x^2 + 2x(y+z)]_{-1}^1\, dy\, dz = \int_{-1}^1 \int_{-1}^1 (4y+4z)\, dy\, dz$

$= \int_{-1}^1 [2y^2 + 4yz]_{-1}^1\, dz = \int_{-1}^1 8z\, dz = [4z^2]_{-1}^1 = 0$

(c) In cylindrical coordinates, Flux $= \iiint_D (2x+2y+2z)\, dx\, dy\, dz$

$= \int_0^1 \int_0^{2\pi} \int_0^2 (2r\cos\theta + 2r\sin\theta + 2z)\, r\, dr\, d\theta\, dz = \int_0^1 \int_0^{2\pi} \left[\frac{2}{3}r^3\cos\theta + \frac{2}{3}r^3\sin\theta + zr^2\right]_0^2 d\theta\, dz$

$= \int_0^1 \int_0^{2\pi} \left(\frac{16}{3}\cos\theta + \frac{16}{3}\sin\theta + 4z\right) d\theta\, dz = \int_0^1 \left[\frac{16}{3}\sin\theta - \frac{16}{3}\cos\theta + 4z\theta\right]_0^{2\pi} dz = \int_0^1 8\pi z\, dz = [4\pi z^2]_0^1 = 4\pi$

7. $\frac{\partial}{\partial x}(y) = 0,\ \frac{\partial}{\partial y}(xy) = x,\ \frac{\partial}{\partial z}(-z) = -1 \Rightarrow \nabla \cdot \mathbf{F} = x - 1;\ z = x^2 + y^2 \Rightarrow z = r^2$ in cylindrical coordinates

$\Rightarrow \text{Flux} = \iiint_D (x-1)\, dz\, dy\, dx = \int_0^{2\pi} \int_0^2 \int_0^{r^2} (r\cos\theta - 1)\, dz\, r\, dr\, d\theta = \int_0^{2\pi} \int_0^2 (r^3\cos\theta - r^2)\, r\, dr\, d\theta$

$= \int_0^{2\pi} \left[\frac{r^5}{5}\cos\theta - \frac{r^4}{4}\right]_0^2 d\theta = \int_0^{2\pi} \left(\frac{32}{5}\cos\theta - 4\right) d\theta = \left[\frac{32}{5}\sin\theta - 4\theta\right]_0^{2\pi} = -8\pi$

8. $\frac{\partial}{\partial x}(x^2) = 2x$, $\frac{\partial}{\partial y}(xz) = 0$, $\frac{\partial}{\partial z}(3z) = 3 \Rightarrow \nabla \cdot \mathbf{F} = 2x + 3 \Rightarrow \text{Flux} = \iiint_D (2x+3)\, dV$

$= \int_0^{2\pi}\int_0^{\pi}\int_0^{2} (2\rho \sin\phi \cos\theta + 3)(\rho^2 \sin\phi)\, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi} \left[\frac{\rho^4}{2}\sin\phi\cos\theta + \rho^3\right]_0^2 \sin\phi\, d\phi\, d\theta$

$= \int_0^{2\pi}\int_0^{\pi} (8\sin\phi\cos\theta + 8)\sin\phi\, d\phi\, d\theta = \int_0^{2\pi} \left[8\left(\frac{\phi}{2} - \frac{\sin 2\phi}{4}\right)\cos\theta - 8\cos\phi\right]_0^{\pi} d\theta = \int_0^{2\pi} (4\pi\cos\theta + 16)\, d\theta = 32\pi$

9. $\frac{\partial}{\partial x}(x^2) = 2x$, $\frac{\partial}{\partial y}(-2xy) = -2x$, $\frac{\partial}{\partial z}(3xz) = 3x \Rightarrow \text{Flux} = \iiint_D 3x\, dx\, dy\, dz$

$= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{2} (3\rho\sin\phi\cos\theta)(\rho^2\sin\phi)\, d\rho\, d\phi\, d\theta = \int_0^{\pi/2}\int_0^{\pi/2} 12\sin^2\phi\cos\theta\, d\phi\, d\theta = \int_0^{\pi/2} 3\pi\cos\theta\, d\theta = 3\pi$

10. $\frac{\partial}{\partial x}(6x^2 + 2xy) = 12x + 2y$, $\frac{\partial}{\partial y}(2y + x^2 z) = 2$, $\frac{\partial}{\partial z}(4x^2y^3) = 0 \Rightarrow \nabla \cdot \mathbf{F} = 12x + 2y + 2$

$\Rightarrow \text{Flux} = \iiint_D (12x + 2y + 2)\, dV = \int_0^3\int_0^{\pi/2}\int_0^2 (12r\cos\theta + 2r\sin\theta + 2)\, r\, dr\, d\theta\, dz$

$= \int_0^3\int_0^{\pi/2} \left(32\cos\theta + \frac{16}{3}\sin\theta + 4\right) d\theta\, dz = \int_0^3 \left(32 + 2\pi + \frac{16}{3}\right) dz = 112 + 6\pi$

11. $\frac{\partial}{\partial x}(2xz) = 2z$, $\frac{\partial}{\partial y}(-xy) = -x$, $\frac{\partial}{\partial z}(-z^2) = -2z \Rightarrow \nabla \cdot \mathbf{F} = -x \Rightarrow \text{Flux} = \iiint_D -x\, dV$

$= \int_0^2\int_0^{\sqrt{16-4x^2}}\int_0^{4-y} -x\, dz\, dy\, dx = \int_0^2\int_0^{\sqrt{16-4x^2}} (xy - 4x)\, dy\, dx = \int_0^2 \left[\frac{1}{2}x(16 - 4x^2) - 4x\sqrt{16-4x^2}\right] dx$

$= \left[4x^2 - \frac{1}{2}x^4 + \frac{1}{3}(16 - 4x^2)^{3/2}\right]_0^2 = -\frac{40}{3}$

12. $\frac{\partial}{\partial x}(x^3) = 3x^2$, $\frac{\partial}{\partial y}(y^3) = 3y^2$, $\frac{\partial}{\partial z}(z^3) = 3z^2 \Rightarrow \nabla \cdot \mathbf{F} = 3x^2 + 3y^2 + 3z^2 \Rightarrow \text{Flux} = \iiint_D 3(x^2 + y^2 + z^2)\, dV$

$= 3\int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^2(\rho^2\sin\phi)\, d\rho\, d\phi\, d\theta = 3\int_0^{2\pi}\int_0^{\pi} \frac{a^5}{5}\sin\phi\, d\phi\, d\theta = 3\int_0^{2\pi} \frac{2a^5}{5}\, d\theta = \frac{12\pi a^5}{5}$

13. Let $\rho = \sqrt{x^2 + y^2 + z^2}$. Then $\frac{\partial\rho}{\partial x} = \frac{x}{\rho}$, $\frac{\partial\rho}{\partial y} = \frac{y}{\rho}$, $\frac{\partial\rho}{\partial z} = \frac{z}{\rho} \Rightarrow \frac{\partial}{\partial x}(\rho x) = \left(\frac{\partial\rho}{\partial x}\right)x + \rho = \frac{x^2}{\rho} + \rho$, $\frac{\partial}{\partial y}(\rho y) = \left(\frac{\partial\rho}{\partial y}\right)y + \rho$

$= \frac{y^2}{\rho} + \rho$, $\frac{\partial}{\partial z}(\rho z) = \left(\frac{\partial\rho}{\partial z}\right)z + \rho = \frac{z^2}{\rho} + \rho \Rightarrow \nabla \cdot \mathbf{F} = \frac{x^2+y^2+z^2}{\rho} + 3\rho = 4\rho$, since $\rho = \sqrt{x^2 + y^2 + z^2}$

$\Rightarrow \text{Flux} = \iiint_D 4\rho\, dV = \int_0^{2\pi}\int_0^{\pi}\int_1^{\sqrt{2}} (4\rho)(\rho^2\sin\phi)\, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi} 3\sin\phi\, d\phi\, d\theta = \int_0^{2\pi} 6\, d\theta = 12\pi$

14. Let $\rho = \sqrt{x^2 + y^2 + z^2}$. Then $\frac{\partial\rho}{\partial x} = \frac{x}{\rho}$, $\frac{\partial\rho}{\partial y} = \frac{y}{\rho}$, $\frac{\partial\rho}{\partial z} = \frac{z}{\rho} \Rightarrow \frac{\partial}{\partial x}\left(\frac{x}{\rho}\right) = \frac{1}{\rho} - \left(\frac{x}{\rho^2}\right)\frac{\partial\rho}{\partial x} = \frac{1}{\rho} - \frac{x^2}{\rho^3}$. Similarly,

$\frac{\partial}{\partial y}\left(\frac{y}{\rho}\right) = \frac{1}{\rho} - \frac{y^2}{\rho^3}$ and $\frac{\partial}{\partial z}\left(\frac{z}{\rho}\right) = \frac{1}{\rho} - \frac{z^2}{\rho^3} \Rightarrow \nabla \cdot \mathbf{F} = \frac{3}{\rho} - \frac{x^2+y^2+z^2}{\rho^3} = \frac{2}{\rho}$

$\Rightarrow \text{Flux} = \iiint_D \frac{2}{\rho}\, dV = \int_0^{2\pi}\int_0^{\pi}\int_1^{2} \left(\frac{2}{\rho}\right)(\rho^2\sin\phi)\, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi} 3\sin\phi\, d\phi\, d\theta = \int_0^{2\pi} 6\, d\theta = 12\pi$

15. $\frac{\partial}{\partial x}(5x^3 + 12xy^2) = 15x^2 + 12y^2$, $\frac{\partial}{\partial y}(y^3 + e^y\sin z) = 3y^2 + e^y\sin z$, $\frac{\partial}{\partial z}(5z^3 + e^y\cos z) = 15z^2 - e^y\sin z$

$\Rightarrow \nabla \cdot \mathbf{F} = 15x^2 + 15y^2 + 15z^2 = 15\rho^2 \Rightarrow \text{Flux} = \iiint_D 15\rho^2\, dV = \int_0^{2\pi}\int_0^{\pi}\int_1^{\sqrt{2}} (15\rho^2)(\rho^2\sin\phi)\, d\rho\, d\phi\, d\theta$

$= \int_0^{2\pi}\int_0^{\pi} \left(12\sqrt{2} - 3\right)\sin\phi\, d\phi\, d\theta = \int_0^{2\pi} \left(24\sqrt{2} - 6\right) d\theta = \left(48\sqrt{2} - 12\right)\pi$

16. $\frac{\partial}{\partial x}[\ln(x^2 + y^2)] = \frac{2x}{x^2+y^2}$, $\frac{\partial}{\partial y}\left(-\frac{2z}{x}\tan^{-1}\frac{y}{x}\right) = \left(-\frac{2z}{x}\right)\left[\frac{\left(\frac{1}{x}\right)}{1 + \left(\frac{y}{x}\right)^2}\right] = -\frac{2z}{x^2+y^2}$, $\frac{\partial}{\partial z}\left(z\sqrt{x^2+y^2}\right) = \sqrt{x^2+y^2}$

$\Rightarrow \nabla \cdot \mathbf{F} = \frac{2x}{x^2+y^2} - \frac{2z}{x^2+y^2} + \sqrt{x^2+y^2} \Rightarrow \text{Flux} = \iiint_D \left(\frac{2x}{x^2+y^2} - \frac{2z}{x^2+y^2} + \sqrt{x^2+y^2}\right) dz\, dy\, dx$

$= \int_0^{2\pi}\int_1^{\sqrt{2}}\int_{-1}^{2}\left(\frac{2r\cos\theta}{r^2} - \frac{2z}{r^2} + r\right) dz\, r\, dr\, d\theta = \int_0^{2\pi}\int_1^{\sqrt{2}}\left(6\cos\theta - \frac{3}{r} + 3r^2\right) dr\, d\theta$
$= \int_0^{2\pi}\left[6\left(\sqrt{2}-1\right)\cos\theta - 3\ln\sqrt{2} + 2\sqrt{2} - 1\right] d\theta = 2\pi\left(-\frac{3}{2}\ln 2 + 2\sqrt{2} - 1\right)$

17. (a) $\mathbf{G} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k} \Rightarrow \nabla \times \mathbf{G} = \text{curl}\,\mathbf{G} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\mathbf{k} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k} \Rightarrow \nabla \cdot \nabla \times \mathbf{G}$
$= \text{div}(\text{curl}\,\mathbf{G}) = \frac{\partial}{\partial x}\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$
$= \frac{\partial^2 P}{\partial x\partial y} - \frac{\partial^2 N}{\partial x\partial z} + \frac{\partial^2 M}{\partial y\partial z} - \frac{\partial^2 P}{\partial y\partial x} + \frac{\partial^2 N}{\partial z\partial x} - \frac{\partial^2 M}{\partial z\partial y} = 0$ if all first and second partial derivatives are continuous

(b) By the Divergence Theorem, the outward flux of $\nabla \times \mathbf{G}$ across a closed surface is zero because outward flux of $\nabla \times \mathbf{G} = \iint_S (\nabla \times \mathbf{G}) \cdot \mathbf{n}\, d\sigma$
$= \iiint_D \nabla \cdot \nabla \times \mathbf{G}\, dV$ [Divergence Theorem with $\mathbf{F} = \nabla \times \mathbf{G}$]
$= \iiint_D (0)\, dV = 0$ [by part (a)]

18. (a) Let $\mathbf{F}_1 = M_1\mathbf{i} + N_1\mathbf{j} + P_1\mathbf{k}$ and $\mathbf{F}_2 = M_2\mathbf{i} + N_2\mathbf{j} + P_2\mathbf{k} \Rightarrow a\mathbf{F}_1 + b\mathbf{F}_2$
$= (aM_1 + bM_2)\mathbf{i} + (aN_1 + bN_2)\mathbf{j} + (aP_1 + bP_2)\mathbf{k} \Rightarrow \nabla \cdot (a\mathbf{F}_1 + b\mathbf{F}_2)$
$= \left(a\frac{\partial M_1}{\partial x} + b\frac{\partial M_2}{\partial x}\right) + \left(a\frac{\partial N_1}{\partial y} + b\frac{\partial N_2}{\partial y}\right) + \left(a\frac{\partial P_1}{\partial z} + b\frac{\partial P_2}{\partial z}\right)$
$= a\left(\frac{\partial M_1}{\partial x} + \frac{\partial N_1}{\partial y} + \frac{\partial P_1}{\partial z}\right) + b\left(\frac{\partial M_2}{\partial x} + \frac{\partial N_2}{\partial y} + \frac{\partial P_2}{\partial z}\right) = a(\nabla \cdot \mathbf{F}_1) + b(\nabla \cdot \mathbf{F}_2)$

(b) Define $\mathbf{F}_1$ and $\mathbf{F}_2$ as in part a $\Rightarrow \nabla \times (a\mathbf{F}_1 + b\mathbf{F}_2)$
$= \left[\left(a\frac{\partial P_1}{\partial y} + b\frac{\partial P_2}{\partial y}\right) - \left(a\frac{\partial N_1}{\partial z} + b\frac{\partial N_2}{\partial z}\right)\right]\mathbf{i} + \left[\left(a\frac{\partial M_1}{\partial z} + b\frac{\partial M_2}{\partial z}\right) - \left(a\frac{\partial P_1}{\partial x} + b\frac{\partial P_2}{\partial x}\right)\right]\mathbf{j}$
$+ \left[\left(a\frac{\partial N_1}{\partial x} + b\frac{\partial N_2}{\partial x}\right) - \left(a\frac{\partial M_1}{\partial y} + b\frac{\partial M_2}{\partial y}\right)\right]\mathbf{k} = a\left[\left(\frac{\partial P_1}{\partial y} - \frac{\partial N_1}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M_1}{\partial z} - \frac{\partial P_1}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N_1}{\partial x} - \frac{\partial M_1}{\partial y}\right)\mathbf{k}\right]$
$+ b\left[\left(\frac{\partial P_2}{\partial y} - \frac{\partial N_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M_2}{\partial z} - \frac{\partial P_2}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N_2}{\partial x} - \frac{\partial M_2}{\partial y}\right)\mathbf{k}\right] = a\nabla \times \mathbf{F}_1 + b\nabla \times \mathbf{F}_2$

(c) $\mathbf{F}_1 \times \mathbf{F}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ M_1 & N_1 & P_1 \\ M_2 & N_2 & P_2 \end{vmatrix} = (N_1P_2 - P_1N_2)\mathbf{i} - (M_1P_2 - P_1M_2)\mathbf{j} + (M_1N_2 - N_1M_2)\mathbf{k} \Rightarrow \nabla \cdot (\mathbf{F}_1 \times \mathbf{F}_2)$
$= \nabla \cdot [(N_1P_2 - P_1N_2)\mathbf{i} - (M_1P_2 - P_1M_2)\mathbf{j} + (M_1N_2 - N_1M_2)\mathbf{k}]$
$= \frac{\partial}{\partial x}(N_1P_2 - P_1N_2) - \frac{\partial}{\partial y}(M_1P_2 - P_1M_2) + \frac{\partial}{\partial z}(M_1N_2 - N_1M_2) = \left(P_2\frac{\partial N_1}{\partial x} + N_1\frac{\partial P_2}{\partial x} - N_2\frac{\partial P_1}{\partial x} - P_1\frac{\partial N_2}{\partial x}\right)$
$- \left(M_1\frac{\partial P_2}{\partial y} + P_2\frac{\partial M_1}{\partial y} - P_1\frac{\partial M_2}{\partial y} - M_2\frac{\partial P_1}{\partial y}\right) + \left(M_1\frac{\partial N_2}{\partial z} + N_2\frac{\partial M_1}{\partial z} - N_1\frac{\partial M_2}{\partial z} - M_2\frac{\partial N_1}{\partial z}\right)$
$= M_2\left(\frac{\partial P_1}{\partial y} - \frac{\partial N_1}{\partial z}\right) + N_2\left(\frac{\partial M_1}{\partial z} - \frac{\partial P_1}{\partial x}\right) + P_2\left(\frac{\partial N_1}{\partial x} - \frac{\partial M_1}{\partial y}\right) + M_1\left(\frac{\partial N_2}{\partial z} - \frac{\partial P_2}{\partial y}\right) + N_1\left(\frac{\partial P_2}{\partial x} - \frac{\partial M_2}{\partial z}\right)$
$+ P_1\left(\frac{\partial M_2}{\partial y} - \frac{\partial N_2}{\partial x}\right) = \mathbf{F}_2 \cdot \nabla \times \mathbf{F}_1 - \mathbf{F}_1 \cdot \nabla \times \mathbf{F}_2$

19. (a) $\text{div}(g\mathbf{F}) = \nabla \cdot g\mathbf{F} = \frac{\partial}{\partial x}(gM) + \frac{\partial}{\partial y}(gN) + \frac{\partial}{\partial z}(gP) = \left(g\frac{\partial M}{\partial x} + M\frac{\partial g}{\partial x}\right) + \left(g\frac{\partial N}{\partial y} + N\frac{\partial g}{\partial y}\right) + \left(g\frac{\partial P}{\partial z} + P\frac{\partial g}{\partial z}\right)$
$= \left(M\frac{\partial g}{\partial x} + N\frac{\partial g}{\partial y} + P\frac{\partial g}{\partial z}\right) + g\left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}\right) = g\nabla \cdot \mathbf{F} + \nabla g \cdot \mathbf{F}$

(b) $\nabla \times (g\mathbf{F}) = \left[\frac{\partial}{\partial y}(gP) - \frac{\partial}{\partial z}(gN)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}(gM) - \frac{\partial}{\partial x}(gP)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(gN) - \frac{\partial}{\partial y}(gM)\right]\mathbf{k}$
$= \left(P\frac{\partial g}{\partial y} + g\frac{\partial P}{\partial y} - N\frac{\partial g}{\partial z} - g\frac{\partial N}{\partial z}\right)\mathbf{i} + \left(M\frac{\partial g}{\partial z} + g\frac{\partial M}{\partial z} - P\frac{\partial g}{\partial x} - g\frac{\partial P}{\partial x}\right)\mathbf{j} + \left(N\frac{\partial g}{\partial x} + g\frac{\partial N}{\partial x} - M\frac{\partial g}{\partial y} - g\frac{\partial M}{\partial y}\right)\mathbf{k}$
$= \left(P\frac{\partial g}{\partial y} - N\frac{\partial g}{\partial z}\right)\mathbf{i} + \left(g\frac{\partial P}{\partial y} - g\frac{\partial N}{\partial z}\right)\mathbf{i} + \left(M\frac{\partial g}{\partial z} - P\frac{\partial g}{\partial x}\right)\mathbf{j} + \left(g\frac{\partial M}{\partial z} - g\frac{\partial P}{\partial x}\right)\mathbf{j} + \left(N\frac{\partial g}{\partial x} - M\frac{\partial g}{\partial y}\right)\mathbf{k}$
$+ \left(g\frac{\partial N}{\partial x} - g\frac{\partial M}{\partial y}\right)\mathbf{k} = g\nabla \times \mathbf{F} + \nabla g \times \mathbf{F}$

20. Let $\mathbf{F}_1 = M_1\mathbf{i} + N_1\mathbf{j} + P_1\mathbf{k}$ and $\mathbf{F}_2 = M_2\mathbf{i} + N_2\mathbf{j} + P_2\mathbf{k}$.

(a) $\mathbf{F}_1 \times \mathbf{F}_2 = (N_1P_2 - P_1N_2)\mathbf{i} + (P_1M_2 - M_1P_2)\mathbf{j} + (M_1N_2 - N_1M_2)\mathbf{k} \Rightarrow \nabla \times (\mathbf{F}_1 \times \mathbf{F}_2)$

$= \left[\frac{\partial}{\partial y}(M_1N_2 - N_1M_2) - \frac{\partial}{\partial z}(P_1M_2 - M_1P_2)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}(N_1P_2 - P_1N_2) - \frac{\partial}{\partial x}(M_1N_2 - N_1M_2)\right]\mathbf{j}$

$+ \left[\frac{\partial}{\partial x}(P_1M_2 - M_1P_2) - \frac{\partial}{\partial y}(N_1P_2 - P_1N_2)\right]\mathbf{k}$

and consider the **i**-component only: $\frac{\partial}{\partial y}(M_1N_2 - N_1M_2) - \frac{\partial}{\partial z}(P_1M_2 - M_1P_2)$

$= N_2\frac{\partial M_1}{\partial y} + M_1\frac{\partial N_2}{\partial y} - M_2\frac{\partial N_1}{\partial y} - N_1\frac{\partial M_2}{\partial y} - M_2\frac{\partial P_1}{\partial z} - P_1\frac{\partial M_2}{\partial z} + P_2\frac{\partial M_1}{\partial z} + M_1\frac{\partial P_2}{\partial z}$

$= \left(N_2\frac{\partial M_1}{\partial y} + P_2\frac{\partial M_1}{\partial z}\right) - \left(N_1\frac{\partial M_2}{\partial y} + P_1\frac{\partial M_2}{\partial z}\right) + \left(\frac{\partial N_2}{\partial y} + \frac{\partial P_2}{\partial z}\right)M_1 - \left(\frac{\partial N_1}{\partial y} + \frac{\partial P_1}{\partial z}\right)M_2$

$= \left(M_2\frac{\partial M_1}{\partial x} + N_2\frac{\partial M_1}{\partial y} + P_2\frac{\partial M_1}{\partial z}\right) - \left(M_1\frac{\partial M_2}{\partial x} + N_1\frac{\partial M_2}{\partial y} + P_1\frac{\partial M_2}{\partial z}\right) + \left(\frac{\partial M_2}{\partial x} + \frac{\partial N_2}{\partial y} + \frac{\partial P_2}{\partial z}\right)M_1$

$- \left(\frac{\partial M_1}{\partial x} + \frac{\partial N_1}{\partial y} + \frac{\partial P_1}{\partial z}\right)M_2$. Now, **i**-comp of $(\mathbf{F}_2 \cdot \nabla)\mathbf{F}_1 = \left(M_2\frac{\partial}{\partial x} + N_2\frac{\partial}{\partial y} + P_2\frac{\partial}{\partial z}\right)M_1$

$= \left(M_2\frac{\partial M_1}{\partial x} + N_2\frac{\partial M_1}{\partial y} + P_2\frac{\partial M_1}{\partial z}\right)$; likewise, **i**-comp of $(\mathbf{F}_1 \cdot \nabla)\mathbf{F}_2 = \left(M_1\frac{\partial M_2}{\partial x} + N_1\frac{\partial M_2}{\partial y} + P_1\frac{\partial M_2}{\partial z}\right)$;

i-comp of $(\nabla \cdot \mathbf{F}_2)\mathbf{F}_1 = \left(\frac{\partial M_2}{\partial x} + \frac{\partial N_2}{\partial y} + \frac{\partial P_2}{\partial z}\right)M_1$ and **i**-comp of $(\nabla \cdot \mathbf{F}_1)\mathbf{F}_2 = \left(\frac{\partial M_1}{\partial x} + \frac{\partial N_1}{\partial y} + \frac{\partial P_1}{\partial z}\right)M_2$.

Similar results hold for the **j** and **k** components of $\nabla \times (\mathbf{F}_1 \times \mathbf{F}_2)$. In summary, since the corresponding components are equal, we have the result

$\nabla \times (\mathbf{F}_1 \times \mathbf{F}_2) = (\mathbf{F}_2 \cdot \nabla)\mathbf{F}_1 - (\mathbf{F}_1 \cdot \nabla)\mathbf{F}_2 + (\nabla \cdot \mathbf{F}_2)\mathbf{F}_1 - (\nabla \cdot \mathbf{F}_1)\mathbf{F}_2$

(b) Here again we consider only the **i**-component of each expression. Thus, the **i**-comp of $\nabla(\mathbf{F}_1 \cdot \mathbf{F}_2)$

$= \frac{\partial}{\partial x}(M_1M_2 + N_1N_2 + P_1P_2) = \left(M_1\frac{\partial M_2}{\partial x} + M_2\frac{\partial M_1}{\partial x} + N_1\frac{\partial N_2}{\partial x} + N_2\frac{\partial N_1}{\partial x} + P_1\frac{\partial P_2}{\partial x} + P_2\frac{\partial P_1}{\partial x}\right)$

i-comp of $(\mathbf{F}_1 \cdot \nabla)\mathbf{F}_2 = \left(M_1\frac{\partial M_2}{\partial x} + N_1\frac{\partial M_2}{\partial y} + P_1\frac{\partial M_2}{\partial z}\right)$,

i-comp of $(\mathbf{F}_2 \cdot \nabla)\mathbf{F}_1 = \left(M_2\frac{\partial M_1}{\partial x} + N_2\frac{\partial M_1}{\partial y} + P_2\frac{\partial M_1}{\partial z}\right)$,

i-comp of $\mathbf{F}_1 \times (\nabla \times \mathbf{F}_2) = N_1\left(\frac{\partial N_2}{\partial x} - \frac{\partial M_2}{\partial y}\right) - P_1\left(\frac{\partial M_2}{\partial z} - \frac{\partial P_2}{\partial x}\right)$, and

i-comp of $\mathbf{F}_2 \times (\nabla \times \mathbf{F}_1) = N_2\left(\frac{\partial N_1}{\partial x} - \frac{\partial M_1}{\partial y}\right) - P_2\left(\frac{\partial M_1}{\partial z} - \frac{\partial P_1}{\partial x}\right)$.

Since corresponding components are equal, we see that

$\nabla(\mathbf{F}_1 \cdot \mathbf{F}_2) = (\mathbf{F}_1 \cdot \nabla)\mathbf{F}_2 + (\mathbf{F}_2 \cdot \nabla)\mathbf{F}_1 + \mathbf{F}_1 \times (\nabla \times \mathbf{F}_2) + \mathbf{F}_2 \times (\nabla \times \mathbf{F}_1)$, as claimed.

21. The integral's value never exceeds the surface area of S. Since $|\mathbf{F}| \le 1$, we have $|\mathbf{F} \cdot \mathbf{n}| = |\mathbf{F}|\,|\mathbf{n}| \le (1)(1) = 1$ and

$$\iiint_D \nabla \cdot \mathbf{F}\, d\sigma = \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma \qquad \text{[Divergence Theorem]}$$

$$\le \iint_S |\mathbf{F} \cdot \mathbf{n}|\, d\sigma \qquad \text{[A property of integrals]}$$

$$\le \iint_S (1)\, d\sigma \qquad [|\mathbf{F} \cdot \mathbf{n}| \le 1]$$

$$= \text{Area of S.}$$

22. Yes, the outward flux through the top is 5. The reason is this: Since $\nabla \cdot \mathbf{F} = \nabla \cdot (x\mathbf{i} - 2y\mathbf{j} + (z+3)\mathbf{k}) = 1 - 2 + 1 = 0$, the outward flux across the closed cubelike surface is 0 by the Divergence Theorem. The flux across the top is therefore the negative of the flux across the sides and base. Routine calculations show that the sum of these latter fluxes is -5. (The flux across the sides that lie in the xz-plane and the yz-plane are 0, while the flux across the xy-plane is -3.) Therefore the flux across the top is 5.

23. (a) $\frac{\partial}{\partial x}(x) = 1,\ \frac{\partial}{\partial y}(y) = 1,\ \frac{\partial}{\partial z}(z) = 1 \Rightarrow \nabla \cdot \mathbf{F} = 3 \Rightarrow \text{Flux} = \iiint_D 3\, dV = 3\iiint_D dV = 3(\text{Volume of the solid})$

(b) If **F** is orthogonal to **n** at every point of S, then $\mathbf{F} \cdot \mathbf{n} = 0$ everywhere $\Rightarrow$ Flux $= \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = 0$. But the flux is 3 (Volume of the solid) $\ne 0$, so **F** is not orthogonal to **n** at every point.

24. $\nabla \cdot \mathbf{F} = -2x - 4y - 6z + 12 \Rightarrow \text{Flux} = \int_0^a \int_0^b \int_0^1 (-2x - 4y - 6z + 12)\, dz\, dy\, dx = \int_0^a \int_0^b (-2x - 4y + 9)\, dy\, dx$ $= \int_0^a (-2xb - 2b^2 + 9b)\, dx = -a^2b - 2ab^2 + 9ab = ab(-a - 2b + 9) = f(a, b);\ \frac{\partial f}{\partial a} = -2ab - 2b^2 + 9b$ and $\frac{\partial f}{\partial b} = -a^2 - 4ab + 9a$ so that $\frac{\partial f}{\partial a} = 0$ and $\frac{\partial f}{\partial b} = 0 \Rightarrow b(-2a - 2b + 9) = 0$ and $a(-a - 4b + 9) = 0 \Rightarrow b = 0$ or $-2a - 2b + 9 = 0$, and $a = 0$ or $-a - 4b + 9 = 0$. Now $b = 0$ or $a = 0 \Rightarrow \text{Flux} = 0$; $-2a - 2b + 9 = 0$ and $-a - 4b + 9 = 0 \Rightarrow 3a - 9 = 0 \Rightarrow a = 3 \Rightarrow b = \frac{3}{2}$ so that $f\left(3, \frac{3}{2}\right) = \frac{27}{2}$ is the maximum flux.

25. $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_D \nabla \cdot \mathbf{F}\, dV = \iiint_D 3\, dV \Rightarrow \frac{1}{3} \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_D dV = \text{Volume of D}$

26. $\mathbf{F} = \mathbf{C} \Rightarrow \nabla \cdot \mathbf{F} = 0 \Rightarrow \text{Flux} = \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_D \nabla \cdot \mathbf{F}\, dV = \iiint_D 0\, dV = 0$

27. (a) From the Divergence Theorem, $\iint_S \nabla f \cdot \mathbf{n}\, d\sigma = \iiint_D \nabla \cdot \nabla f\, dV = \iiint_D \nabla^2 f\, dV = \iiint_D 0\, dV = 0$

(b) From the Divergence Theorem, $\iint_S f \nabla f \cdot \mathbf{n}\, d\sigma = \iiint_D \nabla \cdot f \nabla f\, dV$. Now,

$f \nabla f = \left(f \frac{\partial f}{\partial x}\right)\mathbf{i} + \left(f \frac{\partial f}{\partial y}\right)\mathbf{j} + \left(f \frac{\partial f}{\partial z}\right)\mathbf{k} \Rightarrow \nabla \cdot f \nabla f = \left[f \frac{\partial^2 f}{\partial x^2} + \left(\frac{\partial f}{\partial x}\right)^2\right] + \left[f \frac{\partial^2 f}{\partial y^2} + \left(\frac{\partial f}{\partial y}\right)^2\right] + \left[f \frac{\partial^2 f}{\partial z^2} + \left(\frac{\partial f}{\partial z}\right)^2\right]$ $= f \nabla^2 f + |\nabla f|^2 = 0 + |\nabla f|^2$ since f is harmonic $\Rightarrow \iint_S f \nabla f \cdot \mathbf{n}\, d\sigma = \iiint_D |\nabla f|^2\, dV$, as claimed.

28. From the Divergence Theorem, $\iint_S \nabla f \cdot \mathbf{n}\, d\sigma = \iiint_D \nabla \cdot \nabla f\, dV = \iiint_D \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right) dV$. Now, $f(x, y, z) = \ln \sqrt{x^2 + y^2 + z^2} = \frac{1}{2} \ln (x^2 + y^2 + z^2) \Rightarrow \frac{\partial f}{\partial x} = \frac{x}{x^2 + y^2 + z^2},\ \frac{\partial f}{\partial y} = \frac{y}{x^2 + y^2 + z^2},\ \frac{\partial f}{\partial z} = \frac{z}{x^2 + y^2 + z^2}$ $\Rightarrow \frac{\partial^2 f}{\partial x^2} = \frac{-x^2 + y^2 + z^2}{(x^2 + y^2 + z^2)^2},\ \frac{\partial^2 f}{\partial y^2} = \frac{x^2 - y^2 + z^2}{(x^2 + y^2 + z^2)^2},\ \frac{\partial^2 f}{\partial z^2} = \frac{x^2 + y^2 - z^2}{(x^2 + y^2 + z^2)^2},\ \Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$ $= \frac{x^2 + y^2 + z^2}{(x^2 + y^2 + z^2)^2} = \frac{1}{x^2 + y^2 + z^2} \Rightarrow \iint_S \nabla f \cdot \mathbf{n}\, d\sigma = \iiint_D \frac{dV}{x^2 + y^2 + z^2} = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^a \frac{\rho^2 \sin \phi}{\rho^2}\, d\rho\, d\phi\, d\theta$ $= \int_0^{\pi/2} \int_0^{\pi/2} a \sin \phi\, d\phi\, d\theta = \int_0^{\pi/2} [-a \cos \phi]_0^{\pi/2}\, d\theta = \int_0^{\pi/2} a\, d\theta = \frac{\pi a}{2}$

29. $\iint_S f \nabla g \cdot \mathbf{n}\, d\sigma = \iiint_D \nabla \cdot f \nabla g\, dV = \iiint_D \nabla \cdot \left(f \frac{\partial g}{\partial x}\mathbf{i} + f \frac{\partial g}{\partial y}\mathbf{j} + f \frac{\partial g}{\partial z}\mathbf{k}\right) dV$ $= \iiint_D \left(f \frac{\partial^2 g}{\partial x^2} + \frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + f \frac{\partial^2 g}{\partial y^2} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y} + f \frac{\partial^2 g}{\partial z^2} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial z}\right) dV$ $= \iiint_D \left[f\left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2}\right) + \left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial z}\right)\right] dV = \iiint_D (f \nabla^2 g + \nabla f \cdot \nabla g)\, dV$

30. By Exercise 29, $\iint_S f \nabla g \cdot \mathbf{n}\, d\sigma = \iiint_D (f \nabla^2 g + \nabla f \cdot \nabla g)\, dV$ and by interchanging the roles of f and g, $\iint_S g \nabla f \cdot \mathbf{n}\, d\sigma = \iiint_D (g \nabla^2 f + \nabla g \cdot \nabla f)\, dV$. Subtracting the second equation from the first yields: $\iint_S (f \nabla g - g \nabla f) \cdot \mathbf{n}\, d\sigma = \iiint_D (f \nabla^2 g - g \nabla^2 f)\, dV$ since $\nabla f \cdot \nabla g = \nabla g \cdot \nabla f$.

31. (a) The integral $\iiint_D p(t, x, y, z)\, dV$ represents the mass of the fluid at any time t. The equation says that the instantaneous rate of change of mass is flux of the fluid through the surface S enclosing the region D: the mass decreases if the flux is outward (so the fluid flows out of D), and increases if the flow is inward (interpreting $\mathbf{n}$ as the outward pointing unit normal to the surface).

(b) $\iiint_D \frac{\partial p}{\partial t}\, dV = \frac{d}{dt} \iiint_D p\, dV = -\iint_S p\mathbf{v} \cdot \mathbf{n}\, d\sigma = -\iiint_D \nabla \cdot p\mathbf{v}\, dV \Rightarrow \frac{\partial \rho}{\partial t} = -\nabla \cdot p\mathbf{v}$

Since the law is to hold for all regions D, $\nabla \cdot p\mathbf{v} + \frac{\partial p}{\partial t} = 0$, as claimed

32. (a) ∇T points in the direction of maximum change of the temperature, so if the solid is heating up at the point the temperature is greater in a region surrounding the point $\Rightarrow$ ∇T points away from the point $\Rightarrow -\nabla T$ points toward the point $\Rightarrow -\nabla T$ points in the direction the heat flows.

(b) Assuming the Law of Conservation of Mass (Exercise 31) with $-k\nabla T = p\mathbf{v}$ and $c\rho T = p$, we have $\frac{d}{dt}\iiint_D c\rho T\, dV = -\iint_S -k\nabla T \cdot \mathbf{n}\, d\sigma \Rightarrow$ the continuity equation, $\nabla \cdot (-k\nabla T) + \frac{\partial}{\partial t}(c\rho T) = 0$ $\Rightarrow c\rho \frac{\partial T}{\partial t} = -\nabla \cdot (-k\nabla T) = k\nabla^2 T \Rightarrow \frac{\partial T}{\partial t} = \frac{k}{c\rho}\nabla^2 T = K\nabla^2 T$, as claimed

CHAPTER 16 PRACTICE EXERCISES

1. Path 1: $\mathbf{r} = t\mathbf{i} + t\mathbf{j} + t\mathbf{k} \Rightarrow x = t, y = t, z = t, 0 \le t \le 1 \Rightarrow f(g(t), h(t), k(t)) = 3 - 3t^2$ and $\frac{dx}{dt} = 1, \frac{dy}{dt} = 1,$ $\frac{dz}{dt} = 1 \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = \sqrt{3}\, dt \Rightarrow \int_C f(x,y,z)\, ds = \int_0^1 \sqrt{3}(3 - 3t^2)\, dt = 2\sqrt{3}$

Path 2: $\mathbf{r}_1 = t\mathbf{i} + t\mathbf{j}, 0 \le t \le 1 \Rightarrow x = t, y = t, z = 0 \Rightarrow f(g(t), h(t), k(t)) = 2t - 3t^2 + 3$ and $\frac{dx}{dt} = 1, \frac{dy}{dt} = 1,$ $\frac{dz}{dt} = 0 \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = \sqrt{2}\, dt \Rightarrow \int_{C_1} f(x,y,z)\, ds = \int_0^1 \sqrt{2}(2t - 3t^2 + 3)\, dt = 3\sqrt{2};$

$\mathbf{r}_2 = \mathbf{i} + \mathbf{j} + t\mathbf{k} \Rightarrow x = 1, y = 1, z = t \Rightarrow f(g(t), h(t), k(t)) = 2 - 2t$ and $\frac{dx}{dt} = 0, \frac{dy}{dt} = 0, \frac{dz}{dt} = 1$

$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = dt \Rightarrow \int_{C_2} f(x,y,z)\, ds = \int_0^1 (2 - 2t)\, dt = 1$

$\Rightarrow \int_C f(x,y,z)\, ds = \int_{C_1} f(x,y,z)\, ds + \int_{C_2} f(x,y,z) = 3\sqrt{2} + 1$

2. Path 1: $\mathbf{r}_1 = t\mathbf{i} \Rightarrow x = t, y = 0, z = 0 \Rightarrow f(g(t), h(t), k(t)) = t^2$ and $\frac{dx}{dt} = 1, \frac{dy}{dt} = 0, \frac{dz}{dt} = 0$

$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = dt \Rightarrow \int_{C_1} f(x,y,z)\, ds = \int_0^1 t^2\, dt = \frac{1}{3};$

$\mathbf{r}_2 = \mathbf{i} + t\mathbf{j} \Rightarrow x = 1, y = t, z = 0 \Rightarrow f(g(t), h(t), k(t)) = 1 + t$ and $\frac{dx}{dt} = 0, \frac{dy}{dt} = 1, \frac{dz}{dt} = 0$

$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = dt \Rightarrow \int_{C_2} f(x,y,z)\, ds = \int_0^1 (1 + t)\, dt = \frac{3}{2};$

$\mathbf{r}_3 = \mathbf{i} + \mathbf{j} + t\mathbf{k} \Rightarrow x = 1, y = 1, z = t \Rightarrow f(g(t), h(t), k(t)) = 2 - t$ and $\frac{dx}{dt} = 0, \frac{dy}{dt} = 0, \frac{dz}{dt} = 1$

$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = dt \Rightarrow \int_{C_3} f(x,y,z)\, ds = \int_0^1 (2 - t)\, dt = \frac{3}{2}$

$\Rightarrow \int_{\text{Path 1}} f(x,y,z)\, ds = \int_{C_1} f(x,y,z)\, ds + \int_{C_2} f(x,y,z)\, ds + \int_{C_3} f(x,y,z)\, ds = \frac{10}{3}$

Path 2: $\mathbf{r}_4 = t\mathbf{i} + t\mathbf{j} \Rightarrow x = t, y = t, z = 0 \Rightarrow f(g(t), h(t), k(t)) = t^2 + t$ and $\frac{dx}{dt} = 1, \frac{dy}{dt} = 1, \frac{dz}{dt} = 0$

$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = \sqrt{2}\, dt \Rightarrow \int_{C_4} f(x,y,z)\, ds = \int_0^1 \sqrt{2}(t^2 + t)\, dt = \frac{5}{6}\sqrt{2};$

$\mathbf{r}_3 = \mathbf{i} + \mathbf{j} + t\mathbf{k}$ (see above) $\Rightarrow \int_{C_3} f(x,y,z)\, ds = \frac{3}{2}$

$\Rightarrow \int_{\text{Path 2}} f(x,y,z)\, ds = \int_{C_3} f(x,y,z)\, ds + \int_{C_4} f(x,y,z)\, ds = \frac{5}{6}\sqrt{2} + \frac{3}{2} = \frac{5\sqrt{2}+9}{6}$

Path 3: $\mathbf{r}_5 = t\mathbf{k} \Rightarrow x = 0, y = 0, z = t, 0 \le t \le 1 \Rightarrow f(g(t), h(t), k(t)) = -t$ and $\frac{dx}{dt} = 0, \frac{dy}{dt} = 0, \frac{dz}{dt} = 1$

$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = dt \Rightarrow \int_{C_5} f(x,y,z)\, ds = \int_0^1 -t\, dt = -\frac{1}{2};$

$\mathbf{r}_6 = t\mathbf{j} + \mathbf{k} \Rightarrow x = 0, y = t, z = 1, 0 \le t \le 1 \Rightarrow f(g(t), h(t), k(t)) = t - 1$ and $\frac{dx}{dt} = 0, \frac{dy}{dt} = 1, \frac{dz}{dt} = 0$

$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = dt \Rightarrow \int_{C_6} f(x,y,z)\, ds = \int_0^1 (t - 1)\, dt = -\frac{1}{2};$

$\mathbf{r}_7 = t\mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow x = t, y = 1, z = 1, 0 \le t \le 1 \Rightarrow f(g(t), h(t), k(t)) = t^2$ and $\frac{dx}{dt} = 1, \frac{dy}{dt} = 0, \frac{dz}{dt} = 0$

$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = dt \Rightarrow \int_{C_7} f(x, y, z)\, ds = \int_0^1 t^2\, dt = \frac{1}{3}$

$\Rightarrow \int_{\text{Path 3}} f(x, y, z)\, ds = \int_{C_5} f(x, y, z)\, ds + \int_{C_6} f(x, y, z)\, ds + \int_{C_7} f(x, y, z)\, ds = -\frac{1}{2} - \frac{1}{2} + \frac{1}{3} = -\frac{2}{3}$

3. $\mathbf{r} = (a \cos t)\mathbf{j} + (a \sin t)\mathbf{k} \Rightarrow x = 0, y = a \cos t, z = a \sin t \Rightarrow f(g(t), h(t), k(t)) = \sqrt{a^2 \sin^2 t} = a\,|\sin t|$ and $\frac{dx}{dt} = 0, \frac{dy}{dt} = -a \sin t, \frac{dz}{dt} = a \cos t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = a\, dt$

$\Rightarrow \int_C f(x, y, z)\, ds = \int_0^{2\pi} a^2\,|\sin t|\, dt = \int_0^{\pi} a^2 \sin t\, dt + \int_{\pi}^{2\pi} -a^2 \sin t\, dt = 4a^2$

4. $\mathbf{r} = (\cos t + t \sin t)\mathbf{i} + (\sin t - t \cos t)\mathbf{j} \Rightarrow x = \cos t + t \sin t, y = \sin t - t \cos t, z = 0$

$\Rightarrow f(g(t), h(t), k(t)) = \sqrt{(\cos t + t \sin t)^2 + (\sin t - t \cos t)^2} = \sqrt{1 + t^2}$ and $\frac{dx}{dt} = -\sin t + \sin t + t \cos t$

$= t \cos t, \frac{dy}{dt} = \cos t - \cos t + t \sin t = t \sin t, \frac{dz}{dt} = 0 \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$

$= \sqrt{t^2 \cos^2 t + t^2 \sin^2 t}\, dt = |t|\, dt = t\, dt$ since $0 \le t \le \sqrt{3} \Rightarrow \int_C f(x, y, z)\, ds = \int_0^{\sqrt{3}} t\sqrt{1 + t^2}\, dt = \frac{7}{3}$

5. $\frac{\partial P}{\partial y} = -\frac{1}{2}(x + y + z)^{-3/2} = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = -\frac{1}{2}(x + y + z)^{-3/2} = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = -\frac{1}{2}(x + y + z)^{-3/2} = \frac{\partial M}{\partial y}$

$\Rightarrow M\, dx + N\, dy + P\, dz$ is exact; $\frac{\partial f}{\partial x} = \frac{1}{\sqrt{x + y + z}} \Rightarrow f(x, y, z) = 2\sqrt{x + y + z} + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \frac{1}{\sqrt{x + y + z}} + \frac{\partial g}{\partial y}$

$= \frac{1}{\sqrt{x + y + z}} \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y, z) = h(z) \Rightarrow f(x, y, z) = 2\sqrt{x + y + z} + h(z) \Rightarrow \frac{\partial f}{\partial z} = \frac{1}{\sqrt{x + y + z}} + h'(z)$

$= \frac{1}{\sqrt{x + y + z}} \Rightarrow h'(x) = 0 \Rightarrow h(z) = C \Rightarrow f(x, y, z) = 2\sqrt{x + y + z} + C \Rightarrow \int_{(-1,1,1)}^{(4,-3,0)} \frac{dx + dy + dz}{\sqrt{x + y + z}}$

$= f(4, -3, 0) - f(-1, 1, 1) = 2\sqrt{1} - 2\sqrt{1} = 0$

6. $\frac{\partial P}{\partial y} = -\frac{1}{2\sqrt{yz}} = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = 0 = \frac{\partial M}{\partial y} \Rightarrow M\, dx + N\, dy + P\, dz$ is exact; $\frac{\partial f}{\partial x} = 1 \Rightarrow f(x, y, z)$

$= x + g(y, z) \Rightarrow \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = -\sqrt{\frac{z}{y}} \Rightarrow g(y, z) = -2\sqrt{yz} + h(z) \Rightarrow f(x, y, z) = x - 2\sqrt{yz} + h(z)$

$\Rightarrow \frac{\partial f}{\partial z} = -\sqrt{\frac{y}{z}} + h'(z) = -\sqrt{\frac{y}{z}} \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \Rightarrow f(x, y, z) = x - 2\sqrt{yz} + C$

$\Rightarrow \int_{(1,1,1)}^{(10,3,3)} dx - \sqrt{\frac{z}{y}}\, dy - \sqrt{\frac{y}{z}}\, dz = f(10, 3, 3) - f(1, 1, 1) = (10 - 2 \cdot 3) - (1 - 2 \cdot 1) = 4 + 1 = 5$

7. $\frac{\partial M}{\partial z} = -y \cos z \ne y \cos z = \frac{\partial P}{\partial x} \Rightarrow \mathbf{F}$ is not conservative; $\mathbf{r} = (2 \cos t)\mathbf{i} + (2 \sin t)\mathbf{j} - \mathbf{k}, 0 \le t \le 2\pi$

$\Rightarrow d\mathbf{r} = (-2 \sin t)\mathbf{i} - (2 \cos t)\mathbf{j} \Rightarrow \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} [-(-2 \sin t)(\sin(-1))(-2 \sin t) + (2 \cos t)(\sin(-1))(-2 \cos t)]\, dt$

$= 4 \sin(1) \int_0^{2\pi} (\sin^2 t + \cos^2 t) dt = 8\pi \sin(1)$

8. $\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = 3x^2 = \frac{\partial M}{\partial y} \Rightarrow \mathbf{F}$ is conservative $\Rightarrow \int_C \mathbf{F} \cdot d\mathbf{r} = 0$

9. Let $M = 8x \sin y$ and $N = -8y \cos x \Rightarrow \frac{\partial M}{\partial y} = 8x \cos y$ and $\frac{\partial N}{\partial x} = 8y \sin x \Rightarrow \int_C 8x \sin y\, dx - 8y \cos x\, dy$

$= \iint_R (8y \sin x - 8x \cos y)\, dy\, dx = \int_0^{\pi/2}\int_0^{\pi/2} (8y \sin x - 8x \cos y)\, dy\, dx = \int_0^{\pi/2} (\pi^2 \sin x - 8x)\, dx = -\pi^2 + \pi^2 = 0$

10. Let $M = y^2$ and $N = x^2 \Rightarrow \frac{\partial M}{\partial y} = 2y$ and $\frac{\partial N}{\partial x} = 2x \Rightarrow \int_C y^2\, dx + x^2\, dy = \iint_R (2x - 2y)\, dx\, dy$

$= \int_0^{2\pi}\int_0^2 (2r \cos\theta - 2r \sin\theta)\, r\, dr\, d\theta = \int_0^{2\pi} \frac{16}{3}(\cos\theta - \sin\theta)\, d\theta = 0$

11. Let $z = 1 - x - y \Rightarrow f_x(x, y) = -1$ and $f_y(x, y) = -1 \Rightarrow \sqrt{f_x^2 + f_y^2 + 1} = \sqrt{3} \Rightarrow$ Surface Area $= \iint_R \sqrt{3}\, dx\, dy$ $= \sqrt{3}$(Area of the circular region in the xy-plane) $= \pi\sqrt{3}$

12. $\nabla f = -3\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$, $\mathbf{p} = \mathbf{i} \Rightarrow |\nabla f| = \sqrt{9 + 4y^2 + 4z^2}$ and $|\nabla f \cdot \mathbf{p}| = 3$
$\Rightarrow$ Surface Area $= \iint_R \frac{\sqrt{9+4y^2+4z^2}}{3}\, dy\, dz = \int_0^{2\pi}\int_0^{\sqrt{3}} \frac{\sqrt{9+4r^2}}{3}\, r\, dr\, d\theta = \frac{1}{3}\int_0^{2\pi}\left(\frac{7}{4}\sqrt{21} - \frac{9}{4}\right) d\theta = \frac{\pi}{6}\left(7\sqrt{21} - 9\right)$

13. $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$, $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla f| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2\sqrt{x^2 + y^2 + z^2} = 2$ and $|\nabla f \cdot \mathbf{p}| = |2z| = 2z$ since $z \ge 0 \Rightarrow$ Surface Area $= \iint_R \frac{2}{2z}\, dA = \iint_R \frac{1}{z}\, dA = \iint_R \frac{1}{\sqrt{1-x^2-y^2}}\, dx\, dy = \int_0^{2\pi}\int_0^{1/\sqrt{2}} \frac{1}{\sqrt{1-r^2}}\, r\, dr\, d\theta$
$= \int_0^{2\pi}\left[-\sqrt{1-r^2}\right]_0^{1/\sqrt{2}} d\theta = \int_0^{2\pi}\left(1 - \frac{1}{\sqrt{2}}\right) d\theta = 2\pi\left(1 - \frac{1}{\sqrt{2}}\right)$

14. (a) $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$, $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla f| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2\sqrt{x^2 + y^2 + z^2} = 4$ and $|\nabla f \cdot \mathbf{p}| = 2z$ since $z \ge 0 \Rightarrow$ Surface Area $= \iint_R \frac{4}{2z}\, dA = \iint_R \frac{2}{z}\, dA = 2\int_0^{\pi/2}\int_0^{2\cos\theta} \frac{2}{\sqrt{4-r^2}}\, r\, dr\, d\theta = 4\pi - 8$

(b) $\mathbf{r} = 2\cos\theta \Rightarrow d\mathbf{r} = -2\sin\theta\, d\theta$; $ds^2 = r^2\, d\theta^2 + dr^2$ (Arc length in polar coordinates)
$\Rightarrow ds^2 = (2\cos\theta)^2\, d\theta^2 + dr^2 = 4\cos^2\theta\, d\theta^2 + 4\sin^2\theta\, d\theta^2 = 4\, d\theta^2 \Rightarrow ds = 2\, d\theta$; the height of the cylinder is $z = \sqrt{4 - r^2} = \sqrt{4 - 4\cos^2\theta} = 2|\sin\theta| = 2\sin\theta$ if $0 \le \theta \le \frac{\pi}{2} \Rightarrow$ Surface Area $= \int_{-\pi/2}^{\pi/2} h\, ds$
$= 2\int_0^{\pi/2} (2\sin\theta)(2\, d\theta) = 8$

15. $f(x, y, z) = \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \Rightarrow \nabla f = \left(\frac{1}{a}\right)\mathbf{i} + \left(\frac{1}{b}\right)\mathbf{j} + \left(\frac{1}{c}\right)\mathbf{k} \Rightarrow |\nabla f| = \sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}$ and $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = \frac{1}{c}$ since $c > 0 \Rightarrow$ Surface Area $= \iint_R \frac{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}}{\left(\frac{1}{c}\right)}\, dA = c\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}\iint_R dA = \frac{1}{2}abc\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}$,
since the area of the triangular region R is $\frac{1}{2}ab$. To check this result, let $\mathbf{v} = a\mathbf{i} + c\mathbf{k}$ and $\mathbf{w} = -a\mathbf{i} + b\mathbf{j}$; the area can be found by computing $\frac{1}{2}|\mathbf{v} \times \mathbf{w}|$.

16. (a) $\nabla f = 2y\mathbf{j} - \mathbf{k}$, $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla f| = \sqrt{4y^2 + 1}$ and $|\nabla f \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = \sqrt{4y^2 + 1}\, dx\, dy$
$\Rightarrow \iint_S g(x, y, z)\, d\sigma = \iint_R \frac{yz}{\sqrt{4y^2+1}}\sqrt{4y^2 + 1}\, dx\, dy = \iint_R y(y^2 - 1)\, dx\, dy = \int_{-1}^{1}\int_0^3 (y^3 - y)\, dx\, dy$
$= \int_{-1}^{1} 3(y^3 - y)\, dy = 3\left[\frac{y^4}{4} - \frac{y^2}{2}\right]_{-1}^{1} = 0$

(b) $\iint_S g(x, y, z)\, d\sigma = \iint_R \frac{z}{\sqrt{4y^2+1}}\sqrt{4y^2 + 1}\, dx\, dy = \int_{-1}^{1}\int_0^3 (y^2 - 1)\, dx\, dy = \int_{-1}^{1} 3(y^2 - 1)\, dy$
$= 3\left[\frac{y^3}{3} - y\right]_{-1}^{1} = -4$

17. $\nabla f = 2y\mathbf{j} + 2z\mathbf{k}$, $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla f| = \sqrt{4y^2 + 4z^2} = 2\sqrt{y^2 + z^2} = 10$ and $|\nabla f \cdot \mathbf{p}| = 2z$ since $z \ge 0$
$\Rightarrow d\sigma = \frac{10}{2z}\, dx\, dy = \frac{5}{z}\, dx\, dy = \iint_S g(x, y, z)\, d\sigma = \iint_R (x^4 y)(y^2 + z^2)\left(\frac{5}{z}\right) dx\, dy$
$= \iint_R (x^4 y)(25)\left(\frac{5}{\sqrt{25-y^2}}\right) dx\, dy = \int_0^4\int_0^1 \frac{125y}{\sqrt{25-y^2}}\, x^4\, dx\, dy = \int_0^4 \frac{25y}{\sqrt{25-y^2}}\, dy = 50$

18. Define the coordinate system so that the origin is at the center of the earth, the z-axis is the earth's axis (north is the positive z direction), and the xz-plane contains the earth's prime meridian. Let S denote the surface which is Wyoming so then S is part of the surface $z = (R^2 - x^2 - y^2)^{1/2}$. Let R_{xy} be the projection of S onto the xy-plane. The surface area of

Wyoming is $\iint_S 1\, d\sigma = \iint_{R_{xy}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA = \iint_{R_{xy}} \sqrt{\frac{x^2}{R^2 - x^2 - y^2} + \frac{y^2}{R^2 - x^2 - y^2} + 1}\, dA = \iint_{R_{xy}} \frac{R}{(R^2 - x^2 - y^2)^{1/2}}\, dA$
$= \int_{\theta_1}^{\theta_2} \int_{R \sin 45°}^{R \sin 49°} R\,(R^2 - r^2)^{-1/2}\, r\, dr\, d\theta$ (where θ_1 and θ_2 are the radian equivalent to 104°3′ and 111°3′, respectively)
$= \int_{\theta_1}^{\theta_2} -R\,(R^2 - r^2)^{1/2} \Big|_{R \sin 45°}^{R \sin 49°} = \int_{\theta_1}^{\theta_2} R\,(R^2 - R^2 \sin^2 45°)^{1/2} - R\,(R^2 - R^2 \sin^2 49°)^{1/2}\, d\theta$
$= (\theta_2 - \theta_1) R^2 (\cos 45° - \cos 49°) = \frac{7\pi}{180} R^2 (\cos 45° - \cos 49°) = \frac{7\pi}{180} (3959)^2 (\cos 45° - \cos 49°) \approx 97{,}751$ sq. mi.

19. A possible parametrization is $\mathbf{r}(\phi, \theta) = (6 \sin \phi \cos \theta)\mathbf{i} + (6 \sin \phi \sin \theta)\mathbf{j} + (6 \cos \phi)\mathbf{k}$ (spherical coordinates); now $\rho = 6$ and $z = -3 \Rightarrow -3 = 6 \cos \phi \Rightarrow \cos \phi = -\frac{1}{2} \Rightarrow \phi = \frac{2\pi}{3}$ and $z = 3\sqrt{3} \Rightarrow 3\sqrt{3} = 6 \cos \phi$ $\Rightarrow \cos \phi = \frac{\sqrt{3}}{2} \Rightarrow \phi = \frac{\pi}{6} \Rightarrow \frac{\pi}{6} \le \phi \le \frac{2\pi}{3}$; also $0 \le \theta \le 2\pi$

20. A possible parametrization is $\mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} - \left(\frac{r^2}{2}\right)\mathbf{k}$ (cylindrical coordinates); now $r = \sqrt{x^2 + y^2} \Rightarrow z = -\frac{r^2}{2}$ and $-2 \le z \le 0 \Rightarrow -2 \le -\frac{r^2}{2} \le 0 \Rightarrow 4 \ge r^2 \ge 0 \Rightarrow 0 \le r \le 2$ since $r \ge 0$; also $0 \le \theta \le 2\pi$

21. A possible parametrization is $\mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + (1 + r)\mathbf{k}$ (cylindrical coordinates); now $r = \sqrt{x^2 + y^2} \Rightarrow z = 1 + r$ and $1 \le z \le 3 \Rightarrow 1 \le 1 + r \le 3 \Rightarrow 0 \le r \le 2$; also $0 \le \theta \le 2\pi$

22. A possible parametrization is $\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + \left(3 - x - \frac{y}{2}\right)\mathbf{k}$ for $0 \le x \le 2$ and $0 \le y \le 2$

23. Let $x = u \cos v$ and $z = u \sin v$, where $u = \sqrt{x^2 + z^2}$ and v is the angle in the xz-plane with the x-axis $\Rightarrow \mathbf{r}(u, v) = (u \cos v)\mathbf{i} + 2u^2\mathbf{j} + (u \sin v)\mathbf{k}$ is a possible parametrization; $0 \le y \le 2 \Rightarrow 2u^2 \le 2 \Rightarrow u^2 \le 1$ $\Rightarrow 0 \le u \le 1$ since $u \ge 0$; also, for just the upper half of the paraboloid, $0 \le v \le \pi$

24. A possible parametrization is $\left(\sqrt{10} \sin \phi \cos \theta\right)\mathbf{i} + \left(\sqrt{10} \sin \phi \sin \theta\right)\mathbf{j} + \left(\sqrt{10} \cos \phi\right)\mathbf{k}$, $0 \le \phi \le \frac{\pi}{2}$ and $0 \le \theta \le \frac{\pi}{2}$

25. $\mathbf{r}_u = \mathbf{i} + \mathbf{j}$, $\mathbf{r}_v = \mathbf{i} - \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{vmatrix} = \mathbf{i} - \mathbf{j} - 2\mathbf{k} \Rightarrow |\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{6}$
$\Rightarrow$ Surface Area $= \iint_{R_{uv}} |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv = \int_0^1 \int_0^1 \sqrt{6}\, du\, dv = \sqrt{6}$

26. $\iint_S (xy - z^2)\, d\sigma = \int_0^1 \int_0^1 [(u + v)(u - v) - v^2] \sqrt{6}\, du\, dv = \sqrt{6} \int_0^1 \int_0^1 (u^2 - 2v^2)\, du\, dv$
$= \sqrt{6} \int_0^1 \left[\frac{u^3}{3} - 2uv^2\right]_0^1 dv = \sqrt{6} \int_0^1 \left(\frac{1}{3} - 2v^2\right) dv = \sqrt{6} \left[\frac{1}{3} v - \frac{2}{3} v^3\right]_0^1 = -\frac{\sqrt{6}}{3} = -\sqrt{\frac{2}{3}}$

27. $\mathbf{r}_r = (\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j}$, $\mathbf{r}_\theta = (-r \sin \theta)\mathbf{i} + (r \cos \theta)\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta & \sin \theta & 0 \\ -r \sin \theta & r \cos \theta & 1 \end{vmatrix}$
$= (\sin \theta)\mathbf{i} - (\cos \theta)\mathbf{j} + r\mathbf{k} \Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{\sin^2 \theta + \cos^2 \theta + r^2} = \sqrt{1 + r^2} \Rightarrow$ Surface Area $= \iint_{R_{r\theta}} |\mathbf{r}_r \times \mathbf{r}_\theta|\, dr\, d\theta$
$= \int_0^{2\pi} \int_0^1 \sqrt{1 + r^2}\, dr\, d\theta = \int_0^{2\pi} \left[\frac{r}{2} \sqrt{1 + r^2} + \frac{1}{2} \ln\left(r + \sqrt{1 + r^2}\right)\right]_0^1 d\theta = \int_0^{2\pi} \left[\frac{1}{2} \sqrt{2} + \frac{1}{2} \ln\left(1 + \sqrt{2}\right)\right] d\theta$
$= \pi \left[\sqrt{2} + \ln\left(1 + \sqrt{2}\right)\right]$

28. $\iint_S \sqrt{x^2+y^2+1}\,d\sigma = \int_0^{2\pi}\int_0^1 \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + 1}\,\sqrt{1+r^2}\,dr\,d\theta = \int_0^{2\pi}\int_0^1 (1+r^2)\,dr\,d\theta$
$= \int_0^{2\pi}\left[r+\frac{r^3}{3}\right]_0^1 d\theta = \int_0^{2\pi}\frac{4}{3}\,d\theta = \frac{8}{3}\pi$

29. $\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z},\ \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x},\ \frac{\partial N}{\partial x} = 0 = \frac{\partial M}{\partial y} \Rightarrow$ Conservative

30. $\frac{\partial P}{\partial y} = \frac{-3zy}{(x^2+y^2+z^2)^{-5/2}} = \frac{\partial N}{\partial z},\ \frac{\partial M}{\partial z} = \frac{-3xz}{(x^2+y^2+z^2)^{-5/2}} = \frac{\partial P}{\partial x},\ \frac{\partial N}{\partial x} = \frac{-3xy}{(x^2+y^2+z^2)^{-5/2}} = \frac{\partial M}{\partial y} \Rightarrow$ Conservative

31. $\frac{\partial P}{\partial y} = 0 \neq ye^z = \frac{\partial N}{\partial z} \Rightarrow$ Not Conservative

32. $\frac{\partial P}{\partial y} = \frac{x}{(x+yz)^2} = \frac{\partial N}{\partial z},\ \frac{\partial M}{\partial z} = \frac{-y}{(x+yz)^2} = \frac{\partial P}{\partial x},\ \frac{\partial N}{\partial x} = \frac{-z}{(x+yz)^2} = \frac{\partial M}{\partial y} \Rightarrow$ Conservative

33. $\frac{\partial f}{\partial x} = 2 \Rightarrow f(x,y,z) = 2x + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = 2y + z \Rightarrow g(y,z) = y^2 + zy + h(z)$
$\Rightarrow f(x,y,z) = 2x + y^2 + zy + h(z) \Rightarrow \frac{\partial f}{\partial z} = y + h'(z) = y + 1 \Rightarrow h'(z) = 1 \Rightarrow h(z) = z + C$
$\Rightarrow f(x,y,z) = 2x + y^2 + zy + z + C$

34. $\frac{\partial f}{\partial x} = z\cos xz \Rightarrow f(x,y,z) = \sin xz + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = e^y \Rightarrow g(y,z) = e^y + h(z)$
$\Rightarrow f(x,y,z) = \sin xz + e^y + h(z) \Rightarrow \frac{\partial f}{\partial z} = x\cos xz + h'(z) = x\cos xz \Rightarrow h'(z) = 0 \Rightarrow h(z) = C$
$\Rightarrow f(x,y,z) = \sin xz + e^y + C$

35. Over Path 1: $\mathbf{r} = t\mathbf{i} + t\mathbf{j} + t\mathbf{k},\ 0 \le t \le 1 \Rightarrow x = t, y = t, z = t$ and $d\mathbf{r} = (\mathbf{i}+\mathbf{j}+\mathbf{k})\,dt \Rightarrow \mathbf{F} = 2t^2\mathbf{i} + \mathbf{j} + t^2\mathbf{k}$
$\Rightarrow \mathbf{F}\cdot d\mathbf{r} = (3t^2+1)\,dt \Rightarrow \text{Work} = \int_0^1 (3t^2+1)\,dt = 2;$
Over Path 2: $\mathbf{r}_1 = t\mathbf{i} + t\mathbf{j},\ 0 \le t \le 1 \Rightarrow x = t, y = t, z = 0$ and $d\mathbf{r}_1 = (\mathbf{i}+\mathbf{j})\,dt \Rightarrow \mathbf{F}_1 = 2t^2\mathbf{i} + \mathbf{j} + t^2\mathbf{k}$
$\Rightarrow \mathbf{F}_1\cdot d\mathbf{r}_1 = (2t^2+1)\,dt \Rightarrow \text{Work}_1 = \int_0^1 (2t^2+1)\,dt = \frac{5}{3};\ \mathbf{r}_2 = \mathbf{i} + \mathbf{j} + t\mathbf{k},\ 0 \le t \le 1 \Rightarrow x = 1, y = 1, z = t$ and
$d\mathbf{r}_2 = \mathbf{k}\,dt \Rightarrow \mathbf{F}_2 = 2\mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{F}_2\cdot d\mathbf{r}_2 = dt \Rightarrow \text{Work}_2 = \int_0^1 dt = 1 \Rightarrow \text{Work} = \text{Work}_1 + \text{Work}_2 = \frac{5}{3} + 1 = \frac{8}{3}$

36. Over Path 1: $\mathbf{r} = t\mathbf{i} + t\mathbf{j} + t\mathbf{k},\ 0 \le t \le 1 \Rightarrow x = t, y = t, z = t$ and $d\mathbf{r} = (\mathbf{i}+\mathbf{j}+\mathbf{k})\,dt \Rightarrow \mathbf{F} = 2t^2\mathbf{i} + t^2\mathbf{j} + \mathbf{k}$
$\Rightarrow \mathbf{F}\cdot d\mathbf{r} = (3t^2+1)\,dt \Rightarrow \text{Work} = \int_0^1 (3t^2+1)\,dt = 2;$
Over Path 2: Since f is conservative, $\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$ around any simple closed curve C. Thus consider
$\int_{\text{curve}} \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} \mathbf{F}\cdot d\mathbf{r} + \int_{C_2} \mathbf{F}\cdot d\mathbf{r}$, where C_1 is the path from $(0,0,0)$ to $(1,1,0)$ to $(1,1,1)$ and C_2 is the path from $(1,1,1)$ to $(0,0,0)$. Now, from Path 1 above, $\int_{C_2} \mathbf{F}\cdot d\mathbf{r} = -2 \Rightarrow 0 = \int_{\text{curve}} \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} \mathbf{F}\cdot d\mathbf{r} + (-2)$
$\Rightarrow \int_{C_1} \mathbf{F}\cdot d\mathbf{r} = 2$

37. (a) $\mathbf{r} = (e^t\cos t)\,\mathbf{i} + (e^t\sin t)\,\mathbf{j} \Rightarrow x = e^t\cos t, y = e^t\sin t$ from $(1,0)$ to $(e^{2\pi},0) \Rightarrow 0 \le t \le 2\pi$
$\Rightarrow \frac{d\mathbf{r}}{dt} = (e^t\cos t - e^t\sin t)\,\mathbf{i} + (e^t\sin t + e^t\cos t)\,\mathbf{j}$ and $\mathbf{F} = \frac{x\mathbf{i}+y\mathbf{j}}{(x^2+y^2)^{3/2}} = \frac{(e^t\cos t)\,\mathbf{i} + (e^t\sin t)\,\mathbf{j}}{(e^{2t}\cos^2 t + e^{2t}\sin^2 t)^{3/2}}$
$= \left(\frac{\cos t}{e^{2t}}\right)\mathbf{i} + \left(\frac{\sin t}{e^{2t}}\right)\mathbf{j} \Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = \left(\frac{\cos^2 t}{e^t} - \frac{\sin t\cos t}{e^t} + \frac{\sin^2 t}{e^t} + \frac{\sin t\cos t}{e^t}\right) = e^{-t}$
$\Rightarrow \text{Work} = \int_0^{2\pi} e^{-t}\,dt = 1 - e^{-2\pi}$

(b) $\mathbf{F} = \frac{x\mathbf{i}+y\mathbf{j}}{(x^2+y^2)^{3/2}} \Rightarrow \frac{\partial f}{\partial x} = \frac{x}{(x^2+y^2)^{3/2}} \Rightarrow f(x,y,z) = -(x^2+y^2)^{-1/2} + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = \frac{y}{(x^2+y^2)^{3/2}} + \frac{\partial g}{\partial y}$
$= \frac{y}{(x^2+y^2)^{3/2}} \Rightarrow g(y,z) = C \Rightarrow f(x,y,z) = -(x^2+y^2)^{-1/2}$ is a potential function for $\mathbf{F} \Rightarrow \int_C \mathbf{F}\cdot d\mathbf{r}$
$= f(e^{2\pi},0) - f(1,0) = 1 - e^{-2\pi}$

38. (a) $\mathbf{F} = \nabla(x^2ze^y) \Rightarrow \mathbf{F}$ is conservative $\Rightarrow \oint_C \mathbf{F}\cdot d\mathbf{r} = 0$ for <u>any</u> closed path C

(b) $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{(1,0,0)}^{(1,0,2\pi)} \nabla(x^2ze^y)\cdot d\mathbf{r} = (x^2ze^y)|_{(1,0,2\pi)} - (x^2ze^y)|_{(1,0,0)} = 2\pi - 0 = 2\pi$

39. $\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & -y & 3z^2 \end{vmatrix} = -2y\mathbf{k}$; unit normal to the plane is $\mathbf{n} = \frac{2\mathbf{i}+6\mathbf{j}-3\mathbf{k}}{\sqrt{4+36+9}} = \frac{2}{7}\mathbf{i} + \frac{6}{7}\mathbf{j} - \frac{3}{7}\mathbf{k}$

$\Rightarrow \nabla\times\mathbf{F}\cdot\mathbf{n} = \frac{6}{7}y$; $\mathbf{p} = \mathbf{k}$ and $f(x,y,z) = 2x + 6y - 3z \Rightarrow |\nabla f\cdot\mathbf{p}| = 3 \Rightarrow d\sigma = \frac{|\nabla f|}{|\nabla f\cdot\mathbf{p}|}\,dA = \frac{7}{3}\,dA$

$\Rightarrow \oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_R \frac{6}{7}y\,d\sigma = \iint_R \left(\frac{6}{7}y\right)\left(\frac{7}{3}\,dA\right) = \iint_R 2y\,dA = \int_0^{2\pi}\int_0^1 2r\sin\theta\,r\,dr\,d\theta = \int_0^{2\pi} \frac{2}{3}\sin\theta\,d\theta = 0$

40. $\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y & x+y & 4y^2-z \end{vmatrix} = 8y\mathbf{i}$; the circle lies in the plane $f(x,y,z) = y + z = 0$ with unit normal

$\mathbf{n} = \frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k} \Rightarrow \nabla\times\mathbf{F}\cdot\mathbf{n} = 0 \Rightarrow \oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_R \nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint_R 0\,d\sigma = 0$

41. (a) $\mathbf{r} = \sqrt{2}t\mathbf{i} + \sqrt{2}t\mathbf{j} + (4-t^2)\mathbf{k},\ 0 \le t \le 1 \Rightarrow x = \sqrt{2}t,\ y = \sqrt{2}t,\ z = 4 - t^2 \Rightarrow \frac{dx}{dt} = \sqrt{2},\ \frac{dy}{dt} = \sqrt{2},\ \frac{dz}{dt} = -2t$

$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt = \sqrt{4+4t^2}\,dt \Rightarrow M = \int_C \delta(x,y,z)\,ds = \int_0^1 3t\sqrt{4+4t^2}\,dt = \left[\frac{1}{4}(4+4t)^{3/2}\right]_0^1$

$= 4\sqrt{2} - 2$

(b) $M = \int_C \delta(x,y,z)\,ds = \int_0^1 \sqrt{4+4t^2}\,dt = \left[t\sqrt{1+t^2} + \ln\left(t + \sqrt{1+t^2}\right)\right]_0^1 = \sqrt{2} + \ln\left(1+\sqrt{2}\right)$

42. $\mathbf{r} = t\mathbf{i} + 2t\mathbf{j} + \frac{2}{3}t^{3/2}\mathbf{k},\ 0 \le t \le 2 \Rightarrow x = t,\ y = 2t,\ z = \frac{2}{3}t^{3/2} \Rightarrow \frac{dx}{dt} = 1,\ \frac{dy}{dt} = 2,\ \frac{dz}{dt} = t^{1/2}$

$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt = \sqrt{t+5}\,dt \Rightarrow M = \int_C \delta(x,y,z)\,ds = \int_0^2 3\sqrt{5+t}\,\sqrt{t+5}\,dt$

$= \int_0^2 3(t+5)\,dt = 36$; $M_{yz} = \int_C x\delta\,ds = \int_0^2 3t(t+5)\,dt = 38$; $M_{xz} = \int_C y\delta\,ds = \int_0^2 6t(t+5)\,dt = 76$;

$M_{xy} = \int_C z\delta\,ds = \int_0^2 2t^{3/2}(t+5)\,dt = \frac{144}{7}\sqrt{2} \Rightarrow \overline{x} = \frac{M_{yz}}{M} = \frac{38}{36} = \frac{19}{18},\ \overline{y} = \frac{M_{xz}}{M} = \frac{76}{36} = \frac{19}{9},\ \overline{z} = \frac{M_{xy}}{M} = \frac{\left(\frac{144}{7}\sqrt{2}\right)}{36}$

$= \frac{4}{7}\sqrt{2}$

43. $\mathbf{r} = t\mathbf{i} + \left(\frac{2\sqrt{2}}{3}t^{3/2}\right)\mathbf{j} + \left(\frac{t^2}{2}\right)\mathbf{k},\ 0 \le t \le 2 \Rightarrow x = t,\ y = \frac{2\sqrt{2}}{3}t^{3/2},\ z = \frac{t^2}{2} \Rightarrow \frac{dx}{dt} = 1,\ \frac{dy}{dt} = \sqrt{2}t^{1/2},\ \frac{dz}{dt} = t$

$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt = \sqrt{1+2t+t^2}\,dt = \sqrt{(t+1)^2}\,dt = |t+1|\,dt = (t+1)\,dt$ on the domain given.

Then $M = \int_C \delta\,ds = \int_0^2 \left(\frac{1}{t+1}\right)(t+1)\,dt = \int_0^2 dt = 2$; $M_{yz} = \int_C x\delta\,ds = \int_0^2 t\left(\frac{1}{t+1}\right)(t+1)\,dt = \int_0^2 t\,dt = 2$;

$M_{xz} = \int_C y\delta\,ds = \int_0^2 \left(\frac{2\sqrt{2}}{3}t^{3/2}\right)\left(\frac{1}{t+1}\right)(t+1)\,dt = \int_0^2 \frac{2\sqrt{2}}{3}t^{3/2}\,dt = \frac{32}{15}$; $M_{xy} = \int_C z\delta\,ds$

$= \int_0^2 \left(\frac{t^2}{2}\right)\left(\frac{1}{t+1}\right)(t+1)\,dt = \int_0^2 \frac{t^2}{2}\,dt = \frac{4}{3} \Rightarrow \overline{x} = \frac{M_{yz}}{M} = \frac{2}{2} = 1;\ \overline{y} = \frac{M_{xz}}{M} = \frac{\left(\frac{32}{15}\right)}{2} = \frac{16}{15};\ \overline{z} = \frac{M_{xy}}{M}$

$= \frac{\left(\frac{4}{3}\right)}{2} = \frac{2}{3}$; $I_x = \int_C (y^2+z^2)\,\delta\,ds = \int_0^2 \left(\frac{8}{9}t^3 + \frac{t^4}{4}\right)dt = \frac{232}{45}$; $I_y = \int_C (x^2+z^2)\,\delta\,ds = \int_0^2 \left(t^2 + \frac{t^4}{4}\right)dt = \frac{64}{15}$;

$I_z = \int_C (y^2+x^2)\,\delta\,ds = \int_0^2 \left(t^2 + \frac{8}{9}t^3\right)dt = \frac{56}{9}$

44. $\overline{z} = 0$ because the arch is in the xy-plane, and $\overline{x} = 0$ because the mass is distributed symmetrically with respect to the y-axis; $\mathbf{r}(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j},\ 0 \le t \le \pi \Rightarrow ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$

$= \sqrt{(-a\sin t)^2 + (a\cos t)^2}\,dt = a\,dt$, since $a \ge 0$; $M = \int_C \delta\,ds = \int_C (2a - y)\,ds = \int_0^\pi (2a - a\sin t)\,a\,dt$

$= 2a^2\pi - 2a^2$; $M_{xz} = \int_C y\delta\,dt = \int_C y(2a - y)\,ds = \int_0^{\pi} (a\sin t)(2a - a\sin t)\,dt = \int_0^{\pi} (2a^2\sin t - a^2\sin^2 t)\,dt$

$= \left[-2a^2\cos t - a^2\left(\frac{t}{2} - \frac{\sin 2t}{4}\right)\right]_0^{\pi} = 4a^2 - \frac{a^2\pi}{2} \Rightarrow \overline{y} = \frac{\left(4a^2 - \frac{a^2\pi}{2}\right)}{2a^2\pi - 2a^2} = \frac{8-\pi}{4\pi-4} \Rightarrow (\overline{x}, \overline{y}, \overline{z}) = \left(0, \frac{8-\pi}{4\pi-4}, 0\right)$

45. $\mathbf{r}(t) = (e^t\cos t)\,\mathbf{i} + (e^t\sin t)\,\mathbf{j} + e^t\mathbf{k}$, $0 \le t \le \ln 2 \Rightarrow x = e^t\cos t,\ y = e^t\sin t,\ z = e^t \Rightarrow \frac{dx}{dt} = (e^t\cos t - e^t\sin t)$,

$\frac{dy}{dt} = (e^t\sin t + e^t\cos t)$, $\frac{dz}{dt} = e^t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$

$= \sqrt{(e^t\cos t - e^t\sin t)^2 + (e^t\sin t + e^t\cos t)^2 + (e^t)^2}\,dt = \sqrt{3e^{2t}}\,dt = \sqrt{3}\,e^t\,dt$; $M = \int_C \delta\,ds = \int_0^{\ln 2} \sqrt{3}\,e^t\,dt$

$= \sqrt{3}$; $M_{xy} = \int_C z\delta\,ds = \int_0^{\ln 2} \left(\sqrt{3}\,e^t\right)(e^t)\,dt = \int_0^{\ln 2} \sqrt{3}\,e^{2t}\,dt = \frac{3\sqrt{3}}{2} \Rightarrow \overline{z} = \frac{M_{xy}}{M} = \frac{\left(\frac{3\sqrt{3}}{2}\right)}{\sqrt{3}} = \frac{3}{2}$;

$I_z = \int_C (x^2 + y^2)\,\delta\,ds = \int_0^{\ln 2} (e^{2t}\cos^2 t + e^{2t}\sin^2 t)\left(\sqrt{3}\,e^t\right)dt = \int_0^{\ln 2} \sqrt{3}\,e^{3t}\,dt = \frac{7\sqrt{3}}{3}$

46. $\mathbf{r}(t) = (2\sin t)\mathbf{i} + (2\cos t)\mathbf{j} + 3t\mathbf{k}$, $0 \le t \le 2\pi \Rightarrow x = 2\sin t,\ y = 2\cos t,\ z = 3t \Rightarrow \frac{dx}{dt} = 2\cos t,\ \frac{dy}{dt} = -2\sin t$,

$\frac{dz}{dt} = 3 \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt = \sqrt{4+9}\,dt = \sqrt{13}\,dt$; $M = \int_C \delta\,ds = \int_0^{2\pi} \delta\sqrt{13}\,dt = 2\pi\delta\sqrt{13}$;

$M_{xy} = \int_C z\delta\,ds = \int_0^{2\pi} (3t)\left(\delta\sqrt{13}\right)dt = 6\delta\pi^2\sqrt{13}$; $M_{yz} = \int_C x\delta\,ds = \int_0^{2\pi} (2\sin t)\left(\delta\sqrt{13}\right)dt = 0$;

$M_{xz} = \int_C y\delta\,ds = \int_0^{2\pi} (2\cos t)\left(\delta\sqrt{13}\right)dt = 0 \Rightarrow \overline{x} = \overline{y} = 0$ and $\overline{z} = \frac{M_{xy}}{M} = \frac{6\delta\pi^2\sqrt{13}}{2\delta\pi\sqrt{13}} = 3\pi \Rightarrow (0, 0, 3\pi)$ is the center of mass

47. Because of symmetry $\overline{x} = \overline{y} = 0$. Let $f(x, y, z) = x^2 + y^2 + z^2 = 25 \Rightarrow \nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$

$\Rightarrow |\nabla f| = \sqrt{4x^2 + 4y^2 + 4z^2} = 10$ and $\mathbf{p} = \mathbf{k} \Rightarrow |\nabla f \cdot \mathbf{p}| = 2z$, since $z \ge 0 \Rightarrow M = \iint_R \delta(x, y, z)\,d\sigma$

$= \iint_R z\left(\frac{10}{2z}\right)dA = \iint_R 5\,dA = 5(\text{Area of the circular region}) = 80\pi$; $M_{xy} = \iint_R z\delta\,d\sigma = \iint_R 5z\,dA$

$= \iint_R 5\sqrt{25 - x^2 - y^2}\,dx\,dy = \int_0^{2\pi}\int_0^4 \left(5\sqrt{25 - r^2}\right) r\,dr\,d\theta = \int_0^{2\pi} \frac{490}{3}\,d\theta = \frac{980}{3}\pi \Rightarrow \overline{z} = \frac{\left(\frac{980}{3}\pi\right)}{80\pi} = \frac{49}{12}$

$\Rightarrow (\overline{x}, \overline{y}, \overline{z}) = \left(0, 0, \frac{49}{12}\right)$; $I_z = \iint_R (x^2 + y^2)\,\delta\,d\sigma = \iint_R 5(x^2 + y^2)\,dx\,dy = \int_0^{2\pi}\int_0^4 5r^3\,dr\,d\theta = \int_0^{2\pi} 320\,d\theta = 640\pi$

48. On the face $z = 1$: $g(x, y, z) = z = 1$ and $\mathbf{p} = \mathbf{k} \Rightarrow \nabla g = \mathbf{k} \Rightarrow |\nabla g| = 1$ and $|\nabla g \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dA$

$\Rightarrow I = \iint_R (x^2 + y^2)\,dA = 2\int_0^{\pi/4}\int_0^{\sec\theta} r^3\,dr\,d\theta = \frac{2}{3}$; On the face $z = 0$: $g(x, y, z) = z = 0 \Rightarrow \nabla g = \mathbf{k}$ and $\mathbf{p} = \mathbf{k}$

$\Rightarrow |\nabla g| = 1 \Rightarrow |\nabla g \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dA \Rightarrow I = \iint_R (x^2 + y^2)\,dA = \frac{2}{3}$; On the face $y = 0$: $g(x, y, z) = y = 0$

$\Rightarrow \nabla g = \mathbf{j}$ and $\mathbf{p} = \mathbf{j} \Rightarrow |\nabla g| = 1 \Rightarrow |\nabla g \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dA \Rightarrow I = \iint_R (x^2 + 0)\,dA = \int_0^1\int_0^1 x^2\,dx\,dz = \frac{1}{3}$;

On the face $y = 1$: $g(x, y, z) = y = 1 \Rightarrow \nabla g = \mathbf{j}$ and $\mathbf{p} = \mathbf{j} \Rightarrow |\nabla g| = 1 \Rightarrow |\nabla g \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dA$

$\Rightarrow I = \iint_R (x^2 + 1^2)\,dA = \int_0^1\int_0^1 (x^2 + 1)\,dx\,dz = \frac{4}{3}$; On the face $x = 1$: $g(x, y, z) = x = 1 \Rightarrow \nabla g = \mathbf{i}$ and $\mathbf{p} = \mathbf{i}$

$\Rightarrow |\nabla g| = 1 \Rightarrow |\nabla g \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dA \Rightarrow I = \iint_R (1^2 + y^2)\,dA = \int_0^1\int_0^1 (1 + y^2)\,dy\,dz = \frac{4}{3}$; On the face

$x = 0$: $g(x, y, z) = x = 0 \Rightarrow \nabla g = \mathbf{i}$ and $\mathbf{p} = \mathbf{i} \Rightarrow |\nabla g| = 1 \Rightarrow |\nabla g \cdot \mathbf{p}| = 1 \Rightarrow d\sigma = dA$

$\Rightarrow I = \iint_R (0^2 + y^2)\,dA = \int_0^1\int_0^1 y^2\,dy\,dz = \frac{1}{3} \Rightarrow I_z = \frac{2}{3} + \frac{2}{3} + \frac{1}{3} + \frac{4}{3} + \frac{4}{3} + \frac{1}{3} = \frac{14}{3}$

49. $M = 2xy + x$ and $N = xy - y \Rightarrow \frac{\partial M}{\partial x} = 2y + 1,\ \frac{\partial M}{\partial y} = 2x,\ \frac{\partial N}{\partial x} = y,\ \frac{\partial N}{\partial y} = x - 1 \Rightarrow \text{Flux} = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy$
$= \iint_R (2y + 1 + x - 1)\,dy\,dx = \int_0^1 \int_0^1 (2y + x)\,dy\,dx = \frac{3}{2}$; $\text{Circ} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$
$= \iint_R (y - 2x)\,dy\,dx = \int_0^1 \int_0^1 (y - 2x)\,dy\,dx = -\frac{1}{2}$

50. $M = y - 6x^2$ and $N = x + y^2 \Rightarrow \frac{\partial M}{\partial x} = -12x,\ \frac{\partial M}{\partial y} = 1,\ \frac{\partial N}{\partial x} = 1,\ \frac{\partial N}{\partial y} = 2y \Rightarrow \text{Flux} = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy$
$= \iint_R (-12x + 2y)\,dx\,dy = \int_0^1 \int_y^1 (-12x + 2y)\,dx\,dy = \int_0^1 (4y^2 + 2y - 6)\,dy = -\frac{11}{3}$;
$\text{Circ} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R (1 - 1)\,dx\,dy = 0$

51. $M = -\frac{\cos y}{x}$ and $N = \ln x \sin y \Rightarrow \frac{\partial M}{\partial y} = \frac{\sin y}{x}$ and $\frac{\partial N}{\partial x} = \frac{\sin y}{x} \Rightarrow \oint_C \ln x \sin y\,dy - \frac{\cos y}{x}\,dx$
$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R \left(\frac{\sin y}{x} - \frac{\sin y}{x}\right) dx\,dy = 0$

52. (a) Let $M = x$ and $N = y \Rightarrow \frac{\partial M}{\partial x} = 1,\ \frac{\partial M}{\partial y} = 0,\ \frac{\partial N}{\partial x} = 0,\ \frac{\partial N}{\partial y} = 1 \Rightarrow \text{Flux} = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy$
$= \iint_R (1 + 1)\,dx\,dy = 2 \iint_R dx\,dy = 2(\text{Area of the region})$

(b) Let C be a closed curve to which Green's Theorem applies and let $\mathbf{n}$ be the unit normal vector to C. Let $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ and assume $\mathbf{F}$ is orthogonal to $\mathbf{n}$ at every point of C. Then the flux density of $\mathbf{F}$ at every point of C is 0 since $\mathbf{F} \cdot \mathbf{n} = 0$ at every point of C $\Rightarrow \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = 0$ at every point of C
$\Rightarrow \text{Flux} = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy = \iint_R 0\,dx\,dy = 0$. But part (a) above states that the flux is 2(Area of the region) $\Rightarrow$ the area of the region would be 0 $\Rightarrow$ contradiction. Therefore, $\mathbf{F}$ cannot be orthogonal to $\mathbf{n}$ at every point of C.

53. $\frac{\partial}{\partial x}(2xy) = 2y,\ \frac{\partial}{\partial y}(2yz) = 2z,\ \frac{\partial}{\partial z}(2xz) = 2x \Rightarrow \nabla \cdot \mathbf{F} = 2y + 2z + 2x \Rightarrow \text{Flux} = \iiint_D (2x + 2y + 2z)\,dV$
$= \int_0^1 \int_0^1 \int_0^1 (2x + 2y + 2z)\,dx\,dy\,dz = \int_0^1 \int_0^1 (1 + 2y + 2z)\,dy\,dz = \int_0^1 (2 + 2z)\,dz = 3$

54. $\frac{\partial}{\partial x}(xz) = z,\ \frac{\partial}{\partial y}(yz) = z,\ \frac{\partial}{\partial z}(1) = 0 \Rightarrow \nabla \cdot \mathbf{F} = 2z \Rightarrow \text{Flux} = \iiint_D 2z\,r\,dr\,d\theta\,dz$
$= \int_0^{2\pi} \int_0^4 \int_3^{\sqrt{25 - r^2}} 2z\,dz\,r\,dr\,d\theta = \int_0^{2\pi} \int_0^4 r(16 - r^2)\,dr\,d\theta = \int_0^{2\pi} 64\,d\theta = 128\pi$

55. $\frac{\partial}{\partial x}(-2x) = -2,\ \frac{\partial}{\partial y}(-3y) = -3,\ \frac{\partial}{\partial z}(z) = 1 \Rightarrow \nabla \cdot \mathbf{F} = -4$; $x^2 + y^2 + z^2 = 2$ and $x^2 + y^2 = z \Rightarrow z = 1$
$\Rightarrow x^2 + y^2 = 1 \Rightarrow \text{Flux} = \iiint_D -4\,dV = -4 \int_0^{2\pi} \int_0^1 \int_{r^2}^{\sqrt{2 - r^2}} dz\,r\,dr\,d\theta = -4 \int_0^{2\pi} \int_0^1 \left(r\sqrt{2 - r^2} - r^3\right) dr\,d\theta$
$= -4 \int_0^{2\pi} \left(-\frac{7}{12} + \frac{2}{3}\sqrt{2}\right) d\theta = \frac{2}{3}\pi\left(7 - 8\sqrt{2}\right)$

56. $\frac{\partial}{\partial x}(6x + y) = 6,\ \frac{\partial}{\partial y}(-x - z) = 0,\ \frac{\partial}{\partial z}(4yz) = 4y \Rightarrow \nabla \cdot \mathbf{F} = 6 + 4y$; $z = \sqrt{x^2 + y^2} = r$
$\Rightarrow \text{Flux} = \iiint_D (6 + 4y)\,dV = \int_0^{\pi/2} \int_0^1 \int_0^r (6 + 4r\sin\theta)\,dz\,r\,dr\,d\theta = \int_0^{\pi/2} \int_0^1 (6r^2 + 4r^3 \sin\theta)\,dr\,d\theta$
$= \int_0^{\pi/2} (2 + \sin\theta)\,d\theta = \pi + 1$

57. $\mathbf{F} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k} \Rightarrow \nabla \cdot \mathbf{F} = 0 \Rightarrow \text{Flux} = \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_D \nabla \cdot \mathbf{F}\, dV = 0$

58. $\mathbf{F} = 3xz^2\mathbf{i} + y\mathbf{j} - z^3\mathbf{k} \Rightarrow \nabla \cdot \mathbf{F} = 3z^2 + 1 - 3z^2 = 1 \Rightarrow \text{Flux} = \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_D \nabla \cdot \mathbf{F}\, dV$

$= \int_0^4 \int_0^{\sqrt{16-x^2}/2} \int_0^{y/2} 1\, dz\, dy\, dx = \int_0^4 \left(\frac{16-x^2}{16}\right) dx = \left[x - \frac{x^3}{48}\right]_0^4 = \frac{8}{3}$

59. $\mathbf{F} = xy^2\mathbf{i} + x^2y\mathbf{j} + y\mathbf{k} \Rightarrow \nabla \cdot \mathbf{F} = y^2 + x^2 + 0 \Rightarrow \text{Flux} = \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_D \nabla \cdot \mathbf{F}\, dV$

$= \iiint_D (x^2 + y^2)\, dV = \int_0^{2\pi} \int_0^1 \int_{-1}^1 r^2\, dz\, r\, dr\, d\theta = \int_0^{2\pi} \int_0^1 2r^3\, dr\, d\theta = \int_0^{2\pi} \frac{1}{2}\, d\theta = \pi$

60. (a) $\mathbf{F} = (3z + 1)\mathbf{k} \Rightarrow \nabla \cdot \mathbf{F} = 3 \Rightarrow$ Flux across the hemisphere $= \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_D \nabla \cdot \mathbf{F}\, dV = \iiint_D 3\, dV$

$= 3\left(\frac{1}{2}\right)\left(\frac{4}{3}\pi a^3\right) = 2\pi a^3$

(b) $f(x, y, z) = x^2 + y^2 + z^2 - a^2 = 0 \Rightarrow \nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow |\nabla f| = \sqrt{4x^2 + 4y^2 + 4z^2} = \sqrt{4a^2} = 2a$ since $a \ge 0 \Rightarrow \mathbf{n} = \frac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{2a} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a} \Rightarrow \mathbf{F} \cdot \mathbf{n} = (3z + 1)\left(\frac{z}{a}\right)$; $\mathbf{p} = \mathbf{k} \Rightarrow \nabla f \cdot \mathbf{p} = \nabla f \cdot \mathbf{k} = 2z$

$\Rightarrow |\nabla f \cdot \mathbf{p}| = 2z$ since $z \ge 0 \Rightarrow d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} = \frac{2a}{2z}\, dA = \frac{a}{z}\, dA \Rightarrow \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_{R_{xy}} (3z + 1)\left(\frac{z}{a}\right)\left(\frac{a}{z}\right) dA$

$= \iint_{R_{xy}} (3z + 1)\, dx\, dy = \iint_{R_{xy}} \left(3\sqrt{a^2 - x^2 - y^2} + 1\right) dx\, dy = \int_0^{2\pi} \int_0^a \left(3\sqrt{a^2 - r^2} + 1\right) r\, dr\, d\theta$

$= \int_0^{2\pi} \left(\frac{a^2}{2} + a^3\right) d\theta = \pi a^2 + 2\pi a^3$, which is the flux across the hemisphere. Across the base we find $\mathbf{F} = [3(0) + 1]\mathbf{k} = \mathbf{k}$ since $z = 0$ in the xy-plane $\Rightarrow \mathbf{n} = -\mathbf{k}$ (outward normal) $\Rightarrow \mathbf{F} \cdot \mathbf{n} = -1 \Rightarrow$ Flux across the base $= \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_{R_{xy}} -1\, dx\, dy = -\pi a^2$. Therefore, the total flux across the closed surface is $(\pi a^2 + 2\pi a^3) - \pi a^2 = 2\pi a^3$.

CHAPTER 16 ADDITIONAL AND ADVANCED EXERCISES

1. $dx = (-2 \sin t + 2 \sin 2t)\, dt$ and $dy = (2 \cos t - 2 \cos 2t)\, dt$; Area $= \frac{1}{2}\oint_C x\, dy - y\, dx$

$= \frac{1}{2} \int_0^{2\pi} [(2 \cos t - \cos 2t)(2 \cos t - 2 \cos 2t) - (2 \sin t - \sin 2t)(-2 \sin t + 2 \sin 2t)]\, dt$

$= \frac{1}{2} \int_0^{2\pi} [6 - (6 \cos t \cos 2t + 6 \sin t \sin 2t)]\, dt = \frac{1}{2} \int_0^{2\pi} (6 - 6 \cos t)\, dt = 6\pi$

2. $dx = (-2 \sin t - 2 \sin 2t)\, dt$ and $dy = (2 \cos t - 2 \cos 2t)\, dt$; Area $= \frac{1}{2} \oint_C x\, dy - y\, dx$

$= \frac{1}{2}\int_0^{2\pi} [(2 \cos t + \cos 2t)(2 \cos t - 2 \cos 2t) - (2 \sin t - \sin 2t)(-2 \sin t - 2 \sin 2t)]\, dt$

$= \frac{1}{2} \int_0^{2\pi} [2 - 2(\cos t \cos 2t - \sin t \sin 2t)]\, dt = \frac{1}{2} \int_0^{2\pi} (2 - 2 \cos 3t)\, dt = \frac{1}{2} \left[2t - \frac{2}{3} \sin 3t\right]_0^{2\pi} = 2\pi$

3. $dx = \cos 2t\, dt$ and $dy = \cos t\, dt$; Area $= \frac{1}{2}\oint_C x\, dy - y\, dx = \frac{1}{2} \int_0^{\pi} \left(\frac{1}{2} \sin 2t \cos t - \sin t \cos 2t\right) dt$

$= \frac{1}{2} \int_0^{\pi} [\sin t \cos^2 t - (\sin t)(2 \cos^2 t - 1)]\, dt = \frac{1}{2} \int_0^{\pi} (-\sin t \cos^2 t + \sin t)\, dt = \frac{1}{2} \left[\frac{1}{3} \cos^3 t - \cos t\right]_0^{\pi} = -\frac{1}{3} + 1 = \frac{2}{3}$

4. $dx = (-2a \sin t - 2a \cos 2t)\, dt$ and $dy = (b \cos t)\, dt$; Area $= \frac{1}{2} \oint_C x\, dy - y\, dx$

$= \frac{1}{2} \int_0^{2\pi} [(2ab \cos^2 t - ab \cos t \sin 2t) - (-2ab \sin^2 t - 2ab \sin t \cos 2t)]\, dt$

$= \frac{1}{2}\int_0^{2\pi}\left[2ab - 2ab\cos^2 t \sin t + 2ab(\sin t)(2\cos^2 t - 1)\right] dt = \frac{1}{2}\int_0^{2\pi}(2ab + 2ab\cos^2 t\sin t - 2ab\sin t)\, dt$

$= \frac{1}{2}\left[2abt - \frac{2}{3}ab\cos^3 t + 2ab\cos t\right]_0^{2\pi} = 2\pi ab$

5. (a) $\mathbf{F}(x,y,z) = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$ is $\mathbf{0}$ only at the point $(0,0,0)$, and curl $\mathbf{F}(x,y,z) = \mathbf{i} + \mathbf{j} + \mathbf{k}$ is never $\mathbf{0}$.
   (b) $\mathbf{F}(x,y,z) = z\mathbf{i} + y\mathbf{k}$ is $\mathbf{0}$ only on the line $x = t$, $y = 0$, $z = 0$ and curl $\mathbf{F}(x,y,z) = \mathbf{i} + \mathbf{j}$ is never $\mathbf{0}$.
   (c) $\mathbf{F}(x,y,z) = z\mathbf{i}$ is $\mathbf{0}$ only when $z = 0$ (the xy-plane) and curl $\mathbf{F}(x,y,z) = \mathbf{j}$ is never $\mathbf{0}$.

6. $\mathbf{F} = yz^2\mathbf{i} + xz^2\mathbf{j} + 2xyz\mathbf{k}$ and $\mathbf{n} = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{\sqrt{x^2+y^2+z^2}} = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{R}$, so $\mathbf{F}$ is parallel to $\mathbf{n}$ when $yz^2 = \frac{cx}{R}$, $xz^2 = \frac{cy}{R}$, and $2xyz = \frac{cz}{R} \Rightarrow \frac{yz^2}{x} = \frac{xz^2}{y} = 2xy \Rightarrow y^2 = x^2 \Rightarrow y = \pm x$ and $z^2 = \pm\frac{c}{R} = 2x^2 \Rightarrow z = \pm\sqrt{2}x$. Also, $x^2+y^2+z^2 = R^2 \Rightarrow x^2 + x^2 + 2x^2 = R^2 \Rightarrow 4x^2 = R^2 \Rightarrow x = \pm\frac{R}{2}$. Thus the points are: $\left(\frac{R}{2}, \frac{R}{2}, \frac{\sqrt{2}R}{2}\right)$, $\left(\frac{R}{2}, \frac{R}{2}, -\frac{\sqrt{2}R}{2}\right)$, $\left(-\frac{R}{2}, -\frac{R}{2}, \frac{\sqrt{2}R}{2}\right)$, $\left(-\frac{R}{2}, -\frac{R}{2}, -\frac{\sqrt{2}R}{2}\right)$, $\left(\frac{R}{2}, -\frac{R}{2}, \frac{\sqrt{2}R}{2}\right)$, $\left(\frac{R}{2}, -\frac{R}{2}, -\frac{\sqrt{2}R}{2}\right)$, $\left(-\frac{R}{2}, \frac{R}{2}, \frac{\sqrt{2}R}{2}\right)$, $\left(-\frac{R}{2}, \frac{R}{2}, -\frac{\sqrt{2}R}{2}\right)$

7. Set up the coordinate system so that $(a,b,c) = (0,R,0) \Rightarrow \delta(x,y,z) = \sqrt{x^2 + (y-R)^2 + z^2}$
$= \sqrt{x^2+y^2+z^2-2Ry+R^2} = \sqrt{2R^2 - 2Ry}$; let $f(x,y,z) = x^2+y^2+z^2-R^2$ and $\mathbf{p} = \mathbf{i}$
$\Rightarrow \nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \Rightarrow |\nabla f| = 2\sqrt{x^2+y^2+z^2} = 2R \Rightarrow d\sigma = \frac{|\nabla f|}{|\nabla f\cdot \mathbf{i}|}\, dz\, dy = \frac{2R}{2x}\, dz\, dy$
$\Rightarrow \text{Mass} = \iint_S \delta(x,y,z)\, d\sigma = \iint_{R_{yz}} \sqrt{2R^2-2Ry}\left(\frac{R}{x}\right) dz\, dy = R\iint_{R_{yz}} \frac{\sqrt{2R^2-2Ry}}{\sqrt{R^2-y^2-z^2}}\, dz\, dy$
$= 4R\int_{-R}^{R}\int_0^{\sqrt{R^2-y^2}} \frac{\sqrt{2R^2-2Ry}}{\sqrt{R^2-y^2-z^2}}\, dz\, dy = 4R\int_{-R}^{R}\sqrt{2R^2-2Ry}\,\sin^{-1}\left(\frac{z}{\sqrt{R^2-y^2}}\right)\Big|_0^{\sqrt{R^2-y^2}}\, dy$
$= 2\pi R\int_{-R}^{R}\sqrt{2R^2-2Ry}\, dy = 2\pi R\left(\frac{-1}{3R}\right)(2R^2-2Ry)^{3/2}\Big|_{-R}^{R} = \frac{16\pi R^3}{3}$

8. $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + \theta\mathbf{k}$, $0 \le r \le 1$, $0 \le \theta \le 2\pi \Rightarrow \mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 1 \end{vmatrix}$
$= (\sin\theta)\mathbf{i} - (\cos\theta)\mathbf{j} + r\mathbf{k} \Rightarrow |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{1+r^2}$; $\delta = 2\sqrt{x^2+y^2} = 2\sqrt{r^2\cos^2\theta + r^2\sin^2\theta} = 2r$
$\Rightarrow \text{Mass} = \iint_S \delta(x,y,z)\, d\sigma = \int_0^{2\pi}\int_0^1 2r\sqrt{1+r^2}\, dr\, d\theta = \int_0^{2\pi}\left[\frac{2}{3}(1+r^2)^{3/2}\right]_0^1 d\theta = \int_0^{2\pi}\frac{2}{3}\left(2\sqrt{2}-1\right) d\theta$
$= \frac{4\pi}{3}\left(2\sqrt{2}-1\right)$

9. $M = x^2 + 4xy$ and $N = -6y \Rightarrow \frac{\partial M}{\partial x} = 2x + 4y$ and $\frac{\partial N}{\partial x} = -6 \Rightarrow \text{Flux} = \int_0^b\int_0^a (2x+4y-6)\, dx\, dy$
$= \int_0^b (a^2 + 4ay - 6a)\, dy = a^2b + 2ab^2 - 6ab$. We want to minimize $f(a,b) = a^2b + 2ab^2 - 6ab = ab(a+2b-6)$.
Thus, $f_a(a,b) = 2ab + 2b^2 - 6b = 0$ and $f_b(a,b) = a^2 + 4ab - 6a = 0 \Rightarrow b(2a+2b-6) = 0 \Rightarrow b = 0$ or $b = -a+3$.
Now $b = 0 \Rightarrow a^2 - 6a = 0 \Rightarrow a = 0$ or $a = 6 \Rightarrow (0,0)$ and $(6,0)$ are critical points. On the other hand, $b = -a+3$ $\Rightarrow a^2 + 4a(-a+3) - 6a = 0 \Rightarrow -3a^2 + 6a = 0 \Rightarrow a = 0$ or $a = 2 \Rightarrow (0,3)$ and $(2,1)$ are also critical points. The flux at $(0,0) = 0$, the flux at $(6,0) = 0$, the flux at $(0,3) = 0$ and the flux at $(2,1) = -4$. Therefore, the flux is minimized at $(2,1)$ with value -4.

10. A plane through the origin has equation $ax + by + cz = 0$. Consider first the case when $c \ne 0$. Assume the plane is given by $z = ax + by$ and let $f(x,y,z) = x^2 + y^2 + z^2 = 4$. Let C denote the circle of intersection of the plane with the sphere. By Stokes's Theorem, $\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F}\cdot\mathbf{n}\, d\sigma$, where $\mathbf{n}$ is a unit normal to the plane. Let

$\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + (ax + by)\mathbf{k}$ be a parametrization of the surface. Then $\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & a \\ 0 & 1 & b \end{vmatrix} = -a\mathbf{i} - b\mathbf{j} + \mathbf{k}$

$\Rightarrow\ d\sigma = |\mathbf{r}_x \times \mathbf{r}_y|\, dx\, dy = \sqrt{a^2 + b^2 + 1}\, dx\, dy$. Also, $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{n} = \frac{a\mathbf{i} + b\mathbf{j} - \mathbf{k}}{\sqrt{a^2 + b^2 + 1}}$

$\Rightarrow \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_{R_{xy}} \frac{a + b - 1}{\sqrt{a^2 + b^2 + 1}} \sqrt{a^2 + b^2 + 1}\, dx\, dy = \iint_{R_{xy}} (a + b - 1)\, dx\, dy = (a + b - 1) \iint_{R_{xy}} dx\, dy$. Now $x^2 + y^2 + (ax + by)^2 = 4 \Rightarrow \left(\frac{a^2 + 1}{4}\right) x^2 + \left(\frac{b^2 + 1}{4}\right) y^2 + \left(\frac{ab}{2}\right) xy = 1 \Rightarrow$ the region R_{xy} is the interior of the ellipse $Ax^2 + Bxy + Cy^2 = 1$ in the xy-plane, where $A = \frac{a^2 + 1}{4}$, $B = \frac{ab}{2}$, and $C = \frac{b^2 + 1}{4}$. The area of the ellipse is $\frac{2\pi}{\sqrt{4AC - B^2}} = \frac{4\pi}{\sqrt{a^2 + b^2 + 1}} \Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = h(a, b) = \frac{4\pi(a + b - 1)}{\sqrt{a^2 + b^2 + 1}}$. Thus we optimize $H(a, b) = \frac{(a + b - 1)^2}{a^2 + b^2 + 1}$:

$\frac{\partial H}{\partial a} = \frac{2(a + b - 1)(b^2 + 1 + a - ab)}{(a^2 + b^2 + 1)^2} = 0$ and $\frac{\partial H}{\partial b} = \frac{2(a + b - 1)(a^2 + 1 + b - ab)}{(a^2 + b^2 + 1)^2} = 0 \Rightarrow a + b - 1 = 0$, or $b^2 + 1 + a - ab = 0$ and $a^2 + 1 + b - ab = 0 \Rightarrow a + b - 1 = 0$, or $a^2 - b^2 + (b - a) = 0 \Rightarrow a + b - 1 = 0$, or $(a - b)(a + b - 1) = 0$ $\Rightarrow a + b - 1 = 0$ or $a = b$. The critical values $a + b - 1 = 0$ give a saddle. If $a = b$, then $0 = b^2 + 1 + a - ab$ $\Rightarrow a^2 + 1 + a - a^2 = 0 \Rightarrow a = -1 \Rightarrow b = -1$. Thus, the point $(a, b) = (-1, -1)$ gives a local extremum for $\oint_C \mathbf{F} \cdot d\mathbf{r}$ $\Rightarrow z = -x - y \Rightarrow x + y + z = 0$ is the desired plane, if $c \neq 0$.

Note: Since $h(-1, -1)$ is negative, the circulation about $\mathbf{n}$ is clockwise, so $-\mathbf{n}$ is the correct pointing normal for the counterclockwise circulation. Thus $\iint_S \nabla \times \mathbf{F} \cdot (-\mathbf{n})\, d\sigma$ actually gives the maximum circulation.

If $c = 0$, one can see that the corresponding problem is equivalent to the calculation above when $b = 0$, which does not lead to a local extreme.

11. (a) Partition the string into small pieces. Let $\Delta_i s$ be the length of the i^{th} piece. Let (x_i, y_i) be a point in the i^{th} piece. The work done by gravity in moving the i^{th} piece to the x-axis is approximately $W_i = (g x_i y_i \Delta_i s) y_i$ where $x_i y_i \Delta_i s$ is approximately the mass of the i^{th} piece. The total work done by gravity in moving the string to the x-axis is $\sum_i W_i = \sum_i g x_i y_i^2 \Delta_i s \Rightarrow \text{Work} = \int_C gxy^2\, ds$

(b) $\text{Work} = \int_C gxy^2\, ds = \int_0^{\pi/2} g(2 \cos t)(4 \sin^2 t)\sqrt{4 \sin^2 t + 4 \cos^2 t}\, dt = 16g \int_0^{\pi/2} \cos t \sin^2 t\, dt$ $= \left[16g\left(\frac{\sin^3 t}{3}\right)\right]_0^{\pi/2} = \frac{16}{3} g$

(c) $\overline{x} = \frac{\int_C x(xy)\, ds}{\int_C xy\, ds}$ and $\overline{y} = \frac{\int_C y(xy)\, ds}{\int_C xy\, ds}$; the mass of the string is $\int_C xy\, ds$ and the weight of the string is $g \int_C xy\, ds$. Therefore, the work done in moving the point mass at $(\overline{x}, \overline{y})$ to the x-axis is $W = \left(g \int_C xy\, ds\right) \overline{y} = g \int_C xy^2\, ds = \frac{16}{3} g$.

12. (a) Partition the sheet into small pieces. Let $\Delta_i \sigma$ be the area of the i^{th} piece and select a point (x_i, y_i, z_i) in the i^{th} piece. The mass of the i^{th} piece is approximately $x_i y_i \Delta_i \sigma$. The work done by gravity in moving the i^{th} piece to the xy-plane is approximately $(g x_i y_i \Delta_i \sigma) z_i = g x_i y_i z_i \Delta_i \sigma \Rightarrow \text{Work} = \iint_S gxyz\, d\sigma$.

(b) $\iint_S gxyz\, d\sigma = g \iint_{R_{xy}} xy(1 - x - y)\sqrt{1 + (-1)^2 + (-1)^2}\, dA = \sqrt{3}g \int_0^1 \int_0^{1-x} (xy - x^2y - xy^2)\, dy\, dx$ $= \sqrt{3}g \int_0^1 \left[\frac{1}{2} xy^2 - \frac{1}{2} x^2y^2 - \frac{1}{3} xy^3\right]_0^{1-x} dx = \sqrt{3}g \int_0^1 \left[\frac{1}{6} x - \frac{1}{2} x^2 + \frac{1}{2} x^3 - \frac{1}{6} x^4\right] dx$ $= \sqrt{3}g \left[\frac{1}{12} x^2 - \frac{1}{6} x^3 + \frac{1}{6} x^4 - \frac{1}{30} x^5\right]_0^1 = \sqrt{3}g\left(\frac{1}{12} - \frac{1}{30}\right) = \frac{\sqrt{3}g}{20}$

(c) The center of mass of the sheet is the point $(\overline{x}, \overline{y}, \overline{z})$ where $\overline{z} = \frac{M_{xy}}{M}$ with $M_{xy} = \iint_S xyz\, d\sigma$ and $M = \iint_S xy\, d\sigma$. The work done by gravity in moving the point mass at $(\overline{x}, \overline{y}, \overline{z})$ to the xy-plane is $gM\overline{z} = gM\left(\frac{M_{xy}}{M}\right) = gM_{xy} = \iint_S gxyz\, d\sigma = \frac{\sqrt{3}g}{20}$.

13. (a) Partition the sphere $x^2 + y^2 + (z-2)^2 = 1$ into small pieces. Let $\Delta_i\sigma$ be the surface area of the i^{th} piece and let (x_i, y_i, z_i) be a point on the i^{th} piece. The force due to pressure on the i^{th} piece is approximately $w(4 - z_i)\Delta_i\sigma$. The total force on S is approximately $\sum_i w(4 - z_i)\Delta_i\sigma$. This gives the actual force to be $\iint_S w(4-z)\, d\sigma$.

(b) The upward buoyant force is a result of the **k**-component of the force on the ball due to liquid pressure. The force on the ball at (x, y, z) is $w(4-z)(-\mathbf{n}) = w(z-4)\mathbf{n}$, where **n** is the outer unit normal at (x, y, z). Hence the **k**-component of this force is $w(z-4)\mathbf{n}\cdot\mathbf{k} = w(z-4)\mathbf{k}\cdot\mathbf{n}$. The (magnitude of the) buoyant force on the ball is obtained by adding up all these **k**-components to obtain $\iint_S w(z-4)\mathbf{k}\cdot\mathbf{n}\, d\sigma$.

(c) The Divergence Theorem says $\iint_S w(z-4)\mathbf{k}\cdot\mathbf{n}\, d\sigma = \iiint_D \operatorname{div}(w(z-4)\mathbf{k})\, dV = \iiint_D w\, dV$, where D is $x^2 + y^2 + (z-2)^2 \le 1 \Rightarrow \iint_S w(z-4)\mathbf{k}\cdot\mathbf{n}\, d\sigma = w\iiint_D 1\, dV = \frac{4}{3}\pi w$, the weight of the fluid if it were to occupy the region D.

14. The surface S is $z = \sqrt{x^2+y^2}$ from $z = 1$ to $z = 2$. Partition S into small pieces and let $\Delta_i\sigma$ be the area of the i^{th} piece. Let (x_i, y_i, z_i) be a point on the i^{th} piece. Then the magnitude of the force on the i^{th} piece due to liquid pressure is approximately $F_i = w(2 - z_i)\Delta_i\sigma \Rightarrow$ the total force on S is approximately $\sum_i F_i = \Sigma\, w(2-z_i)\Delta_i\sigma \Rightarrow$ the actual force is $\iint_S w(2-z)\, d\sigma = \iint_{R_{xy}} w\left(2 - \sqrt{x^2+y^2}\right)\sqrt{1 + \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}}\, dA$ $= \iint_{R_{xy}} \sqrt{2}\, w\left(2 - \sqrt{x^2+y^2}\right) dA = \int_0^{2\pi}\int_1^2 \sqrt{2}w(2-r)\, r\, dr\, d\theta = \int_0^{2\pi} \sqrt{2}w\left[r^2 - \frac{1}{3}r^3\right]_1^2 d\theta = \int_0^{2\pi} \frac{2\sqrt{2}w}{3}\, d\theta = \frac{4\sqrt{2}\pi w}{3}$

15. Assume that S is a surface to which Stokes's Theorem applies. Then $\oint_C \mathbf{E}\cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{E})\cdot\mathbf{n}\, d\sigma$ $= \iint_S \left(-\frac{\partial \mathbf{B}}{\partial t}\right)\cdot\mathbf{n}\, d\sigma = -\frac{\partial}{\partial t}\iint_S \mathbf{B}\cdot\mathbf{n}\, d\sigma$. Thus the voltage around a loop equals the negative of the rate of change of magnetic flux through the loop.

16. According to Gauss's Law, $\iint_S \mathbf{F}\cdot\mathbf{n}\, d\sigma = 4\pi GmM$ for any surface enclosing the origin. But if $\mathbf{F} = \nabla \times \mathbf{H}$ then the integral over such a closed surface would have to be 0 by the Divergence Theorem since $\operatorname{div}\mathbf{F} = 0$.

17. $\oint_C f\nabla g\cdot d\mathbf{r} = \iint_S \nabla \times (f\nabla g)\cdot\mathbf{n}\, d\sigma$ (Stokes's Theorem)

$= \iint_S (f\nabla \times \nabla g + \nabla f \times \nabla g)\cdot\mathbf{n}\, d\sigma$ (Section 16.8, Exercise 19b)

$= \iint_S [(f)(\mathbf{0}) + \nabla f \times \nabla g]\cdot\mathbf{n}\, d\sigma$ (Section 16.7, Equation 8)

$= \iint_S (\nabla f \times \nabla g)\cdot\mathbf{n}\, d\sigma$

18. $\nabla \times \mathbf{F}_1 = \nabla \times \mathbf{F}_2 \Rightarrow \nabla \times (\mathbf{F}_2 - \mathbf{F}_1) = \mathbf{0} \Rightarrow \mathbf{F}_2 - \mathbf{F}_1$ is conservative $\Rightarrow \mathbf{F}_2 - \mathbf{F}_1 = \nabla f$; also, $\nabla\cdot\mathbf{F}_1 = \nabla\cdot\mathbf{F}_2$ $\Rightarrow \nabla\cdot(\mathbf{F}_2 - \mathbf{F}_1) = 0 \Rightarrow \nabla^2 f = 0$ (so f is harmonic). Finally, on the surface S, $\nabla f\cdot\mathbf{n} = (\mathbf{F}_2 - \mathbf{F}_1)\cdot\mathbf{n}$ $= \mathbf{F}_2\cdot\mathbf{n} - \mathbf{F}_1\cdot\mathbf{n} = 0$. Now, $\nabla\cdot(f\nabla f) = \nabla f\cdot\nabla f + f\nabla^2 f$ so the Divergence Theorem gives

$\iiint_D |\nabla f|^2 \, dV + \iiint_D f \nabla^2 f \, dV = \iiint_D \nabla \cdot (f \nabla f) \, dV = \iint_S f \nabla f \cdot \mathbf{n} \, d\sigma = 0$, and since $\nabla^2 f = 0$ we have $\iiint_D |\nabla f|^2 \, dV + 0 = 0 \Rightarrow \iiint_D |\mathbf{F}_2 - \mathbf{F}_1|^2 \, dV = 0 \Rightarrow \mathbf{F}_2 - \mathbf{F}_1 = \mathbf{0} \Rightarrow \mathbf{F}_2 = \mathbf{F}_1$, as claimed.

19. False; let $\mathbf{F} = y\mathbf{i} + x\mathbf{j} \neq \mathbf{0} \Rightarrow \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(x) = 0$ and $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & 0 \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$

20. $|\mathbf{r}_u \times \mathbf{r}_v|^2 = |\mathbf{r}_u|^2 |\mathbf{r}_v|^2 \sin^2\theta = |\mathbf{r}_u|^2 |\mathbf{r}_v|^2 (1 - \cos^2\theta) = |\mathbf{r}_u|^2 |\mathbf{r}_v|^2 - |\mathbf{r}_u|^2 |\mathbf{r}_v|^2 \cos^2\theta = |\mathbf{r}_u|^2 |\mathbf{r}_v|^2 - (\mathbf{r}_u \cdot \mathbf{r}_v)^2$
$\Rightarrow |\mathbf{r}_u \times \mathbf{r}_v|^2 = EG - F^2 \Rightarrow d\sigma = |\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv = \sqrt{EG - F^2} \, du \, dv$

21. $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \Rightarrow \nabla \cdot \mathbf{r} = 1 + 1 + 1 = 3 \Rightarrow \iiint_D \nabla \cdot \mathbf{r} \, dV = 3 \iiint_D dV = 3V \Rightarrow V = \frac{1}{3} \iiint_D \nabla \cdot \mathbf{r} \, dV$
$= \frac{1}{3} \iint_S \mathbf{r} \cdot \mathbf{n} \, d\sigma$, by the Divergence Theorem